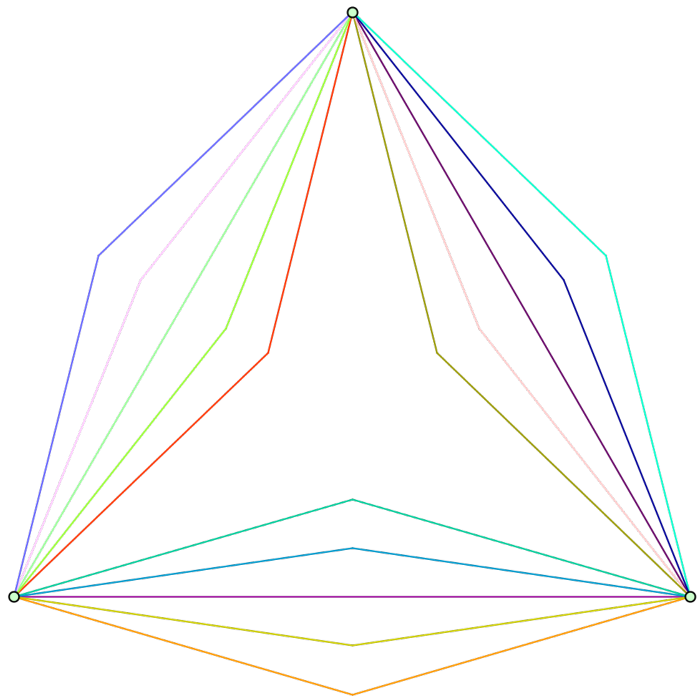
1. Give an example to show that if G is allowed to have multiple edges, then $\chi'(G)$ may exceed $\Delta(G)+1$.

Solution:

Figure 1: Multigraph G with $\Delta(G)+1 = 11$ but $\chi'(G) = 15$



2. Without using Proposition 5.3.1, show that $\chi'(G) = k$ for every k-regular bipartite graph G.

Proof. We will proceed to show that $\chi'(G) = k$ for every k-regular bipartite graph G via induction on k. Consider the base case, where G is a 1-regular bipartite graph. Since every vertex is incident to exactly one edge, we can simply color all the edges the same color, an produce a 1 edge coloring, hence by Vising's Theorem $\chi'(G) = 1$.

Suppose that G is $n+1$-regular bipartite. Since G is a regular bipartite graph, it contains a one factor, M. Note that $G - M$ is n-regular bipartite since each $e \in M$ contributes one degree to exactly one vertex in A and one vertex in B and since M spans the vertices of G. Now by the induction hypothesis there exists an n coloring of $G - M$, call it C. Apply C to G and color the edges of M the $n+1^{th}$ color, call this coloring C'. Note that C' is a valid coloring since M is a one factor. Hence by Vising's Theorem $\chi'(G) = n+1$. $\square$

3. Give an explicit edge-coloring to prove that the n-dimensional cube, Q^n, is Class 1.

Proof. Recall that the n-dimensional cube, Q^n is n- regular and therefore showing Q^n is Class 1 is equivalent to showing $\chi'(Q^n) = n$. Now we will proceed by induction on n. The base case is trivial since $Q^n = K^2$, there is only one edge.... color it. Now consider Q^{n+1} and recall the construction of Q^{n+1} via two copies of Q^n and an independent set of edges call them M (We have discussed this before and I've proven this set is independent in HW1). So $Q^{n+1} - M$ is two components, call them Q_1 and Q_2, they are isomorphic Q^n dimensional cubes and therefore $Q_1 = Q_2 = \chi'(Q^n) = n$. Apply the same edge-coloring C on Q_1 and Q_2, and color the edges of M the $(n+1)^{th}$ color. Thus we have produced an edge-coloring of Q^{n+1} with $n+1$ so by Vising's Theorem $\chi'(Q^{n+1}) = n+1$. $\square$

4. Prove that if G is a regular graph with a cut vertex, then $\chi'(G) > \Delta(G)$.

   Proof. Suppose G is a k-regular connected graph, with $\chi'(G) = \Delta(G) = k$. We will proceed to show that G has no cut vertex, by proving G is 2-connected. Since G is k-regular and k-edge colorable for every $u, v \in G$ and for all $\alpha, \beta \in [k]$ there exists an α, β alternating uv-path. Now let $u, v \in G$ and consider the α, β alternating uv-path. Regardless of the edge color entering v, there exist an edge of the other color leaving v and entering another vertex, not on the path. Since the α, β color classes span the graph G, the path can be extended to eventually enter u via a β edge, forming a cycle. Hence G is 2-connected.

   □

   The rest is for my own good.

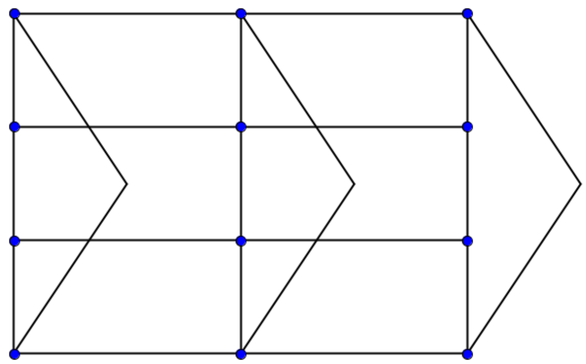
   Since G is k-regular and k-edge colorable we know that for every $\alpha \in [k]$ the set of all α-edges is a one-factor by regularity, and even further for every pair of $\alpha, \beta \in [k]$ the set of all α-edges is disjoint from the set of all β-edges by being k-edge colorable. Thus we can conclude that the subgraph formed by all α-edges and β-edges is connected and spans the vertices of G and clearly any path between two vertices will be alternating α, β.

5. The **cartesian product** of two graph G and H, denoted $G\Box H$, is the graph with vertex set $V(G\Box H) = V(G)\times V(H)$. A pair of vertices $(x_1,y_1),(x_2,y_2)$ are adjacent in $G\Box H$ if and only if $x_1 = x_2$ and $y_1y_2 \in E(H)$ or $x_1x_2 \in E(G)$ and $y_1 = y_2$.

   (a) Draw $P_2\Box C^4$.

   Solution:

   Figure 2: $P_2\Box C^4$.

   (b) Prove that $\Delta(G\Box H) = \Delta(G)+\Delta(H)$

   Proof. Let G and H be simple graphs, and let $(u,v)\in V(G\Box H)$. By definition, each adjacency of u in G, and v in H, induces an adjacency of (u,v) in $G\Box H$, so it follows that $d((u,v)) = d(u)+d(v)$. Clearly it follows that, $\Delta(G\Box H) = \Delta(G)+\Delta(H)$. $\square$

   (c) Prove that if $\chi'(H) = \Delta(H)$, then $\chi'(G\Box H) = \Delta(G\Box H)$.

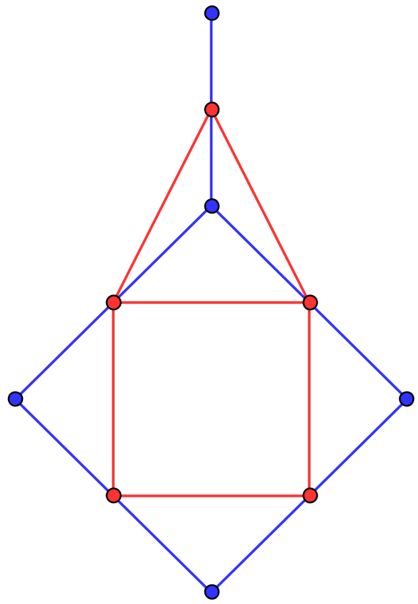
   Proof. Suppose $\chi'(H) = \Delta(H)$. Now consider $(u,v)\in G\Box H$ such that $u\in G$ and $v\in H$ have maximum degree. Color every copy of G in $G\Box H$ with $\Delta(G)+1$ colors. Now note that for every $u\in V(G)$ there exists at least one unused color across all copies of u in $G\Box H$, call this color c_u. For every $u\in V(G)$ color the graph H incident to every copy of u with $\Delta(H)-1$ new colors and the corresponding unused color c_u. Thus we have constructed a $\delta(G)+1+\delta(H)-1 = \Delta(G)+\Delta(H) = \Delta(G\Box H)$ coloring of $G\Box H$.

   $\square$

6. (a) Let G_1 be a 5-cycle with one chord. Show that there exists a graph H such that $L(H) = G_1$.

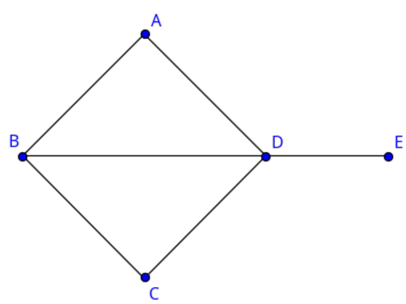
Solution:

Figure 3: Graph G_1 in Red and H in Blue.



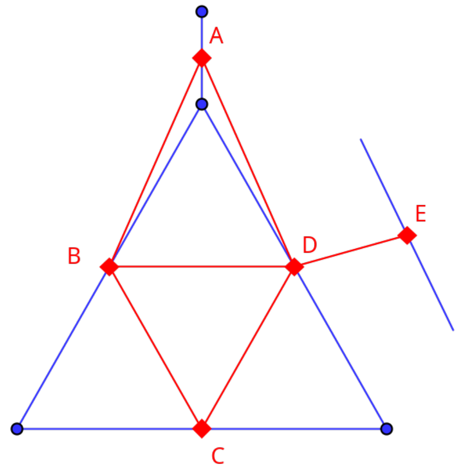
(b) Prove that for the graph G_2 (drawn below) that there does not exist any graph H such that $L(H) = G_2$.

Figure 4: Graph G_2



Proof. Suppose for the sake of contradiction that there exists a graph H such that $L(H) = G_2$. Clearly the part of G_2 which is a 4-cycle with a chord can be a line graph via a similar construction as the previous problem. However to incorporate vertex E into the line graph, would also add an edge CE or AE.

Figure 5: Graph G_2 in Red and H in Blue



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