

1. Let (a_n) and (b_n) be sequences of numbers such that $a_n \leq b_n$ for all n .

a) Give a careful proof that

$$\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n.$$

b) Show that it need not be true that

$$\limsup_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} b_n.$$

2. Carothers 8.53

3. Determine if the following functions are continuous.

a) $F : (C[0, 1], L^\infty) \rightarrow \mathbb{R}, \quad F(f) = f(0)$

b) $W : (C[0, 1], L^2) \rightarrow \mathbb{R}, \quad W(f) = \int_0^1 f$

c) $G : (C[0, 1], L^1) \rightarrow (C[0, 1], L^2), \quad G(f) = f$

d) $H : (C[0, 1], L^2) \rightarrow (C[0, 1], L^1), \quad H(f) = f$

You are welcome to use the Cauchy-Schwartz inequality for integrals.

4. Carothers 10.20

5. Carothers 10.33

6. Let X be a compact metric space, and let \mathcal{X} be the set of non-empty closed subsets of X . If $a \in X$ and $B \subseteq X$, we define $d(a, B) = \inf_{b \in B} d(a, b)$. We define a metric, called the Hausdorff distance, on \mathcal{X} by

$$H(A, B) = \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, A).$$

a) Suppose $X \subseteq \mathbb{R}^2$ is the closed ball of radius 100, A is the closed square with side length 1 centered at the origin, and B is the closed ball of radius 1/4 centered at the point $(1/2, 1/2)$. Draw a picture of the arrangement and compute $H(A, B)$. (No rigor here please!)

b) Show that H is a metric on \mathcal{X} .

7. Show that the space \mathcal{X} in the previous problem is totally bounded.
8. Let $C^{2\pi}$ denote the continuous 2π -periodic functions on \mathbb{R} . Let $f \in C^{2\pi}$, and define

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

for $n = 0, 1, 2, \dots$

- a) Suppose $h \in C^{2\pi}$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) \cos(nx) dx = 0, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) \sin(nx) dx = 0$$

for $n = 0, 1, 2, \dots$. Show that $h = 0$. Hint: See Application 11.6.

- b) Suppose that $\sum_{n=1}^{\infty} |a_n| + |b_n|$ converges. Show that the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

converges uniformly to a function $g \in C^{2\pi}$.

- c) Recall the function g just defined. Let

$$\hat{a}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx) dx, \quad \hat{b}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(nx) dx.$$

Show that $a_n = \hat{a}_n$ and $b_n = \hat{b}_n$ for all n .

- d) Conclude that $f = g$.

Remark: In this problem you've shown that each element of $C^{2\pi}$ determines a different collection of numbers (a_n) and (b_n) . Moreover, if these sequences converge to zero so fast that the series in part 2 converges, then the Fourier series of f converges uniformly to f . This begs the question: given an arbitrary element of $C^{2\pi}$, is it true that the series in part 2 converges? This is food for thought, not food for the exam.

9. Suppose $\sum |a_n|$ converges. By Carothers Exercise 10.26 we can define a function $f \in C(\mathbb{R})$ by $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$.
- a) If there exist constants $M > 0$ and $\alpha > 2$ such that $|a_n| \leq \frac{M}{n^\alpha}$, prove that f is differentiable.
- b) Visit <http://mathworld.wolfram.com/FourierSeriesTriangleWave.html>. Then remark on what that has to do with the current problem.

Remark: This problem illustrates a special case of the general principle that the faster the Fourier coefficients of a function converge to zero, the smoother that function is.

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- You may not discuss the exam with anyone else until after the due date/time.
- You are permitted to reference Carothers but no other text, nor may you consult the internet.
- Each problem is weighted equally.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- The due date/time is absolutely firm.
- We will hold a hint session at the usual time. There may also be a second.