1. Carothers 8.55 Give an example of a bounded continuous map $f : \mathbb{R} to \mathbb{R}$ that is not uniformly continuous. Can an unbounded continuous function $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous? Explain,

Proof. Consider the function $f(x) = \sin(e^x)$, this function is clearly bounded, and continuous as it is the composition of $\sin(x)$ and e^x, continuous functions. Consider the sequences $x_n = ln(n\pi)$ and $y_n = \ln(n\pi + \pi/2)$. Now consider that

$$|y_n - x_n| = |\ln(n\pi + \pi/2) - \ln(n\pi)| = \left|\ln\left(\frac{n\pi + \pi/2}{n\pi}\right)\right| = \left|\ln\left(1 + \frac{2}{n}\right)\right|$$

Note that $|y_n - x_n|$ can be made arbitrarily small, since $\left|\ln\left(1 + \frac{2}{n}\right)\right| \to 0$. Let $\epsilon_0 = 1/2$ and $\delta > 0$, pick $h < \delta$ and choose N such that $|y_N - x_N| < h$, so finally it follows that

$$|\sin(e^{x_N}) - \sin(e^{y_N})| = |\sin(n\pi) - \sin(n\pi + \pi/2)| = 1 > 1/2 = \epsilon_0$$

□

Proof. For an example of an unbounded uniformly continuous function consider the identity map. This function is clearly unbounded, and is Lipschitz continuous with constant $K = 1$. □

2. Carothers 8.57 A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy *Lipschitz condition of order* α, where $\alpha > 0$, if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K|x - y|^\alpha$, for all x, y. Prove that such a function is uniformly continuous,

Proof. Suppose function $f : \mathbb{R} \to \mathbb{R}$ is *Lipschitz condition of order* α. Let $\epsilon > 0$ and choose $\delta = (\epsilon/K^\alpha)^{1/\alpha}$ and note that if $0 < |x - y| < \delta$ it follows that,

$$|f(x) - f(y)| \leq K|x - y|^\alpha < \delta = \epsilon.$$

□

3. Carothers 8.58 Show that any function $f : \mathbb{R} \to \mathbb{R}$ having a bounded derivative is Lipschitz of order 1

Proof. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ with a bounded derivative, therefore for all $x \in \mathbb{R}$ there exists a K such that $|f'(x)| \leq K$. By implication f is differentiable, and therefore continuous. Let $x, y \in \mathbb{R}$ with $y < x$ and note $f : [y, x] \to \mathbb{R}$ is continuous and differentiable over (y, x). Thus by the Mean Value Theorem there exists a $c \in (y, x)$ such that,

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

By substitution it follows that,

$$|f'(c)| \leq K$$
$$|\frac{f(x)-f(y)}{x-y}| \leq K$$
$$\frac{|f(x)-f(y)|}{|x-y|} \leq K$$
$$|f(x)-f(y)| \leq K|x-y|^1$$

Hence $f(x)$ is Lipschitz of order 1. □

4. Carothers 8.66 If $f:(0,1)\to\mathbb{R}$ is uniformly continuous, show that $\lim_{x\to 0^+} f(x)$ exists. Conclude that f is bounded on $(0,1)$.

Proof. Suppose $f:(0,1)\to\mathbb{R}$ is uniformly continuous. Recall that by Theorem 8.16 (and in class) since f is uniformly continuous, $\overline{(0,1)} = [0,1]$ and $\mathbb{R}$ is complete, there exists a unique, uniformly continuous function $\overline{f}:[0,1]\to Y$ such that $\overline{f}|_{(0,1)} = f$. Note that $\lim_{x\to 0^+}\overline{f}(x) = \overline{f}(0)$ since $\overline{f}$ continuous over $[0,1]$ and since $\overline{f}|_{(0,1)} = f$ it follows that $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\overline{f}(x) = \overline{f}(0)$.

Now note that since $\overline{f}$ is continuous and it's domain is compact, $\overline{f}$ is bounded. Since $\overline{f}|_{(0,1)} = f$ it follows that f is bounded. □

5. Carothers 8.76 Fix $y\in\mathbb{R}^n$ and define a linear map $L:\mathbb{R}^n\to\mathbb{R}$ by $L(x) = \langle x, y\rangle$. Show that L is continuous and compute $||L|| = \sup_{x\neq 0}||L(x)||/\,||x||_2$.[Hint: Cauchy-Schwarz!]

Proof. Fix $y\in\mathbb{R}^n$ and define a linear map $L:\mathbb{R}^n\to\mathbb{R}$ by $L(x) = \langle x, y\rangle$. Suppose $x\in\mathbb{R}^n$ and note that by Cauchy-Schwarz it follows that,

$$|L(x)| = \left|\sum_{i=1}^{n} x_i y_i\right| \leq \sum_{i=1}^{n}|x_i y_i| \leq ||y||_2\,||x||_2\,.$$

Therefore $L(x)$ is bounded, and since it is a linear map we conclude that $L(x)$ is continuous. Now considering the operator norm we find that,

$$||L|| = \sup_{x\neq 0}\frac{|L(x)|}{||x||_2} \leq \sup_{x\neq 0}\frac{||y||_2\,||x||_2}{||x||_2} = ||y||_2\,.$$

So $||L|| \leq ||y||_2$,

Now consider $x\in\mathbb{R}^n$ such that $x = y$,

$$\frac{|L(y)|}{||y||_2} = \frac{|\langle y, y\rangle|}{||y||_2} = \frac{||y||_2\,||y||_2}{||y||_2} = ||y||_2\,.$$

So $||L|| = ||y||_2$.

□

6. Carothers 8.77 Fix $k \geq 1$ and define $f : \ell_\infty \to \mathbb{R}$ by $f(x) = x(k)$. Show that f is linear and has $\|f\| = 1$.

Proof. Fix $k \geq 1$ and define $f : \ell_\infty \to \mathbb{R}$ by $f(x) = x(k)$. Let $x_1, x_2 \in \ell_\infty$, and $a, b \in \mathbb{R}$. Clearly $ax_1 + bx_2$ is still a bounded sequence and therefore in ℓ_∞, applying f we get the following,

$$f(ax_1 + bx_2) = ax_1(k) + bx_2(k) = af(x_1) + bf(x_2).$$

Thus f is linear. Now consider $\|f\|$ and note that since $|x(k)| < \|x\|_\infty$, we get the following

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_\infty} = \sup_{x \neq 0} \frac{|x(k)|}{\|x\|_\infty} \leq \sup_{x \neq 0} \frac{\|x\|_\infty}{\|x\|_\infty} = 1$$

Now consider an $x \in \ell_\infty$ such that $|x(k)| = \|x\|_\infty$,

$$\frac{|f(x)|}{\|x\|_\infty} = \frac{|x(k)|}{\|x\|_\infty} = \frac{\|x\|_\infty}{\|x\|_\infty} = 1.$$

So therefore $\|f\| = 1$. □

7. Carothers 8.78 Define a linear map $f : \ell_2 \to \ell_1$ by $f(x) = (x(n)/n)_{n=1}^\infty$. Is f bounded? If so, what is $\|f\|$.

Proof. Let $x \in \ell_2$, and note that by Cauchy-Schwarz inequality,

$$\|f(x)\|_1 = \sum_{i=1}^{\infty} \left|\frac{x(i)}{i}\right| = \sum_{i=1}^{\infty} |x(i)| \left|\frac{1}{i}\right| \leq \|x\|_2 \left\|\frac{1}{n}\right\|_2 = \frac{\pi}{\sqrt{6}} \|x\|_2$$

Hence f is a bounded linear map and we can also conclude that $\|f\| \leq \frac{\pi}{\sqrt{6}}$. Finally note that note that for $x = (1/n)_{n=1}^\infty$ we get the following,

$$\frac{\|f(x)\|_1}{\|x\|_2} = \frac{\sum_{i=1}^\infty \left|\frac{1}{i^2}\right|}{\|x\|_2} = \frac{\sum_{i=1}^\infty \left(\frac{1}{i}\right)^2}{\|x\|_2} = \frac{\|x\|_2^2}{\|x\|_2} = \frac{\pi}{\sqrt{6}}.$$

Hence $\|f\| = \frac{\pi}{\sqrt{6}}$. □

8. Carothers 8.80 Show that the definite integral $I(f) = \int_a^b f(t)dt$ is continuous from $C[a, b]$ into $\mathbb{R}$. What is $\|I\|$

Proof. Let $f, g \in C[a, b]$ and $\alpha, \beta \in \mathbb{R}$. Note that the definite integral is a linear operator,

$$I(\alpha f + \beta g) = \int_a^b \alpha f(t) + \beta g(t)dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt = \alpha I(f) + \beta I(g).$$

Now we will show that the definite integral is a bounded linear functional on $C[a,b]$, and is therefore continuous. Let $f \in C[a,b]$ and recall that since f is a continuous function over a compact interval it must achieve it's minimum and maximum, and hence there exists a $t \in [a,b]$ such that $|f(t)| = ||f||_\infty$. So it follows,

$$|I(f)| = \left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt \leq \int_a^b ||f||_\infty \, dt = (b-a)\,||f||_\infty$$

Thus I is a bounded linear operator, and is therefore continuous. We also conclude that $||I|| \leq (b-a)$. Now let $f \in C[a,b]$ such that $f(x) = C$ and $C \neq 0$, we find that

$$\frac{|I(f)|}{||f||_\infty} = \frac{\left|\int_a^b f(t)dt\right|}{||f||_\infty} = \frac{|(b-a)C|}{|C|} = \frac{(b-a)|C|}{|C|} = (b-a).$$

Therefore $||f|| = (b-a)$.

□

9. Carothers 8.81 Prove that the indefinite integral, defined by $T(f)(x) = \int_a^x f(t)dt$, is continuous as a map from $C[a,b]$ into $C[a,b]$. Estimate $||T||$.

Proof. First we will show that T is a linear operator. Let $f, g \in C(a,b)$ and $\alpha, \beta \in \mathbb{R}$. Note that,

$$T(\alpha f + \beta g)(x) = \int_a^x (\alpha f(t) + \beta g(t))\, dt = \alpha \int_a^x f(t)dt + \beta \int_a^x g(t)dt = \alpha T(f)(x) + \beta T(g)(x).$$

We proceed to show that T is bounded, let $f \in C[a,b]$ and again note that since f is a continuous function over a compact interval there exists $t \in [a,b]$ such that $|f(t)| = ||f||_\infty$. Now consider the following,

$$\begin{aligned} ||T(f)(x)||_\infty = \max_{a\leq x\leq b}\left|\int_a^x f(t)dt\right| &\leq \max_{a\leq x\leq b}\int_a^x |f(t)|dt \\ &\leq \max_{a\leq x\leq b}\int_a^x ||f||_\infty \, dt = \max_{a\leq x\leq b}(x-a)\,||f||_\infty = (b-a)\,||f||_\infty \end{aligned}$$

Thus T is a bounded linear operator and is therefore continuous. We can also conclude that $||T|| \leq (b-a)$. Again consider $f \in C[a,b]$ such that $f(x) = C$ and $C \neq 0$, we find that,

$$\frac{||T(f)(x)||_\infty}{||f||_\infty} = \max_{a\leq x\leq b}\frac{\left|\int_a^x f(t)dt\right|}{||f||_\infty} = \max_{a\leq x\leq b}\frac{|(x-a)C|}{|C|} = \max_{a\leq x\leq b}(x-a) = (b-a).$$

Hence $||T|| = (b-a)$ □

10. Carothers 8.84 Prove that $B(V,W)$ is complete whenever W is complete.

Proof. Let V and W be normed vector spaces, and suppose W is complete. Consider $B(V, W)$ and recall that $B(V, W)$ is complete if $\sum_{n=1}^{\infty} T_n$ converges in $B(V, W)$ whenever $\sum_{n=1}^{\infty} \|T_n\| < \infty$. Consider the following candidate limit, $T(x) = \sum_{n=1}^{\infty} T_n(x)$. Recall that $T_n(x)$ is a sequence in W, a complete space, and therefore $\sum_{n=1}^{\infty} T_n(x)$ converges whenever $\sum_{n=1}^{\infty} \|T_n(x)\|_W$ converges. Recall that by the definition of the operator norm,

$$\|T_n\| \geq \frac{\|T_n(x)\|_W}{\|x\|_V}$$
$$\|T_n\| \, \|x\|_V \geq \|T_n(x)\|_W .$$

Therefore it follows that,

$$\sum_{n=1}^{\infty} \|T_n(x)\|_W \leq \sum_{n=1}^{\infty} \|T_n\| \, \|x\|_V = \|x\|_V \sum_{n=1}^{\infty} \|T_n\| < \infty. \tag{1}$$

Therefore since W is complete $\sum_{n=1}^{\infty} T_n(x)$ converges and thus T is well-defined. Now we must demonstrate that T is a bounded linear operator. Evidently T is linear, to show T is bounded let $C = \sum_{n=1}^{\infty} \|T_n\|$ and we get the following,

$$\|T(x)\|_W \leq \sum_{n=1}^{\infty} \|T_n(x)\|_W \leq \|x\|_V \sum_{n=1}^{\infty} \|T_n\| = C \, \|x\|_V$$

Therefore $T \in B(V, W)$. What is left to show is that $\sum_{n=1}^{\infty} T_n = T$. Consider the sequence of partial sums (S_n) defined by $\sum_{i=1}^{n} T_i$. Let $\epsilon > 0$ and choose N such that the residual of $\sum_{i=N+1}^{\infty} \|T_n\| < \epsilon$, then it follows that for all $n \geq N$ we get the following,

$$\|S_n - T\| = \left\| \sum_{i=1}^{n} T_i - \sum_{i=1}^{\infty} T_i \right\| = \left\| \sum_{i=n+1}^{\infty} T_i \right\| \leq \sum_{i=n+1}^{\infty} \|T_i\| < \epsilon.$$

Therefore $\sum_{n=1}^{\infty} T_n = T$ and thus $B(V, W)$ is complete.

□