1. Prove that for every graph G, there exists an order of the vertex set of G such that a greedy algorithm using this ordering will use $\chi(G)$ colors.

   Proof. Suppose a graph G on n vertices and a coloring C which uses $\chi(G) = k$ colors. Therefore C partitions G into k partite sets, call them $V_1, V_2, \ldots, V_k$. Now consider a vertex ordering which traverses the entirety of a partite set one at a time. Note a greedy algorithm would color all vertices in V_1 with 1, since they are independent, and since it is the lowest color available. We also find that the steps of the algorithm which are coloring vertices from V_m where $1 < m \leq k$, the color m is always available since V_m is an independent set of vertices, thus the algorithm will terminate using k colors.

   $\square$

2. For every $n \geq 3$, construct a bipartite graph on $2n$ vertices and an ordering of the vertex set such that the greedy algorithm will use n colors (as opposed to the optimal 2 colors). Give a justification.

Proof. Consider bipartite graph $G = (A, B)$ with equally sized partite sets on $2n$ vertices such that a vertex a_i is adjacent to b_j for all $i \neq j$ with $i, j \in [n]$. Now consider the vertex ordering $a_1 b_1 a_2 b_2 \ldots a_n b_n$. Note a greedy coloring algorithm will have $c(a_1) = c(b_1) = 1$ since a_1 and b_1 are not adjacent. Now on the k^{th} step of the algorithm, where $1 < k \leq n$ we will find that $c(a_k) = c(b_k) = k$ since a_k is incident to all b_i where $i < k$ which have $c(b_i) = i$ and a similar argument follows or b_k. Therefore a greedy algorithm on G with our given vertex sequence will produce an n-coloring. $\square$

3. A k-chromatic graph G is called **critical** if $\chi(G-v) < k$ for every vertex $v \in G$.

   (a) Characterize critical 2-chromatic graphs.

   Proof. Firstly all 2-chromatic graphs are bipartite. For a graph 2-chromatic graph G, with $\chi(G-v) < 2$ it must be the case that $\chi(G-v) = 1$ and therefore $\chi(G-v)$ is class of graphs where each component is a K_1. For this to be true for all $v \in G$ it must be the case that $G = K_2$. □

   (b) Find an example of a critical 3-chromatic graph.

   Proof. A K_3 is clearly critically 3-chromatic. □

   (c) Prove that for $k \geq 3$ every critical k-chromatic graph is $(k-1)$-edge-connected.

   Proof. Let G be a critical k-chromatic graph with $k \geq 3$, such that G is not $(k-1)$-edge-connected. Then there exists an edge-cut set C such that $|C| \leq k-2$. Note that C separates G into two components A and B, and since $|A| < |G|$ and $|B| < |G|$, we know that there exists a $(k-1)$-coloring of both A and B call them $c_A()$ and $c_B()$. Now we will show that there exists a permutation of $c_A()$ such that for each $ab \in C$, $c_A(a) \neq c_B(b)$.

   This must be possible since the number of total color permutations of $c_A()$ are $(k-1)!$ and the number of permutations, in which at least one color class is mapped to itself is given by $|C|(k-2)!$, there are $|C|$ mappings and for each mapping $(k-2)!$ permutations which fix an element. Therefore since $|C|(k-2)! \leq (k-2)(k-2)! < (k-1)!$, there must exists $(k-1)$-colorings of A and B such that for every $ab \in C$, $c_A(a) \neq c_B(b)$ and thus we have constructed a $k-1$-coloring of G, a contradiction. □

   (d) Characterize the set of critical 3-chromatic graphs.

   Proof. Let G be a critical 3-chromatic graph. By the previous argument it follows that every critical 3-chromatic graph is 2-edge connected and therefore must be a cycle. However even cycles are clearly 2-chromatic so it follows that G is an odd cycle. □

4. The **clique number** of a graph, denoted by $\omega(G)$, is the largest r such that $K^r \subseteq G$. The **independence number** of a graph, denoted by $\alpha(G)$, is the largest r such that G contains an independent set of vertices of cardinality r.

   (a) Determine $\omega(G)$ and $\alpha(G)$ for the graphs below. Answers are sufficient. No justification required.

      i. P^m for $m \geq 1$

         Solution:
         $\omega(P^m) = 2$ and $\alpha(P^m) = \lceil \frac{m}{2} \rceil$

      ii. C^k

         Solution:
         When $k = 3$ then $\omega(C^k) = 3$ otherwise $\omega(C^k) = 2$. Also $\alpha(C^k) = \lfloor \frac{k}{2} \rfloor$.

      iii. $K_{m,n}$ where $m \leq n$

         Solution:
         $\omega(K_{m,n}) = 2$ and $\alpha(K_{m,n}) = n$.

      iv. K^n

         Solution:
         $\omega(K^n) = n$ and $\alpha(K^n) = 1$.

   (b) Prove that $\chi(G) \geq \max\{\omega(G), |G|/\alpha(G)\}$.

      Proof. Clearly $\chi(G) \geq \omega(G)$ as any $K^{\omega(G)}$ subgraph of G will require at least $\omega(G)$ colors to color. Since any k-coloring of G can be thought of as a partition of G by k independent sets, a possible lower bound on the number of independent sets is given by $|G|/\alpha(G)$ and therefore $\chi(G) \geq |G|/\alpha(G)$. □

5. Prove or Disprove: Every k-chromatic graph G has a k-coloring in which some color class has at least $\alpha(G)$ vertices.

Proof. Let G be a k-chromatic graph and suppose for the sake of contradiction that every k-coloring of G has no color class with more than $\alpha(G)-1$ vertices. Now let C be a k-coloring of G and let A be the independent set of vertices of size $\alpha(G)$. Note that since no color class has more than $\alpha(G)-1$ vertices, A must be partitioned among 2 or more color classes. Choose two color classes and call them C_1 and C_2. Now since C_1 and C_2 are independent we can recolor $C_1 \cup C_2$ a single color and produce a $k-1$ coloring of G, a contradiction.

□

6. Assume that H is a k-chromatic triangle-free graph and the G is obtained from H by Mycielski's Construction.

   (a) Prove that G is also triangle-free.

   Proof. Suppose for the sake of contradiction that G has a triangle. Let $V(H)'$ be the copy vertex ste from Mycielski's Contruction and note that since $V(H)'$ is an independent set, no triangles can be formed using $V(H)' \cup \{z\}$. Therefore the triangle in G uses $V(H) \cup V(H)'$ however again since $V(H)'$ is independent and H is triangle free, only one vertex $y_i \in V(H)$ is used in the triangle in G. Let $x_j, x_k \in V(H)$ which form a triangle in G with y_i and therefore $x_j x_k \in E(H)$. However by construction $x_j, x_k \in N(x_i)$ and therefore there exists a triangle in H via vertices x_i, x_j, x_k. $\square$

   (b) Prove that G is $(k+1)$-colorable.

   Proof. Let C be a k coloring of H. Let $\hat(c)$ be a coloring of G such that $\hat{C}(y_i) = C(x_i)$ and $\hat{C}(z) = k+1$. Clearly $G-z$ is a valid k-coloring since $V(H)'$ are independent and $N(x_i) = N(y_i)$ by construction if x_i has no neighbors of the same color, then neither will y_i. Since $G-z$ is colored using only k-colors, we can color z with the $(k+1)^{th}$ color. $\square$

7. Describe the topic of your project and what source(s) you have found.

For my project I would like to discuss a varying family of flow problems such as the transportation problem, assignment problem, and shortest path problem, both in the graph framework, and the linear programming framework. Discussion of the primal-dual relationship of the max-flow min-cut problem through the lens of linear programming could also be interesting. I could also talk about the Network Simplex method, which is an algorithm for solving flow problems adapted from a more general purpose linear programming algorithm that uses ideas about spanning trees to be substantially more performant.

Finally I have also thought briefly about very fun, application of these graph flow problems in image segmentation. The tool in question is called lazy snapping and I've linked a paper below about how it works. Big picture you just turn your image into a graph, have the user select a couple pixels for the source (subject) and the since (background) and then run a graph cut algorithm to identify the boundary.

Sources:

Linear and Nonlinear Optimization, Griva, Nash & Sofer
`http://home.cse.ust.hk/~cktang/sample_pub/lazy_snapping.pdf`