1. Find (with proof) a function in $\mathcal{R}[a, b]$ that is not a uniform limit of step functions.

Proof. Consider χ_Δ and suppose for the sake of contradiction that there exists a sequence of functions $f_n \in \text{Step}[0, 1]$ such that $f_n \to \chi_\Delta$ uniformly. Note that foror all $\epsilon > 0$ there exists an N such that for all $x \in [0, 1]$ it follows that,

$$\left|f_N(x) - \chi_\Delta(x)\right| < \epsilon.$$

Since f_N is a step function with a finite step partition $\mathcal{P}$, and since Δ is uncountable, there exists an $x \in \Delta$ such that $x \in I$ where I is an open interval in $\mathcal{P}$. Let $y \in I$ such that $y \neq x$. Now there are two cases which each lead to a contradiction

Suppose $y \in \Delta$ then since Δ is nowhere dense there exists a $z \in I$ such that $z \notin \Delta$. Then by uniformly continuity we get the following,

$$\left|\chi_\Delta(x) - f_N(x)\right| < \epsilon,$$

$$\left|f_N(z) - \chi_\Delta(z)\right| < \epsilon.$$

Since $x, z \in I$ it follows that $f_N(z) = f_N(x)$ and therefore the triangle inequality it follows that,

$$\left|\chi_\Delta(x) - \chi_\Delta(z)\right| \leq \left|\chi_\Delta(x) - f_N(x)\right| + \left|f_N(z) - \chi_\Delta(z)\right| < 2\epsilon.$$

This is clearly a contradiction as $\left|\chi_\Delta(x) - \chi_\Delta(z)\right| = 1$.

Now suppose $y \notin \Delta$ and note that since $x, y \in I$, the same argument to get a contradiction.

□

2. Suppose $\ell : \mathcal{P}(\mathbb{R}) \to [0, \infty]$. Show that ℓ is countably additive if and only if ℓ is finitely additive and countably subadditive.

Proof. ($\to$) Suppose $\ell : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is countably additive. Clearly ℓ is also finitely additive, simply consider your countable collection to be some disjoint sets A and B and the rest to be the emptyset.

To show that ℓ is countably subadditive let $\{A_i\}_{i=1}^\infty$ such that A_i are not necessarily disjoint. Construct another collection of sets, $\{B_i\}_{i=1}^\infty$ such that $B_1 = A_1$ and $B_i = A_i \setminus \left(\cup_{k=1}^{i-1} A_k\right)$ for all $i \geq 2$. We have proven that $B_k \subseteq A_k$ and $\cup_{k=1}^n B_k = \cup_{k=1}^n A_k$, for all n. Therefore countable additivity and monotonicity it follows that,

$$\ell\left(\bigcup_{k=1}^\infty A_k\right) = \ell\left(\bigcup_{k=1}^\infty B_k\right) = \sum_{k=1}^\infty \ell(B_k) \leq \sum_{k=1}^\infty \ell(A_k).$$

□

Proof. ($\leftarrow$) Suppose $\ell : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is finitely additive and countably subadditive. Now consider a disjoint collection of sets $\{A_i\}_{i=1}^\infty$. By countable subadditivity we know,

$$\ell\left(\cup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty \ell\left(A_i\right).$$

Note that for each n we know that $\cup_{i=1}^{n} A_i \subseteq \cup_{i=1}^{\infty} A_i$ and therefore by monotonicity it follows that,

$$\ell\left(\cup_{i=1}^{n} A_i\right) \leq \ell\left(\cup_{i=1}^{\infty} A_i\right).$$

Since our sets A_i are dijoint, finite additivity for all n it follows that,

$$\sum_{i=1}^{n} \ell\left(A_i\right) = \ell\left(\cup_{i=1}^{n} A_i\right) \leq \ell\left(\cup_{i=1}^{\infty} A_i\right).$$

Hence,

$$\sum_{i=1}^{\infty} \ell\left(A_i\right) \leq \ell\left(\cup_{i=1}^{\infty} A_i\right).$$

So finally we can conclude that,

$$\ell\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \ell\left(A_i\right).$$

□

3. Carothers 16.4 Given any subset E of $\mathbb{R}$ and any $h \in \mathbb{R}$, show that $m^*(E+h) = m^*(E)$, where $E + h = \{x + h : x \in E\}$.

Proof. Let $\{I_n\}_{n=1}^{\infty}$ be a measuring cover for E. Note that

$$\sum_{n=1}^{\infty} m^*\left(I_n\right) = \sum_{n=1}^{\infty} m^*\left(I_n + h\right). \tag{1}$$

We want to show that $\{I_n + h\}_{n=1}^{\infty}$ is a measuring cover for $E + h$ since by (1) we get that $m^*(E) \leq m^*(E + h)$, and by symmetry we conclude that $m^*(E + h) = m^*(E)$.

To do that we need to show that $E + h \subseteq \cup_{n=1}^{\infty} I_n + h$. Let $x \in E + h$, then by definition $x - h \in E$ and since $\{I_n\}_{n=1}^{\infty}$ is a measuring cover for E, there exists an $I_N \in \{I_n\}_{n=1}^{\infty}$ such that $x - h \in I_N$, and therefore $x \in I_N + h$. So $x \in \cup_{n=1}^{\infty} I_n + h$ and thus $E + h \subseteq \cup_{n=1}^{\infty} I_n + h$.

□

4. Carothers 16.12 Prove that $m^*(E) = \inf\{m^*(U) : U \text{ is open and } E \subset U\}$.

Proof. First note that by monotonicity we know immediately that,

$$m^*(E) \leq \inf\{m^*(U) : U \text{ is open and } E \subset U\}.$$

Now we must establish,

$$m^*(E) \geq \inf\{m^*(U) : U \text{ is open and } E \subset U\}. \tag{2}$$

Consider the case where $m^*(E) < \infty$ (otherwise the result is trivial) and note that to demonstrate (2) we must find a sequence of open sets U_n such that for a given $\epsilon > 0$ there exists an N where for all $n \geq N$,

$$m^*(U_n) \leq m^*(E) + \epsilon.$$

Let $\epsilon > 0$. By definition of outer measure there exists a sequence of measuring covers of E, sequenced by n, denoted $\{I_{i,n}\}_{i=1}^{\infty}$ for which there exists an N such that for all $n \geq N$ it follows that,

$$\sum_{i=1}^{\infty} \ell(I_{i,n}) \leq m^*(E) + \epsilon.$$

By our definition in class of a measuring cover each $I_{i,n}$ is an open interval, and further it follows if $J_n = \cup_{i=1}^{\infty} I_{i,n}$, then J_n is open, $E \subseteq J_n$, and $m^*(J_n) = \sum_{i=1}^{\infty} I_{i,n}$. Thus,

$$m^*(J_n) = \sum_{i=1}^{\infty} \ell(I_{i,n}) \leq m^*(E) + \epsilon.$$

Having demonstrated a sequence of open sets (J_n) with our desired quality (2), we can conclude that $m^*(E) = \inf\{m^*(U) : U \text{ is open and } E \subset U\}$.

□

5. Carothers 16.16 If $m^*(E) = 0$, show that $m^*(E \cup A) = m^*(A) = m^*(A \setminus E)$ for any A.

Proof. Note that by monotonicity, since $A \cup E \subseteq A \subseteq A \setminus E$ we know that $m^*(A \cup E) \geq m^*(A) \geq m^*(A \setminus E)$. By countable subadditivity it follows that,

$$m^*(A \cup E) \leq m^*(A) + m^*(E) = m^*(A).$$

Since $A = (A \cap E) \cup (A \cap E^c)$, by countable subadditivity we also get,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(E) + m^*(A \cap E^c) = m^*(A \cap E^c).$$

Therefore $m^*(E \cup A) = m^*(A) = m^*(A \setminus E)$.

□

6. Carothers 16.22 Let $E = \cup_{n=1}^{\infty} E_n$. Show that $m^*(E) = 0$ if and only if $m^*(E_n) = 0$ for every n.

Proof. Let $E = \cup_{n=1}^{\infty} E_n$ and suppose $m^*(E) = 0$. Since $E_n \subseteq E$ it follows by monotonicity that $m^*(E_n) \leq m^*(E) = 0$ and thus $m^*(E_n) = 0$ for every n. □

Proof. Let $E = \cup_{n=1}^{\infty} E_n$ and suppose $m^*(E_n) = 0$ for every n. By countable subadditivity it follows that,

$$m^*(E) \leq \sum_{i=1}^{\infty} m^*(E_i) = 0.$$

Thus $m^*(E)$.

$\square$

7. Carothers 16.24 Given a subset E of $\mathbb{R}$, prove that there is a G_δ-set G, containing E such that $m^*(G) = m^*(E)$.

Proof. Let E be a subset of $\mathbb{R}$ and consider the case with $m^*(E) < \infty$ (otherwise let $G = \mathbb{R}$ and the result follows). By Problem 4 there exists a sequence of open sets U_n with $E \subset U_n$, such that $m^*(U_n) \to m^*(E)$. Now let $G = \cap_{n=1}^{\infty} U_n$ and note that clearly $E \subset G$. By monotonicity $m^*(G) = \inf\{m^*(U) : U \text{ is open and } E \subset U\}$ and by Problem 4 $m^*(G) = m^*(E)$. $\square$

8. Carothers 16.25 Suppose that $m^*(E) > 0$. Give $0 < \alpha < 1$, show that there exists an open interval I such that $m^*(E \cap I) > \alpha m^*(I)$.[Hint: It is enough to consider the case $m^*(E) < \infty$. Now suppose that the conclusion fails.]

Proof. Let $m^*(E) > 0$ and suppose there exists an α with $0 < \alpha < 1$ such that for every open interval I, $m^*(E \cap I) \leq \alpha\ell(I)$. Now let $\epsilon > 0$ and note that there exists a measuring cover $\{I_n\}_{n=1}^{\infty}$ such that,

$$\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(E) + \epsilon.$$

Now it follows that,

$$\alpha \sum_{n=1}^{\infty} \ell(I_n) \leq \alpha\,(m^*(E) + \epsilon)$$

$$\sum_{n=1}^{\infty} \alpha\ell(I_n) \leq \alpha\,(m^*(E) + \epsilon)$$

$$\sum_{n=1}^{\infty} \alpha m^*(E \cap I_n) \leq \alpha\,(m^*(E) + \epsilon)$$

Since $E \subseteq \cup_{n=1}^{\infty}(E \cap I_n)$, by countable subadditivity it follows that,

$$m^*(E) \leq \alpha\,(m^*(E) + \epsilon)\,.$$

Further we find that,

$$0 < m^*(E)(1 - \alpha) \leq \alpha\epsilon. \tag{3}$$

However clearly an ϵ can be chosen such that the (3) does not hold.

$\square$

9. Carothers 16.28 Fix α with $0 < \alpha < 1$ and repeat our "middle thirds" construction of the Cantor set except that now, at the nth stage, each of the 2^{n-1} open intervals we discard from $[0, 1]$ is to have length $(1 - \alpha)3^{-n}$. (We still want to remove each open interval from the 'middle' of a closed interval in the current level- it is important that the closed intervals that remain turn out to be nested.) The limit of this process, a set that we will name Δ_α, is called the generalized Cantor set and is very much like the ordinary Cantor set. Note that Δ_α is uncountable, compact, nowhere dense, and so on but has nonzero outer measure. Indeed check that $m^*(\Delta_\alpha) = \alpha$. (See Chapter two for an example.)[Hint: you only need upper estimates for $m^*(\Delta_\alpha)$ and $m^*({\Delta_\alpha}^c)$]

Proof. By definition we know that ${\Delta_\alpha}^c = \cap_{n=1}^{\infty} J_n$ where J_n is itself the union of 'middle thirds' taken from the nth step in the set recurrence relation definition of Δ_α . Note that each J_n is a union of 2^{n-1} disjoint intervals of length $(1 - \alpha)3^{-n}$, and therefore by countable subadditivity we conclude that,

$$m^*({\Delta_\alpha}^c) \leq \sum_{n=1}^{\infty} m^*(J_n) = \sum_{n=1}^{\infty}(1 - \alpha)2^{n-1}3^{-n} = \frac{(1-\alpha)}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = 1 - \alpha < 1.$$

Since $[0, 1] = {\Delta_\alpha}^c \cup \Delta_\alpha$ by countable subadditivity we also know that,

$$m^*([0, 1]) \leq m^*({\Delta_\alpha}^c) + m^*(\Delta_\alpha)$$

Clearly since $m^*([0, 1]) = 1$ and $m^*({\Delta_\alpha}^c) < 1$ it must be the case that $m^*(\Delta_\alpha) > 0$ as desired.

□