1. Carothers 1.4 Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Show that there is a sequence x_n of elements of A that converge to $\sup A$.

Proof. Suppose A is a nonempty subset of $\mathbb{R}$ that is bounded above. Let $s = \sup A$ and consider the sequence $x_n = s - \frac{1}{n}$.

Note that $x_n \to s$, let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $1/N < \epsilon$ and note that for all $n \geq N$,

$$\begin{aligned} |x_n - s| &= |s - \frac{1}{n} - s| \\ &= \frac{1}{n} \leq \frac{1}{N} < \epsilon. \end{aligned}$$

Since s is the supremum of A we know that for all $x_n < s$ there exists some $a_n \in A$ such that $x_n < a_n \leq s$. Hence the sequence $a_n \to S$. □

2. Carothers 1.11 Fix $a > 0$ and let $x_1 > \sqrt{a}$. For $n \geq 1$, define,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Show that (x_n) converges and that $\lim_{n\to\infty} x_n = \sqrt{a}$.

Proof. Suppose $a > 0$ and let $x_1 > \sqrt{a}$. Since $x_1 > \sqrt{a} > 0$ it follows that $x_n > 0$. We will proceed to show that x_n is bounded below by $\sqrt{a}$. Note that,

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \\ 2x_{n+1} &= x_n + \frac{a}{x_n}, \\ -x_n + 2x_{n+1} &= \frac{a}{x_n}, \\ -x_n^2 + 2x_{n+1}x_n &= a, \\ -x_n^2 + 2x_{n+1}x_n - a &= 0, \\ x_n^2 - 2x_{n+1}x_n + a &= 0. \end{aligned}$$

Arriving at an equation that is quadratic with respect to x_n it follows that it's discriminant, $(-2x_{n+1})^2 - 4a = 4x_{n+1}^2 - 4a \geq 0$ which implies that $x_{n+1} \geq \sqrt{a}$ and therefore $x_n \geq \sqrt{a}$.

Now we will demonstrate that $x_{n+1} \leq x_n$. From the previous result we find that,

$$\begin{aligned}
\sqrt{a} &\leq x_n, \\
a &\leq x_n^2, \\
\frac{a}{x_n} &\leq x_n, \\
\frac{a}{2x_n} &\leq \frac{x_n}{2}, \\
\frac{x_n}{2} + \frac{a}{2x_n} &\leq x_n, \\
\frac{1}{2}\left(x_n + \frac{a}{x_n}\right) &\leq x_n, \\
\frac{1}{2}\left(x_n + \frac{a}{x_n}\right) &\leq x_n, \\
x_{n+1} &\leq x_n.
\end{aligned}$$

Thus x_n is a monotone decreasing bounded sequence and therefore converges to some limit $\lim_{n\to\infty} x_n = L$. By substitution we find that,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Note that $L \neq 0$ since that would force the $\frac{a}{x_n}$ to infinity, contradiction $x_n \geq x_{n+1}$. Therefore,

$$\begin{aligned}
L &= \frac{1}{2}\left(L + \frac{a}{L}\right), \\
L &= \frac{a}{L}, \\
L^2 &= a, \\
L &= \sqrt{a}.
\end{aligned}$$

□

3. Carothers 1.15 Show that a Cauchy sequence with a convergent subsequence actually converges.

Proof. Suppose (x_n) is a Cauchy sequence and $(x_n)_i \to a$ is a convergent subsequence. Let $\epsilon > 0$. Since $(x_n)_i \to a$ there exists an $I \in \mathbb{N}$ such that for all $i \geq I$ it follows that,

$$|x_{n_i} - a| < \frac{\epsilon}{2}.$$

Similarly since (x_n) is Cauchy we know that there exists an $\hat{N} \geq 1$ such that when $n, m \geq \hat{N}$ it follows that,

$$|x_n - x_m| < \frac{\epsilon}{2}.$$

Let $\epsilon > 0$, and note that for $N = \max\{n_I, \hat{N}\}$ all $n \geq N$ have the property that,

$$\begin{aligned} |x_n - a| &= |x_n - x_{n_N} + x_{n_N} - a|, \\ &\leq |x_n - x_{n_N}| + |x_{n_N} - a|, \end{aligned}$$

Note that we use x_{n_N} to ensure that $n_N \geq \hat{N}, n_I$. So we conclude with,

$$|x_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

4. Carothers 1.21 Let $p \geq 2$ be a fixed integer, and let $0 < x < 1$. If x has a finite-length base p decimal expansion, that is, if $x = a_1/p + \ldots a_n/p^n$ with $a_n \neq 0$, prove that x has precisely two base p decimal expansions. Otherwise, show that the base p decimal expansion for x is unique. Characterize the numbers $0 < x < 1$ that have repeating base p decimal expansions. How about eventually repeating?

5. Carothers 1.24 Show that $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

Proof. Let (a_n) be a bounded sequence of real numbers. Note that by definition,

$$\limsup_{n\to\infty}(-a_n) = \overline{\lim_{n\to\infty}} - a_n = \inf_{n\geq 1}\left(\sup\{-a_n, -a_{n+1}, -a_{n+2}, \ldots\}\right)$$

Let $-a_i$ be an eventual upper bound for the sequence and note that for all $n \geq i$ we know that $-a_i \geq -a_n$. Multiplying both sides by -1, we reverse the inequality to get $a_i \leq a_n$ and conclude that a_i is an eventual lower bound for the sequence a_n. Hence it follows that

$$\inf_{n\geq 1}\left(\sup\{-a_n, -a_{n+1}, -a_{n+2}, \ldots\}\right) = -\sup_{n\geq 1}\left(\inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}\right)$$

□

6. Suppose $\limsup_{n\to\infty} x_n = -\infty$, as defined in terms of eventual upper bounds. Show that

$$\overline{\lim_{n\to\infty}} x_n = -\infty,$$

as defined in the text.

Proof. Suppose that $\limsup_{n\to\infty} x_n = -\infty$. Defined in terms of eventual upper bounds,

$$\limsup_{n\to\infty}{}^{*}\, x_n = \inf\{M : M \text{ is an eventual upper bound for } (x_n)\} = -\infty$$

Recall the definition of $\limsup_{n\to\infty} x_n$ from the text,

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 1}\left(\sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}\right)$$

Note that $\sup\{a_n, a_{n+1}, ...\}$ describes an eventual upper bound with index $M \geq n$ which is also an element in the sequence (a_n). Therefore the text's definition $\limsup_{n\to\infty} x_n$ describes an infimum over a subset of eventual upper bounds.

Suppose for the sake of contradiction that $\limsup_{n\to\infty} x_n \geq L$ with $L \in \mathbb{R}$. By $\limsup^*_{n\to\infty} x_n = -\infty$ we know there exists an eventual upper bound L^* such that $L^* < L$. If $L^* \in (x_n)$ then it follows that $\limsup_{n\to\infty} x_n = L^*$ a contradiction. Otherwise if $L^* \notin (x_n)$ it follows from the definition of E.U.B that for some $M \in \mathbb{N}$ all $n \geq M$, $x_n < L$. Finally we arrive at a contradiction

$$\sup\{a_M, a_{M+1}, a_{n+2}, \dots\} < L^* < L.$$

Therefore $\limsup_{n\to\infty} x_n = -\infty$.

□

7. Let (r_n) be an enumeration of $\mathbb{Q} \cap [0, 1]$. Show that $\limsup n \to \infty = 1$.

Proof. Suppose (r_n) be an enumeration of $\mathbb{Q} \cap [0, 1]$. Clearly 1 is an upper bound for (r_n) and therefore an eventual least upper bound. We will proceed to show that 1 is an infimum on the set of eventual upper bounds, and therefore $\limsup n \to \infty = 1$.

Suppose for the sake of contradiction that an eventual upper bound M such that $M < 1$. Therefore there exists some N where for all $n \geq N$ we know that $r_n \leq M$. However by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $N + 1$ rational numbers in $(M, 1)$ and hence there must exists some $r_i > M$ where $i > N$, a contradiction. □

8. Prove that

$$\limsup x_n + \liminf y_n \leq \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

so long as neither of the right- or left-hand sides are of the form $\infty - \infty$.

9. Carothers 1.36 Let $a_n \geq 0$.

(i) If $\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$, show that $\sum_{n=1}^{\infty} a_n < \infty$.

Proof. Suppose $\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$ and let $m = \limsup_{n\to\infty} \sqrt[n]{a_n}$. Note that there exists an eventual upper bound to the sequence, M such that $m < M < 1$. Therefore for some $N \in \mathbb{N}$, for $n \geq N$ we know that $\sqrt[n]{a_n} < M$. Therefore by substitution we get,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n < \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} (M)^n.$$

Clearly $\sum_{n=N+1}^{\infty}(M)^n < \infty$ since it is a convergent geometric series, and $\sum_{n=1}^{N} a_n < \infty$ since it is a finite sum. Therefore $\sum_{n=1}^{\infty} a_n < \infty$. □

(ii) If $\liminf_{n\to\infty} \sqrt[n]{a_n} > 1$, show that $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Suppose $M = \liminf_{n\to\infty} \sqrt[n]{a_n} > 1$ and for the sake of contradiction we suppose $\sum_{n=1}^{\infty} a_n$ converges. For the infinite series to converge it is necessary for $\lim_{n\to\infty}(a_n) \to 0$ and therefore $\liminf_{n\to\infty} a_n = 0$. Note that there exists some eventual lower bound m for the sequence $\sqrt[n]{a_n}$ such that $1 < m < M$. Therefore for some $N \in \mathbb{N}$, for $n \geq N$ we know that $\sqrt[n]{a_n} > 1$ and thus $a_n > 1$. To conclude, we have shown that there exists a tail of the sequence (a_n) which is above 1, yet still convergent to 0, a contradiction. □

(iii) Find examples of both a convergent and divergent series having $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$.