**Exercise P5:** This question requires nothing but calculus as a prerequisite. It shows a major source of linear systems from applications.

a. Consider these three equation, chosen for visualizability:

$$x^2 + y^2 + z^2 = 4$$

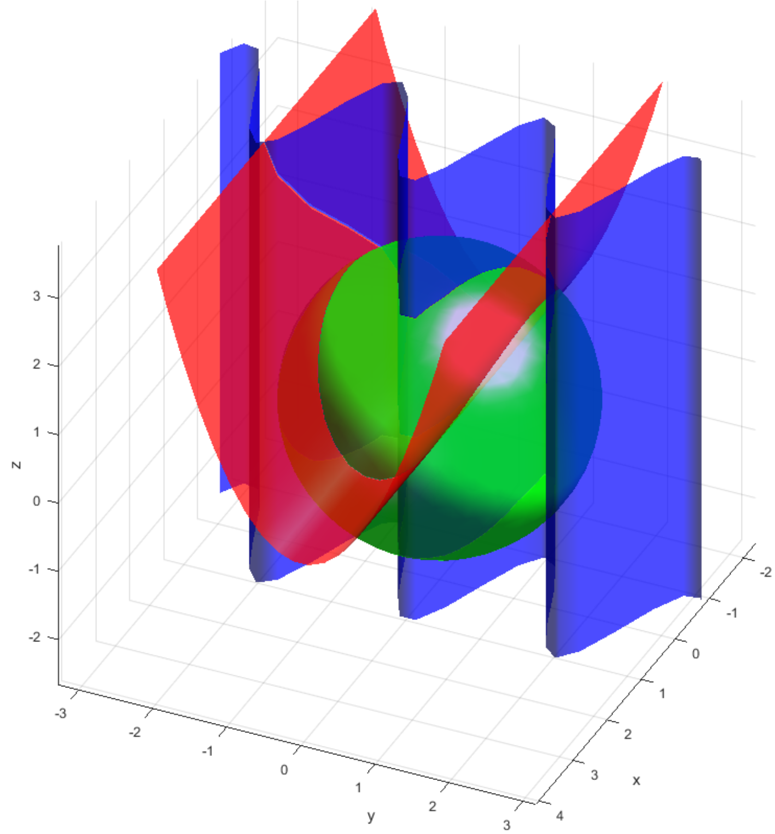
$$x = cos(\pi y)$$

$$z = y^2$$

Provide a sketch of each equation individually as a surface in $\mathbb{R}^3$. Consider where all three surfaces intersect, describe informally why there are two solutions. Explain why both solutions are inside the closed box $-1 \leq x \leq 1$, $-2 \leq y \leq 2$, and $0 \leq z \leq 2$.

**Solution:**
First let's consider a graph of each of the surfaces described above.

Figure 1: The three proposed surfaces.




Describing the solutions we can see that the ellipsoid and the parabolic cylinder intersect at a curve that looks like a taco shell (Don't know how else to describe the curve). When this curve is projected to the $x-y$ plane it only intersects the function $x = cos(\pi y)$ in two places. The solutions lie within $-1 \leq x \leq 1$ because we are bounded by $x = cos(\pi y)$, $0 \leq z$ because we are bounded by $z = y^2$, and $-2 \leq y \leq 2$ and $z \leq 2$ because we are bounded by $x^2 + y^2 + z^2 = 4$.

b. Newton's method for a system of nonlinear equations is an iterative, approximate, and sometimes very fast, method for solving systems like the one above. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Suppose there are three scalar functions $f_i(x)$ forming a column vector function, and consider the system,

$$f(x) = 0,$$

Also let,

$$J_{ij} = \frac{\delta f_i}{\delta x_j}$$

Be the Jacobian matrix: $J \in \mathbb{R}^{3x3}$. The Jacobian generally depends on location and it generalizes the ordinary scalar derivative.
Newton's method itself is,

$$J(x_n)s = -f(x_n) \quad (1)$$

$$x_{n+1} = x_n + s \quad (2)$$

Where $s = (s_1, s_2, s_3)$ is the step and $x_0$ is the initial iterate. Using $x_0 = (-1, 1, 1)$ write out equation (1) for the $n = 0$ for the surfaces in part $a$, as a concrete linear system of three equation for the three unknown components of step $s = (s_1, s_2, s_3)$.

**Solution:**
First let's form the vector function $f(x)$ with the surfaces in part a,

$$f(x) = \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 - 4 \\ cos(\pi x_2) - x_1 \\ x_2^2 - x_3 \end{bmatrix}$$

Now we can consider the associated Jacobian for this system,

$$J(x) = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ -1 & -\pi sin(\pi x_2) & 0 \\ 0 & 2x_2 & -1 \end{bmatrix}$$

Solving each for $x_0 = (-1, 1, 1)$,

$$f(x_0) = \begin{bmatrix} -1^2 + 1 + 1 - 4 \\ cos(\pi) - (-1) \\ 1^2 - 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

$$J(x_0) = \begin{bmatrix} 2(-1) & 2(1) & 2(1) \\ -1 & -\pi sin(\pi) & 0 \\ 0 & 2(1) & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ -1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

Putting everything together to form the system to solve for $s$,

$$\begin{bmatrix} -2 & 2 & 2 \\ -1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = - \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

c. Implement Newton's method in Matlab to solve the part ($a$) nonlinear system. Show your script and generate at least five iterations. Use $x_0 = (-1, 1, 1)$ as an initial iterate to find one solution, and also find the other solution using a different initial iterate. Note that $format\ long$ is appropriate here.

**Solution:**
**Code:**

```
function [Hist, xfinal] = NewtonsMethodP5(f1,f2,f3,xnot,n)
%This function takes the three surfaces from P5a f1,f2,and f3
%an initial iterate xnot, and the number of Newton Methed
%iterations n. Symbolic Math Toolbox is required

%Initializing symbolic variables and history
syms x y z
Hist = [xnot];

%Initializing vector functions
NonLinearSystem = [f1;f2;f3];
Jacobian = jacobian(NonLinearSystem,[x y z]);

    for i = 1:n
        %Computing jacobian for current iterate
        J = double(subs(Jacobian,[x y z],xnot));
        %Computing vector function value for current iterate
        f = -1*double(subs(NonLinearSystem,[x y z],xnot));
        %Solving for s
        s = J\f;

        %Newton Step
        xx = xnot + s';
        Hist = [Hist; xx];
        %Computed vector becomes next iterate
        xnot = xx;
    end
%Setting Final
xfinal = xnot;
end
```

**Console:**

```
>> f1
x^2 + y^2 + z^2 - 4

>> f2
cos(pi*y) - x

>> f3
y^2 - z

>> xnot
   -1     1     1

>> [Hist xfinal]=NewtonsMethodP5(f1,f2,f3,xnot,6)

Hist =

  -1.000000000000000   1.000000000000000   1.000000000000000
  -1.000000000000000   1.166666666666667   1.333333333333333
  -0.850071809562076   1.176823040188952   1.384809315996443
  -0.856411365815418   1.172732213485454   1.375284109683375
  -0.856360744297454   1.172720052382338   1.375272321111742
  -0.856360744261663   1.172720052019146   1.375272320407789
  -0.856360744261663   1.172720052019146   1.375272320407789


xfinal =

  -0.856360744261663   1.172720052019146   1.375272320407789


>> [Hist xfinal]=NewtonsMethodP5(f1,f2,f3,[-1,-1,1], 6)

Hist =

  -1.000000000000000  -1.000000000000000   1.000000000000000
  -1.000000000000000  -1.166666666666667   1.333333333333333
  -0.850071809562076  -1.176823040188952   1.384809315996443
  -0.856411365815418  -1.172732213485454   1.375284109683375
  -0.856360744297454  -1.172720052382338   1.375272321111742
  -0.856360744261663  -1.172720052019146   1.375272320407789
  -0.856360744261663  -1.172720052019146   1.375272320407789
```

```
xfinal =

   -0.856360744261663   -1.172720052019146    1.375272320407789
```

d. In calculus you likely learned Newton's method as a memorized formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$. rewrite equations (1) and (2) for $\mathbb{R}^1$ to derive this formula.

**Solution:**
We can see that equation (1) for $\mathbb{R}^1$ is,

$$f'(x_n)s = -f(x_n).$$

We also get that equation (2) for $\mathbb{R}^1$ is,

$$x_{n+1} = x_n + s$$

Solving the first equation for $s$ and substituting into the second equation we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**Exercise P6:** Its likely that you have learned a recursive method for computing determinants called "expansion by minors".

a. Compute the following determinant by hand to demonstrate that you can apply expansion in minors:

$$\left\| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right\|$$

**Solution:**
From what I remember we called this method co-factor expansion,

$$\left\| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} \right\| - 2 \left\| \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \right\| + 3 \left\| \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \right\|$$

$$\left\| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right\| = 5 * 9 - 8 * 6 - 2(4 * 9 - 7 * 6) + 3(4 * 8 - 7 * 5) = 0$$