Lemma 1: The Lebesgue integral for nonnegative measurable functions is translation invariant. If f is a nonnegative measurable function, then for all $c \in \mathbb{R}$, we know $\int f = \int f(x+c)$.

Proof. Suppose f is nonnegative measurable function and $c \in \mathbb{R}$. Let $f' = f(x+c)$ and note that it is also a nonnegative measurable function. By definition of the Lebesgue integral we know that,

$$\int f = \sup\left\{\int \varphi : 0 \leq \varphi \leq f,\ \varphi, \text{ is simple and integrable}\right\}.$$

$$\int f' = \sup\left\{\int \psi : 0 \leq \psi \leq f(x+c),\ \psi, \text{ is simple and integrable}\right\}.$$

Let,

$$S_f = \left\{\int \varphi : 0 \leq \varphi \leq f, \varphi \text{ simple and integrable}\right\}$$

and define $S_{f'}$ analogously. Let $\int \varphi \in S_f$, and by definition we know that $0 \leq \varphi \leq f$. We have shown that $\int \varphi = \int \varphi(x+c)$, noting that $0 \leq \varphi(x+c) \leq f'$, we find $\int \varphi \in S_{f'}$. Hence $S_f \subseteq S_{f'}$. Now consider $\int \psi \in S_{f'}$, by definition we know that $0 \leq \psi \leq f(x+c)$ and therefore $0 \leq \psi(x-c) \leq f$ so we find that $\int \psi = \int \psi(x-c)$ so $\int \psi \in S_f$. Thus $S_f = S_{f'}$ and hence $\int f = \int f'$. □

1. In your last homework you showed that Riemann integrable functions are measurable. Now show that the Riemann integral and the Lebesgue integral agree for such functions.

Proof. (Preface: Riemann integrals are denoted with (R), we also recall that the Lebesgue and Riemann integrals for step functions agree, therefore integrals of step functions will be denoted with just $\int$. Also this is Theorem 18.16 in the book) Let $f \geq 0$ be a Riemann integrable function defined on an interval $[a, b]$. As stated, we have shown f to be measurable in a previous homework. Recall the definition of a Riemann integrable function, there exists two sequences of step function (ℓ_n) and (u_n) with $\ell_n \leq f \leq u_n$ such that,

$$\sup_n \int_a^b \ell_n = (R)\int_a^b f = \inf_n \int_a^b u_n$$

However by monotonicity of the Lebesgue integral it also follows that,

$$\sup_n \int_a^b \ell_n \leq \int_a^b f \leq \inf_n \int_a^b u_n.$$

Hence,

$$\int_a^b f = (R)\int_a^b f.$$

□

2. Carothers 18.21 Suppose that f, f_n are non-negative measurable functions, that $f_n \to f$ and that $f_n \leq f$ for all n. Show that $\int f = \lim_{n\to\infty} \int f_n$

Proof. Suppose hat f, f_n are non-negative measurable functions, that $f_n \to f$ and that $f_n \leq f$ for all n. By the definition of the Lebesgue integral for non-negative measurable functions,

$$\int f = \sup\left\{\int \varphi : 0 \leq \varphi \leq f, \varphi \text{ simple and integrable}\right\}.$$

Define the following set for a given function f,

$$S_f = \left\{\int \varphi : 0 \leq \varphi \leq f, \varphi \text{ simple and integrable}\right\}.$$

Fix n, and let φ be a measurable simple function with the property that $0 \leq \varphi \leq f_n$, since $f_n \leq f$ it follows that that $0 \leq \varphi \leq f$. Hence $S_{f_n} \subseteq S_f$, and therefore $\int f \geq \int f_n$ for all n, so therefore $\int f \geq \lim_{n\to\infty} \int f_n$.

Now by Fatou's Lemma it follows,

$$\begin{aligned}
\int \left(\liminf_{n\to\infty} f_n\right) &\leq \liminf_{n\to\infty} \int f_n \\
\int \left(\lim_{n\to\infty} f_n\right) &\leq \liminf_{n\to\infty} \int f_n \\
\int f &\leq \liminf_{n\to\infty} \int f_n \\
\int f &\leq \lim_{n\to\infty} \int f_n
\end{aligned}$$

□

3. Carothers 18.26 Let $f(x) = x^{-\frac{1}{2}}$ for $0 < x < 1$ and $f(x) = 0$ otherwise. Let (r_n) be an enumeration of $\mathbb{Q}$, and let $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$. Show that:

(a) $g \in L^1$, and in particular g is finite a.e.

Proof. First note that, since f is measurable, and $2^{-n} > 0$ for all n we know that $2^{-n} f(x - r_n)$ is a sequence of nonnegative measurable functions. Now by Corollary 18.11 it follows that,

$$\int g(x) = \int \sum_{n=1}^{\infty} 2^{-n} f(x - r_n) = \sum_{n=1}^{\infty} \int 2^{-n} f(x - r_n).$$

So therefore by translation invariance of the Lebesgue integral it follows that $\int f(x) = \int f(x - r_n)$ and therefore,

$$\int g(x) = \sum_{n=1}^{\infty} 2^{-n} \int f(x - r_n) = \sum_{n=1}^{\infty} 2^{-n} \int f(x) = \sum_{n=0}^{\infty} 2^{-n} = 2.$$

Therefore $g \in L^1$, and g is finite a.e. □

(b) g is discontinuous at every point and is unbounded on every interval; it remains so even after modification on an arbitrary set of measure zero;

Proof. □

(c) g^2 is finite a.e., but g^2 is not integrable on any interval.

Proof. □

4. Carothers 18.36 Suppose that $f, (f_n)$ are measurable and uniformly bounded on $[a, b]$. If $f_n \to f$ on $[a, b]$, prove that $\int_a^b |f_n - f| \to 0$.

Proof. Suppose that $f, (f_n)$ are measurable and uniformly bounded by constant K on $[a, b]$. Let $\epsilon > 0$ and by Egorov's Theorem choose a measurable set $E \subset [a, b]$ such that $m(E) < \frac{\epsilon}{2K}$ and $f_n \to f$ uniformly on $E' = [a, b] \setminus E$. On E' choose N such that for all $n \geq N$, we have $|f - f_n| < \frac{\epsilon}{(b-a)}$. Now note that since $f, (f_n)$ uniformly bounded on $[a, b]$, we know that $|f - f_n| < 2K$, so it follows that for all $n \geq N$,

$$\begin{aligned}
\int \chi_{[a,b]}|f - f_n| &= \int \chi_E |f - f_n| + \int \chi_{E'} |f - f_n| \\
&< \int \chi_E |f - f_n| + \int \chi_{[a,b]} |f - f_n| \\
&< \frac{\epsilon}{2K} 2K + (b - a) \frac{\epsilon}{(b - a)} = 2\epsilon
\end{aligned}$$

□

,

5. Carothers 18.39 Compute $\sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2}} \left(1 - \sqrt{\sin(x)}\right)^n \cos(x) dx$.

Proof. The series rewritten, with an integral over $\mathbb{R}$,

$$\sum_{n=0}^{\infty} \int \chi_{[0,\frac{\pi}{2}]} \left(1 - \sqrt{\sin(x)}\right)^n \cos(x) dx$$

Now note that since $0 \leq \sqrt{\sin(x)} \leq 1$, and $0 \leq \cos(x) \leq 1$ on $[0, \frac{\pi}{2}]$ we find that $\chi_{[0,\frac{\pi}{2}]}\left(1-\sqrt{\sin(x)}\right)^n\cos(x)$ is a sequence of nonnegative measurable functions. By Corollary 18.11 we find that,

$$\sum_{n=0}^{\infty}\int \chi_{[0,\frac{\pi}{2}]}\left(1-\sqrt{\sin(x)}\right)^n\cos(x)dx = \int \chi_{[0,\frac{\pi}{2}]}\sum_{n=0}^{\infty}\left(1-\sqrt{\sin(x)}\right)^n\cos(x)dx$$

Now note that $|(1-\sqrt{\sin(x)})^n| < 1$ on $(0, \frac{\pi}{2})$, which in terms of our integral is an equivalent support. With this fact, we note that the sum is an geometric series an conclude that,

$$\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}}\left(1-\sqrt{\sin(x)}\right)^n\cos(x)dx = \int_0^{\frac{\pi}{2}}\frac{\cos(x)}{\sqrt{\sin(x)}}dx.$$

This function however is unbounded, since as $x \to 0$ we have $\frac{1}{\sqrt{\sin(x)}} \to \infty$. Consider the following sequence of nonnegative measurable functions,

$$f_n = \chi_{(\frac{1}{n},\frac{\pi}{2})}\frac{\cos(x)}{\sqrt{\sin(x)}}$$

Note that clearly f_n converges pointwise to our desired integrand, with $f_n \leq f_{n+1}$. By the Monotone Convergence Theorem, it follows that

$$\int_0^{\frac{\pi}{2}}\frac{\cos(x)}{\sqrt{\sin(x)}} = \lim_{n\to\infty}\int_{\frac{1}{n}}^{\frac{\pi}{2}}\frac{\cos(x)}{\sqrt{\sin(x)}} = \lim_{n\to\infty}\left(2\sqrt{\sin\left(\frac{\pi}{2}\right)} - 2\sqrt{\sin\left(\frac{1}{n}\right)}\right) = 2.$$

□

6. Carothers 18.40 Let (f_n), (g_n), and g be integrable, and suppose $f_n \to f$ almost everywhere, $g_n \to g$ almost everywhere, $|f_n| \leq g_n$ almost everywhere for all n, and that $\int g_n \to \int g$. Prove that $f \in L^1$ and $\int f_n \to \int f$.

Proof. Let (f_n), (g_n), and g be integrable, and suppose $f_n \to f$ almost everywhere, $g_n \to g$ almost everywhere, $|f_n| \leq g_n$ almost everywhere for all n, and that $\int g_n \to \int g$.

Let E be the set where $g_n \not\to g$, $f_n \not\to f$ and $|f_n| \leq g_n$. Note that E is a union of null sets and is therefore a null set. Define a new sequence $\tilde{g_n} \to \chi_{E^c}g_n$ where $\tilde{g} = \chi_{E^c}g$. Define $\tilde{f_n}$ and $\tilde{f}$ analogously. Note that $|\tilde{f_n}| = \tilde{g_n} = 0$ on E and $|\tilde{f_n}| = |f_n| \leq g_n = \tilde{g_n}$ on E^cf and therefore $|\tilde{f_n}| \leq \tilde{g_n}$ everywhere.

$$|\tilde{f_n}| \leq \tilde{g_n}$$
$$-\tilde{g_n} \leq \tilde{f_n} \leq \tilde{g_n}$$

Since $\tilde{f_n} \to \tilde{f}$, and $\tilde{g_n} \to \tilde{g}$ we know

$$-\tilde{g} \leq \tilde{f} \leq \tilde{g}$$

So we conclude that $\left|\tilde{f}\right| \leq \tilde{g}$ a.e so since g is integrable we find that,

$$\int |f| = \int \left|\tilde{f}\right| \leq \int |g| = \int g < \infty.$$

So $f \in L^1$.

The following inequalities apply everywhere as a consequence of $\left|\tilde{f_n}\right| \leq \tilde{g_n}$.

$$\begin{aligned} -\tilde{g_n} \leq& \tilde{f_n} \leq \tilde{g_n} \\ 0 \leq& \tilde{g_n} + \tilde{f_n} \leq 2\tilde{g_n}, \end{aligned}$$

$$\begin{aligned} \tilde{g_n} \geq& -\tilde{f_n} \geq -\tilde{g_n} \\ 2\tilde{g_n} \geq& \tilde{g_n} - \tilde{f_n} \geq 0. \end{aligned}$$

Therefore $(\tilde{g_n} + \tilde{f_n}), (\tilde{g_n} - \tilde{f_n})$ are nonnegative everywhere. By Fatou's lemma,

$$\begin{aligned} \int \lim_{n\to\infty} \tilde{g_n} + \int \lim_{n\to\infty} \tilde{f_n} &= \int \lim_{n\to\infty} (\tilde{g_n} + \tilde{f_n}), \\ &\leq \liminf_{n\to\infty} \int (\tilde{g_n} + \tilde{f_n}), \\ &= \liminf_{n\to\infty} \int \tilde{g_n} + \int \tilde{f_n}, \\ &= \liminf_{n\to\infty} \int \tilde{g_n} + \liminf_{n\to\infty} \int \tilde{f_n}, \\ &= \int \tilde{g} + \liminf_{n\to\infty} \int \tilde{f_n}. \end{aligned}$$

$$\begin{aligned} \int \lim_{n\to\infty} \tilde{g_n} - \int \lim_{n\to\infty} \tilde{f_n} &= \int \lim_{n\to\infty} (\tilde{g_n} - \tilde{f_n}), \\ &\leq \liminf_{n\to\infty} \int (\tilde{g_n} - \tilde{f_n}), \\ &= \liminf_{n\to\infty} \int \tilde{g_n} - \int \tilde{f_n}, \\ &= \liminf_{n\to\infty} \int \tilde{g_n} - \limsup_{n\to\infty} \int \tilde{f_n}, \\ &= \int \tilde{g} - \limsup_{n\to\infty} \int \tilde{f_n}. \end{aligned}$$

Thus we conclude that $\lim_{n\to\infty} \int \tilde{f_n} = \int \tilde{f}$. Since $\tilde{f_n}, \tilde{f}$ and $(f_n), f$ differ on a set of measure zero, we also conclude that $f \in L^1$ and $\int f_n \to \int f$. $\square$

7. Carothers 18.41 Let (f_n),f be integrable, and suppose that $f_n \to f$ almost everywhere. Prove that $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

Proof. Suppose $\int |f_n - f| \to 0$. By the reverse triangle inequality,

$$||f_n| - |f|| \leq |f_n - f|\,. \tag{1}$$

Let $\epsilon > 0$ and choose N such that for all $n \geq N$ we have $\int |f_n - f| < \epsilon$, and note that

$$\left|\int |f_n| - \int |f|\right| = \left|\int |f_n| - |f|\right| \leq \int ||f_n| - |f|| \leq \int |f_n - f| < \epsilon.$$

Hence $\int |f_n| \to \int |f|$.

□

Proof. Let (f_n),f be integrable, and suppose that $f_n \to f$ almost everywhere. Suppose $\int |f_n| \to \int |f|$. Define $g_n = |f_n| + |f|$, and $h_n = |f_n - f|$. We find that g_n, h_n are integrable for all n. Note that $h_n \to 0$ a.e and $g_n \to 2\,|f|$ a.e since $f_n \to f$ a.e .Now we see that since, (g_n) is a sequence of nonnegative measurable functions we find by Fatou's Lemma that

$$\int g \leq \lim_{n\to\infty} \int |f_n| + |f| = 2\int |f| < \infty$$

Therefore g is integrable. Now note that $|h_n| = ||f_n - f|| \leq |f_n| + |f| = g_n$. We also know that $\int g_n \to \int g$ since,

$$\lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \int |f_n| + |f| = \lim_{n\to\infty} \int |f_n| + \lim_{n\to\infty} \int |f| = 2\int |f| = \int g.$$

Having satisfied the hypothesis for problem 18.40 it follows that $\int h_n \to \int h$ and therefore $\int |f_n - f| \to 0$ □

8. Carothers 18.55 Prove the Riemann-Lebesgue Lemma: If f is integrable on $\mathbb{R}$, then $f(x)\cos(nx)$ is integrable and $\lim_{n\to\infty} \int f(x)\cos(nx)dx = 0$. The same is true with $\sin(nx)$ instead of $\cos(nx)$.

Proof. Let f be integrable on $\mathbb{R}$. Note that since $|\cos nx| \leq 1$ for all n it also follows that,

$$\int |f(x)\cos(nx)| = \int |f(x)|\,|\cos n(x)| = \int |f(x)|\,|\cos n(x)| \leq \int |f(x)| < \infty$$

for all n. Therefore $f(x)\cos(nx)$ is integrable.

Consider the case where $f = \chi[a,b]$, and note that,

$$\lim_{n\to\infty} \int \chi_{[a,b]} \cos(nx)dx = \lim_{n\to\infty} \int_a^b \cos(nx)dx = \lim_{n\to\infty} \frac{1}{n}\left(\sin(nb) - \sin(na)\right) \to 0.$$

Recall that step function has finite step partition $\mathcal{P}$ and can be represented by the following sum, where $a_p < \infty$ is the value along $p \in \mathcal{P}$,

$$h = \sum_{p \in \mathcal{P}} \chi_p a_p.$$

Considering the case where $f = h$ we find,

$$\lim_{n\to\infty} \int \sum_{p\in\mathcal{P}} \chi_p a_p \cos(nx)dx = \lim_{n\to\infty} \sum_{p\in\mathcal{P}} a_p \int \chi_p \cos(nx)dx \to 0.$$

A finite sum of integrals which converge to zero, clearly converges to zero.

Finally to the main result, recall that f is an integrable function on $\mathbb{R}$ and let $\epsilon > 0$. By Theorem 18.27 there exists an integrable step function h such that $\int |f - h| < \epsilon$. Choose N such that for all $n \geq N$ we have $\int h \cos(nx) < \epsilon$. Now note that,

$$\begin{aligned}
\left|\int f(x)\cos(nx)dx\right| &= \left|\int f(x)\cos(nx) - h(x)\cos(nx) + h(x)\cos(nx)dx\right| \\
&= \left|\int (f(x) - h(x))\cos(nx) + \int h(x)\cos(nx)dx\right| \\
&\leq \left|\int (f(x) - h(x))\cos(nx)\right| + \left|\int h(x)\cos(nx)dx\right| \\
&< \int |(f(x) - h(x))| \cos(nx) + \epsilon \\
&< 2\epsilon.
\end{aligned}$$

□

9. For $t \in \mathbb{R}$ and $f \in L^1$, let $f_t(x) = f(x - t)$. Show that $f_t(x) \in L^1$ and that the map $t \to f_t$ is continuous from $\mathbb{R}$ to L^1.

Proof. Suppose $t \in \mathbb{R}$ and $f \in L^1$. Note that by Theorem 18.27 there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g = 0$ outside of an interval $[a, b]$ and $\int |g - f| < \epsilon$. Note that g is a continuous function on compact support and is therefore uniformly continuous. Note that by translation invariance we also have $\int |g_t - f_t| < \epsilon$ for all t.

Let $\epsilon > 0$, and consider a $\delta > 0$ such that if $|x - y| < \delta$ then $|g(x) - g(y)| < \frac{\epsilon}{b-a}$. Now note that by triangle inequality we get,

$$\begin{aligned}
\left\|f_x - f_y\right\|_{L^1} &\leq \|f_x - g_x\|_{L_1} + \left\|g_x - g_y\right\|_{L_1} + \left\|g_y - f_y\right\|_{L_1}, \\
&< 2\epsilon + \left\|g_x - g_y\right\|_{L_1},
\end{aligned}$$

Let $x < y$ and note that, the function $g_x - g_y$ has nonzero support over a region $[a + x, b + y]$ and therefore we get the following,

$$\left\|f_x - f_y\right\|_{L^1} \leq 2\epsilon + ((b + y) - (a + x))\frac{\epsilon}{(b - a)} = 2\epsilon((b - a) + (y - x))\frac{\epsilon}{(b - a)} \leq 3\epsilon + \delta\epsilon.$$

Clearly δ can be taken to be less than zero, and hence we have continuity of the map $t \to f_t$ from $\mathbb{R}$ to L^1. ◻

10. Carothers 19.23 Let $1 < p < \infty$ and let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$, then $\left|f^{p-1}\right| \in L^q$ and

$$\left\||f|^{p-1}\right\|_q = \|f\|_p^{p-1}.$$

Proof. Let $1 < p < \infty$, where q is defined by $\frac{1}{p} + \frac{1}{q} = 1$, and suppose that $f \in L^p$. Consider the following,

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} &= 1, \\ 1 + \frac{p}{q} &= p, \\ \frac{p}{q} &= p - 1, \\ p &= (p-1)q. \end{aligned}$$

Now note that,

$$\begin{aligned} \int |f|^{(p-1)q} &= \int |f|^p \\ \left\||f|^{p-1}\right\|_q^q &= \|f\|_p^p \\ \left\||f|^{p-1}\right\|_q &= \|f\|_p^{\frac{p}{q}} = \|f\|_p^{p-1}. \end{aligned}$$

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