1. Carothers 4.14 Show that the set $A = \{x \in \ell^2 : |x_n| \leq 1/n, n = 1, 2, \dots\}$ is a closed set in ℓ_2 but that $B = \{x \in \ell_2 : |x_n| < 1/n, n = 1, 2, \dots\}$ is not an open set.

Proof. Suppose $(x_n) \subseteq A$ with $x_n \to x$. Let $\epsilon > 0$ and note that there exists some N such that if $n \geq N$ we have,

$$\left(\sum_{k=1}^{\infty}(x_n(k) - x(k))^2\right)^{\frac{1}{2}} < \epsilon$$

Note that for a fixed k we know,

$$|x_n(k) - x(k)| \leq \left(\sum_{i=1}^{\infty}(x_n(i) - x(i))^2\right)^{\frac{1}{2}} < \epsilon.$$

Therefore the sequence (x_n) converges term-wise. Recall that by the definition of A, for each k it follows that $|x_n(k)| \leq \frac{1}{k}$ or equivalently $-\frac{1}{k} \leq x_n(k) \leq \frac{1}{k}$. Since $x_n(k) \to x(k)$ it follows that $-\frac{1}{k} \leq x(k) \leq \frac{1}{k}$, and therefore $|x(k)| \leq \frac{1}{k}$. Hence $x \in A$ and A is therefore closed. □

Proof. First note that the 0 sequence is clearly in B. To show that B is not open, we will show that for all $\epsilon > 0$ there exists an $x \in B_\epsilon(0)$ such that $x \notin B$.

Let $\epsilon > 0$ and find n such that $\frac{1}{n} < \epsilon$, then construct the sequence x such that $x(k) = 0$ when $k \neq n$ and $x(k) = \frac{1}{n}$ when $k = n$. Note that,

$$d(0, x) = \|x\|_2 = \frac{1}{n} < \epsilon.$$

Therefore $x \in B_\epsilon(0)$, but clearly $|x(n)| = \frac{1}{n}$ and not less than, so $x \notin B$. Hence B is not open.

□

2. Carothers 4.19 Show that $\text{diam}(A) = \text{diam}(\overline{A})$.

Proof. Let $D(A) = \{d(a, b) : a, b \in A\}$ and $D(\overline{A}) = \{d(a, b) : a, b \in \overline{A}\}$.

Let $x \in \overline{D(A)}$. Then by definition there exists a sequence $(d(a, b)_n) \subseteq D(A)$ where $d(a, b)_n \to x$. Choose (a_n) and (b_n) in A such that $d(a_n, b_n) = d(a, b)_n$. Since d is a continuous function and $d(a, b)_n \to x$ it follows that $a_n \to a$ and $b_n \to b$ where $d(a, b) = x$. By definition $a, b \in \overline{A}$ and therefore $x \in D(\overline{A})$. Hence $\overline{D(A)} \subseteq D(\overline{A})$.

let $x \in D(\overline{A})$. By definition $x = d(a, b)$ where $a, b \in \overline{A}$. Therefore there exists sequences (a_n) and (b_n) in A such that $a_n \to a$ and $b_n \to b$. Since d is a continuous function, it follows that $d(a_n, b_n) \to d(a, b) = x$, therefore $x \in \overline{D(A)}$. Thus $D(\overline{A}) \subseteq \overline{D(A)}$.

Since $\overline{D(A)} = D(\overline{A})$ we know that $D(A)$ and $D(\overline{A})$ have the same limit points and therefore $\sup D(A) = \sup D(\overline{A})$ and hence $\text{diam}(A) = \text{diam}(\overline{A})$.

□

3. Carothers 5.17 Let $f, g : (M, d) \to (N, \rho)$ be continuous, and let D be a dense subset of M. If $f(x) = g(x)$ for all $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If f is onto, show that $f(D)$ is dense in N.

Proof. Let $x \in M$. Since D is dense, $\overline{D} = M$ so we can find a sequence $(a_n) \subseteq D$ such that $a_n \to x$. Since f and g are continuous functions it follows that $g(a_n) \to g(x)$ and $f(a_n) \to f(x)$. Since $(a_n) \subseteq D$ we know that $g(a_n) = f(a_n)$ and since N is a metric space with unique limits, $f(x) = g(x)$.

Now suppose f is onto, let $y \in N$ and therefore there exists some $x \in M$ such that $f(x) = y$. Since D is dense in M there exists a sequence $(a_n) \subseteq D$ which converges $a_n \to x$. Since f is continuous it follows that $f(a_n) \to y$. Note $f(a_n) \subseteq f(D)$ and therefore $N \subseteq \overline{f(D)}$. Clearly $\overline{f(D)} \subseteq N$ so hence $N = \overline{f(D)}$. Thus $f(D)$ is dense in N. □

4. Carothers 5.19 A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a Lipschitz condition if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. More economically, we may say that f is Lipschitz. Show that $\sin(x)$ is Lipschitz with constant $K = 1$. Prove that a Lipschitz function is (uniformly) continuous.

Proof. Since $\sin(x)$ is continuous and infinitely differentiable the mean value theorem applies. Let $a, b \in \mathbb{R}$ such that $a < b$ and by the MVT there exists some $c \in (a, b)$ such that,

$$\cos(c) = \frac{\sin(b) - \sin(a)}{b - a}.$$

Taking the absolute value of both sides, we find that

$$|\cos(c)| = \frac{|\sin(b) - \sin(a)|}{|b - a|}.$$

Note that $|\cos(c)| \leq 1$ and therefore by substitution we conclude that,

$$|\sin(b) - \sin(a)| \leq 1|b - a|.$$

□

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant K. Let $\epsilon > 0$, $x, y \in \mathbb{R}$ and choose $\delta = \frac{\epsilon}{K}$. Then when $|x - y| < \delta$ it follows that,

$$|f(x) - f(y)| \leq K|x - y| < K\delta = \epsilon.$$

Since δ was chosen solely as a function of ϵ, f is uniformly continuous. □

5. Carothers 5.24 Let V be a normed vector space. If $y \in V$ is fixed, show that the maps $f : \alpha \to \alpha y$ from $\mathbb{R}$ into V, and $g : x \to x + y$, from V into V, are continuous.

Proof. Let $\alpha \in \mathbb{R}$ and $(\alpha_n) \subseteq \mathbb{R}$ such that $\alpha_n \to \alpha$. Let $\epsilon > 0$. Then since $\alpha_n \to \alpha$, choose N such that if $n \geq N$, then $|\alpha_n - \alpha| < \frac{\epsilon}{||y||_2}$, it follows that

$$||\alpha_n y - \alpha y|| = ||(\alpha_n - \alpha)y|| = |(\alpha_n - \alpha)|| \cdot ||y|| < \epsilon.$$

Therefore $f(\alpha_n) \to f(\alpha)$. □

Proof. Let $x \in V$ and $(x_n) \subseteq V$ such that $x_n \to x$. Let $\epsilon > 0$. Then since $x_n \to x$, choose N such that $n \geq N$, then $||x_n - x|| < \epsilon$, it follows that

$$||(x_n + y) - (x + y)|| \leq ||x_n - x|| < \epsilon.$$

Therefore $g(x_n) \to g(x)$.

□

6. Carothers 5.25 A function $f : (M, d) \to (N, \rho)$ is called Lipschitz if there is a constant $k < \infty$ such that $\rho(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in M$. Prove that a Lipschitz mapping is continuous.

Proof. Let $x \in M$ and $(x_n) \subseteq M$ such that $x_n \to x$. Let $\epsilon > 0$. Since $x_n \to x$, choose L such that if $n \geq L$, then $d(x_n, x) \leq \epsilon/k$, it follows that,

$$\rho(f(x_n), f(x)) \leq Kd(x_n, x) < \epsilon.$$

□

7. Carothers 5.28 Define $g : \ell^2 \to \mathbb{R}$ by $g(x) = \sum_{n=1}^{\infty} x(k)/k$. Is g continuous.

Proof. Suppose $x \in \ell^2$ and $(x_n) \subseteq \ell^2$ such that $x_n \to x$.

$$\begin{aligned} |g(x_n) - g(x)| &= \Big|\sum_{k=1}^{\infty} x_n(k)\frac{1}{k} - \sum_{k=1}^{\infty} x(k)\frac{1}{k}\Big| \\ &= \Big|\sum_{k=1}^{\infty} (x_n(k) - x(k))\,\frac{1}{k}\Big| \\ &\leq \sum_{k=1}^{\infty} \Big|(x_n(k) - x(k))\,\frac{1}{k}\Big| \end{aligned}$$

By the Cauchy-Schwartz inequality we conclude that,

$$|g(x_n) - g(x)| \leq ||x_n - x||_2 \left\|\frac{1}{k}\right\|_2$$

Since $x_n \to x$ it follows that $||x_n - x||_2 \to 0$. Note that $\left(\frac{1}{k}\right) \in \ell^2$ so we know that $\left\|\frac{1}{k}\right\|_2 < \infty$. Therefore $|g(x_n) - g(x)| \to 0$ and thus $g(x)$ is continuous.

□

8. Carothers 5.32 Prove that f is lower semicontinuous if and only if $f(x) \leq \liminf_{n\to\infty} f(x_n)$ whenever $x_n \to x$ in M.

Proof. Suppose f is a lower semicontinuous function and $x \in M$ with $(x_n) \subseteq M$ such that $x_n \to x$. Let $m = \liminf_{n\to\infty} f(x_n) < \infty$, otherwise concluding $f(x) \leq \liminf_{n\to\infty} f(x_n)$ is trivial. By the definition of the limit infimum there are finitely many n such that $m - \epsilon > f(x_n)$ and infinitely many n such that $f(x_n) < m + \epsilon$. Hence it follows that the set $D = \{x \in M : f(x_n) < m + \epsilon\}$ contains infinitely many terms of x_n. Consider the subsequence (x_{n_j}) of elements of (x_n) which are contained in D. Since f is lower semicontinuous, D is closed and since $x_{n_j} \to x$ it must be the case that $x \in D$. Thus $f(x) \leq \liminf_{n\to\infty} f(x_n)$. □

Proof. Suppose that $f(x) \leq \liminf_{n\to\infty} f(x_n)$ whenever $x_n \to x$ in M. For the sake of contradiction suppose there exists an $\alpha \in \mathbb{R}$ such that the set $D = \{x \in M : f(x_n) \leq \alpha\}$ is not closed. Therefore there exists $(x_n) \subseteq D$ such that $x_n \to x$ with $x \notin D$. However since $x_n \to x$ in M it follows that $f(x) \leq \liminf_{n\to\infty} f(x_n)$. Since $(x_n) \subseteq D$ then we know that $f(x_n) \leq \alpha$ so therefore $\inf_{n\to\infty} f(x_n) \leq \alpha$. Finally we conclude that

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \leq \alpha.$$

Thus $x \in D$, a contradiction. □

9. Carothers 7.5 Prove that A is totally bounded if and only if $\overline{A}$ is totally bounded.

Proof. Suppose A is totally bounded. Given $\epsilon > 0$, there exists $A_1, \ldots, A_n \subset A$, with $\text{diam}(A_i) < \epsilon$ for all i, such that $A \subset \cup_{i=1}^n A_i$. Note that by Problem 4.1 $\text{diam}(\overline{A_i}) = \text{diam}(A_i) < \epsilon$.

Let $x \in \overline{A}$. Then there exists $(x_n) \subseteq A$ such that $x_n \to x$. But $A \subseteq \cup_{i=1}^n A_i$, so $(x_n) \subseteq \cup_{i=1}^n A_i$. Since the set $\{A_i\}_{i=1}^n$ is finite there exists an A_k with a tail of (x_n). Hence there exists $(x_{n_j}) \subset A_k$ such that $x_{n_j} \to x$ and therefore by definition $x \in \overline{A_j}$. Therefore $\overline{A} \subseteq \cup_{i=1}^n \overline{A_i}$.

□

Proof. Suppose $\overline{A}$ is totally bounded. Given $\epsilon > 0$ there exists $A_1, \ldots, A_n \subset \overline{A}$, with $\text{diam}(A_i) < \epsilon$ for all i, such that $\overline{A} \subseteq \cup_{i=1}^n A_i$. Note that $A \subseteq \overline{A} \subseteq \cup_{i=1}^n A_i$, hence A is totally bounded. □

10. Carothers 7.10 Prove that a totally bounded metric space M is separable.

Proof. Let M be a totally bounded metric space. Let $\epsilon > 0$ and choose n such that $1/n < \epsilon$. Since M is totally bounded there exists a finite $(1/n)$-net corresponding to balls of radius $1/n$, call it D_n. Let $D = \cup_{n=1}^\infty D_n$, and note that D is a countable union of finite sets, and is therefore countable. What is left to show is that D is dense in M.

Let $x \in M$ and with $\epsilon > 0$ consider $B_\epsilon(x)$. Choose n such that $1/n < \epsilon$, and note that there exists an element $x' \in D_n$ such that $d(x, x') < 1/n < \epsilon$ and hence $x' \in B_\epsilon(x)$. Therefore D is dense in M. $\square$