1. Carothers 19.3

2. Compute $\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \cos(x/n)\, dx$.

3. Let $\{f_n\}$ be a sequence of measurable real-valued functions. Let $E = \{x : (f_n(x)) \text{ converges}\}$. Show that E is measurable. [Hint: If the sequence does not converge at some x, the sequence $(f_n(x))$ is not Cauchy; try to give a description of the places where the sequence is not Cauchy in terms of a countable collection of set operations.]

4. Let

$$X_K = \left\{ f \in C([0,1]) : f \text{ is Lipschitz with constant } K \text{ and } \int_0^1 |f| \le 1 \right\}.$$

Show that X_K is compact in $C([0,1])$. Is X_K also compact in $L_1([0,1])$?

5. (Riemann integrable functions are continuous almost everywhere.)

a) On a domain $I = [a,b]$, let (ψ_n) be an increasing sequence of step functions with $|\psi_n| \le M$ for some M. Show that $\lim \psi_n$ is lower semicontinuous almost everywhere. That is, show that for almost every $x \in [a,b]$, if $x_n \to x$ then $f(x) \le \liminf_{n\to\infty} f(x_n)$.

b) Suppose g, f and G are measurable functions on $[a,b]$, $g \le f \le G$, $g = G$ almost everywhere, g is lower semicontinuous, and G is upper semicontinuous. Show that f is continuous almost everywhere.

c) Show that Riemann integrable functions are continuous almost everywhere. Hint: Find functions g and G with $g \le f \le G$ where $G = g$ almost everywhere and where g and G are continuous almost everywhere.

6. Carothers 8.20

7. (Cauchy-Schwartz inequality for integrals.)

a) Use the (ℓ_2) Cauchy-Schwartz inequality to prove that if f and g are simple and integrable, then

$$\int |f||g| \le \left[\int |f|^2\right]^{\frac{1}{2}} \left[\int |g|^2\right]^{\frac{1}{2}}$$

b) Suppose that f and g are measurable functions such that $|f|^2, |g|^2 \in L^1$. Show that $fg \in L^1$ and $\int |fg| \le [\int |f|^2]^{\frac{1}{2}} [\int |g|^2]^{\frac{1}{2}}$.

8. (The approximate with wild abandon problem.)

Suppose $f \in L^1[a,b]$ and $\int_a^b fg = 0$ for every polynomial g. Show that $f = 0$ almost everywhere.

Hint: First show that $\int_I f = 0$ for every interval in $[a, b]$. Then show that $\int_E f = 0$ for every measurable set in $[a, b]$. You might find Exercise 18.35 (the "even more is true" part) to be handy, as well.

9. A sequence (f_n) is Cauchy in measure if for every $\epsilon > 0$ there is an index N such that if $n, m \geq M$ then $m(\{|f_n - f_m| > \epsilon\}) < \epsilon$.

Show that if (f_n) is Cauchy in measure and has a subsequence that is convergent in measure, then the full sequence is Cauchy in measure.

10. Consider the series $\sum_{k=1}^{\infty} a_k \sin(kx)$ on the domain $[0, 2\pi]$. Suppose that $\sum_{k=1}^{\infty}(a_k)^2$ converges. Prove that the series converges in $L^2([0, 2\pi])$.

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- You may not discuss the exam with anyone else until after the due date/time.
- You are permitted to reference Carothers but no other text, nor may you consult the internet.
- Each problem is weighted equally.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- The due date/time is absolutely firm.
- We will hold a hint session during finals week, TBA.