I WANT USE MY ONE TIME, THIS HOMEWORK IS MOSTLY DONE BUT NOT FINISHED

1. Carothers 7.15 Prove or disprove: if M is complete and $f : (M, d) \to (N\rho)$ is continuous, then $f(M)$ is complete.

Proof. Consider $\mathbb{R}$ under the usual metric, and let f be the tan^{-1} function, so therefore $f(\mathbb{R}) = (-\pi/2, \pi/2)$ which is certainly not complete since $\pi/2 - 1/n$ is a Cauchy sequence which does not converge in $f(\mathbb{R})$ □

2. Carothers 7.18 (Note Done) Fill in the details of the proofs that ℓ_1 and ℓ_∞ are complete.

3. Carothers 7.19 Prove that c_0 is complete by showing that c_0 is closed in ℓ_∞.

Proof. Let $(x_n) \subseteq c_0$ and $x_n \to x$ with $x \in \ell_\infty$. For $\epsilon > 0$ there exists an N such that,

$$\|x_N - x\|_\infty = \sup_k\{|x_N(k) - x(k)|\} < \epsilon/2.$$

Note that since $x_N \in c_0$ we know that $x_N(k) \to 0$, so there exists a K such that if $k \geq K$ then,

$$|x_N(k) - 0| < \epsilon/2.$$

Let $\epsilon > 0$ and note that for $k \geq K$, by the triangle inequality it follows that,

$$\begin{aligned} |x(k) - 0| &\leq |x(k) - x_N(k)| + |x_N(k) - 0|, \\ &\leq \epsilon/2 + \epsilon/2, \\ &= \epsilon. \end{aligned}$$

□

4. Carothers 7.22 Let D be a dense subset of a metric space M, and suppose that every Cauchy sequence from D converges to some point of M. Prove that M is complete.

Proof. Suppose $(x_n) \subseteq M$ and that (x_n) is a Cauchy sequence. Construct the following sequence $(a_n) \subseteq D$ such that $a_n \in B_{1/n}(x_n)$. We will proceed to show that (a_n) is a Cauchy sequence.

Let $\epsilon > 0$ and choose N_1 such that if $n \geq N_1$, then $1/n < \epsilon/3$, and also choose N_2 such that if $n, m \geq N_2$ then $d(x_n, x_m) < \epsilon/3$, note that for $N = \max\{N_1, N_2\}$ applying the triangle inequality shows that,

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, x_n) + d(a_m, x_m) + d(x_n, x_m), \\ &< \epsilon. \end{aligned}$$

Since (a_n) is Cauchy, by our hypothesis it converges $a_n \to a$ in M. Now we will show that (x_n) converges to a, a point in M satisfying completeness. Let $\epsilon > 0$ and choose $N = \max\{N_1, N_2\}$ such that if $n \geq N_1$, then $d(a_n, a) < \epsilon/2$ and if $n \geq N_2$, then $1/n < \epsilon/2$, then we apply the triangle inequality to find

$$\begin{aligned} d(x_n, a) &\leq d(x_n, a_n) + d(a_n, a), \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

□

5. Carothers 7.32 Use Theorem 7.12 to prove that ℓ_1 is complete.

Theroem 7.12. *A normed vector space X is complete if and only if every absolutly summable series in X is summable. That is, X is complete if and only if $\sum_{n=1}^{\infty} x_n$ converges in X whenever $\sum_{n=1}^{\infty} \|x_n\| < \infty$.*

As a preliminary, note that parentheticals like $S(k)$ or $x_n(k)$ denote the k^{th} term in a sequence of real numbers and subscripts like x_n denote the n^{th} sequence in a sequence of sequences.

Proof. Let $(x_n) \subseteq \ell_1$ and suppose $\sum_{n=1}^{\infty} \|x_n\|_1 < \infty$. Now to show ℓ_1 is complete using Theorem 7.12 we must show that $\sum_{n=1}^{\infty} x_n$ converges in ℓ_1.

We will first construct a candidate limit, let S be the sequence of real numbers defined by

$$S(k) = \sum_{n=1}^{\infty} x_n(k).$$

Note each term in S is in fact a real number and bounded above by $S(k) < \sum_{n=1}^{\infty} \|x_n\|_1 < \infty$. Now we will demonstrate that $S \in \ell_1$. Note that by definition,

$$\sum_{k=1}^{\infty} |S(k)| = \sum_{k=1}^{\infty} |\sum_{n=1}^{\infty} x_n(k)| \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |x_n(k)|.$$

Note that by our hypothesis the following series converges absolutely,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |x_n(k)| = \sum_{n=1}^{\infty} \|x_n\| < \infty,$$

and therefore any rearrangement of the series will converge to the same limit. Therefore it follows that

$$\sum_{k=1}^{\infty} |S(k)| \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |x_n(k)| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |x_n(k)| < \infty.$$

Hence $S \in \ell_1$.

We will proceed to show that $\sum_{n=1}^{\infty} x_n = S$. We define the sequence of partial sums by the following,

$$S_m = \sum_{n=1}^{m} x_n.$$

Now we must show that $S_m \to S$ in ℓ_1. Let $\epsilon > 0$, then choose M such that $\sum_{M}^{\infty} \|x_n\|_1 < \epsilon$, note that by our hypothsis this sum is the residual of a convergent series and can be made as small as desired, so then it follows that for all $m \geq M$,

$$\begin{aligned}
\|S - S_m\| &= \sum_{k}^{\infty} |S(k) - S_m(k)| \\
&= \sum_{k}^{\infty} |\sum_{n=1}^{\infty} x_n(k) - \sum_{n=1}^{m} x_n(k)| \\
&= \sum_{k}^{\infty} |\sum_{n=m+1}^{\infty} x_n(k)| \\
&\leq \sum_{k}^{\infty} \sum_{n=m+1}^{\infty} |x_n(k)| \\
&= \sum_{n=m+1}^{\infty} \sum_{k}^{\infty} |x_n(k)| \\
&= \sum_{n=m+1}^{\infty} \|x_n\|_1 \\
&< \epsilon.
\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} x_n = S$. Now since $\sum_{n=1}^{\infty} x_n$ converges whenever $\sum_{n=1}^{\infty} \|x_n\| < \infty$ we apply Theorem 7.12 to conclude that ℓ_1 is complete.

□

6. Carothers 8.4 If A and B are compact sets in M, show that $A \cup B$ is compact.

Proof. Let $(x_n) \subseteq A \cup B$. In order to show that $A \cup B$ is compact we must demonstrate a subsequence of (x_n) which converges to a value in $A \cup B$. It is either the case that infinitely many terms of (x_n) reside in A, B, or both A and B so without loss of generality suppose infinitely many of the terms of (x_n) exists in A. Denote such terms in A via subsequence (x_{n_k}). Note that since A is compact there exists a further subsequence $(x_{n_{k_j}})$ which converges to a value in A and therefore also $A \cup B$. Thus we have found a subsequence of (x_n) which converges to a value in $A \cup B$, as desired.

□

7. Carothers 8.13 Given $c(n) \geq 0$ for all n, prove that the set $A = \{x \in \ell_2 : |x(n)| \leq c(n), n \geq 1\}$ is compact in ℓ_2 if and only if $\sum_{n=1}^{\infty} c(n)^2 < \infty$.

Proof. ($\leftarrow$) Let c be a sequence such that $c(n) \geq 0$ for all n and suppose that the set $A = \{x \in \ell_2 : |x(n)| \leq c(n), n \geq 1\}$ is compact in ℓ_2. We will proceed to show that $c \in \ell_2$ and therefore $\sum_{n=1}^{\infty} c_n^2 < \infty$ by exhibiting a sequence in A, a compact and therefore closed set, such that whcih converges to c in ℓ_2.

Consider the sequence (x_n) where the terms are defined by the following,

$$x_n = (c(1), c(2), c(3), \dots, c(n), 0, \dots).$$

For a fixed n consider the sequence x_n, and note that by construction $x_n(k) \to 0$ and by continuouity of the x^2 we know that $(x_n(k))^2 \to 0$ so then it follows that $\sum_{k=1}^{\infty}(x_n(k))^2 < \infty$ and $x_n \in \ell_2$.

By construction we know that $|x_n(k)| \leq c(k)$ for all $k \geq 1$. Hence $(x_n) \subseteq A$, a compact set. Now since A is compact there exists a convergent subsequence (x_{n_j}). Since this (x_{n_j}) converges in ℓ_2 it converges coordinatewise and must therefore converge to c as desired. □

Proof. Let c be a sequence such that $c(n) \geq 0$ for all n, and suppose that $\sum_{n=1}^{\infty} c(n)^2 < \infty$. Consider $A = \{x \in \ell_2 : |x(n)| \leq c(n), n \geq 1\}$, let $(x_n) \subseteq A$ such that $x_n \to x$. Since $(x_n) \subseteq \ell_2$, a metric space we also have coordinatewise convergence. By our definition of A is folows that for each k we have $|x_n(k)| < c(k)$, or equivalently $-c(k) < x_n(k) < c(k)$. Since $x_n(k) \to x(k)$ it follows that $-c(k) \leq x(k) \leq c(k)$, and therefore $|x(k)| \leq c(k)$ hence $x \in A$. Thus A is closed and since ℓ_2 is a metric space A is complete.

Now we wish to show that A is totally bounded. Let $\epsilon > 0$ and since $\sum_{n=1}^{\infty} c(n)^2 < \infty$ there exists an N such that,

$$\sum_{n=N+1}^{\infty} c(n)^2 < \epsilon/2.$$

Note that for all $x \in A$, since $|x(n)| \leq c(n)$ for all $x \geq 1$ it follows that,

$$\sum_{n=N+1}^{\infty} x(n)^2 \leq \sum_{n=N+1}^{\infty} c(n)^2 < \epsilon/2.$$

Now consider $\hat{A} \subset \mathbb{R}^n$ defined by,

$$\hat{A} = \{x(n)_{n=1}^{N} : x(n) \in A\}.$$

Now consider $C = \max c(n)_{1 \leq n \leq N}$ and note that $\hat{A}$ is bounded by the closed set $[-C, C]^n$ since again $|x(n)| \leq c(n)$ for all $x \geq 1$. Note that $\mathbb{R}^n$ is totally bounded so consider an $\epsilon/2$-net and intersect it with $\hat{A}$, call it $\hat{W}$. Now consider $W \subseteq \ell_2$ defined by

$$W = \{x \in \ell_2 : \{x(n)\}_{n=1}^{N} \in \hat{W}\text{and } \{x(n)\}_{n=N+1}^{\infty} = 0\}$$

I'm not certain how we choose the finite number of elements to do the argument, however it is clear to me that we can choose an N (to rule them all) such that,

$$\sum_{k=N}^{\infty} |c(k)| < \epsilon/2$$

and that it then follows that this N works to bound every element x_n in the net,

$$\sum_{k=N}^{\infty} |x_n(k)| \leq \sum_{k=N}^{\infty} |c(k)| < \epsilon/2.$$

Oh my goodness, ok now it's clear to me that the elements in the net for A come from the finite diminsional vector space.

So we bound EVERY element in A by some N so the residual of their squared sum is size less than $\epsilon/2$ then we reduce to a finite dimensional vector space $\mathbb{R}^N$, this new set in $\mathbb{R}^N$ is totally bounded so we produce an $\epsilon/2$ net. We bring that net back into ℓ_2. Note that it is in A again, and then we show that every element in A is within ϵ of our net. The first N terms will be within $\epsilon/2$ of each other, then the remaining residual sum will be within $\epsilon/2$.

□

8. Carothers 8.17 If M is compact show that M is also seperable.

Proof. Suppose M is a compact metric space. By definition of compactness, M is also a totally bounded metric space. We have shown in the previous homework that totally bounded metric spaces are separable. □

9. Carothers 8.29 Let M be a compact metric space and suppose that $f : M \to M$ satisfies $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. Show that f has a fixed point.

Proof. Let M be a compact metric space and suppose $f : M \to M$ satisfies $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. Consider the function $g : M \to \mathbb{R}_{\geq 0}$ defined by $g(x) = d(x, f(x))$. Note g is continuous since it is a composition of continuous function f, d and the identity map. Since M is compact and g continuous we have shown that g is bounded and attains its minimum value. Let $x_0 \in M$ such that $g(x_0)$ is the minimum value. Suppose for the sake of contradiction that $g(x_0) \neq 0$ then $d(x_0, f(x_0)) > 0$ and therefore $x_0 \neq f(x_0)$. Let $y = f(x_0)$ and note that by our hypothesis,

$$d(y, f(y)) = d(f(x_0), f(y)) < d(x_0, y) = d(x_0, f(x_0)).$$

However since x_0 is a minimizer its clear that $d(y, f(y)) = g(y) > g(x_0) = d(x_0, f(x_0))$. This is a contradiction, so we conclude that $g(x_0) = 0$ noting that $d(x_0, f(x_0)) = 0$ implies $f(x_0) = x_0$ and hence f contains a fixed point. □

10. Carothers 8.38 If M is compact, prove that every lower semicontinuous function on M is bounded below and attains a minimum value.

Proof. Let M be a compact space and let $f : M \to \mathbb{R}$ be lower semicontinuous. Consider the set $f(M) \subseteq \mathbb{R}$ which achieves an infimum, call it m. Since $f(M)$ is nonempty there exists an $f(x_n) \to m$, where $(f(x_n)) \subseteq f(M)$. Note that since $(f(x_n)) \subseteq f(M)$ there exists some corresponding $(x_n) \subseteq M$. Since M is compact it there exists a convergent subsequence $(x_{n_j}) \to x$. Since f is lower semicontinuous, a previous homework shows that $f(x) \leq \liminf_{n\to\infty} f(x_{n_j})$. Note that since $f(x_{n_j})$ is a subsequence of $f(x_n)$ it follows that,

$$f(x) \leq \liminf_{n\to\infty} f(x_{n_j}) = \liminf_{n\to\infty} f(x_n) = m$$

Since $m = \inf(M)$ it must follow that $f(x) = m$.

□

11. Carothers 8.40 Let M be compact and let $f : M \to M$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. Show that f is onto.

Proof. Let M be compact and let $f : M \to M$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. First note that clearly f is continuous, choose $\delta = \epsilon$ and the argument follows trivially. Since f is continuous $f(M)$ is compact and since $f(M) \subseteq M$, a metric space then $f(M)$ is closed and bounded.

Suppose for the sake of contradiction that f is not onto. Then there exists an $x_0 \in M$ such that $x_0 \in f(M)^c$, an open set. Therefore there exists $B_\epsilon(x_0)$ such that $B_\epsilon(x_0) \subseteq f(M)^c$. We define a new sequence $(x_n) \subseteq M$ by $x_n = f^n(x_0)$ where $f^n(x_0)$ denotes n compositions of f applied to x_0. Let $n < m$ and note that since f is an isometry we find that,

$$d(x_n, x_m) = d(f^n(x_0), f^m(x_0)) = d(x_0, f^{m-n}(x_0)) > \epsilon,$$

since $f^{m-n}(x_0) \in f(M)$ and $B_\epsilon(x_0) \subseteq f(M)^c$. Therefore we have shown that (x_n) is not Cauchy, however since $(x_n) \subseteq M$ which is a compact space, there exists a convergent subsequence (x_{n_j}) which must be Cauchy, a contradiction.

□