1. Carothers 18.1 If ψ is nonnegative, integral simple function, check that,

$$\int \psi = \sup\left\{\int \varphi : 0 \leq \varphi \leq \psi, \varphi \text{ simple and integrable}\right\}$$

Proof. Clearly since ψ is itself a nonnegative, integrable simple function we can conclude that,

$$\int \psi \leq \sup\left\{\int \varphi : 0 \leq \varphi \leq \psi, \varphi \text{ simple and integrable}\right\}.$$

What is left to show is that for all $\epsilon > 0$, there exists a simple and integral simple function $0 \leq \varphi \leq \psi$ such that,

$$\int \psi - \epsilon = \int \varphi.$$

Now recall that by definition the Lebesgue integral of ψ is given by,

$$\int \psi = \sum_{i=1}^{n} a_i m(A_i)$$

where a_i are distinct real numbers and A_i are pairwise disjoint measurable sets. So it follows that,

$$\int \psi = \sum_{i=1}^{n} a_i m(A_i) - \epsilon = \sum_{i=1}^{n} (a_i m(A_i) - \frac{\epsilon}{n}) = \sum_{i=1}^{n} (a_i - \frac{\epsilon}{nm(A_i)}) m(A_i).$$

Simply take $b_i = a_i - \frac{\epsilon}{nm(A_i)}$ redefining any non-distinct $b_1, b_2, \ldots$ as a single b_i, and combining their there associated support $B_i = A_1 \cup A_2 \cup \ldots$, otherwise simply take $B_i = A_i$. Note that each b_i is distinct, B_i a union of disjoint measurable sets, and therefore also disjoint and measurable so hence,

$$\varphi = \sum_{i=1}^{j} b_i \chi_{E_i}$$

defined a new simple function with the desired Lebesgue integral. Thus

$$\int \psi = \sup\left\{\int \psi : 0 \leq \varphi \leq \psi, \varphi \text{ simple and integrable}\right\}$$

□

I am aware that this problem is immensely simplified, if we recall that the integrals of nonnegative simple functions are monotonic.

2. Carothers 18.3 Prove that $\int_1^\infty (\frac{1}{x}) dx = \infty$ as a Lebesgue integral.

Proof. First note that the following is a class of simple and integrable functions which are nonnegative and lie underneath $\frac{1}{x}$,

$$\varphi_n = \sum_{i=2}^{n} \frac{1}{n} \chi[i-1, i].$$

Now by definition we know that,

$$\int_1^\infty \frac{1}{x}dx = \sup\left\{\int \varphi : 0 \le \varphi \le \frac{1}{x}, \varphi \text{ simple and integrable}\right\}$$

So clearly it should follow that,

$$\int_1^\infty \frac{1}{x}dx \ge \sup\left\{\int \varphi_n : 0 \le \varphi_n \le \frac{1}{x}, \varphi_n \text{ as previously defined}\right\}$$

However we find that,

$$\lim_{n\to\infty}\int \varphi_n = \lim_{n\to\infty}\sum_{i=2}^{n}\frac{1}{n} = \infty.$$

So we conclude that $\int_1^\infty \frac{1}{x}dx = \infty$ as desired. □

3. Carothers 18.4 Find a sequence (f_n) of nonnegative measurable function such that $\lim_{n\to\infty} f_n = 0$, but $\lim_{n\to\infty}\int f_n = 1$. In fact, show that (f_n) can be chosen to converge uniformly to 0.

Proof. Consider the sequence of simple functions,

$$f_n = \frac{1}{n}\chi_{[0,n]}$$

Clearly the pointwise limit of f_n is zero, but since $\int f_n = 1$ for all n we get that,

$$\lim_{n\to\infty}\int \frac{1}{n}\chi_{[0,n]} = 1.$$

Now let $\epsilon > 0$ and note that we can choose N such that $\frac{1}{N} < \epsilon$ and for all $n \ge N$ and for all x we get,

$$\left|\frac{1}{n}\chi_{[0,n]}(x)\right| \le \frac{1}{n} < \epsilon.$$

□

4. Carothers 18.6 Suppose that f and (f_n) are nonnegative measurable functions, that (f_n) decreases pointwise to f, and that $\int f_k < \infty$ for some k. Prove that $\int f = \lim_{n\to\infty}\int f_n$.[Hint: Consider $(f_k - f_n)$ for $n > k$.] Give an example showing that this fails without the assumption that $\int f_k < \infty$ for some k.

Proof. Suppose that f and (f_n) are nonnegative measurable functions, that (f_n) decreases pointwise to f, and that $\int f_k < \infty$ for some k.

First let $g_n = \chi_E(f_k - f_n)$ for all $n > k$ where $E = \{f_k < \infty\}$, note that g_n are measurable, since (f_n) and χ_E are measurable. Note that since $f_n \ge f_{n+1} \ge f \ge 0$ we get that $g_n = \chi_E(f_k - f_n) \le \chi_E(f_k - f_{n+1}) = g_{n+1}$ and hence g_n is an increasing sequence of nonnegative measurable functions. Also note that $\int(\chi_E f_k - \chi_E f_n) = \int \chi_E f_k - \chi_E \int f_n$ since f_k and f_n are nonnegative measurable functions. So applying the Monotone Convergence Theorem we get,

$$\begin{aligned}
\int \left(\lim_{n\to\infty} g_n\right) &= \lim_{n\to\infty} \int g_n \\
\int \left(\lim_{n\to\infty} \chi_E(f_k - f_n)\right) &= \lim_{n\to\infty} \int \chi_E(f_k - f_n) \\
\int \left(\lim_{n\to\infty} (\chi_E f_k - \chi_E f_n)\right) &= \lim_{n\to\infty} \left(\int \chi_E f_k - \chi_E f_n\right) \\
\int \lim_{n\to\infty} \chi_E f_k - \lim_{n\to\infty} \chi_E f_n &= \lim_{n\to\infty} \left(\int \chi_E f_k - \int \chi_E f_n\right) \\
\int \chi_E f_k - \chi_E f &= \lim_{n\to\infty} \int \chi_E f_k - \lim_{n\to\infty} \int \chi_E f_n \\
\int \chi_E f_k - \chi_E f &= \int \chi_E f_k - \lim_{n\to\infty} \int \chi_E f_n
\end{aligned}$$

Now since f and f_n are all integrable, $m(E^c) = 0$ and therefore $\int \chi_E f = \int f$ and $\int \chi_E f_n = \int f_n$ So we conclude that $\int f = \lim_{n\to\infty} \int f_n$. □

Proof. For an example where this result fails without the assumption that $\int f_k < \infty$ for some k consider the sequence of functions $f_n = \chi_{[n,\infty]}$. Clearly this converges pointwise to zero, so $\int f = 1$ however $\int f_n = \infty$ for all n. □

5. Carothers 18.9 Let f be measurable with $f > 0$ a.e. and f is nonnegative. If $\int_E f = 0$ for some measurable set E, show that $m(E) = 0$.

Proof. Let f be measurable with $f > 0$ a.e and suppose that $\int_E f = 0$ for some measurable set E. Note that since f is measurable

$$\int_E f = \int_E f_+ - \int_E f_- = \int \chi_E f_+ - \int \chi_E f_-$$

Now since $f > 0$ a.e we know that $m(\{f \leq 0\}) = 0$ and therefore $m(E \cap \{f \leq 0\}) = 0$ and thus,

$$\int_{E\cap\{f\leq 0\}} f = \int \chi_E f_- = 0.$$

Hence,

$$\int_E f = \int \chi_E f_+ = 0$$

Note that $\chi_E f_+$ is nonnegative and measurable with $\int \chi_E f_+ = 0$ so by Corollary 18.10 we know that $\chi_E f_+ = 0$ and since $f > 0$ a.e. it must follow that $m(E \cap \{f \geq 0\}) = 0$. Now note that $E \cap \{f \geq 0\}$ and $E \cap \{f \leq 0\}$ are disjoint and $E = E \cap \{f \geq 0\} \cup E \cap \{f \leq 0\}$, we find that,

$$m(E) = m(E \cap \{f \geq 0\}) + m(E \cap \{f \leq 0\}) = 0.$$

□

6. Carothers 18.11 If f is nonnegative and integrable, show that $\int_{-\infty}^{\infty} f = \lim_{n\to\infty} \int_{f\leq n} f$.

Proof. Let f be nonnegative and integrable, and note that by Chebyshev's inequality we know that $m\{f > n\} \leq \frac{1}{n}\int f$ holds for all n, so it follows that $m(\{f = \infty\}) = 0$.

Now note that the support of f can be expressed as the following limit,

$$\{f > 0\} = \left(\lim_{n\to\infty}\{f \leq n\}\right) \cup \{f = \infty\}$$

Now as we have shown f is finite a.e, so the limit of our integral can be rewritten,

$$\int_{-\infty}^{\infty} f = \lim_{n\to\infty} \int_{f\leq n} f + \int_{\{f=\infty\}} f = \lim_{n\to\infty} \int_{f\leq n} f.$$

□

7. Let $f \geq 0$ be Riemann integrable. In this exercise you will show that f is measurable. In your work, you are welcome to use the obvious fact that the Riemann integral and the Lebesgue integral agree for step functions.

(a) Show that there exists a monotone increasing sequence of step function φ_n and a monotone decreasing sequence of step function ψ_n such that $\varphi_n \leq f \leq \psi_n$ for each n and such that,

$$(R)\int_a^b (\psi_n - \varphi_n) \to 0.$$

Proof. Let $f \geq 0$ be Riemann integrable. Recall that from Prop 5 of the Riemann Integral notes, we can construct a sequence of step function φ_n and ψ_n, not necessarily decreasing or increasing such that $\varphi_n \leq f \leq \psi_n$ and,

$$(R)\int_a^b (\psi_n - \varphi_n) < \frac{1}{n}.$$

Now construct new sequences φ_n', ψ_n' such that $\varphi_1' = \varphi_1$ and $\psi_1' = \psi_1$ with the following,

$$\varphi_n' = \max\{\varphi_{n-1}', \varphi_n\},$$
$$\psi_n' = \min\{\psi_{n-1}', \psi_n\}.$$

Clearly φ_n' and ψ_n' continue to be sequences of step functions, and now φ_n' is monotone increasing, and ψ_n' is monotone decreasing. Since $\varphi_n' \geq \varphi_n$ and $\psi_n' \leq \psi_n$ it follows that,

$$(R)\int_a^b (\psi_n' - \varphi_n') \leq (R)\int_a^b (\psi_n - \varphi_n) < \frac{1}{n}.$$

Hence,

$$(R)\int_a^b (\psi_n' - \varphi_n') \to 0.$$

□

(b) Let $\Phi = \sup \varphi_n$ and $\Psi = \inf \phi_n$. Show that $\Psi - \Phi = 0$ almost everywhere.

Proof. Let $\Phi = \sup \varphi_n$ and $\Psi = \inf \psi_n$. Note that since $\varphi_n \leq f \leq \psi_n$, and f is a Riemann integrable function which is also bounded, we know that Ψ and Φ are both bounded, we have also shown that the inf and sup on sets of measurable functions, results in a measurable function, therefore the following integral is well defined,

$$\int \Psi - \Phi = \int \lim_{n\to\infty} \psi_n - \lim_{n\to\infty} \varphi_n = \int \lim_{n\to\infty} \psi_n - \varphi_n = \int \chi_{[a,b]^c} \lim_{n\to\infty} \psi_n - \varphi_n + \int_a^b \lim_{n\to\infty} \psi_n - \varphi_n.$$

Since our step functions are defined on $[a, b]$ it follows that,

$$\int \chi_{[a,b]^c} \lim_{n\to\infty} \psi_n - \varphi_n = 0.$$

So finally since $\psi_n - \varphi_n$ is a nonnegative measurable function we know,

$$\int \Psi - \Phi = \int_a^b \lim_{n\to\infty} \psi_n - \varphi_n = \lim_{n\to\infty} \int_a^b \psi_n - \varphi_n = 0.$$

Hence $\Psi - \Phi = 0$ almost everywhere. □

(c) Conclude that f is measurable.

Solution:
From the previous result it also follows that $\sup \varphi_n = \inf \psi_n$ almost everywhere and by construction we know that $\sup \varphi_n \leq f \leq \inf \psi_n$ everywhere, so $\sup \varphi_n = f = \inf \psi_n$ almost everywhere and since $\sup \varphi_n$ and $\inf \psi_n$ are measurable we know that f is measurable.

8. Carothers 18.16 Let f be nonnegative and integrable. Given $\epsilon > 0$, show that there is a measurable set E with $m(E) < \infty$ such that $\int_E f > \int f - \epsilon$. Moreover, show that E can be chosen so that f is bounded (above) on E.

Proof. Let f be nonnegative and integrable. Construct the following measurable sets $E_k = \{f \geq k\}$, and note that by definition f is bounded on E_k^c. Let $\epsilon > 0$ and consider the following functions $\chi_{E_k^c} f$. Firstly note that since f is nonnegative $\chi_{E_k^c}$ is a nonnegative sequence of functions. Since $E_k^c = \{f < k\}$ we find that $E_k^c \subseteq E_{k+1}^c$ and therefore $\chi_k^c f \leq \chi_{k+1}^c f$. Observe that $\chi_{E_k^c} f \to f$ pointwise a.e. and therefore by the monotone convergence theorem we find that there exists a k such that,

$$\int_{E_k^c} f \geq \int f - \frac{\epsilon}{2}.$$

Now consider the following measurable sets, $A_n = E_k^c \cap [-n, n]$. Again clearly $\chi_{A_n} f \to \chi_{E_k^c} f$ pointwise increasing e.a. so by the monotone convergence theorem there exists an n such that,

$$\int_{A_n} f \geq \int_{E_k^c} f - \frac{\epsilon}{2}.$$

By substitution we conclude that,

$$\int_{A_n} f \geq \int_{E_k^c} f - \frac{\epsilon}{2} \geq \int f - \epsilon.$$

□

9. Carothers 18.17 If f is nonnegative and integrable, prove that the function $F(x) = \int_{-\infty}^{x} f$ is continuous. In fact, even more is true: Given $\epsilon > 0$, show that there is a $\delta > 0$ such that $\int_E f < \epsilon$ whenever $m(E) < \delta$. [Hint: This is easy if f is bounded; see Exercise 16]

Proof. Let f be nonnegative and integrable and also let $\epsilon > 0$. By the previous problem there exists a measurable set E with $m(E) < \infty$, $\int_E f > \int f - \epsilon$ and there exists some constant K such that $K \geq f$ on E. Now choose δ such that $\delta K < \epsilon$ and note that if $0 \leq |x - y| \leq \delta$ it follows that,

$$\begin{aligned} |F(x) - F(y)| &= \left| \int \chi_{(x,y]} f \right| \\ &= \int \left| \chi_{(x,y]} f \right| \\ &< \int_E \left| \chi_{(x,y]} f \right| + \epsilon < \delta K + \epsilon < 2\epsilon. \end{aligned}$$

□

Proof. If f is nonnegative and integrable, let $\epsilon > 0$ and by the previous problem there exists a set E such that $\int_E f > \int f - \frac{\epsilon}{2}$, with $m(E) < \infty$, and $f \leq K$ on E. Now let A be a set such that $m(A) < \delta$ with $\delta K < \frac{\epsilon}{2}$. Now note that,

$$\int_A f = \int_{A \cap E^c} f + \int_{A \cap E} f.$$

Also note,

$$\begin{aligned} \int_E f &> \int f - \frac{\epsilon}{2}, \\ \int_E f &> \int_E f + \int_{E^c} f - \frac{\epsilon}{2}, \\ \frac{\epsilon}{2} &> \int_{E^c} f. \end{aligned}$$

Finally we conclude that,

$$\int_A f = \int_{A\cap E^c} f + \int_{A\cap E} f < \frac{\epsilon}{2} + \delta K < \epsilon.$$

□