1. Carothers 16.40 If A and B are measurable sets, show that $m(A \cup B) + m(A \cap B) = m(A) + m(B)$

Proof. Since A and B are measurable sets by CC it follows that,

$$m(A \cap B) + m(A^c \cap B) = m(B),$$

$$m(B \cap A) + m(B^c \cap A) = m(A).$$

Adding both equalities we get,

$$m(A \cap B) + m(B \cap A) + m(A^c \cap B) + m(B^c \cap A) = m(B) + m(A).$$

Since $A \cup B = (B \cap A) \cup (A^c \cap B) \cap (B^c \cap A)$, a finite union of disjoint sets and by finite additivity we find that, $m(A \cup B) = m(B \cap A) + m(A^c \cap B) + m(B^c \cap A)$. So by substitution we get,

$$m(A \cap B) + m(A \cup B) = m(B) + m(A).$$

□

2. Carothers 16.42 Suppose E is measurable with $m(E) = 1$. Show that:

(a) There is a measurable set $F \subset E$ such that $m(F) = \frac{1}{2}$.[Hint: Consider the function $f(x) = m(E \cap (-\infty, x])$]

Proof. Consider $f(x) = m(E \cap (-\infty, x])$ and let $x, y \in \mathbb{R}$ where without loss of generality $y < x$, and note that by finite additivity and monotonicity the following holds,

$$\begin{aligned} |f(x) - f(y)| &= |m(E \cap (-\infty, x)) - m(E \cap (-\infty, y))| \\ &= |m(E \cap (-\infty, y)) + m(E \cap (y, x)) - m(E \cap (-\infty, y))| \\ &= |m(E \cap (y, x))| \leq |m((y, x))| = |x - y|. \end{aligned}$$

Thus f is Lipschitz and therefore continuous. Now note that $E_n = E \cap (-\infty, n]$ is an increasing sequence of measurable sets with $E_n \subseteq E_{n+1}$, by continuity from below it follows that,

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} m(E \cap (-\infty, n]) = m\left(\bigcup_{n=1}^{\infty} E \cap (-\infty, n]\right) = m\left(E \cap \left(\bigcup_{n=1}^{\infty}(-\infty, n]\right)\right) = m(E) = 1.$$

Similarly we find that $E_n = E \cap (-\infty, -n]$ is a decreasing sequence of measurable sets with $E_{n+1} \subseteq E_n$ and $E_1 < \infty$, so by continuity from above we find,

$$\lim_{n\to\infty} f(-n) = \lim_{n\to\infty} m(E \cap (-\infty, -n]) = m\left(\bigcap_{n=1}^{\infty} E \cap (-\infty, -n]\right) = 0.$$

So there exists $x \in \mathbb{R}$ for which $f(x)$ is arbitrarily close to 0 or 1. By the Intermediate Value Theorem there must exists some $x \in \mathbb{R}$ such that $f(x) = \frac{1}{2}$. □

(b) There is a closed set F consisting entirely of irrationals, such that $F \subset E$ and $m(F) = \frac{1}{2}$.

Proof. First note that by finite additivity we get the following,

$$m(E) = m(E \cap \mathbb{Q}) + m(E \cap \mathbb{Q}^c).$$

Since $m(E \cap \mathbb{Q})$ has measure zero, $m(E) = m(E \cap \mathbb{Q}^c) = 1$. Since $(E \cap \mathbb{Q}^c)$ is measurable there exists a closed and measurable set $F \subset (E \cap \mathbb{Q}^c)$ such that $m(F) = \frac{3}{4}$. By the previous argument the function $f'(x) = m(F \cap (-\infty, x])$ is continuous, and by the IVT (the upper bound will be arbitrarily close to $\frac{3}{4}$ from below and zero from above) also contains a value $x \in \mathbb{R}$ for which $f'(x) = m(F \cap (-\infty, x]) = \frac{1}{2}$. Note for such an x, $F \cap (-\infty, x]$ is closed and consists entirely of irrationals. □

(c) There is a compact set F' with empty interior such that $F' \subset E$ and $m(F') = \frac{1}{2}$.

Proof. Fix $y \in \mathbb{R}$ such that $f'(y) > \frac{1}{2}$. Then consider the function $g(x) = f'(y) - f'(x)$, on the domain $x \leq y$. Clearly this function is continuous, since f' is continuous. By finite additivity we get that for any $x \leq y$,

$$g(x) = f'(y) - f'(x) = m(F \cap (-\infty, y]) - m(F \cap (-\infty, x]) = m(F \cap (x, y]).$$

Note that when $x = y$ clearly $g(y) = 0$ and,

$$\lim_{x \to -\infty} g(x) = f'(y) - \lim_{x \to -\infty} f'(x) > \frac{1}{2}.$$

So by the Intermediate Value Theorem there exists an $x, y \in \mathbb{R}$ such that $m(F \cap (x, y]) = \frac{1}{2}$. Further it follows that $m(F \cap [x, y]) = \frac{1}{2}$ and since $F \cap [x, y]$ is closed and bounded, it is also compact and since F had an empty interior so does $F \cap [x, y]$. □

3. Carothers 16.44 Let E be a measurable set with $m(E) > 0$. Prove that $E - E = \{x - y : x, y \in E\}$ contains an interval centered at 0.

Proof. Let E be a measurable set and suppose $m(E) > 0$. Let $\alpha = \frac{3}{4}$, and note that by a previous homework there exists an open interval I such that $m(E \cap I) > \alpha m(I)$. Now choose an $x \in \mathbb{R}$ such that $|x| < m(I)/2$ and it follows that $m(I \cup (I + x)) \leq 3m(I)/2$. This is clear since translation by at just below half the length of I ensures that $m(I \cap (I + x)) > \frac{1}{2}$ and by inclusion-exclusion and finite additivity $m(I \cup (I + x)) \leq 3m(I)/2$.

Now suppose for the sake of contradiction that $E \cap I$ and $(E \cap I) + x$ are disjoint, then let $A = (E \cap I) \cup ((E \cap I) + x)$ by finite additivity and translation invariance it would follow that $m(A) = 2m(E \cap I)$ and so $m(A) > \frac{3}{2}m(I)$. Note that $A = (E \cap I) \cup ((E \cap I) + x) =$

$(E \cap I) \cup ((E + x \cap I + x))$ and so clearly $A \subseteq (I \cup (I + x))$ and by monotonicity a contradiction would follow,

$$\frac{3}{2}m(I) < m(A) \leq m(I \cup (I + x)) \leq \frac{3}{2}m(I).$$

Hence $E \cap I$ and $(E \cap I) + x$ are not disjoint. Moreover, since set intersection is commutative we find that,

$$\begin{aligned}(E \cap I) \cap ((E \cap I) + x) &\neq \emptyset,\\ (E \cap I) \cap ((E + x) \cap (I + x)) &\neq \emptyset,\\ (E \cap E + x) \cap (I \cap I + x) &\neq \emptyset.\end{aligned}$$

Which implies that $(E \cap E + x) \neq \emptyset$ and further that $x \in E - E$. Since x was chosen such that $|x| < m(I)/2$ we conclude that $(-m(I)/2, m(I)/2) \subset E - E$. □

4. Carothers 16.45 Let $f : X \to Y$ be any function.

(a) If $\mathcal{B}$ is a σ-algebra of subsets of Y, show that $\mathcal{A} = \{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra of subsets of X.

Proof. We must show that $\mathcal{A}$ is closed with respect to compliments, finite unions and countable unions. Let $U \in \mathcal{A}$ and note that by definition there exists some $B \in \mathcal{B}$ such that $U = f^{-1}(B)$. Now note $U^c = (f^{-1}(B))^c = f^{-1}(B^c)$. Since $\mathcal{B}$ is a σ-algebra, $B^c \in \mathcal{B}$ and therefore $U^c \in \mathcal{A}$.

Let $\{U_i\} \subset \mathcal{A}$ be a countable or finite collection, note again that by definition there exists some $B_i \in \mathcal{B}$ such that $U_i = f^{-1}(B_i)$. Therefore $\cup U_i = \cup f^{-1}(B_i) = f^{-1}(\cup B_i)$. Since $\mathcal{B}$ is a σ-algebra, $\cup B_i \in \mathcal{B}$ and therefore $\cup U_i \in \mathcal{A}$. □

(b) If $\mathcal{A}$ is a σ-algebra of subsets of X, show that $\mathcal{B} = \{B : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra of subsets of Y.

Proof. We must show that $\mathcal{B}$ is closed with respect to compliments, finite unions and countable unions. Let $U \in \mathcal{B}$ and note that by definition $f^{-1}(U) \in \mathcal{A}$. Note that $f^{-1}(U^c) = f^{-1}(U)^c \in \mathcal{A}$ since $\mathcal{A}$ is a σ-algebra. Hence $U^c \in \mathcal{B}$.

Let $\{U_i\} \subset \mathcal{B}$ be a countable or finite collection, and again note that for each i, $f^{-1}(U_i) \in \mathcal{A}$. Now it follows that

□

5. Carothers 16.53 Show that $\mathcal{B}$ is generated by each of the following:

(a) The open interval $\mathcal{E}_1 = \{(a, b) : a < b\}$

(b) The closed intervals $\mathcal{E}_2 = \{[a, b] : a < b\}$

(c) The half-open interval $\mathcal{E}_3 = \{(a, b], [a, b) : a < b\}$

(d) The open rays $\mathcal{E}_4 = \{(a, \infty), (-\infty, b) : a, b \in \mathbb{R}\}$

(e) The closed rays $\mathcal{E}_5 = \{[a, \infty), (-\infty, b] : a, b \in \mathbb{R}\}$

Proof. First we will show that $\mathcal{B} = \sigma(\mathcal{E}_1)$. Note that all the elements of $\mathcal{E}_1$ are open sets and therefore $\mathcal{E}_1 \subseteq \mathcal{B}$ and therefore $\sigma(\mathcal{E}_1) \subseteq \mathcal{B}$. Recall that $\mathcal{B}_\infty$ forms a topological basis for the open sets of $\mathbb{R}$ and therefore every open set $U \in \sigma(\mathcal{E}_1)$, moreover since $\mathcal{B}$ is the smallest σ-algebra containing the open sets it follows that $\mathcal{B} \subseteq \mathcal{E}_1$. So we conclude that $\mathcal{B} = \sigma(\mathcal{E}_1)$.

Now we will demonstrate that $\mathcal{E}_1 \subset \sigma(\mathcal{E}_i)$ for $i = 2, 3, 4, 5$. Let $(a, b) \in \mathcal{E}_1$, and consider the following constructions to show:

$$\mathcal{E}_1 \subset \sigma(\mathcal{E}_2), \qquad (a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

$$\mathcal{E}_1 \subset \sigma(\mathcal{E}_3), \qquad (a, b) = (a, b] \cap [a, b)$$

$$\mathcal{E}_1 \subset \sigma(\mathcal{E}_4), \qquad (a, b) = (a, \infty] \cap [-\infty, b)$$

$$\mathcal{E}_1 \subset \sigma(\mathcal{E}_5), \qquad (a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, \infty\right) \cap \left(-\infty, b - \frac{1}{n}\right]$$

So since $\mathcal{E}_1 \subset \sigma(\mathcal{E}_i)$ it follows that $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_i)$

It's clear that each generating set $\mathcal{E}_i \subseteq \mathcal{B}$, so it follows that $\sigma(\mathcal{E}_i) \subseteq \mathcal{B}$. Since $\mathcal{B} = \sigma(\mathcal{E}_1)$ and $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_i)$ it follows that,

$$\mathcal{B} = \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2) \subseteq \sigma(\mathcal{E}_3) \subseteq \sigma(\mathcal{E}_4) \subseteq \sigma(\mathcal{E}_5) \subseteq \mathcal{B}$$

□

6. Carothers 16.58 Suppose that $m(E) < \infty$. Prove that E is measurable if and only if, for every $\epsilon > 0$, there is a finite union of bounded intervals A such that $m(E \triangle A) < \epsilon$ (where $E \triangle A$ is the symmetric difference of E and A).

Proof. Let $m(E) < \infty$, and suppose E is measurable. Let $\epsilon > 0$ and let $\{I_i\}$ be a measuring cover of E such that,

$$\sum_{i=1}^{\infty} \ell(I_i) < m^*(E) + \epsilon$$

Choose N such that $\sum_{N+1}^{\infty} \ell(I_i) < \epsilon$. Let $A = \cup_{i=1}^{N} I_i$, note that since $\sum_{i=1}^{\infty} \ell(I_i)$ it is also the case that each I_i is bounded. Also let $U = \cup_{i=1}^{\infty} I_i$. Note that $U \setminus A = \cup_{i=N+1}^{\infty} I_i$, therefore it follows that,

$$m(U \setminus A) \leq m(\cup_{i=1}^{N} I_i) \leq \sum_{N+1}^{\infty} \ell(I_i) < \epsilon.$$

Since $E \subseteq U$ it follows that $E \setminus A \subset U \setminus A$ and by monotonicity,

$$m(E \setminus A) \leq m(U \setminus A) < \epsilon.$$

Note that since $m(U) \leq \sum_{i=1}^{\infty} \ell(I_i) < m(E) + \epsilon$, and E is measurable we find that,

$$\begin{aligned} m(U) &= m(U \cap E^c) + m(U \cap E), \\ m(U) - m(U \cap E) &= m(U \cap E^c), \\ m(U) - m(E) &= m(U \cap E^c). \end{aligned}$$

So it follows that $m(U \setminus E) < \epsilon$. Since $A \subseteq U$ it follows that $A \setminus E \subseteq U \setminus E$ and therefore by monotonicity,

$$m(A \setminus E) \leq m(U \setminus E) < \epsilon.$$

Since $E \triangle A = A \setminus E \cup E \setminus A$, by subadditivity,

$$m(E \triangle A) \leq m(A \setminus E) + m(E \setminus A) < 2\epsilon.$$

$\square$

Proof. Let $m^*(E) < \infty$ and suppose that for every $\epsilon > 0$, there is a finite union of bounded intervals A such that $m^*(E \triangle A) < \epsilon$. Since $(A \setminus E), (E \setminus A) \subseteq E \triangle A$ it follows that $m^*(E \setminus A) < \epsilon$ and $m^*(A \setminus E) < \epsilon$. Now there exists an open set U such that $U \supseteq E \setminus A$ and

$$m^*(U) < m^*(E \setminus A) + \epsilon.$$

Now define $U' = A \cup U$ and note that U' is open and $E \subseteq U'$. Now note that

$$m^*(U' \setminus E) = m^*((A \cup U) \cap E^c) = m^*((A \cap E^c) \cup (U \cap E^c)).$$

By subadditivity we get that,

$$\begin{aligned} m^*(U' \setminus E) &\leq m^*((A \cap E^c)) + m^*(U \cap E^c) \\ &= m^*(A \setminus E) + m^*(U \setminus E) \end{aligned}$$

Since $(U \setminus E) \subseteq U$ by monotonicity it follows that

$$m^*(U' \setminus E) \leq m^*(A \setminus E) + m^*(U).$$

Now it follows that since $m^*(U) < m^*(E \setminus A) + \epsilon$, $m^*(E \setminus A) < \epsilon$, and $m^*(A \setminus E) < \epsilon$

$$m^*(U' \setminus E) < 3\epsilon.$$

$\square$

7. **Carothers 16.64** Prove Corollary 16.26: Suppose that $m^*(E) < \infty$. Then, E is measurable if and only if, for every $\epsilon > 0$, there exists a compact set $F \subset E$ such that $m(F) > m^*(E) - \epsilon$.

Proof. Let $m^*(E) < \infty$ and suppose that E is measurable. Let $\epsilon > 0$ and note that by Problem 8 there exists a closed set $F \subseteq E$ such that $m^*(E \setminus F) < \frac{\epsilon}{2}$. Since F is closed and therefore measurable by CC, monotonicity, and our previous result we find that,

$$\begin{aligned} m(E) = m(E \cap F) + m(E \cap F^c), \\ m(E) < m(F) + \frac{\epsilon}{2}, \\ m(E) - \frac{\epsilon}{2} < m(F). \end{aligned}$$

Now consider the following collection of increasing measurable sets, $F \cap [n, n]_{n=1}^{\infty}$. By continuity from below we know that,

$$\lim_{n\to\infty} m(F \cap [n, n]) = m(\cup_{n=1}^{\infty} F \cap [n, n]) = m(F).$$

Therefore there exists some N such that $m(F \cap [N, N]) > m(F) - \frac{\epsilon}{2}$. Now it follows that,

$$m(F \cap [N, N]) > m(F) - \frac{\epsilon}{2} > m(E) - \epsilon.$$

Note that $F \cap [N, N]$ is closed and bounded and therefore compact. □

Lemma 1: Let $E \subseteq \mathbb{R}$ be a set, then if for every $\epsilon > 0$ there is a closed set $V \subseteq E$ such that $m^*(E \setminus V) < \epsilon$, then E is measurable.

Proof. Let $E \subseteq \mathbb{R}$ be a set, then suppose that for every $\epsilon > 0$ there is a closed set $V \subseteq E$ such that $m^*(E \setminus V) < \epsilon$. Note that equivalently for every ϵ there exists an open set $V^c \supseteq E^c$ such that $m^*(V^c \setminus E^c) = m^*(V^c \cap (E^c)^c) = m^*(V^c \cap E) = m^*(E \cap V^c) = m^*(E \setminus V) < \epsilon$. So we consclude that E^c is measureable, and since the measerable sets form a σ-algebra E is also measerable. □

Proof. Let $m^*(E) < \infty$ and suppose that for every $\epsilon > 0$ there exists a compact set $F \subset E$ such that $m(F) > m^*(E) - \epsilon$. Let $\epsilon > 0$, and consider a compact set $F \subset E$ such that $m(F) > m^*(E) - \epsilon$. Note F is compact in $\mathbb{R}$ and therefore closed, bounded and measurable. By CC and since $m^*(E \cap F) \leq m^*(E) < \infty$ we find that,

$$\begin{aligned} m^*(E) = m^*(E \cap F) + m^*(E \cap F^c), \\ m^*(E) - m^*(E \cap F) = m^*(E \setminus F). \end{aligned}$$

Since $F \subseteq E$ it follows that,

$$m^*(E) - m(F) = m^*(E \setminus F).$$

So finally we can conclude that

$$m^*(E \setminus F) = m^*(E) - m(F) < \epsilon.$$

□

8. Suppose $E \subseteq \mathbb{R}$. Prove that E is measurable if and only if for any $\epsilon > 0$ there is an opens set G and a closed set F such that $F \subseteq E \subseteq G$ and $m^*(G \setminus F) < \epsilon$.

Proof. Let $E \subseteq \mathbb{R}$ and suppose that E is measurable. Let $\epsilon > 0$, and recall that if E is measurable then there exists an open set G such that $E \subseteq G$ and $m^*(G \setminus E) < \epsilon$. Note that E^c is also measurable since measurable sets form a σ-algebra. It follows that there exists an open set U, with $E^c \subseteq U$ with $m^*(U \setminus E^c) < \epsilon$. Let $F = U^c$ a closed set, and note that since $U \subseteq E^c$ we know $F = U^c \supseteq E$. Finally note that $m^*(E \setminus F) = m^*(E \setminus U^c) = m^*(E \cap U) = m^*(U \cap (E^c)^c) = m^*(U \setminus E^c) < \epsilon$.

Therefore there exists a closed set F and an open set G such that $F \subseteq E \subseteq G$ and $m^*(G \setminus E) < \epsilon$ and $m^*(E \setminus F) < \epsilon$. Now note that $G \setminus F \subseteq (G \setminus E) \cup (E \setminus F)$, and therefore by subadditivity we conclude that,

$$m^*(G \setminus F) \leq m^*(E \setminus F) + m^*(G \setminus E) < 2\epsilon.$$

□

Proof. Let $E \subseteq \mathbb{R}$ and suppose that for any $\epsilon > 0$ there is an opens set G and a closed set F such that $F \subseteq E \subseteq G$ and $m^*(G \setminus F) < \epsilon$. Since $G \setminus F \supseteq G \setminus E$ it follows by subadditivity that,

$$m^*(G \setminus E) \leq m^*(G \setminus F) < \epsilon.$$

□

9. Carothers 16.73 If E is a measurable subset of a nonmeasurable set N (constructed in this section), prove $m(E) = 0$. [Hint: Consider $E_r = E + r(\mod 1)$, for $r \in \mathbb{Q} \cap [0, 1)$.]

Proof. Suppose E is a measurable subset of a nonmeasurable set N. Let $E_r = E + r(\mod 1)$, for $r \in \mathbb{Q} \cap [0, 1)$ and note by translation invariance we find that $m(E) = m(E_r)$. We now claim that E_r are disjoint, we have shown in class that the analogously defined N_r are disjoint, and since $E \subseteq N$ it follows that $E_r \subseteq N_r$ for all $r \in \mathbb{Q} \cap [0, 1)$, and therefore E_r are disjoint. Now by countable additivity, and monotonicity

$$m(\cup E_r) = \sum_{r \in \mathbb{Q} \cap [0,1)} m(E_r) \leq m([0, 1)).$$

Clearly it follows that if $m(E) \neq 0$ then $m(\cup E_r) = \sum_{r \in \mathbb{Q} \cap [0,1)} m(E_r) = \infty \nleq m([0, 1))$. So it must be the case that $m(E) = 0$. □

10. Carothers 16.74 If $m^*(A) > 0$, show that A contains a nonmeasurable set. [Hint: We must have $m^*(A \cap [n, n + 1)) > 0$ for some $n \in \mathbb{Z}$, and so we say suppose that $A \subset [0, 1)$. (How?) it follows from Exercise 73 that one of the sets $E_r = A \cap N_r$ is nonmeasurable. (Why?)]

Proof. Suppose $m^*(A) > 0$ and note that $\cup_{n\in\mathbb{Z}} A \cap [n, n+1) = A$ so we find by countable subadditivity that,

$$0 < m^*(A) = m^*\left(\bigcup_{n\in\mathbb{Z}} A \cap [n, n+1)\right) \leq \sum_{n\in\mathbb{Z}} m^*(A \cap [n, n+1)).$$

Therefore there must exist some $n \in \mathbb{Z}$ such that $m^*(A \cap [n, n+1)) > 0$. Consider a new A to be $(A \cap [n, n+1)) - n$ and note that by translation invariance it still holds that $m^*(A) = m^*(A \cap [n, n+1)) > 0$ and now we can assume $A \subset [0, 1)$. Now consider the sets $E_r = A \cap N_r$ where N_r are the familiar nonmeaserable sets constructed in class and invoked in the previous problem. Now for the sake of contradiction suppose that all such E_r are measurable. By the previous problem $m(E_r) = 0$ moreover since the collection of N_r are disjoint we know that E_r are disjoint so it follows by countable additivity,

$$m\left(\bigcup_{r\in\mathbb{Q}\cap[0,1)} E_r\right) = \sum_{r\in\mathbb{Q}\cap[0,1)} m(E_r) = 0.$$

Further it follows that,

$$m\left(\bigcup_{r\in\mathbb{Q}\cap[0,1)} E_r\right) = m\left(\bigcup_{r\in\mathbb{Q}\cap[0,1)} (A \cap N_r)\right) = m\left(A \cap \left(\bigcup_{r\in\mathbb{Q}\cap[0,1)} N_r\right)\right) = m(A \cap [0, 1)) = m(A).$$

A contradiction, since $m(A) > 0$. □

11. Carothers 16.75 Measurable sets aren't necessarily preserved by continuous maps, not even sets of measure zero. Here's an old example: Recall that the Cantor function $f : [0, 1] \to [0, 1]$ maps the Cantor set Δ onto $[0, 1]$. That is, the Cantor function takes a set of measure zero and 'spreads it out' to a set of measure one. Conclude that f maps some measurable set onto a nonmeasurable set.

Solution:
Since $m(f(\Delta)) = 1$, by the previous result there exists some subset $f(U) \subseteq f(\Delta)$ such that $f(U)$ is a nonmeasurable set. Now it must follow that $U \subseteq \Delta$, and therefore by monotonicity $m^*(U) \leq m^*(\Delta) = 0$, so U is a null set, which are themselves measurable. Hence f maps some measurable set onto a nonmeasurable set.