This was also a very poor showing, very sorry.

1. Carothers 11.65 Let $K(x,t)$ be a continuous function on the square $[a,b] \times [a,b]$.

(a) Given $f \in C[a,b]$, show that $g(x) = \int_a^b f(t)K(x,t)dt$ defines a continuous function $g \in C[a,b]$.

Proof. Let $\epsilon > 0$. Note that since K and f are continuous functions it follows that $f(x)K(x,t)$ is also continuous on $[a,b] \times [a,b]$. Since $f(x)K(x,t)$ is a continuous function on a compact domain, it is also uniformly continuous, therefore there exists a δ, such that for all $c, d \in [a,b] \times [a,b]$ where $0 < \max |x_c - x_d|, |t_c - t_d| < \delta$ we get,

$$|f(x_c)f(x_c,t_c) - f(x_d)f(x_d,t_d)| < \epsilon.$$

Then for a fixed $t \in [a,b]$ it follows that if $0 < |x-y| < \delta$, clearly we get that $\max |x-y|, |t-t| = 0 = |x-y| < \delta$.

$$\begin{aligned} \left| \int_a^b f(t)K(x,t)dt - \int_a^b f(t)K(y,t)dt \right| &= \left| \int_a^b f(t)K(x,t) - f(t)K(y,t)dt \right| \\ &\leq \int_a^b |f(t)K(x,t) - f(t)K(y,t)|\, dt \\ &< \int_a^b \epsilon dt = (b-a)\epsilon. \end{aligned}$$

□

(b) Define $T : C[a,b] \to C[a,b]$ by $(Tf)(x) = \int_a^b f(t)K(x,t)dt$. Show that T maps bounded sets into equicontinuous sets. In particular, T is continuous.

Proof. Let $\mathcal{F} \subset C[a,b]$ be uniformly bounded. We wish to prove that $T(\mathcal{F})$ is equicontinuous, and we will proceed almost exactly as the previous problem. Let $\epsilon > 0$. Since $\mathcal{F}$ is uniformly bounded there exists an f_m such that $|f(t)| \leq f_m$ for all $f \in \mathcal{F}$ and $t \in [a,b]$. Now again, $f_m K(x,t)$ is continuous on a compact domain, it is also uniformly continuous. Therefore there exists a δ, such that for all $c, d \in [a,b] \times [a,b]$ where $0 < \max\{|x_c - x_d|, |t_c - t_d|\} < \delta$ we get,

$$|f_m K(x_c,t_c) - f_m K(x_d,t_d)| = |f_m \left(K(x_c,t_c) - K(x_d,t_d)\right)| < \epsilon.$$

Fix $t \in [a,b]$ and we find that when $0 < |x-y| < \delta$ clearly we get that $\max\{|x -$

$y|, |t - t| = 0\} = |x - y| < \delta$ so it follows that,

$$\begin{aligned}
\left|\int_a^b f(t)K(x,t)dt - \int_a^b f(t)K(y,t)dt\right| &= \left|\int_a^b f(t)K(x,t) - f(t)K(y,t)dt\right| \\
&= \left|\int_a^b f(t)(K(x,t) - K(y,t))dt\right| \\
&\leq \int_a^b |f(t)|\,|(K(x,t) - K(y,t))|\,dt \\
&\leq \int_a^b |f_m|\,|K(x,t) - K(y,t)|\,dt \\
&< (b-a)\epsilon.
\end{aligned}$$

Since we've found a single δ which satisfies the $\epsilon - \delta$ definition of uniform continuity for all $f \in \mathcal{F}$ simultaneously we can conclude that $\mathcal{F}$ is equicontinuous.

Now note that T is a linear operator, since for $f, g \in C[a,b]$ and $\alpha, \beta \in \mathbb{R}$ we find that,

$$\begin{aligned}
T(\alpha f + \beta g) &= \int_a^b (\alpha f(t) + \beta g(t))\, K(x,t)dt \\
&= \int_a^b \alpha f(t)K(x,t) + \beta g(t)K(x,t)dt \\
&= \alpha \int_a^b f(t)K(x,t)dt + \beta \int_a^b g(t)K(x,t)dt = \alpha T(f) + \beta T(g)
\end{aligned}$$

Now let $f \in C[a,b]$, and since $K(x,t)$ is a continuous function on the compact domain $[a,b] \times [a,b]$ so K achieves a maximum at some K_M. Finally we show boundedness with,

$$\begin{aligned}
\|T(f)\|_\infty = \left\|\int_a^b f(t)K(x,t)dt\right\|_\infty &\leq \left\|\int_a^b \|f\|_\infty\, K_M dt\right\|_\infty \\
&= \left\|\|f\|_\infty \int_a^b K_M dt\right\|_\infty \\
&= \|f\|_\infty \left\|\int_a^b K_M dt\right\|_\infty \\
&= \|f\|_\infty\, (b-a)K_m
\end{aligned}$$

□

(c) Show that if $\int_a^b |K(x,t)|\,dt \leq 1$ for all $x \in [a,b]$ then the Arzela-Ascoli Theorem implies that given any $f \in C[a,b]$, the sequence $(T^{(n)}f)_n$ has a subsequence that converges in $C[a,b]$.

Proof. Let $\mathcal{F}$ be the set $(T^{(n)}f)_n$. We will show that $T(\mathcal{F})$ is equicontinuous by proving that $(T^{(n)}f)_n$ is uniformly bounded, and applying the previous result. Since $f \in C[a,b]$ we know that there exists an f_m such that $|f(x)| < f_m$ for all $x \in [a,b]$. We will proceed by induction to show that $\mathcal{F}$ is uniformly bounded by f_m. Note that $T^{(1)}f$ can be written as the following,

$$|T^{(1)}f| = \left|\int_a^b f(x)K(x,t)dt\right| \leq \int_a^b |f(x)|\,|K(x,t)|\,dt \leq f_m \int_a^b |K(x,t)|\,dt \leq f_m.$$

Now suppose $|T^{(n)}f| \leq f_m$ and consider $T^{(n+1)}$ can be written as the following,

$$|T^{(n+1)}| = \left|\int_a^b (T^{(n)}f(x))K(x,t)dt\right| \leq \int_a^b \left|T^{(n)}f(x)\right|\,|K(x,t)|\,dt \leq f_m \int_a^b |K(x,t)|\,dt \leq f_m.$$

Thus we have show that $\mathcal{F}$ is uniformly bounded and by the previous result we know that $T(\mathcal{F})$ is an equicontinuous set of functions. It also follows that since $T(\mathcal{F}) \subseteq \mathcal{F}$ we know that $T(\mathcal{F})$ is uniformly bounded. Now consider $\overline{T(\mathcal{F})}$ a closed, bounded, equicontinuous set (the limit of a uniformly convergent sequence of equicontinuous functions must be uniformly continuous by the same δ, and its also probably bounded. Definitely need to spend some more time on this result.), we conclude be Arzela-Ascoli that $\overline{T(\mathcal{F})}$ is compact. Since $T(\mathcal{F})$ is a sequence contained in a compact set, $\overline{T(\mathcal{F})}$ it must have a convergent subsequence, and therefore since $T(\mathcal{F}) \subseteq \mathcal{F}$, we can conclude that $\mathcal{F}$ has a convergent subsequence. □

2. Suppose $f \in \mathcal{R}[a,b]$ and $\alpha \in \mathbb{R}$. Show that $\alpha f \in \mathcal{R}[a,b]$ and,

$$\int_a^b \alpha f = \alpha \int_a^b f.$$

Proof. First we will show that $\alpha f \in \mathcal{R}[a,b]$. Since $f \in \mathcal{R}[a,b]$ there exists step functions H_n with $H_n \geq f$ such that,

$$\int_a^b H_n \leq \int_a^b f + \frac{1}{n}.$$

Similarly we there exists step functions h_n such that $h_n \leq f$ with the analogous property that,

$$\int_a^b h_n \geq \int_a^b f - \frac{1}{n}.$$

Clearly $\int_a^b H_n \to \int_a^b f$ and $\int_a^b h_n \to \int_a^b f$. So for $\alpha > 0$ it follows that for all $x \in [a,b]$,

$$\alpha h_n(x) \leq \alpha f(x) \leq \alpha H_n(x).$$

Integrating over $[a, b]$ we find that

$$\int_a^b \alpha h_n(x) \leq \int_a^b \alpha f(x) \leq \int_a^b \alpha H_n(x).$$

Since H_n and h_n are step functions by linearity it follows that,

$$\alpha \int_a^b f - \frac{1}{n} \leq \alpha \int_a^b h_n(x) \leq \int_a^b \alpha f(x) \leq \alpha \int_a^b H_n(x) \leq \alpha \int_a^b f + \frac{1}{n}.$$

Taking the limit we find that,

$$\alpha \int_a^b f \leq \int_a^b \alpha f(x) \leq \alpha \int_a^b f.$$

So we conclude that, $\alpha f \in \mathcal{R}[a, b]$ and,

$$\int_a^b \alpha f = \alpha \int_a^b f.$$

□

3. Show that the uniform limit of Riemann integrable functions is Riemann integrable. Conclude that $\mathcal{R}[a, b]$ is a closed subspace of $B[a, b]$.

Proof. let $f_n \to f$ converge uniformly such that $f_n \in \mathcal{R}[a, b]$. Let $\epsilon > 0$. Since $f_n \to f$ choose N such that, $|f(x) - f_N(x)| < \epsilon$ for all $x \in [a, b]$. Now since $f_N \in \mathcal{R}[a, b]$ there exists $H, h \in \text{Step}[a, b]$ with $h \leq f_N \leq H$ such that,

$$\int_a^b H(x) - h(x)dx < \epsilon.$$

Now note that since $|f(x) - f_N(x)| < \epsilon$ it follows that,

$$\begin{aligned} -\epsilon <& f(x) - f_N(x) < \epsilon, \\ f_N(x) - \epsilon <& f(x) < f_N(x) + \epsilon, \\ h(x) - \epsilon <& f(x) < H(x) + \epsilon. \end{aligned}$$

We see that a step function plus a constant is simply another step function so $h(x) - \epsilon, H(x) + \epsilon \in \text{Step}[a, b]$ and we also find that by linearity of the integral of step functions,

$$\begin{aligned} \int_a^b (H(x) + \epsilon) - (h(x) - \epsilon)dx &= \int_a^b H(x) - h(x) + 2\epsilon dx, \\ &= \int_a^b H(x) - h(x)dx + \int_a^b 2\epsilon dx, \\ &< \epsilon + (b - a)2\epsilon = (1 + 2b - 2a)\epsilon. \end{aligned}$$

Therefore $f \in \mathcal{R}[a,b]$. Recall that by definition $\mathcal{R}[a,b] \subseteq B[a,b]$ and therefore by our result it follows that $\mathcal{R}[a,b]$ is a closed subspace of $B[a,b]$.

□

4. Determine if $\chi_\Delta \in \mathcal{R}[0,1]$, where Δ is the Cantor set.

Proof. Recall the that Δ can be defined by the following recurrence relation on sets.

$$\begin{aligned} I_0 &= [0,1] \\ I_{k+1} &= \frac{1}{3}I_k \cup \left(\frac{2}{3} + \frac{1}{3}I_k\right) \\ \Delta &:= \bigcap I_k \end{aligned}$$

Now let H_n be the characteristic function on the set I_k. We know that $H_n \in \text{Step}[a,b]$ since each I_k is a union of finitely many intervals. Also we have that $H_n \geq \chi(\Delta)$ since $\Delta \subset I_k$. Since H_n is the characteristic function on the set I_k its clear that,

$$\int_0^1 H_n = \left(\frac{2}{3}\right)^n.$$

Note that this integral converges to zero. Therefore we know that,

$$\overline{\int_0^1} \chi(\delta) \leq \int_0^1 H_n \to 0.$$

Now clearly we can consider an $h \in \text{Step}[0,1]$ where $h = 0$ and therefore $h \leq \chi\Delta$. So it follows, that

$$\underline{\int_0^1} \chi(\delta) \geq 0.$$

Thus $\chi_\Delta \in \mathcal{R}[0,1]$ and $\int_0^1 \chi(\delta) = 0$.

□