1. Use Euler's Formula to prove that $K_{3,3}$ is not planar.

Proof. Suppose for the sake of contradiction that $K_{3,3}$ is planar. Let G be a plane graph, embedding $K_{3,3}$ and note that since $K_{3,3}$ is connected, so is G. By Euler's Formula it follows that,

$$6-9+\ell=2$$

Which implies that G must have 5 faces. However since $K_{3,3}$ is bipartite, there are no odd cycles, so each face in G must contain at least 4 edges. Therefore the number of edges in G is at least,

$$|E(G)| \geq \frac{(4)(5)}{2} = 10$$

Since the number of edges stays constant via an embedding and $|E(K_{3,3})| = |E(G)| = 9$ we have a contradiction. $\square$

2. Show that every connected planar graph with minimum degree at most 3 is a union of three forests.

   Proof. □

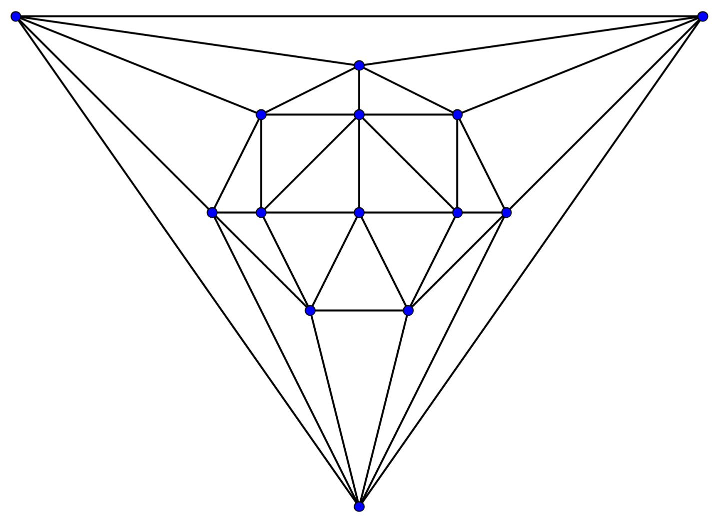
3. Show that every planar graph contains a vertex of degree at most 5. Give an example of a planar graph G such that $\delta(G) \geq 5$

Proof. Suppose G is a planar graph on n vertices, and suppose for the sake of contradiction that for every $v \in G$, $d(v) \geq 6$. First note that $n \geq 3$ since any 2 vertex simple graph cannot achieve $d(v) \geq 6$. Recall that,

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v) \geq \frac{1}{2}(6n) = 3n$$

However, we have shown that a plane triangulation on $n \geq 3$ vertices has at most $3n-6$ edges. Since plane triangulations are edge maximal plane graphs G cannot be planar, since $|E(G)| \geq 3n > 3n-6$. □

Figure 1: Example of a planar G with $\delta(G) \geq 5$.



4. A graph is called **outerplanar** if it has a drawing in which every vertex lies on the boundary of the outer (or infinite) face. Show that a graph is outerplanar if and only if it does not contains a K^4 minor or a $K_{2,3}$ minor.

   Proof. $\rightarrow$ Suppose a graph G is outerplanar and for the sake of contradiction suppose G contains a $K_{2,3}$ or K_4 minor. Let $\iota(G)$ be a plane graph via an embedding ι. Note that adding a vertex to the interior of a face and arcs to all the vertices on the boundary of that face, maintains planarity. Therefore, since G is outerplanar we can construct a plane graph $G' = \iota(G) + v$ where v is a vertex on the interior of the infinite face, such that v is incident to every vertex of $\iota(G)$. Since G' is still a plane graph, Kuratowski's Thoerem applies and we find that G' contains no K_5 or $K_{3,3}$ minor. However clearly a $K_{2,3}$ minor in $\iota(G)$ can be made into a $K_{3,3}$ minor in G' by adding v to the partite set with 2 vertices, and clearly a K_4 minor in $\iota(G)$ can be made into a K_5 in G' by considering v as the fifth vertex. Thus a contradiction. □

   Proof. $\leftarrow$ Suppose G does not contain a K^4 minor or a $K_{2,3}$ minor. Let $G' = G + v$ such that v is a vertex incident to all vertices in G. Note G' cannot have a $K_{3,3}$ or K_5 minor, since any such minor would necessarily have to include vertex v, as G has no K^4 minor or a $K_{2,3}$ minor however, such a minor in G' would imply the existence of a K^4 minor or a $K_{2,3}$ minor in G. By Kuratowski's Theorem G' is planar, furthermore there exists an embedding of G', call it ι, where v is contained in the outer face of G (this can be shown via a composition of homeomorphisms $\phi : \mathbb{R}^2 \rightarrow S^{2,*}$ and $\rho : S^{2,*} \rightarrow S^{2,*}$ where ρ moves the hole to the outer face of G). Now consider removing v from $\iota(G')$, and since v is contained in the outer face of G, $\iota(G') - v$ is now an outerplanar embedding of G.

   □

5. Let G be a 2-connected plane graph. Show G is bipartite if and only if every face is bounded by an even cycle.

Proof. $(\leftarrow)$ Let G be a 2-connected plane graph, and suppose that every face is bounded by an even cycle. Suppose for the sake of contradiction that G is not biparite. Then, G contains an odd cycle call it C. Either C bounds a face of G and we've found a contradiction or, C bounds multiple faces of G. Therefore there exists an arc of G contained in C which splits C into an odd and even cycle. Proceeding iteratively there must exists an odd cycle bounding a face, a contradiction. □

Proof. $(\rightarrow)$ Let G be a 2-connected plane graph, and suppose G is bipartite. Since G is bipartite, it has no odd cycles, so all its cycles are even. Every face is bounded by a cycle which must necessarily be even. □

6. Given a plane graph G, the **dual graph** G^*, of G is a plane graph whose vertices correspond to the faces of G. The edges of G^* are defined as follows: for every edge $e \in E(G)$ on the boundary of faces X and Y in G, edge $\{X,Y\} \in E(G^*)$. Note that the dual graph of a simple plane graph may or may not be simple.

   (a) Describe the dual graphs of P^m, C^k, and K^4.

   Solution:
   The dual graph of P^m looks like a single vertex with $m-1$ loops, since each edge of P^m bounds the infinite face twice.

   The dual graph of C^k is a pair of vertices, one representing the enclosed face, and the other representing the infinite face with k arcs in between.

   The dual graph of K^4 is another K^4.

   (b) Prove that if the n-vertex plane graph G is isomorphic to its dual, G^*, then $||G|| = 2n-2$.

   Proof. Suppose that G is an n-vertex plane graph, which is isomorphic to it's dual G^*. By the construction of G^* the number of faces of G and is equal to the vertices of G^*, since G and G^* are isomorphic, they have the same number of vertices, and therefore, G has the same number of vertices and faces. Applying Euler's Formula to G with $n = f$ we find that,

$$\begin{aligned} n - e + f &= 2, \\ n - e + n &= 2, \\ 2n - 2 &= e. \end{aligned}$$

□