1. Let $m, n \in \mathbb{N}$, and assume that $m-1$ divides $n-1$. Show that every tree T, of order m satisfies $R(T, K_{1,n}) = m+n-1$.

Proof. Let $m, n \in \mathbb{N}$, and assume that $m-1$ divides $n-1$. So $(m-1)J = (n-1)$ for some $J \in \mathbb{N}$. Note that,

$$(J+1)(m-1) = \left(\frac{(n-1)}{(m-2)} + 1\right)(m-1) = (n-1) + (m-1) = n+m-2$$

Construct a complete graph K^{m+n-2} by $J+1$ copies of red K^{m-1} to avoid T. Let v be some vertex, and note that by construction it must have $J(m-1) = n-1$ incident blue edges since there are J copies of k^{m-1} which we must connect to v via blue edges, so by construction we avoid a $K_{1,n}$. Hence $R(T, K_{1,n}) > m+n-2$

Suppose T is a tree with order M, and consider some edge 2-coloring of K^{m+n-1}. Suppose there does not exists a blue $K_{1,n}$. Note that every vertex has $m+n-2$ incident edges. Let v be a vertex and note that v must have strictly less than $n-1$ incident blue edges, and therefore v must have strictly more than $m+n-2-n+2 = m-1$ incident red edges. Now clearly the red coloring of K^{m+n-1} forms a subgraph H such that $\delta(H) \geq m-1$. By Corollary 1.5.4 we find that $T \subset H$.

□

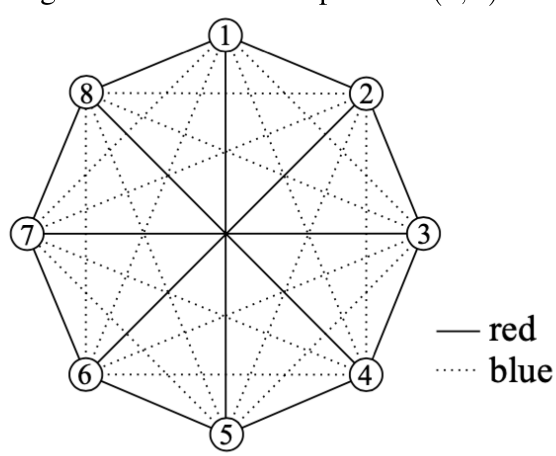
2. Prove that $R(3,4) = R(K^3, K^4) = 9$. (Proof from Mazur)

Proof. Consider a K^9 with some edge 2-coloring. First note that the subgraph induced by the red edges, of course has an odd number of vertices, and therefore there must be a vertex of even degree. Let $v \in K^9$ with an even number of incident red edges. Now consider two cases:

There are least 3 or more red edges incident to v, in which case v has at least 4 incident red edges. Consider the K^4 induced by 4 neighbors of v adjacent by red edges. Since $R(K^2, K^4) = 4$ then such a K^4 is either blue, in which case we are done, or it contains a red K^2. Such a red K^2 would form a red K^3 with v.

There are at most two red edges incident to v, in which case there are at least 6 incident blue edges. Consider the K^6 induced by 6 neighbors of v adjacent by blue edges. Since $R(K^3, K^3) = 6$, then such a K^6 contains either a red K^3 in which case we are done, or a blue K^3. Such a blue K^3 would form a blue K^4 with v.

Figure 1: Counter example for $R(3,4) > 8$



□

3. An oriented complete graph is called a tournament. A Hamilton path is a path through every vertex of the graph. Show that every tournament contains a directed Hamilton path.

Proof. We will proceed by induction on the number of vertices n, in our tournament T. The base case is trivial, $n = 1$ the statement hold vacuously and $n = 2$ the only path to choose from is a Hamilton path.

Suppose we have a tournament, T on $n+1$ vertices. Consider a vertex v and note by the induction hypothesis there exists a length n hamiltonian path in $T \setminus v$, we'll call it $v_0, v_1, v_2, \ldots v_n$. Now consider two cases:

every arc incident to v is of the form (v, v_i) or (v_i, v), in which case v can be added to the beginning or end of induced hamiltonian path to form a new hamiltonian path on T.

There exists both types of arcs (v, v_i) or (v_i, v) incident to vertex v. Now consider the arc of the form (v_j, v) with maximal j. If $j = n$ then we append v to the induced hamiltonian path and from a new hamiltonian path for T. Otherwise there must exists an arc of the form (v, v_{j+1}), in which case we form the following hamiltonian path on T,

$$v_0, \ldots v_j, v, v_{j+1}, \ldots v_n.$$

$\square$

4. Show that every uniquely 3-edge-colorable cubic graph is hamiltonian. By **uniquely** 3-edge-colorable, we mean that every 3-edge coloring induces the same edge partition.

Proof. Suppose G is a uniquely 3-edge-colorable cubic graph. By definition the 3-edge coloring induces an edge partition, into 3 parts E_1, E_2, and E_3. Now since G is 3-regular each E_i must forms a one-factor. Note that $E_1 \cup E_2$ is a graph of disjoint even cycles. Consider creating a path starting at v, note it must alternate E_1, E_2 edges, since the larger graph is cubic, such a path must terminate at v via a free edge forming a cycle. This cycle is either contains every vertex in the graph, or it doesn't, in the case that it doesn't the rest of the vertices must lie on a similar even cycle.

Now should $E_1 \cup E_2$ form a hamiltonian cycle, we are done. If they do not then there exists an even cycle $C \subset E_1 \cup E_2$ for which we can swap edges and form a new edge partition, which is distinct from our original partition, a contradiction.

□

5. (a) Prove Ore's Lemma stated below:

   Let G be a graph on n vertices. If u and v are distinct nonadjacent vertices in G such that $d(u)+d(v) \geq n$, then G is hamiltonian if and only if $G+uv$ is hamiltonian.

   Proof. The forward direction is trivial, if G is hamiltonian, then $G+e$ any edge will still be hamiltonian, via the same hamiltonian cycle on G. □

   Proof. Suppose G is graph on n vertices let u and v are distinct nonadjacent vertices in G such that $d(u)+d(v) \geq n$ and suppose $G+uv$ is hamiltonian. Clearly if a hamiltonian cycle on $G+uv$ fails to use edge uv then we have a hamiltonian cycle on G in which case we are done. Now let C be a hamiltonian cycle on $G+uv$ which contains the edge uv. Let uPv be the hamiltonian path which excludes uv, and therefore $P \subseteq U$. Since $d(u)+d(v) \geq n$ suppose without loss of generality that $d(u) \geq \frac{n}{2}$. If for some $p_i \in P$, there exists vp_{i-1} and up_i then we can form a hamiltonian cycle in G (We have seen this reasoning before). Counting the set of vertices which do not form such a hamiltonian cycle,

$$\underbrace{\underbrace{(n-2)}_{\text{\# of possible neighbors}} \quad - \quad \underbrace{d(u)}_{\text{number of vertices 'behind' a neighbor of } u}}_{\text{\# of bad vertices which don't produce a hamiltonian cycle when incident to } v} \quad .$$

   Now clearly it follows by our hypothesis that,

$$\begin{aligned} d(u)+d(v) &\geq (n-2) \\ d(v) &\geq (n-2)-d(u) \end{aligned}$$

   So u and v must have neighbors in P which produce a hamiltonian cycle in G. □

(b) Use Ore's Lemma to prove that if G is a graph on n vertices such that $d(u)+d(v) \geq n$ for all nonadjacent vertices, then G is hamiltonian.

   Proof. Suppose G is a graph on n vertices such that $d(u)+d(v) \geq n$ for all nonadjacent vertices and for the sake of contradiction suppose G is not hamiltonian. Suppose we add edges to form G' a maximally a-hamiltonian graph. Thus $G'+uv$ is hamiltonian and by the reverse direction of Ore's Lemma G' is also hamiltonian, a contradiction. □

(c) Show that the hypothesis in part (b) is weaker than the hypothesis in Dirac's Theorem.

Proof. Suppose a graph G with $n \geq 3$ vertices and $\delta(G) \geq \frac{n}{2}$. Well clearly for any pair of vertices u, v (not just the non adjacent ones) we have that,

$$d(u)+d(v) \geq \delta(G)+\delta(G) = n.$$

□

6. Show that a connected graph G is countable if all its vertices have countable degrees.

Proof. Let G be a connected graph, such that all its vertices have countable degrees.Let $v \in G$ and consider the following, recursive union of neighborhoods

$$\begin{aligned} N_1 &= \{v\}, \\ N_2 &= \{N(v)\}, \\ N_3 &= \{N(N_2) \setminus N_2\}, \\ N_i &= \{N(N_{i-1}) \setminus N_{i-1}\}, \end{aligned}$$

$$N^* = \bigcup_{i=1}^{\infty} N_i$$

Clearly $N^* \subseteq V$, and we assert $N^* \supseteq V$ and that that N^* is countable. Note that for all $u \in G$ there exists a finite, length n path between v and u, in which case we can be certain that $u \in N_n$ and hence $N^* \supseteq V$, so $|V| = |N^*|$. Note that, N^* is a countable union of countable sets and is therefore countable. Hence G is countable. $\square$

7. Let G be an infinite graph and $A,B \subseteq V(G)$. Show that if no finite set of vertices separates A from B in G, then G contains an infinite set of disjoint A-B paths.

Proof. By Theorem 8.4.2 we know that G contains a set of $\mathscr{P}$ disjoint $A-B$ paths and an $A-B$ separator on $\mathscr{P}$. (I have skimmed the discussion about this Theorem and recognize that it is not a trivial thing to cite, Menger's Theorem does not easily apply to infinite graphs).

Now by our hypothesis we know that the $A-B$ separator on G is not finite, and therefore $\mathscr{P}$ must contain an infinite number of paths.

$\square$