1. Let (a_n) and (b_n) be sequences of numbers such that $a_n \leq b_n$ for all n.

(a) Give a carful proof that,

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$$

Proof. Let (a_n) and (b_n) be sequences of numbers such that $a_n \leq b_n$ for all n. Now let $A_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$ and $B_n = \sup\{b_n, b_{n+1}, b_{n+2}, \ldots\}$. Note that since $a_n \leq b_n$ for all n it follows from the definition of the supremum that $A_n \leq B_n$ for all n. By the definition of lim sup we know that,

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} A_n, \qquad \limsup_{n\to\infty} b_n = \inf_{n\geq 1} B_n.$$

Since $A_n \leq B_n$ for all n it must follows that

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} A_n \leq \inf_{n\geq 1} B_n = \limsup_{n\to\infty} b_n.$$

□

(b) Show that it need not be true that

$$\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$

Solution:
Consider the following pair of sequences,

$$b_n = (-1)^n \qquad a_n = \begin{cases} -\frac{1}{n}, n \text{ is odd} \\ -1, n \text{ is even} \end{cases}$$

Note these sequences satisfy $a_n \leq b_n$ for all n with the property that $\limsup_{n\to\infty} a_n = 0 > -1 = \liminf_{n\to\infty} b_n$.

2. Carothers 8.53 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f(x) \to 0$ as $x \to \pm\infty$. Prove that f is uniformly continuous.

Proof. Let $\epsilon > 0$. Since $f(x) \to 0$ as $x \to \pm\infty$ there exists a $K \in \mathbb{R}$ such that for all $|x| \geq K$ we know that,

$$|f(x)| < \epsilon.$$

Now recall that $f|_{[-K,K]}$ is a continuous function on a compact interval, and therefore it must be uniformly continuous. Consider the appropriate $\delta > 0$ for our given ϵ. To extend uniform continuity to all of $\mathbb{R}$ without loss of generality consider $x \in [-K, K]$ and $|y| \geq K$, such that $|x - y| \leq \delta$

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(K)| + |f(K) - f(y)| \\ &\leq |f(x) - f(K)| + |f(K) - 0| + |0 - f(y)| < 3\epsilon. \end{aligned}$$

$\square$

3. Determine if the following function are continuous. [You are welcome to use the Cauchy-Schwarz inequality for integrals].

(a) $F : (C[0,1], L^\infty) \to \mathbb{R}$ defined by $F(f) = f(0)$.

Solution:
First we will demonstrate that F is a linear operator. Let $f, g \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$, and note that,

$$F(\alpha f + \beta g) = \alpha f(0) + \beta g(0) = \alpha F(f) + \beta F(g).$$

Now to demonstrate continuity we will show that this linear operator is bounded. Let $f \in C[0,1]$ and note that,

$$|F(f)| = |f(0)| \leq \max_{a \leq x \leq b} |f(x)| = \|f\|.$$

Thus f is a bounded linear operator and is therefore continuous.

(b) $W : (C[0,1], L^2) \to \mathbb{R}$ defined by $W(f) = \int_0^1 f$.

Solution:
Again we demonstrate that W is a linear functional. Let $f, g \in (C[0,1], L^2)$ and $\alpha, \beta \in \mathbb{R}$, and note that,

$$W(\alpha f + \beta g) = \int_0^1 \alpha f(x) + \beta g(x) dx = \alpha \int_0^1 f(x) dx + \beta \int_0^1 g(x) dx = \alpha W(f) + \beta W(g).$$

Let $g(x)$ be a constant function with a value of 1 over $C[0,1]$ and by the Cauchy-Schwarz for integral we get the following,

$$\begin{aligned}\left|\int_0^1 f(x)dx\right| &= \left(\left(\int_0^1 f(x)dx\right)^2\right)^{1/2} = \left(\left(\int_0^1 f(x)g(x)dx\right)^2\right)^{1/2} \\ &\leq \left(\int_0^1 f(x)^2 dx \int_0^1 g(x)^2 dx\right)^{1/2} = \left(\int_0^1 f(x)^2 dx\right)^{1/2} = \|f\|_2\end{aligned}$$

Thus W is a bounded linear map and is therefore continuous.

(c) $G : (C[0,1], L^1) \to (C[0,1], L^2)$ defined by $G(f) = f$

Solution:

(d) $H : (C[0,1], L^2) \to (C[0,1], L^1)$ defined by $H(f) = f$

Solution:

4. Carothers 10.20 $C^{(1)}[a,b]$ is the vector space of all functions $f : [a,b] \to \mathbb{R}$ having a continuous first derivative on $[a,b]$. Show that $C^{(1)}[a,b]$ is complete under the norm $\|f\|_{C^{(1)}} = \max_{a\le x\le b}|f(x)| + \max_{a\le x\le b}|f'(x)|$.

Proof. Let $(f_n) \subseteq (C^{(1)}[a,b], \|\cdot\|_{C^{(1)}})$ be a Cauchy sequence of functions. Note that $(f_n') \subset C[a,b]$, and further note that since f_n is Cauchy with respect to $\|\cdot\|_{C^{(1)}}$ it follows that for all $\epsilon > 0$ there exists an N such that if $n, m \geq N$ we know that,

$$\begin{aligned}
\|f_n - f_m\|_{C^{(1)}} &= \max_{a\le x\le b}|f_n(x) - f_m(x)| + \max_{a\le x\le b}|f_n'(x) - f_m'(x)|, \\
&= \|f_n(x) - f_m(x)\|_\infty + \left\|f_n'(x) - f_m'(x)\right\|_\infty, \\
&< \epsilon.
\end{aligned}$$

Therefore an N can be chosen to make, $\|f_n(x) - f_m(x)\|_\infty$ and $\left\|f_n'(x) - f_m'(x)\right\|_\infty$ arbitrarily small in $C[a,b]$ with respect to L^∞. Therefore (f_n) and (f_n') are Cauchy in $C[a,b]$. Since $C[a,b]$ is complete there exists an $f, g \in C[a,b]$ such that $f_n \to f$ and $f_n' \to g$ uniformly with respect to L^∞. It now follows by Theorem 10.7 that f is a differentiable function on $[a,b]$, and that $f' = g$. Now we must show that $f_n \to f$ uniformly under $\|\cdot\|$

What is left to show is that $f \in C^{(1)}[a,b]$ and $f_n \to f$ with respect to $C^{(1)}[a,b]$. Let $\epsilon > 0$ and note that since $f_n \to f$ and $f_n' \to f'$ uniformly with respect to L^∞ there exists an N such that if $n \geq N$ then for all x, $\|f_n(x) - f(x)\|_\infty < \epsilon$ and $\left\|f_n'(x) - f'(x)\right\|_\infty < \epsilon$ so it follows that,

$$\begin{aligned}
\|f_n - f\|_{C^{(1)}[a,b]} &= \|f_n(x) - f(x)\|_\infty + \left\|f_n'(x) - f'(x)\right\|_\infty \\
&< 2\epsilon.
\end{aligned}$$

$\square$

5. Carothers 10.33 Define $I(x) = 0$ for $x \leq 0$ and $I(x) = 1$ for $x > 0$. Given sequences (x_n) and (c_n) in $\mathbb{R}$, with $\sum_{n=1}^{\infty} |c_n| < \infty$, show that $f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$ defines a bounded function on $\mathbb{R}$ that is continuous except, possibly at the x_n.

Proof. Let $\sum_{n=1}^{\infty} |c_n| = M$ and note that for all $x \in \mathbb{R}$ it follows that,

$$|f(x)| = \left|\sum_{n=1}^{\infty} c_n I(x - x_n)\right| \leq \left|\sum_{n=1}^{\infty} c_n\right| \leq \sum_{n=1}^{\infty} |c_n| = M.$$

Note that we have shown f is a well-defined and bounded function. What is left to show is that f is continuous on $\mathbb{R} \setminus (x_n)$.

First let $x \in \mathbb{R} \setminus (x_n)$, $\epsilon > 0$ and note that since $\sum_{n=1}^{\infty} |c_n| < \infty$ there exists an N large enough such that $\sum_{n=N+1}^{\infty} |c_n| < \epsilon$, also note that each function $c_n I(x - x_n)$ is continuous at x, so there exists a δ_n such that when $0 < |x - y| < \delta_n$, it follows that,

$$|c_n I(x - x_n) - c_n I(y - x_n)| < \epsilon.$$

Considering $\delta = \min_{1 \leq n \leq N}\{\delta_n\}$ we find that when $0 < |x - y| < \delta$,

$$\begin{aligned}
|f(x) - f(y)| &= \left|\sum_{n=1}^{\infty} c_n I(x - x_n) - \sum_{n=1}^{\infty} c_n I(y - x_n)\right| \\
&= \left|\sum_{n=1}^{\infty} c_n I(x - x_n) - c_n I(y - x_n)\right| \\
&\leq \sum_{n=1}^{\infty} |c_n I(x - x_n) - c_n I(y - x_n)| \\
&= \sum_{n=1}^{N} |c_n I(x - x_n) - c_n I(y - x_n)| + \sum_{n=N+1}^{\infty} |c_n I(x - x_n) - c_n I(y - x_n)| < (N + 1)\epsilon.
\end{aligned}$$

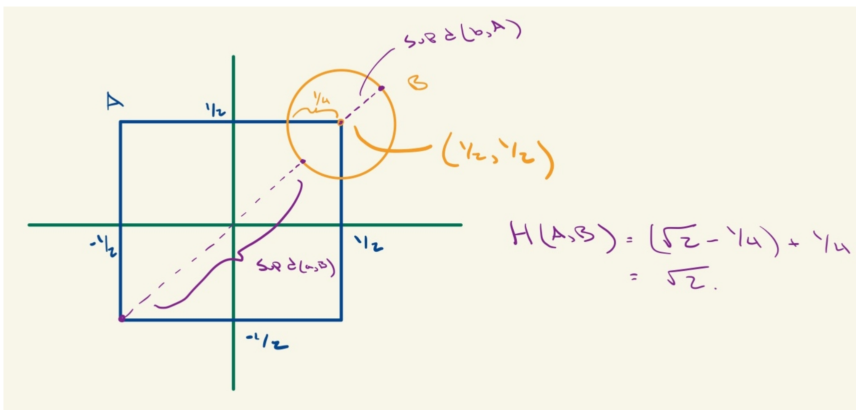
□

6. Let X be a compact metric space, and let χ be the set of non-empty closed subsets of X. If $a \in X$ and $B \subseteq X$, we define $d(a, B) = \inf_{b \in B} d(a, b)$. We define a metric, called the Hausdorff distance, on χ by,

$$H(A, B) = \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, A).$$

(a) Suppose $X \subseteq \mathbb{R}^2$ is the closed ball of radius 100, A is the closed square with side length 1 centered at the origin, and B is the closed ball of radius 1/4 centered at the point (1/2, 1/2). Draw a picture of the arrangement and compute $H(A, B)$. (No rigor here please!)

Solution:



(b) Show that H is a metric on χ.

(a) $0 \leq H(A, B) < \infty$ for all pairs $A, B \in \chi$

Proof. This property is inherited from X being a metric space. For a given pair of closed sets $A, B \in \chi$ we know that since $0 \leq d(b, a) < \infty$ for all $a, b \in X$,

$$H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b) + \sup_{b \in B} \inf_{a \in A} d(b, a) < \infty.$$

□

(b) $H(A, B) = 0$ if and only if $A = B$

Proof. Suppose $A = B$, then clearly the set $\{d(a, B) : a \in A\}$ is all zeros and similarly with $\{d(b, A) : b \in B\}$. So therefore,

$$H(A, B) = \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, A) = 0$$

□

Proof. Suppose $A \neq B$, then without loss of generality there exists some $x \in B$ such that $x \notin A$. Now clearly $0 < \inf_{a \in A} d(x, a) \leq \sup_{b \in B} d(b, A)$ and therefore $H(A, B) \neq 0$ □

(c) $H(B, A) = H(A, B)$ for all $A, B \in \chi$,

Proof. Clearly this follows since,

$$H(A, B) = \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, A) = \sup_{b \in B} d(b, A) + \sup_{a \in A} d(a, B) = H(B, A).$$

□

(d) $H(A, B) \leq H(A, C) + H(C, B)$ for all $A, B, C \in \chi$.

Proof. First note that since X is a metric space it follows that, if we let $c \in C$ then for all $a \in A$ and $b \in B$, □

7. Show that the previous space χ in the previous problem is totally bounded.

Proof. Let X be a compact metric space, and let χ be the set of non-empty closed subsets of X. Let $\epsilon > 0$ and note that since X is compact it is also totally bounded so there exists ϵ-net with finitely many points $\{x_i\}_{i=1}^N$ such that $X \subseteq \cup_{i=1}^N B_\epsilon(x_i)$.

It is not clear to me how to write/ describe the ϵ-net for χ from here. I know I want each point of the ϵ-net for χ to look like a union of ϵ-balls from my ϵ-net of X. I want this since any point in χ, is really a closed set in X which can be covered within ϵ-Hausdorff distance by a union of ϵ-balls from my ϵ-net of X. I guess what would be left to show is that the set of all such points for my proposed ϵ-net for χ is finite, which would have to come from the fact that ϵ-net for X is finite. Then finally we would have that χ is totally bounded. $\square$

8. Let $C^{2\pi}$ denote the continuous 2π-periodic functions on $\mathbb{R}$. Let $f \in C^{2\pi}$, and define

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

for $n = 0, 1, 2, \ldots$.

(a) Suppose $h \in C^{2\pi}$ and,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\cos(nx)dx = 0, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\sin(nx)dx = 0$$

for $n = 0, 1, 2, \ldots$ Show that $h = 0$. Hint: See Application 11.6

Proof. Since $h \in C^{2\pi}$, applying Weierstrass' Second Theorem, we know that there exists a sequence of trig polynomial (T_n) such that $T_n \to h$ uniformly on $\mathbb{R}$. Now note that $(h)(T_n) \to h^2$ uniformly on $\mathbb{R}$. To see this let $\epsilon > 0$ and note that since $T_n \to h$ uniformly there exists an N such that for all $n \geq N$ it follows that for all $x \in \mathbb{R}$,

$$|T_n(x) - h(x)| < \epsilon.$$

However since $h \in C^{2\pi}$, we know that h is bounded so $||h||_\infty = M < \infty$ and therefore it also follows that for all $x \in \mathbb{R}$,

$$|h(x)T(x) - (h(x))^2| = |h(x)(T(x) - h(x))| = |h(x)||T_n(x) - h(x)| < M\epsilon.$$

Now since $(h)(T_n) \to h^2$ uniformly on $\mathbb{R}$ it also follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)T_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} h^2(x)dx$$

Note that we can define $T_n(x)$ by some pair of sequences of real numbers a_n and b_n where,

$$T(x) = \sum_{k=0}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right).$$

By substitution we get the following,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\left(\sum_{k=0}^{n} (a_k \cos(kx) + b_k \sin(kx))\right)dx =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=0}^{n} \left(a_k h(x)\cos(kx) + b_k h(x)\sin(kx)\right)dx$$

and by linearity of the integral it follows that,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)T_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=1}^{n} \left(a_k h(x)\cos(kx) + b_k h(x)\sin(kx)\right)dx =$$

$$\sum_{k=1}^{n} a_k \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\cos(kx)dx\right) + b_k \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\sin(kx)dx\right) = 0$$

Thus it follows that,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h^2(x)dx = 0.$$

Since h is continuous, non-zero function values are forced to contribute to the size of integral and since the integral of h^2 on $[-\pi, \pi]$ is 0 it must be the case that $h(x) = 0$ on $[-\pi, \pi]$ and since $h \in C^{2\pi}$ we know that $h(x) = 0$ on $\mathbb{R}$.

□

(b) Suppose that $\sum_{n=1}^{\infty} |a_n| + |b_n|$ converges. Show that the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

converges uniformly to a function $g \in C^{2n}$.

Proof. First let,

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$

Note each g_n is well-defined, since for a fixed $x \in \mathbb{R}$,

$$\begin{aligned}
g_n(x) &\leq \left|\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)\right|, \\
&\leq \left|\frac{a_0}{2}\right| + \left|\sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)\right|, \\
&\leq \left|\frac{a_0}{2}\right| + \sum_{k=1}^{n} |a_k|\,|\cos(kx)| + |b_k|\,|\sin(kx)|, \\
&\leq \left|\frac{a_0}{2}\right| + \sum_{k=1}^{n} |a_k| + |b_k| < \infty.
\end{aligned}$$

Let g be the pointwise limit of g_n, which is also well defined since for a fixed $x \in \mathbb{R}$,

$$\begin{aligned}
g(x) &\leq \left|\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)\right|, \\
&\leq \left|\frac{a_0}{2}\right| + \left|\sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)\right|, \\
&\leq \left|\frac{a_0}{2}\right| + \sum_{n=1}^{\infty} |a_n|\,|\cos(nx)| + |b_n|\,|\sin(nx)|, \\
&\leq \left|\frac{a_0}{2}\right| + \sum_{n=1}^{\infty} |a_n| + |b_n| < \infty.
\end{aligned}$$

Note that since each $g_n \in C^{2\pi}$ we get that,

$$g(-\pi) = \lim_{n\to\infty} g_n(-\pi) = \lim_{n\to\infty} g_n(\pi) = g(\pi),$$

So we find that $g \in C^{2\pi}$. Now what is left to show is that $g_n \to g$ uniformly on $\mathbb{R}$. Let $\epsilon > 0$ and choose N such that $\sum_{k=N}^{\infty} |a_k| + |b_k| < \epsilon$ and note that for all $n \geq N$ and all $x \in \mathbb{R}$ it follows that,

$$\begin{aligned}
|g_n(x) - g(x)| &= \left|\left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)\right) - \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)\right)\right| \\
&= \left|\sum_{k=n+1}^{\infty} a_k \cos(kx) + b_k \sin(kx)\right| \\
&\leq \sum_{k=n+1}^{\infty} |a_k \cos(kx) + b_k \sin(kx)| \\
&\leq \sum_{k=n+1}^{\infty} |a_k|\,|\cos(kx)| + |b_k|\,|\sin(kx)| \\
&\leq \sum_{k=n+1}^{\infty} |a_k| + |b_k| \leq \epsilon
\end{aligned}$$

□

(c) Recall the function g just defined. Let

$$\hat{a}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)dx, \quad \hat{b}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)dx.$$

Show that $a_n = \hat{a}_n$, and $b_n = \hat{b}_n$ for all n.

Proof. We will begin by demonstrating $a_n = \hat{a}_n$ noting that by a similar argument we can also conclude that $b_n = \hat{b}_n$. Recall $g_n(x)$ and define the following,

$$\hat{a}_{n,N} = \frac{1}{\pi}\int_{-\pi}^{\pi} g_N(x)\cos(nx)dx.$$

First we will demonstrate that whenever $N \geq n$ it is the case that $\hat{a}_{n,N} = a_n$ and then we will demonstrate that $\lim_{N\to\infty} \hat{a}_{n,N} = \hat{a}_n$ which will then allow us to conclude that $\hat{a}_n = a_n$.

Now consider the following expansion of $\hat{a}_{n,N}$ by the linearity of the integral,

$$\hat{a}_{n,N} = \frac{1}{\pi}\left(\frac{a_0}{2}\int_{-\pi}^{\pi}\cos(nx)dx + \sum_{k=1}^{N}\left(a_k\int_{-\pi}^{\pi}\cos(kx)\cos(nx)dx\right) + \sum_{k=1}^{N}\left(b_k\int_{-\pi}^{\pi}\sin(kx)\cos(nx)dx\right)\right)$$

We will proceed with an understanding that integrals of the following forms where $m \in \mathbb{Z}$, evaluate to zero,

$$\int_{-\pi}^{\pi}\sin(mx)dx = \frac{1}{m}[-\cos(m\pi) + \cos(-m\pi)] = \frac{1}{m}[-\cos(m\pi) + \cos(m\pi)] = 0.$$

$$\int_{-\pi}^{\pi} \cos(mx)dx = \frac{1}{m}[\sin(m\pi) - \sin(-m\pi)] = \frac{1}{m}[0] = 0.$$

Now consider that when $k \neq n$ we get the following by the angle sum identities,

$$\begin{aligned}
a_k \int_{-\pi}^{\pi} \cos(kx)\cos(nx)dx &= \frac{a_k}{2}\int_{-\pi}^{\pi} \cos((k+n)x) + \cos((k-n)x)dx \\
&= \frac{a_k}{2}\left(\int_{-\pi}^{\pi} \cos((k+n)x)dx + \int_{-\pi}^{\pi} \cos((k-n)x)dx\right) \\
&= \frac{a_k}{2}(0) = 0,
\end{aligned}$$

$$\begin{aligned}
b_k \int_{-\pi}^{\pi} \sin(kx)\cos(nx)dx &= \frac{b_k}{2}\int_{-\pi}^{\pi} \sin((k+n)x) + \sin((k-n)x)dx \\
&= \frac{b_k}{2}\left(\int_{-\pi}^{\pi} \sin((k+n)x)dx + \int_{-\pi}^{\pi} \sin((k-n)x)dx\right) \\
&= \frac{b_k}{2}(0) = 0.
\end{aligned}$$

Now when $k = n$, we get the following by the half-angle identity,

$$\begin{aligned}
a_n \int_{-\pi}^{\pi} \cos^2(nx)dx &= \frac{a_k}{2}\int_{-\pi}^{\pi} 1 + \cos(2nx), \\
&= \frac{a_n}{2}\left(\int_{-\pi}^{\pi} 1 + \int_{-\pi}^{\pi} \cos(2nx)\right), \\
&= \frac{a_n}{2}(2\pi), \\
&= a_n\pi.
\end{aligned}$$

Applying the angle sum identities again we also find that,

$$b_n \int_{-\pi}^{\pi} \sin(nx)\cos(nx)dx = \frac{b_n}{2}\int_{-\pi}^{\pi} \sin(2nx)dx = \frac{b_n}{2}(0) = 0.$$

Putting our findings together we get that whenever $N \geq n$ it follows that,

$$\hat{a}_{n,N} = \frac{1}{\pi}(a_n\pi) = a_n.$$

Now we must show that $\lim_{N\to\infty} \hat{a}_{n,N} = \hat{a}_n$. Let $\epsilon > 0$ and recall that $g_n \to g$ uniformly and therefore there exists an M such that for all $n \geq M$ and $x \in \mathbb{R}$ we know that,

$$|g_n(x) - g(x)| = \left|\sum_{k=n+1}^{\infty} a_k\cos(kx) + b_k\sin(kx)\right| < \epsilon$$

so consider $N \geq M$ and we can conclude that,

$$\begin{aligned}
\left|\hat{a}_{n,N} - \hat{a}_n\right| &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} g_N(x)\cos(nx)dx - \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)dx\right| \\
&= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} (g_N(x) - g(x))\cos(nx)dx\right| \\
&\leq \left|\frac{1}{\pi}\right|\int_{-\pi}^{\pi} |g_N(x) - g(x)|\,|\cos(nx)|\,dx \\
&< \left|\frac{1}{\pi}\right|\int_{-\pi}^{\pi} \epsilon dx \\
&= \left|\frac{1}{\pi}\right| 2\pi\epsilon \\
&= 2\epsilon
\end{aligned}$$

Therefore $\hat{a}_n = a_n$ for all n. □

(d) Conclude that $f = g$.

Proof. First note that if $\hat{a}_n = a_n$ for all n it also follows that $\hat{a}_n - a_n = 0$ for all n, and analogously for $\hat{b}_n$ and b_n. Now applying this identity we can conclude that,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(g(x) - f(x))\cos(nx)dx = 0, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}(g(x) - f(x))\sin(nx)dx = 0$$

By part a) we can conclude that $g(x) - f(x) = 0$ for all $x \in \mathbb{R}$ and therefore $g(x) = f(x)$. □

Remark. *In this problem you've shown that each element of $C^{2\pi}$ determines a different collection of numbers (a_n) and (b_n). Moreover, if these sequences converge to zero so fast that the series in part 2 converges, then the Fourier series of f converges uniformly to f. This begs the question: given an arbitrary element of $C^{2\pi}$, is it true that the series in part 2 converges? This is food for thought, not food for the exam.*

9. Suppose $\sum |a_n|$ converges. By Carothers Exercise 10.26 we can define a function $f \in C(\mathbb{R})$ by $f(x) = \sum_{n=0}^{\infty} a_n \sin(nx)$.

(a) If there exists constants $M > 0$ and $\alpha > 2$ such $|a_n| \leq \frac{M}{n^\alpha}$, prove that f is differentiable.

Proof. Recall the statement of Exercise 10.26 which states that if $\sum_{n=1}^{\infty} |a_n| < \infty$, then $\sum_{n=1}^{\infty} a_n \sin(nx)$ and $\sum_{n=1}^{\infty} a_n \cos(nx)$ are uniformly convergent on $\mathbb{R}$. Define the following,

$$f_n(x) = \sum_{k=1}^{n} a_k \sin(kx)$$

and by Exercise 10.26 we know that $f_n \to f$ converges uniformly. We will proceed to show that f is differentiable by applying Theorem 10.7. First we must show that each f_n is continuous and differentiable on $\mathbb{R}$. Clearly this is the case as each f_n is a finite sum of continuous and differentiable functions on $\mathbb{R}$. Now we must show that each f_n' is continuous on $\mathbb{R}$. Note that

$$f_n'(x) = \sum_{k=1}^{n} a_k \cos(kx)(k),$$

which is a finite sum of continuous functions, so clearly each f_n' is also continuous on $\mathbb{R}$. Now we must show that $f_n' \to g$ uniformly on $\mathbb{R}$, or equivalently that $\sum_{k=1}^{\infty} a_k \cos(kx)(k)$ converges uniformly. Let $M_n = \frac{M}{n^{\alpha-1}}$ and it follows by our hypothesis that for all $x \in \mathbb{R}$,

$$|a_n \cos(nx)(n)| \leq |a_n||\cos(nx)|n \leq |a_n|n \leq \frac{M}{n^\alpha} n = M_n$$

Note that since $\alpha \geq 3$ we know that $\alpha - 1 \geq 2$ and therefore,

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{M}{n^{\alpha-1}} = M \sum_{n=1}^{\infty} \frac{1}{n^{\alpha-1}}$$

is bounded above by a convergent p-series and therefore by the Weirstrass M-test, $\sum_{k=1}^{\infty} a_k \cos(kx)(k)$ converges uniformly. Applying Theorem 10.7 we can conclude that f is differentiable on $\mathbb{R}$.

□

(b) Visit http://mathworld.wolfram.com/FourierSeriesTriangleWave.html. Then remark on what that has to do with the current problem.

Solution:
The Fourier series for a symmetric triangle wave with period $2L$ is given by,

$$f(x) = \frac{8}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(\frac{n\pi x}{L}\right)$$

Applying what we've shown in part a) there is a sense that f' is relatively well behaved since the fourier coefficients of f converge,

$$\frac{8}{\pi^2}\sum_{n=1,3,5...}^{\infty}\left|\frac{(-1)^{(n-1)/2}}{n^2}\right| \leq \frac{8}{\pi^2}\sum_{n=1,3,5...}^{\infty}\left|\frac{1}{n^2}\right| < \infty.$$

Now, in the examples illustrated in the article with the asymmetric triangle wave we see that derivative of the first $(1/m)th$ and last $(1/m)th$ distance on the $2L$ interval are increasing/decreasing towards $\pm\infty$. We see this represented in the equation for the fourier coefficients of these asymmetric triangle waves, We note that as we increase m the series of fourier coefficients converge slower.

Remark. *This problem illustrates a special case of the general principle that the faster the Fourier coefficients of a function converge to zero, the smoother that function is. 2*