1. **Carothers 17.3** Let $f : D \to \mathbb{R}$, where D is measurable. Show that f is measurable if and only if the function $g : \mathbb{R} \to \mathbb{R}$ is measurable, where $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$.

Proof. Let $f : D \to \mathbb{R}$, where D is measurable and suppose that f is measurable. Define the function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)$ when $x \in D$ and $g(x) = 0$ otherwise. Let $\alpha \in \mathbb{R}$ and consider the set $\{g \geq \alpha\}$, and note that if $\alpha > 0$ then $\{g \geq \alpha\} = \{f \geq \alpha\}$ which is measurable, since f is measurable. Now if $\alpha \leq 0$ we find that $\{g \geq \alpha\} = \{f \geq \alpha\} \cup D^c$ a union of measurable sets, which is also measurable. Hence g is a measurable function. □

Proof. Let $f : D \to \mathbb{R}$, where D is measurable and suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is measurable, where $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$. Note that $\{f \geq \alpha\} = \{g \geq \alpha\} \cap D$ which is always measurable.

□

2. **Carothers 17.4** Prove that χ_E is measurable if and only if E is measurable.

Proof. Suppose χ_E is measurable, then clearly it follows that $\{\chi_E \geq \frac{1}{2}\} = E$ and must also be measurable. Now suppose that E is measurable, and consider the function χ_E. Note that

$$\{\chi_E \geq \alpha\} = \begin{cases} \emptyset & \alpha > 1 \\ E & 0 < \alpha \leq 1 \\ \mathbb{R} & \alpha \leq 0 \end{cases}$$

Hence χ_E is a measurable function. □

3. **Carothers 17.8** Suppose that $D = A \cup B$, where A and B are measurable. Show that $f : D \to \mathbb{R}$ is measurable if and only if $f|_A$ and $f|_B$ are measurable (relative to their respective domains A and B, of course).

Proof. let $D = A \cup B$, where A and B are measurable and suppose that $f : D \to \mathbb{R}$ is measurable. Without loss of generality consider the function $f|_A$ and note that $\{f|_A \geq \alpha\} = \{f|_A \geq \alpha\} \cap D$ which is measurable, and hence $f|_A$ is measurable. Now suppose $f|_A$ and $f|_B$ are measurable (relative to their respective domains), and consider $f : D \to \mathbb{R}$. Note that since $D = A \cup B$ we find that $\{f \geq \alpha\} = \{f|_A \geq \alpha\} \cup \{f|_B \geq \alpha\}$ is measurable and therefore f is measurable. □

4. **Carothers 17.18** Let $f : [0, 1] \to [0, 1]$ be the Cantor function, and set $g(x) = f(x) + x$. Prove that:

   a g is a homeomorphism of $[0, 1]$ onto $[0, 2]$. In particular, $h = g^{-1}$ is continuous.

Proof. First recall that the Cantor function is continuous via Corollary 2.19 and since g is a sum of two continuous functions, g is also continuous. Clearly g is a closed map as $[0, 1]$ is compact and $[0, 2]$ is Hausdorff. What is left to show to prove that g is a homeomorphism we will use the closed map lemma so what is left to show that that g is bijective. Let $x, y \in [0, 1]$ such that $x \neq y$ and note that g is strictly increasing, since $f(x) \geq 0$ and monotone increasing and x is strictly increasing it must follows that $g(x) \neq g(y)$.

Let $y \in [0, 2]$ and note that since f is continuous and $g(0) = 0$ and $g(1) = 2$ we find by the intermediate value theorem that there exists an $x \in [0, 1]$ such that $g(x) = y$. □

b $g(\Delta)$ is measurable and $m(g(\Delta)) = 1$. In particular, $g(\Delta)$ contains a nonmeasurable set A.

Proof. Consider Δ^c constructed as a countable union of disjoint middle third intervals I_i. Since f is a constant c_i, on each interval I_i and since g is bijective and each I_i is disjoint in the domain, each $g(I_i) = I_i + c_i$ is disjoint.

$$g(\Delta^c) = g\left(\bigcup_{i=1}^{\infty} I_i\right) = \left(\bigcup_{i=1}^{\infty} I_i + c_i\right).$$

Clearly this demonstrates that $g(\Delta^c)$ is measurable set, and since G is bijective we find that $g(\Delta^c)^c = g(\Delta)$ is also measurable. It then follows by countable additivity and translation invariance that,

$$m(g(\Delta^c)) = m\left(\bigcup_{i=1}^{\infty} I_i + c_i\right) = \sum_{i=1}^{\infty} m(I_i + c_i) = \sum_{i=1}^{\infty} m(I_i) = 1$$

Since g is bijective we know that $[0, 2] = g(\Delta) \cup g(\Delta^c)$ with $g(\Delta)$ and $g(\Delta^c)$ disjoint we get by additivity that,

$$m(g(\Delta)) = 2 - m(g(\Delta^c)) = 1.$$

The conclusion that $g(\Delta)$ contains a non-measurable set follows from noting that $m(g(\Delta)) > 0$ and recalling problem 10 from homework 10.

□

c g maps some measurable set onto a nonmeasurable set.

Proof. As we have shown in the previous part there exists a nonmeasurable set $N \subseteq g(\Delta)$. Since g is bijection, there exists a set $g^{-1}(N) \subseteq \Delta$ and therefore by monotonicity $m^*(g^{-1}(N)) \leq m^*(\Delta) = 0$ so $m^*(g^{-1}(N))$ is a null set and is therefore measurable. □

d $B = g^{-1}(A)$ is Lebesgue measurable but not Borel set.

Proof. We have shown that $h = g^{-1}$ is a continuous function, and is therefore measurable. Consider $B = g^{-1}(N)$ from the previous problem. We have shown that B is lebesgue measurable, suppose for the sake of contradiction that B is a Borel set, then it would follow that since h is measurable, it follows that $h^{-1}(B) = g(g^{-1}(N)) = N$ is measurable, which is a contradiction. □

e There is a Lebesgue measurable function F and a continuous function G such that $F \circ G$ is not Lebesgue measurable.

Proof. We have show that g^{-1} is a continuous function, and previously we have also shown that indicator functions on measurable sets are also Lebesgue measurable. Now let $G = g^{-1}$ and $F = \chi_B$ note that $F \circ G$ is well defined since $g^{-1} : [0,2] \to [0,1]$ and $\chi_B : [0,1] \to \mathbb{R}$. Note that for an open set $U = (\frac{1}{2}, \frac{3}{2})$ we get the following,

$$(F \circ G)^{-1}(U) = G^{-1}(F^{-1}(U)) = g^{-1}(B) = A$$

a non-measurable set. □

5. Carothers 17.32 Check that the conclusion of Theorem 17.8 holds (with the same proof) if 'measurable' is everywhere interpreted as 'Borel measurable'. :Do the same for the four corollaries. What modifications, if any are needed in Corollary 17.12

Proof. □

6. Carothers 17.33 If $f : (a,b) \to \mathbb{R}$ is differentiable, show that f' is Borel measurable. If f is only differentiable a.e, show that f' is still Lebesgue measurable.[Hint: Write f' as the limit of a sequence of continuous functions.]

Proof. Suppose $f : (a,b) \to \mathbb{R}$ is differentiable and note that by the definition of the derivative, f' can be written as the limit of a sequence of continuous functions

$$f'(x) = \lim_{n\to\infty} \frac{f(x + \frac{b-x}{n}) - f(x)}{\frac{b-x}{n}} = \lim_{n\to\infty} \frac{n}{b-x}\left(f\left(x + \frac{b-x}{n}\right) - f(x)\right)$$

Note that since $x \in (a,b)$ the pointwise limit of f' is well defined. Note that the function,

$$f_n'(x) = \frac{n}{b-x}\left(f\left(x + \frac{b-x}{n}\right) - f(x)\right),$$

is itself a Borel measurable function and therefore by Corollary 17.12 f' is Borel measurable. □

7. Carothers 17.35 Give an example showing that the requirement that $m(D) < \infty$ cannot be dropped from Egorov's Theorem.

Proof. □

8. Carothers 17.36 If (f_n) converges almost uniformly to f, prove that (f_n) converges almost everywhere to f. [Hint: For each k, choose a set E_k such that $m(E_k) < \frac{1}{k}$ and $f_n \to f$ uniformly off of E_k. Then $m(\cap_{k=1}^{\infty} E_k) = 0$].

Proof. Let (f_n) be a sequence of functions which converge almost uniformly to f. Therefore we can construct a collection of E_k such that $m(E_k) < \frac{1}{k}$ and $f_n \to f$ uniformly on $\mathbb{R} \setminus E_k$. Let $E = \cap_{k=1}^{\infty} E_k$ and note that $E \subseteq E_k$ for all k it follows that $m(E) \leq m(E_k)$ and since this holds for all k, it follows that $m(E) = 0$. Now let $x \in D \setminus E$, so by definition, there exists some k for which $x \in D \setminus E_k$ and since $f_n \to f$ uniformly on $D \setminus E_k$ we know that $f_n(x) \to f(x)$. □

9. Carothers 17.44 Let $f : [a, b] \to [-\infty, \infty]$ be measurable and finite a.e. Prove that there is a sequence of continuous function (g_n) on $[a, b]$ such that $g_n \to f$ a.e. on $[a, b]$. In fact the g_n can be taken to be polynomials

Proof. Let $f : [a, b] \to [-\infty, \infty]$ be measurable and finite a.e. By Theorem 17.20 we can construct a continuous (g_n) so that $E_n = \{|f - g_n| \geq 2^{-n-1}\}$. Now since each g_n are continuous on a closed interval $[a, b]$ we know, by the Weirstrauss Approximation Theorem that there exists a polynomial p_n such that $\|g_n - p_n\| \leq 2^{-n-1}$. Now note that for each $x \in E_n^c$ it follows that by the triangle inequality,

$$|f(x) - p_n(x)| \leq |f - g_n| + |g_n - p_n| < 2^{-n}.$$

So we can redefine $E_n = \{|f - p_n| \geq 2^{-n}\}$ and by Theorem 17.20 we have that $m(E_n) \leq 2^{-n}$.

Now define ,

$$E = \limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} E_k \right).$$

Note that if $x \in [a, b] \setminus E$ then there exists some k such that such that $x \notin E_n$ for all $n \geq k$,

$$|f(x) - p_n(x)| < 2^{-n}.$$

□

10. Carothers 17.45 Let $f : [a, b] \to \overline{\mathbb{R}}$ be measurable and finite a.e and let $\epsilon > 0$. Show that there is a continuous function g on $[a, b]$ with $m\{f \neq g\} < \epsilon$.

Proof. Let $f : [a, b] \to \overline{\mathbb{R}}$ be measurable and finite a.e and let $\epsilon > 0$. By the previous problem there exists a sequence of continuous (g_n) on $[a, b]$ such that $g_n \to f$ a.e. on $[a, b]$. By Egorov's Theorem there exists a measurable set $E \subseteq [a, b]$ such that $m(E) < \frac{\epsilon}{2}$ and (g_n) converges uniformly to f on $[a, b] \setminus E$. Now since E is measurable there exists an open $E \subseteq U$ such that $U \setminus E < \frac{\epsilon}{2}$.

Note that $m([a,b] \setminus F) = m(U) < \epsilon$, and since $E \subseteq U$ we find that (g_n) converges uniformly to f on $[a,b] \setminus U$ as well. Note that $F = [a,b] \setminus U$ is a closed subset of $\mathbb{R}$ and by Problem 41 there exists a continuous function $g : [a,b] \to \mathbb{R}$ for which $g(x) = f(x)$ for all $x \in F$ and clearly $mf \neq g = m([a,b] \setminus F) = m(U) < \epsilon$.

$\square$