1. Carothers 3.18 Show that $||x||_\infty \leq ||x||_2 \leq ||x||_1$ for any $x \in \mathbb{R}^n$. Also check that $||x||_1 \leq n||x||_\infty$ and $||x||_1 \leq \sqrt{n}||x||_2$.

Proof. Suppose $x \in \mathbb{R}^n$ and recall that by definition it clearly follows that,

$$||x||_\infty = \max_{1\leq i\leq n} |x_i| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = ||x||_2.$$

Also note that,

$$||x||_1^2 = \sum_{i=1}^{n} |x_i| \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} |x_i|^2 + \sum_{i\neq j} |x_i||x_j| \geq ||x||_2^2.$$

Since $\sqrt{x}$ is a monotone function we can conclude that $||x||_2 \leq ||x||_1$.

Note that,

$$||x||_1 = \sum_{i=1}^{n} |x_i| \leq n \max_{1\leq i\leq n} |x_i| = n||x||_\infty.$$

Finally by Cauchy-Schwarz Inequality we get the following,

$$||x||_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} |x_i 1_i| \leq ||1||_2||x||_2 = \sqrt{n}||x||_2.$$

□

2. Carothers 3.23 The subset of ℓ_∞ consisting of all sequences that converge to 0 is denoted by c_0. Show that we have the following proper set inclusions: $\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$

Proof. Let $(x_n) \in \ell_1$. Note that by definition it follows $|x_n| \to 0$, since x^2 is a monotone over positive reals it follows that $|x_n|^2 = x_n^2 \to 0$. Hence $(x_n) \in \ell_2$. It follows similarly that since $\sqrt{(x)}$ is monotone $x_n \to 0$ and therefore $(x_n) \in c_0$.

To show that these are proper subset relations, consider the sequence $x_n = \frac{1}{n}$. Note that $||x_n||_1$ is a harmonic series so $x_n \notin \ell_1$ but $||x_n||_2$ is a convergent p-series, so $x_n \in \ell_2$.

Similarly we see that $x_n = \frac{1}{\sqrt{n}}$ makes $||x_n||_2$ into a harmonic series so $x_n \notin \ell_2$ but again x_n alone is a bounded convergent sequence.

Finally consider a constant sequence of 1s, which is bounded and therefore contained in ℓ_∞ yet it doesn't converge to zero so it's not in c_0.

□

3. Young's Inequality Let $p \in (1, \infty)$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $a, b \geq 0$. Show,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

and that the inequality is strict unless either $a^{p-1} = b$ or $b^{q-1} = a$.

Proof. Let $p \in (1, \infty)$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$ and suppose $a, b \geq 0$. Note that to Young's inequality it is equivalent to show that the following inequality for a function $f(a, b)$ over a domain $a, b \geq 0$,

$$f(a, b) = ab - \frac{a^p}{p} - \frac{b^q}{q} \geq 0.$$

We fix $b \in [0, \infty)$ and proceed to find f' and f'',

$$f'(a) = b - a^{p-1},$$
$$f''(a) = -(p-1)a^{p-2}.$$

Note that since $p \in (1, \infty)$ and $a \in [0, \infty)$ it follows that $f''(a) \leq 0$ on our domain and therefore any critical point is a maximum. Setting $f'(a) = 0$ we get that $a = b^{\frac{1}{p-1}}$. Now to prove our inequality holds, it is sufficient to show that $f(b^{\frac{1}{p-1}}) \leq 0$,

$$\begin{aligned} f(b^{\frac{1}{p-1}}) &= b^{\frac{q}{p}+1} - \left(\frac{b^q}{p} + \frac{b^q}{q}\right) \\ &= b^{\frac{q}{p}+1} - b^q\left(\frac{1}{p} + \frac{1}{q}\right) \\ &= b^{q-1+1} - b^q \\ &= 0 \end{aligned}$$

□

4. Carothers 3.34 if $x_n \to x$ in (M, d), show that $d(x_n, y) \to d(x, y)$ for any $y \in M$. More generally, if $x_n \to x$ and $y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$

Proof. Suppose $x_n \to x$ in (M, d). By definition, for $\epsilon > 0$ there exists an N such that for all $n \geq N$, implies $d(x_n, x) < \epsilon$. Now let $y \in M$ and consider that by the triangle inequality we get the following,

$$\begin{aligned} d(x_n, y) - d(x, y) &\leq (d(x_n, x) + d(x, y)) - d(x, y), \\ &= d(x_n, x), \\ &< \epsilon. \end{aligned}$$

Hence $d(x_n, y) \to d(x, y)$. □

5. Carothers 3.36 A convergent sequence is Cauchy, and a Cauchy sequence is bounded (that is, the set $\{x_n : n \geq 1\}$ is bounded).

Proof. Suppose $x_n \subseteq M$ with metric d and $x_n \to x$. Let $\epsilon > 0$ then there exists an N such that for all $n \geq N$, we have that $d(x_n, x) < \frac{\epsilon}{2}$. Let $m \geq N$, by the triangle inequality it follows that,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x_m, x), \\ &\leq d(x_n, x) + d(x_m, x), \\ &< \epsilon. \end{aligned}$$

□

Proof. Suppose that $x_n \in M$ with metric d and (x_n) is Cauchy. Since (x_n) is Cauchy there exists some N such that for all $m, n \geq N$ it follows that $d(x_n, x_m) < 1$. Let $c \in M$, $m \geq n$, and note that,

$$\begin{aligned} d(c, x_n) &\leq d(c, x_m) + d(x_m, x_n), \\ &< \max\{d(c, x_i) : 1 \leq i \leq m\} + 1. \end{aligned}$$

Let $R = \max\{d(c, x_i) : 1 \leq i \leq m\} + 1$ and note that $x_n \subseteq B_R(c)$. □

6. Carothers 3.39 If every subsequence of (x_n) has a further subsequence that converges to x, then (x_n) converges to x.

Proof. Suppose $(x_n) \subseteq M$ with metric d and $x_n \not\to x$. We wish to produce a subsequence of (x_n) whose further subsequences don't converge to x. By definition, there exists an $r > 0$, where for every N, when $n \geq N$ it follows that $x_n \not\subset B_r(x)$. Since this statement is true for every N, there are infinitely many such $x_n \notin B_r(x)$, so we can construct a subsequence $(x_{n_k}) = \{x_n : x_n \notin B_r(x)\}$. Clearly any further subsequence will not have a tail in $B_r(x)$ and therefore won't converge to x. □

7. Carothers 4.3 Show that two metrics are equivalent if they generate the same open sets.

Proof. Suppose a space M with two metrics d_1 and d_2 which generate the same open sets. Let $(x_n) \subseteq M$ such that $x_n \underset{d_1}{\longrightarrow} x$. Let $r > 0$ and note that there exists some N such that for all $n \geq N$ we know that $(x_n) \subseteq B_r^1(x)$. Since d_1 and d_2 generate the same open sets it follows that for all $n \geq N$ we know that $(x_n) \subseteq B_r^2(x)$, hence $x_n \underset{d_2}{\longrightarrow} x$. □

Proof. Suppose a space M with two metrics d_1 and d_2 such that for every $(x_n) \subseteq M$ it follows that $x_n \underset{d_1}{\longrightarrow} x$ and $x_n \underset{d_2}{\longrightarrow} x$. Let A be a closed set with respect to d_1. Let $(x_n) \subseteq A$ such that $x_n \underset{d_2}{\longrightarrow} x$, by our hypothesis it follows that $x_n \underset{d_1}{\longrightarrow} x$ and since A is closed with respect to d_1 we know that $x \in A$. Hence A is closed with respect to d_2. □

8. Carothers 4.11 Let $e^{(k)} = (0, \ldots, 0, 1, 0, \ldots)$, where the kth entry is 1 and the rest are 0s. Show that $A = \{e^{(k)} : k \leq 1\}$ is closed as a subset of ℓ_1.

Proof. Let $e^{(k)} = (0, \ldots, 0, 1, 0, \ldots)$, where the kth entry is 1 and the rest are 0s. Suppose $(x_n) \subset A$ with $x_n \to x$ and therefore also Cauchy. By definition, for all $\epsilon > 0$ there exists an N such that for all $m, n \geq N$ we have,

$$d(x_m, x_n) = \|x_m - x_n\|_1 < \epsilon.$$

Let $a = x_m - x_n$ and therefore it follows that,

$$\sum_{i=1}^{\infty} |a| < \epsilon.$$

Note $|a|$ can either be a sequence of all zeros, or a sequence with exactly two 1s. It must follow that $|a_i|$ is a sequence of all zeros, otherwise $\|a\| = 2 \not< \epsilon$. Hence $x_n = x_m$ and the sequence converges to a constant sequence in A.

$\square$