1. Show that every automorphism of a tree fixes a vertex or an edge.

Proof. Suppose T is a tree and ϕ an automorphism. We will proceed to show that ϕ has a fixed vertex or edge by induction. Clearly K_0 will have a fixed point. and a tree on two vertices is guaranteed to have a fixed edge.

Suppose every automorphism of a tree on n vertices has a fixed vertex or edge. Let T be a tree on $n+1$ vertices and ϕ an automorphism. Let L be the set of leaves in T and note that the image of L under any automorphism will have to be L. If $\phi|_L$ has a fixed vertex or edge we are done, otherwise we note that $\phi|_{L^c}$ is itself an automorphism on a tree $T-L$ with less than n vertices and by the induction hypothesis must have a fixed vertex or edge. □

2. Show that a graph is bipartite if and only if every induced cycle has even length.

Proof. $(\leftarrow)$ Suppose a graph G where every induced cycle has even length. Let v be some vertex in G. Note that it must be the case that all $N_1(v)$ must be non-adjacent otherwise an induced 3-cycle would form. If all $N_1(v)$ are non-adjacent it would again follows that all $N_2(v)$ are all non-adjacent otherwise an induced 5-cycle would form. Repeat inductively until the whole graph is traversed. We conclude that all $N_i(v)$ neighborhoods are non-adjacent and graph can be partitioned by even and odd neighborhoods. $\square$

Proof. $(\rightarrow)$ The forward direction is a direct consequence of Proposition 1.6.1 which was shown in class. $\square$

3. Prove or disprove that a graph is bipartite if and only if no two adjacent vertices have the same distance from any other vertex.

   Proof. ($\rightarrow$) Suppose G is a graph and there exists a pair of adjacent vertices x and y whose distance from every vertex in the graph is the same. Note that there must exists some v such that the shortest paths xPv and yPv are disjoint. Clearly these paths, along with edge xy will form an odd cycle, so G is not bipartite. □

   Proof. ($\leftarrow$) Suppose G is not bipartite. Then by Proposition 1.6.1 and the previous problem there exists an smallest induced odd cycle C_{2n+1}. Pick x and y incident in said cycle and note that there exists a v on the cycle a distance of n away from both x and y. There is no shorter path between vertices x, y and v in G as C_{2n+1} was induced and chosen to be the smallest. For clarity, suppose there exist a shorter path xPv in G, not on the cycle. Then up to parity of paths either $xC_{2n+1}vPx$ forms a smaller odd cycle, or $xyC_{2n+1}vPx$ forms a smaller cycle. □

4. Prove or disprove that every connected graph contains a walk that traverses each of its edges exactly once in each direction.

   Proof. Suppose G is a connected graph. Since it is connected it G contains a spanning tree T. We will proceed to show by induction on the number of vertices that all trees have such a walk.

   Clearly a tree on two vertices has such a walk. Suppose a tree T on n vertices. Let x be a leaf adjacent vertex y via edge e. Note that $T - x$ is a tree on $n-1$ vertices, by the induction hypothesis we know that $T - x$ has a desired walk W. We construct a new walk W^* on graph T by inserting exe after the first (or only) instance of vertex y. Note that W^* has the desired property.

   Let W be the walk on T and let E' be the set of all edges in G, not in T. We construct a new walk W^*, for each $e \in E'$ choose one vertex which is incident, call it x and we insert ex into the walk sequence after the first instance of x.

   □

5. Prove that if X is a topological minor of Y and Y is a topological minor of Z, then X is a topological minor of Z.

Proof. Suppose X is a topological minor of Y and Y is a topological minor of Z. Then there exists a subdivision TX of X such that $TX \subseteq Y$. Note that since Y is a topological minor of Z we also know that there exists a subdivision TY such that $TY \subseteq Z$. Since $TX \subseteq Y$ we know there exists a $TX' \subseteq TY$ which is possibly a further subdivision of TX, and most importantly still a subdivision of X. Hence $TX' \subseteq TY \subseteq Z$. $\square$

6. Prove that if G contains a walk from vertex u to vertex v, then it must contain a uv-path.

Proof. Let G be a graph containing a walk from vertex u to vertex v. Let walk W have a walk sequence,

$$W = v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k.$$

If neither v_i or e_i repeats, then our walk is a path. Otherwise for each pair of vertices $v_i = v_j$ with $i \neq j$ remove $e_{i+1} v_{i+1} \dots v_j$ from the walk. Clearly this operation removes repeated vertices, and since any repeated edge is a consequence of a repeated vertex, repeated edges are removed as well. □