1. **Carothers 2.21** Show that any ternary decimal of the form $0.a_1a_2\dots a_n11$(base 3) i.e., any finite-length decimal ending in two (or more) 1s, is not an element of Δ.

Proof. Let x be a ternary decimal of the form $0.a_1a_2\dots a_n11$(base 3). Recall that Δ was defined by the following recurrence relation,

$$A_0 = [0,1]$$
$$A_{k+1} = \frac{1}{3}A_k \cup \left(\frac{2}{3} + \frac{1}{3}A_k\right)$$
$$\Delta := \bigcap A_k$$

We also can describes x in base 10 via,

$$x = \sum_{i=1}^{n+2} \frac{a_i}{3^i} = \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}}.$$

Written in this form it is clear that x lies in middle third of A_{n+1}, more specifically x is $\frac{1}{3^{n+1}}$ greater than the right most endpoint of $\frac{1}{3}A_n$. Hence $x \notin \Delta$. □

2. **Carothers 2.22** Show that Δ contains no (nonempty) open interval. In particular, show that if $x, y \in \Delta$ with $x < y$, then there is some $z \in [0,1] \setminus \Delta$ with $x < z < y$.

Proof. Suppose that $x, y \in \Delta$ with $x < y$. Let x_n and y_n be a sequence of 0 and 2 (the ternary decimal form) such that the following equations satisfy $x < y$,

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}, \qquad y = \sum_{i=1}^{\infty} \frac{y_i}{3^i}.$$

Since $x < y$ there exists some i in the both series where $x_i \neq y_i$ and $x_i = 2$ and $y_1 = 0$. This entry describes how both $x, y \in A_i$ but $x \in \left(\frac{2}{3} + \frac{1}{3}A_{i-1}\right)$ and $y \in \frac{1}{3}A_{i-1}$. Simply choose $z \in \left(\frac{1}{3} + \frac{1}{3}A_{i-1}\right)$ and note that by definition $z \in [0,1] \setminus \Delta$ and $x < z < y$.

□

3. **Carothers 2.25** Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 1$ if $x \in \Delta$, and $g(x) = 0$ otherwise. At which points of $\mathbb{R}$ is g continuous?

Proof. I assert that the function is not continuous on Δ. Suppose the contrary and let $a \in \mathbb{R}, \Delta$ and with $x_n \in \Delta$ such that $x_n \to a$ and by continuity $g(x_n) \to g(a)$. We apply the previous result to produce a new sequence of $z_n \in \Delta^c$ such that $x_n < z_n < x_{n+1}$ and clearly by sandwich theorem it follows that $z_n \to a$ however by construction we know that $f(z_n) \not\to a$, a contradiction.

I assert that the function is continuous on Δ^c. Let $a \in \mathbb{R}, \Delta^c$ and consider some $x_n \to a$. By definition we can describe Δ^c as an arbitrary union of open sets and therefore it is open.

Thus for some $\epsilon > 0$ we can construct a $B_\epsilon(a) \subseteq \Delta^c$. Now note that since $x_n \to a$ there exists some N such that for all $n \geq N$ we know that $(x_n) \subseteq B_\epsilon(a)$. Therefore by definition of G it must follow that for all $n \geq N$, $g(x_n) = 0$.

□

4. Carothers 2.16 The algebraic numbers are those real or complex number that are the roots of polynomials having integer coefficients. Prove that the set of algebraic numbers is countable. [Hint: First show that the set of polynomials having integer coefficients is countable.]

Proof. Let P_n be the set of all polynomials of degree n. Note that there $|P_n| = \mathbb{Z}^{n+1}$ polynomials of such degree, a countable set. Recall that a countable union of countable sets is countable, hence the set of all polynomials having integer coefficients, $P = \cup_{n=1}^{\infty} P_n$ is countable.

Let $p \in P_i$ and note that by the fundamental theorem of algebra P has exactly i roots, a countable number. We define the set of algebraic numbers, A with,

$$A = \bigcup_{P_i \subset P} \bigcup_{p \in P_i} \{x \in \mathbb{R}, \mathbb{C} : p(x) = 0\}$$

Since the number of roots in p is countable, the number of polynomials in P_i is countable and the number of P_i in P is countable, A is also countable. □

5. Carothers 3.7 Here is a generalization of Exercises 5 and 6. Let $f : [0, \infty) \to [0, \infty)$ be increasing and satisfy $f(0) = 0$ and $f(x) > 0$ for all $x > 0$. If f also satisfies $f(x+y) \leq f(x)+f(y)$ for all $x, y \geq 0$, then $f \circ d$ is a metric whenever d is a metric. Show that each of the following conditions is sufficient to endure that $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$:

(a) f has a second derivative satisfying $f'' \leq 0$;

(b) f has a decreasing first derivative;

(c) $f(x)/x$ is decreasing for $x > 0$.

Proof. Let $f : [0, \infty) \to [0, \infty)$ be increasing and satisfy $f(0) = 0$ and $f(x) > 0$ for all $x > 0$. Suppose f has a second derivative satisfying $f'' \leq 0$. Let $[a, b] \subseteq [0, \infty)$. By the mean value theorem it follows that for some $x \in (a, b)$,

$$f''(x) = \frac{f'(b) - f'(a)}{b - a}$$
$$(b - a)f''(x) = f'(b) - f'(a).$$

Since $f''(x) \leq 0$ and $b > a$ it follows that $f'(b) - f'(a) \leq 0$ and therefore f' is decreasing.
□

Proof. Let $f : [0, \infty) \to [0, \infty)$ be increasing and satisfy $f(0) = 0$ and $f(x) > 0$ for all $x > 0$. Suppose f has a decreasing first derivative.

Let $x, y \in (0, \infty)$ such that $x \leq y$. Note that for some $a \in (0, x)$ the mean value theorem applies,

$$f'(a) = \frac{f(x) - f(0)}{x - 0}.$$

Similarly for some $b \in (x, y)$ the mean value theorem applies,

$$f'(b) = \frac{f(y) - f(x)}{y - x}.$$

Since f' is decreasing we know that $f'(a) \geq f'(b)$. By substitution it follows that for all $0 < x \leq y$,

$$\begin{aligned}
\frac{f(x) - f(0)}{x - 0} &\geq \frac{f(y) - f(x)}{y - x}, \\
\frac{f(x)}{x} &\geq \frac{f(y) - f(x)}{y - x}, \\
(y - x)\frac{f(x)}{x} &\geq f(y) - f(x), \\
y\frac{f(x)}{x} - f(x) &\geq f(y) - f(x), \\
y\frac{f(x)}{x} &\geq f(y), \\
\frac{f(x)}{x} &\geq \frac{f(y)}{y}.
\end{aligned}$$

□

Proof. Let $f : [0, \infty) \to [0, \infty)$ be increasing and satisfy $f(0) = 0$ and $f(x) > 0$ for all $x > 0$. Suppose $f(x)/x$ is decreasing for $x > 0$. Let $x, y \geq 0$ and note that,

$$\begin{aligned}
\frac{f(x)}{x} &\geq \frac{f(x + y)}{x + y}, \\
f(x) &\geq f(x + y)\frac{x}{x + y}.
\end{aligned}$$

Similarly we find that,

$$\begin{aligned}
\frac{f(y)}{y} &\geq \frac{f(x + y)}{x + y}, \\
f(y) &\geq f(x + y)\frac{y}{x + y}.
\end{aligned}$$

Summing both inequalities we get the following,

$$\begin{aligned}
f(x) + f(y) &\geq f(x + y)\left(\frac{x}{x + y} + \frac{y}{x + y}\right), \\
f(x) + f(y) &\geq f(x + y).
\end{aligned}$$

□

6. Carothers 3.15 We define the *diameter* of a nonempty subset A of M by $\text{diam}(A) = \sup\{d(a,b); a, b \in A\}$. Show that A is bounded if and only if $\text{diam}(A)$ is finite.

Proof. Suppose that A is a bounded nonempty subset of M. Then there exists an $x_0 \in M$ and some $C < \infty$ such that $d(a, x_0) \leq C$ for all $a \in A$. let $a, b \in A$ and note that by the triangle inequality,

$$d(a,b) \leq d(a, x_0) + d(x_0, b) \leq 2C.$$

Therefore it follows that

$$\text{diam}(A) \leq 2C < \infty.$$

Now let $\text{diam}(A)$ is finite, and for the sake of contradiction suppose A is not bounded. Therefore by definition there exists an $a \in A$ such that for all $x_0 \in M$ we know $d(a, x_0) = \infty$.

Clearly this is a contradiction, the supremum on the set of all distances between points can't be both finite and infinite. $\square$