1. Show that every 2-connected graph contains a cycle.

Proof. Let G be a 2-connected graph and consider vertex x. Note that since G is 2-connected x must have at least two neighbor y, z, otherwise removal of the single neighbor would disconnect x from he graph. Note $G-x$ is connected, so let yPz be the path in $G-x$. Clearly we can form a cycle with yPz and x. □

2. Determine $\kappa(G)$ and $\lambda(G)$ for $G = P^m, C^n, K^n, K_{m,n}$ and the n-dimensional cube.

Proof. For P^m, a path on m vertices is clearly $\kappa(P^m) = \lambda(P^m) = 1$. Removal of any vertex or edge disconnects the graph. □

Proof. For C^n, a cycle on n vertices. Note that the removal of any edge or vertex from C^n results in a graph that is still connected. Removal of any two edges, will disconnect the graph, and removal of any two non-adjacent vertices will disconnect the graph. Therefore $\kappa(C^n) = \lambda(C^n) = 2$ □

Proof. For a K^n note that the removal of any vertex from a graph results in a K^{n-1} which is clearly still connected. Since a K^1 is defined as being disconnected we conclude that $\kappa(K^n) = n-1$. Since $\kappa(K^n) \leq \lambda(K^n) \leq \delta(K^n)$ and $\delta(K^n) = n-1$ it must follow that $\lambda(K^n) = n-1$. □

Proof. For a $K_{m,n}$ and suppose $m \leq n$. Note that for the smaller partition we can remove at most $m-1$ vertices and still be a connected $K_{1,n}$ graph (a star graph). Removing the last vertex disconnects the graph, therefore $\kappa(K_{m,n}) = m-1$. Note that $\delta(K_{m,n}) = m$ and removing m edges from an m partition vertex will disconnect the graph, so $\lambda(K_{m,n}) = m-1$. □

Proof. For the case where G_d is a d-dimensional cube first note that by definition it is d regular and therefore $\delta(G_d) = d$. We will proceed to show that $\kappa(G) = d$ by induction.

Let G_1 be a 1-dimensional cube. Clearly it can be made into a K^1 by removing a single vertex. Hence $\kappa(G) = 1$.

Let G_n be an n-dimensional cube and recall that by a similar construction from homework 1 we can construct G_n from two copies of G_{n-1}. Now suppose A is some minimal disconnecting set. Clearly $|A| \leq n$ as G_n is n-regular.

If A is contained inside a G_{n-1} subgraph we know that removing $n-1$ vertices disconnects the subgraph into two components, yet each of those components is adjacent to the other G_{n-1} subgraph in G_n. Therefore to form a disconnection with vertices in an G_{n-1} subgraph $|A| > n-1$.

Note that A must be contained in some G_{n-1} subgraph. Suppose A is not contained in a G_{n-1} subgraph, then it follows that A has some vertices in both G_{n-1} but not enough in either to form a disconnection because $n+1 < |A \cap G_{n-1}| < n$. Therefore a disconnection has to be formed by disconnecting the G_{n-1} subgraphs. To do so $|A| = 2^n \leq n$ a contradiction.

Therefore we conclude that $n-1 < |A| \leq n$ and that $\kappa(G_n) = n$. Since $\kappa(G_n) \leq \lambda(G_n) \leq \delta(G)$ it follows that $\lambda(G) = n$.

p.s. This is terrible and I hate this but I couldn't find a slicker way. $\square$

3. Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $k \in \mathbb{N}$, every graph of minimum degree at least $f(k)$ is k-connected?

Proof. I assert that such a function does not exists. Consider a function $f : \mathbb{N} \to \mathbb{N}$, and some $k \in \mathbb{N}$. We want to show that there exists a graph of minimum degree greater than or equal to $f(k)$ that is not k-connected. Note that a graph G defined by two disjoint copies $K^{f(k)+1}$ is such a graph. $\square$

4. Prove that for every non-complete, connected graph G, if $F \subseteq E(G)$ is a separating set of edges of minimum order (i.e. $|F| = \lambda(G)$), then $G - F$ has exactly two components.

Proof. Let G be a non-complete, connected graph and suppose $F \subseteq E(G)$ is a separating set of edges such that $|F| = \lambda(G)$. Let $e \in F$ such that x, y are incident to e. Note that $G - F + e$ must be connected because $|F| = \lambda(G)$. Removing e must disconnect the graph into exactly two components. Any less and it would still be connected and any more and $G - F + e$ wouldn't have been connected. $\square$

5. Prove Theorem 1.5.1

The following are equivalent for a graph T

(a) T is a tree.

(b) Any two vertices of T are linked by a unique path.

(c) T is minimally connected.

(d) T is maximally acyclic.

Proof. $(a \to b)$ Suppose T is a tree and let $x, y \in V(T)$. Note that since T is connected there exists a path P between them. For the sake of contradiction suppose there exists another such path P'. Note that the set of vertices $V(P) \cap V(P') \setminus V(P) \cup V(P')$ and it's neighbors in P form a cycle in T, a contradiction. $\square$

Proof. $(b \to c)$ Suppose any two vertices of T are linked by a unique path. Let $e \in E(T)$. Let e be incident to vertices x and y. We know that the only path between x and y goes through e and therefore $T - e$ is disconnected. Hence T is minimally connected. $\square$

Proof. $(b \to d)$ Suppose any two vertices of T are linked by a unique path. Let $x, y \in V(T)$ are non-adjacent and P the unique path with which they are connected. Consider an edge e incident to x and y. Note $P + e$ will form a cycle in T, hence T is maximally connected. □

Proof. $(c \to a)$ Suppose T is minimally connected. Suppose for the sake of contradiction that T has a cycle. Removing an edge from the cycle would still result in a connected graph, a contradiction. Hence T is a tree. □

Proof. $(d \to a)$ Suppose T is maximally acyclic. Suppose for the sake of contradiction that T is disconnected. Then there exists at least two acyclic components, and connecting them via an edge would not produce a cycle, a contradiction. Hence T is a tree. □

6. Let F and F' be forests on the same vertex set such that $||F|| < ||F'||$. Show that F' has an edge e such that $F + e$ is still a forest.

Proof. Suppose F and F' are forests on the same vertex set such that $||F|| < ||F'||$. Note that F cannot be a tree since it's edge set is not maximal and therefore F has more than 1 component. Let F have $h \geq 2$ components, each with n_i vertices. Counting the edges of F we get that,

$$\begin{aligned} ||F|| &= \sum_{i=1}^{h} n_i - h \\ &= n - h \end{aligned}$$

Now suppose we partition F' by the components of F. For the sake of contradiction suppose no edge exists across partitions. Then there are at most $n_i - 1$ edges in each partition, otherwise we would form a cycle and F' would not be a forest, however summing all the possible edges across each partition gives $n - h$ or $||F||$ edges. Since $||F|| < ||F'||$ there must exists an edge e, across partitions. Note this edge cannot exists in F and $F + e$ is still acyclic. □