1. (a) Find a bipartite graph with a set of preferences such that no matching of maximum size is stable and no stable matching has maximum size.

Solution:
Consider the following bipartite graph and preferences,

Figure 1: Bipartite Graph, G

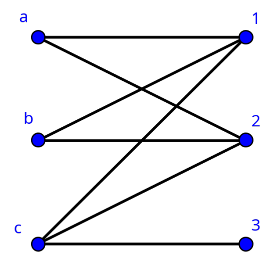


Table 1: Preferences for G

a:	2	>	1				1:	c	>	b	>	a
b:	1	>	2				2:	c	>	a	>	b
c:	1	>	2	>	3		3:	c				

Suppose M is a matching of maximum size. Note m must contain the edge $c3$ yet any matching that contains this edge must be unstable, since c prefers vertices 1 and 2 over 3 and vertices 1 and 2 prefer c over a and b. Therefore no matching of maximum size is stable. Let M be a stable matching, note that c will always be saturated by this matching by edge $c1$ and hence M does not contain $c3$ and is therefore not a maximum matching

(b) Find a non-bipartite graph with a set of preferences that has no stable matching.

Proof. Consider a love triangle. □

P.S: Just want to preface that this is not my best work, and very rushed, very sorry.

2. Give an infinite family of examples that demonstrates that the **bridgeless** hypothesis in Corollary 2.2.2 is necessary.

   Proof. Let $n \in \mathbb{N}$ be an odd, such that $n > 3$. Now consider the rooted tree on n vertices where every parent vertex, has two child vertices, unless they are leaves. Now connect the leaves by a cycle. Call this graph, S_n. Note that except for the root which has degree 2, this graph is 3-regular, and has odd degree. Now we construct a family of graphs which demonstrate the **bridgeless** hypothesis in Corollary 2.2.2 is necessary. Let G_n be a vertex c, adjacent to the root vertices of 3 S_n graphs. Again note that now G_n is 3-regular, c has degree 3 and the degree of the root vertices in each S_n component has been increased by 1 and are now 3. This graph also has 3 bridges incident to c. Applying Tutte's Theorem with $S = \{c\}$ we find that $q(G_n - S) = 3 \not\leq 1 = |S|$ and hence G_n cannot have a 1-factor. $\square$

3. A graph G is called **critically 2-connected** if G is 2-connected but for every edge $e \in E$, $G - e$ is no longer 2-connected.

   (a) Find an infinite family of critically 2-connected graphs that are not cycles.

   Proof. Consider a cycle C_n where $n \geq 4$, choose two antipodal vertices u, v and form a path between them through a new vertex c. Call this new graph C_n^*, and note that this graph is clearly 2-connected, as you could remove any single vertex and remain connected. Note that C_n^* is critically two connected, since of one were to remove any edge on the original cycle or on the new path the graph could be easily disconnected by removing either u or v.

   □

   (b) Prove that if G is 2-connected, then the statements below are equivalent:

   - G is critically 2-connected.

     Proof. Let G be two connected and suppose that G is critically 2-connected. Suppose for the sake of contradiction that there exists an cycle C, in G with a chord, call it uv. Note that in the cycle C which is 2-connected, $C - uv$ remains 2-connected, since $C - uv$ is still a cycle. Thus $G - uv$ is 2-connected.

     □

   - No cycle in G has a chord.

     Proof. Let G be 2-connected and suppose that no cycle in G has a chord. Suppose for the sake of contradiction that G is not critically 2-connected. Therefore there exists an edge e such that $G - e$ is still 2-connected. Let $x, y \in G$ incident to e. Since $G - e$ is 2 connected, $G - e$ is at least 2-edge connected. By Menger's Theroem there exists at least 2 edge-disjoint paths between a and b in $G - e$. These paths form a cycle, and hence e was a chord in a cycle in G a contradiction. □

4. Prove that the block graph of any connected graph is a tree.

Proof. Suppose G is a connected graph, and let $\mathscr{B}(G)$ be it's block graph. Let $u, v \in \mathscr{B}(G)$ and note that u and v can represent a block or a cut vertex in G. Since each block in G contains a cut vertex, every vertex in $\mathscr{B}(G)$ which represents a block in G, is adjacent to a vertex which represents a cut vertex in G therefore to show that $\mathscr{B}(G)$ is connected, it is sufficient to show that there exists a path between any two vertices which represent a cut vertex in G. Let u and v represent cut vertices such that $u \neq v$. Since G is connected there exists a uv path in G. In $\mathscr{B}$ this path is represented by a path for some $n \geq 1$ such that $uB_1a_1B_2, \ldots, B_nv$, where B_n are vertices which represent blocks and a_n are cut vertices in G.

We wish to show that that $\mathscr{B}(G)$ is a cyclic. By construction $\mathscr{B}(G)$ is a bipartite graph, and therefore has no odd cycles. Suppose for the sake of contradiction that there exists an even cycle C, in $\mathscr{B}(G)$. Let without loss of generality let $C = a_1B_1a_2, ...B_na_1$, now note that this cycle can be lifted into G by simply choosing a path $P_i \subseteq B_i$ which between a_i and a_{i+1}. Therefore there exists a cycle in G, C_G such that $a_1P_1a_2...P_na_1$. Since this is a cycle, not contained in a single block we have contradicted that the cycle of G are the cycles of its blocks.

$\square$

5. Use Menger's Theorem to prove that the statements below are equivalent:

   - G is 2-connected.
   - Every pair of vertices of G lie on a common cycle.
   - Every pair of edges of G lie on a common cycle, G is connected

Proof. $(a \to b)$ Suppose G is 2-connected and let $u, v \in V(G)$, by (global) Menger's Theorem, G contains 2 independent paths between u and v, call them P_1 and P_2. Note that P_1 and P_2 form a common cycle in G. □

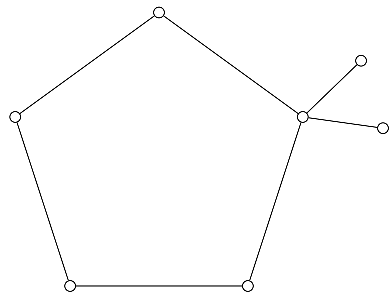
Proof. $(b \to a)$ Suppose every pair of vertices of G lie on a common cycle. Let $u, v \in V(G)$ such that both vertices lie on C, a common cycle in G. Then there exists two vertex disjoint paths P_1 and P_2. By (global) Menger's Theorem G is 2-connected. □

Proof. $(a \to c)$. Suppose G is 2-connected, and consider edges u_0v_0 and u_1v_1 in G. Add the following H-paths u_0av_0 and u_1bv_1 to the graph and call it G'. Note since we added H-paths to a 2-connected graph G' is still 2-connected. Now we apply Menger's Theorem to G', choosing disconnecting sets $\{a\}$ and $\{b\}$. Again since G' is two connected the minimum number ov vertices separating a and b is 2, and therefore by Menger's Theorem the maximum number of disjoint $a-b$ paths in G' is 2, call them P_1 and P_2. Since a and b only have two neighbors, these disjoint $a-b$ paths must contain the following, without loss of generality $v_0, v_1 \in P_1$ and $u_0, u_1 \in P_2$. It follows that, in G, edges u_0v_0 and u_1v_1 form a cycle with $P_1 - \{a, b\}$ and $P_2 - \{a, b\}$. Hence any two edges lie on a cycle in G. □

Proof. $(c \to b)$ Suppose G is connected an any two edges lie on a common cycle. Let $a, b \in G$, and note that since G is connected we can let e_1 be incident to a and e_2 be incident to b such that $e_1 \neq e_2$. Let C be the common cycle containing both e_1 and e_2. This cycle must contain a and b □

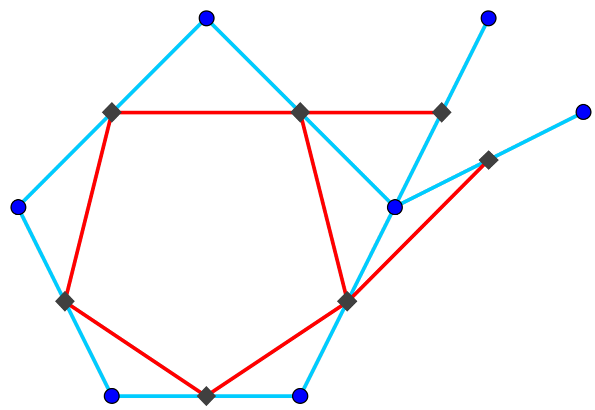
6. Given a graph $G = (V, E)$, the **line graph of** G, denoted $L(G)$ has vertex set E and two vertices $e, f \in E$ are adjacent in $L(G)$ if and only if e and f are incident in G.

   (a) Determine $L(G)$ for P^m, C^k and H (graphed below).



Proof. Note that in the $L(P^m)$ the interior vertices of P^m get mapped to edges and the edges get mapped to vertices. Hence $L(P^m) = P^{m-1}$. A similar argument gets us that $L(C^k) = C^k$, since there are k edges in a C^k. Below is $L(H)$ denoted by the red edges and black square vertices,

Figure 2: $L(H)$ in red, and H in blue.



□

   (b) Show that a cut set of vertices in $L(G)$ must correspond to a cut set of edges of G, but that the reverse does not necessarily hold.

Proof. Let G be a connected graph and let S be a cut set of vertices in $L(G)$ suppose for the sake of contradiction that S corresponds to a set of edges E, that does not disconnect G. Note that by definition $L(G-E) = L(G) - S$ however $G - E$ is connected and $L(G) - S$ is disconnected, a contradiction. □

Proof. As a counter example to show that the reverse does not hold, consider a C_5 and note that $L(C_5) = C_5$. Choose a cut set of edges to be a pair of edges e_1 and e_2 which are incident to the same vertex in C_5. Note that in $L(C_5)$, e_1 and e_2 correspond to two adjacent vertices a cycle, which once removed the graph remains connected. □