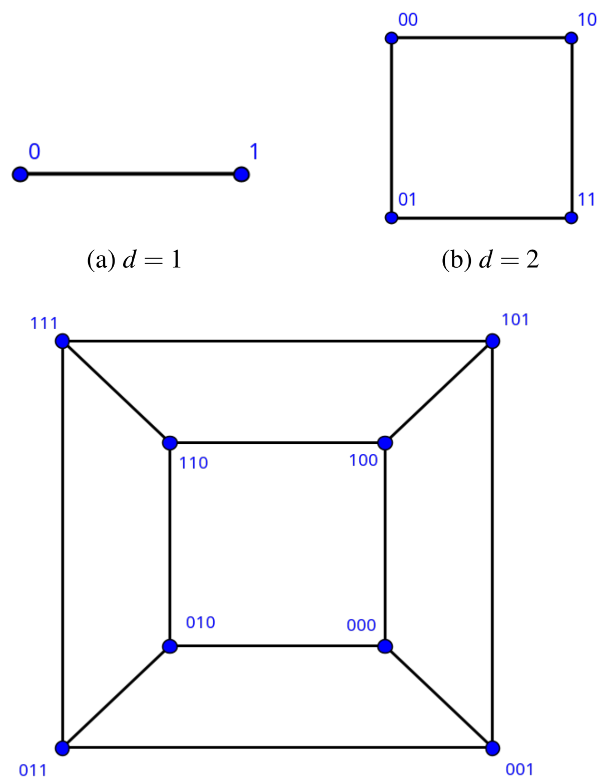
1. Determine the number of edges in a complete graph on n vertices.

Proof. Suppose G is a complete graph on n vertices. There exists $\binom{n}{2}$ ways of pairing vertices. Since G is a complete graph, each pair of vertices correspond to an edge. Hence there are $\binom{n}{2}$, or $\frac{n(n-1)}{2}$ edges. □

2. Let $d \in \mathbb{N}$ and $V = \{0,1\}^d$. That is, V is the set of all binary sequences of length d. Define a graph on V in which two sequences form an edge if and only if they differ in exactly one position. (This graph is called the **d-dimensional cube**.)

(a) Draw and label the vertices of the 1-, 2-, and 3-dimensional cube.

Solution:



(a) $d = 1$ (b) $d = 2$

(c) $d = 3$

Figure 1: d-dimensional cubes.

(b) Determine the average degree, number of edges, diameter, girth and circumference of the n-dimensional cube.

Solution:
Let $n \in \mathbb{N}$ and $V = \{0,1\}^n$. Suppose a graph G, on V in which two sequences form an edge if and only if they differ in exactly one position.

Proof. To determine $d(G)$ note that for a given, length n binary sequence there are n unique ways to differ exactly one position. Hence G is n-regular and thus $d(G) = n$. □

Proof. Clearly $|V| = 2^n$, and since G is n-regular it follows $|E| = \frac{1}{2}2^n n = 2^{n-1}n$. □

Proof. To find the $g(G)$ first note that if $n = 1$ then $g(G) = 0$ since there exists no cycle. Now note that for $n \geq 2$ there will exist vertices which correspond to the sequences $11\ldots, 01\ldots, 10\ldots, 00\ldots$ with trailing zeros after the first two indices. These vertices, by definition must form a 4-cycle. Hence $g(G) = 4$. □

Proof. I assert $\text{diam}(G) = n$. Let x and y be vertices in G and suppose k is the number of differing positions between x and y. We assert that $d(x,y) = k$. If $d(x,y) < k$ then there would be an edge incident to vertices that were more than one position apart. Since two vertices can differ in at most n positions $\text{diam}(G) = n$. □

Proof. I assert that the $\text{circ}(G) = |V(G)| = 2^n$. We will proceed to show that the n-dimension cube has a Hamiltonian cycle, for all $n \geq 2$. The base case is trivial as the 2-dimension cube is a four cycle.

Let G_0 and G_1 be two disjoint but identical n-dimensional cubes. By the induction hypothesis G_0 and G_1 have 2 identical Hamiltonian cycles H_0 and H_1 respectively. We construct a new graph G by,

$$\begin{aligned} V(G) &= V(G_0) \cup V(G_1) \\ E(G) &= E(G_0) \cup E(G_1) \cup \{xy : x \in V(G_0), y \in V(G_1), x = y\} \end{aligned}$$

Now relabel each vertex from G based on the following map $f : G \to G'$ where G' is an $n+1$ dimensional cube,

$$f(x) = \begin{cases} x0 & x \in V(G_0) \\ x1 & x \in V(G_1) \end{cases}$$

This map is clearly a bijection between vertex sets. Now consider some $xy \in E(G)$. We know that either $x, y \in V(G_0)$, $x, y \in V(G_1)$, or without loss of generality $x \in V(G_0)$ and $y \in V(G_1)$. For the former two, since f appends the **same** bit to **both** vertices, $f(x)$ and $f(y)$ differ by only one position and therefore $f(x)f(y) \in G'$.
For the latter case since $xy \in E(G)$ and $x \in V(G_0)$ and $y \in V(G_1)$ it follows by our definition of $E(G)$ that $x = y$ and therefore since f appends a **different** bit to **identical** vertices x and y we know that $f(x)$ and $f(y)$ differ in the last position, so $f(x)f(y) \in G'$.

Thus G is an $n+1$ dimensional cube. Finally we construct a new Hamiltonian cycle by swapping a pair of edges. Note there exists some edge in H_0 incident to vertices $x10$ and $x00$, and similarly H_1 has an edge incident to vertices $x11$ and $x01$. Remove them, and replace them with the edge incident to $x10$ and $x10$ and the edge incident $x00$ and $x01$. These new edges must exists by construction of G and connect to form a Hamiltonian cycle on G. □

3. Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.

   Proof. Let G be a graph containing a cycle C, let x_0 and x_k be vertices of C and suppose that G contains a path P of length at least k between two vertices x_0 and x_k. Now consider a subset of vertices, $V(P) \cap V(C)$ and note that if $|V(P) \cap V(C)| \geq \sqrt{k}$ it follows that C is a desired cycle of length at least $\sqrt{k}$.

   Otherwise suppose $|V(P) \cap V(C)| < \sqrt{k}$. Note that the vertices of $V(P) \cap V(C)$ partition the k edges of P, via $|V(P) \cap V(C)| - 1$ subpaths. Therefore must exists a subpath P_i with terminal vertices, x_i and x_{i+1} in C with length at least

$$\frac{k}{|V(P) \cap V(C)| - 1} \geq \frac{k}{\sqrt{k} - 1} > \sqrt{k}.$$

   It is clear that the path between x_i and x_{i+1} through C, and P_i form a cycle with length greater than $\sqrt{k}$. □

4. Proposition 1.3.2 states that Every graph G containing a cycle satisfies,

$$g(G) \leq 2\text{diam}(G) + 1$$

   Is this bound best possible? Prove your answer is correct.

   Proof. I assert that $2\text{diam}(G) + 1$ is the best possible bound. Consider a cycle C^5, clearly $g(C^5)$ and $\text{diam}(G) = 2$ so we conclude that,

$$g(C^5) = 5 = 2(2) + 1 = 2(\text{diam}(C^5)) + 1.$$

   □

5. Show that for every graph G, $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$.

   Proof. Let P be some path from x_0 to x_k, such that $|P| = \text{diam}(G) = k$. Let x_c be a central vertex, P_0 to be a minimal path between x_0 and x_c and, P_k be the minima path between x_c and x_k. It follows, since x_c is central that $|P_0|, |P_k| \leq \text{rad}(G)$. Construct a walk $W = x_0 P_0 x_c P_k x_k$ and note that since P is a shortest path we get,

$$\text{diam}(G) = |P| \leq |W| = |P_0| + |P_k| \leq 2\text{rad}(G).$$

   Note that $\text{rad}(G) \leq \text{diam}(G)$ is attained by definition, since

$$\text{rad}(G) = \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y),$$

$$\text{diam}(G) = \max_{x \in V(G)} \max_{y \in V(G)} d_G(x, y).$$

   □