1. Carothers 19.3 Prove that $f_n \underset{m}{\longrightarrow} f$ if and only if $(f_n - f) \underset{m}{\longrightarrow} 0$.

Proof. Let $(f_n), f$ be measurable, real-valued functions such that $f_n \underset{m}{\longrightarrow} f$. By definition we know that for all $\epsilon > 0$, there exists an N such that for all $n \geq N$ we have,

$$m\{|f_n - f| \geq \epsilon\} < \epsilon.$$

Note that since $|(f_n - f) - 0| = |f_n - f|$ we also have that for all $n \geq N$

$$m\{|(f_n - f) - 0| \geq \epsilon\} < \epsilon.$$

Hence it follows that $(f_n - f) \underset{m}{\longrightarrow} 0$. □

Proof. Let $(f_n), f$ be measurable, real-valued functions such that $(f_n - f) \underset{m}{\longrightarrow} 0$. By definition we know that for all $\epsilon > 0$, there exists an N such that for all $n \geq N$ we have,

$$m\{|(f_n - f) - 0| \geq \epsilon\} < \epsilon.$$

Now again since $|(f_n - f) - 0| = |f_n - f|$ we also have that for all $n \geq N$

$$m\{|f_n - f| \geq \epsilon\} < \epsilon.$$

Hence it follows that $f_n \underset{m}{\longrightarrow} f$. □

2. Compute $\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \cos(x/n)dx$.

Proof. First consider rewriting the limit as,

$$\lim_{n\to\infty} \int \chi_{(0,n)} \left(1+\frac{x}{n}\right)^{-n} \cos(x/n)dx.$$

Now consider the sequence of functions for $n \geq 2$ which is needed to have the dominating function be in L_1,

$$f_n = \chi_{(0,n)} \left(1+\frac{x}{n}\right)^{-n} \cos(x/n).$$

Now fix n and consider the following,

$$\int |f_n| = \int \left|\chi_{(0,n)} \left(1+\frac{x}{n}\right)^{-n} \cos(x/n)\right| dx \leq \int \chi_{(0,n)} \left|\left(1+\frac{x}{n}\right)^{-n}\right| dx \leq \int_1^n 1 < \infty.$$

Hence $(f_n) \in L_1$. Now it also follows that the pointwise limit gives,

$$\lim_{n\to\infty} \chi_{(0,n)} \frac{1}{(1+\frac{x}{n})^n} \cos\left(\frac{x}{n}\right) = \chi_{(0,\infty)} \frac{1}{e^x}(1).$$

Note that since $\left|\cos\left(\frac{x}{n}\right)\right| \leq 1$ we know that for all $n \geq 2$ it follows that,

$$|f_n| \leq \chi_{[0,\infty]} \left(1+\frac{x}{2}\right)^{-1} = g.$$

Now note that $g \in L_1$ since,

$$\int \chi_{[0,\infty]} \left(1+\frac{x}{2}\right)^{-2} dx = \int_0^\infty \left(1+\frac{x}{2}\right)^{-2} = 2$$

By the Dominated Convergence Theorem we conclude that,

$$\lim_{n\to\infty} \int \chi_{(0,n)} \left(1+\frac{x}{n}\right)^{-n} \cos(x/n)dx = \int \chi_{(0,\infty)} \frac{1}{e^x} dx = \int_0^\infty e^{-x} = 1$$

□

3. Let $\{f_n\}$ be a sequence of measurable real-valued functions. Let $E = \{x : (f_n(x)) \text{ converges}\}$. Show that E is measurable. [Hint: If the sequence does not converge at some x, the sequence $(f_n(x))$ is not Cauchy; try to give a description of the places where the sequence is not Cauchy in terms of a countable collection of set operations.]

Proof. Let $\{f_n\}$ be a sequence of measurable real-valued functions and suppose $E = \{x : (f_n(x)) \text{ converges}\}$. Note that E^c can be characterized as the following,

$$E^c = \{x : (f_n(x)) \text{ does not converge}\}$$
$$E^c = \{x : (f_n(x)) \text{ is not Cauchy}\}$$

Now describing everywhere in the domain where $f_n(x)$ is not a Cauchy sequence we consider the following,

There exists an $\epsilon > 0$ such that for all N, there exists an $n, m \geq N$ such that $|f_n(x) - f_m(x)| > \epsilon$.

$$E^c = \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{\infty}\left\{x : |f_n(x) - f_m(x)| > \frac{1}{2^k}, \quad n, m \geq i\right\}$$

$$E^c = \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{\infty}\bigcup_{n=i}^{\infty}\bigcup_{m=n}^{\infty}\left\{x : |f_n(x) - f_m(x)| > \frac{1}{2^k}\right\}$$

$$E^c = \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{\infty}\bigcup_{n=i}^{\infty}\bigcup_{m=n}^{\infty}\left\{x : -\frac{1}{2^k} > f_n(x) - f_m(x) > \frac{1}{2^k}\right\}$$

Since (f_n) are measurable functions, a difference such as $f_n(x) - f_m(x)$ is also measurable, and note that $\left\{x : -\frac{1}{2^k} > f_n(x) - f_m(x) > \frac{1}{2^k}\right\}$ describes the intersection of a pre-image of a pair of rays, and is hence measurable. Thus E^c is measurable and by the closure of compliments E is also measurable. □

4. Let

$$X_K = \left\{ f \in C([0,1]) : f \text{ is Lipschitz with constant } K \text{ and } \int_0^1 |f| \leq 1 \right\}.$$

Show that X_K is compact in $C([0,1])$. Is X_K also compact in $L^1([0,1])$?

Proof. We will proceed to show that X_K is compact by apply the Arzela-Ascoli Theorem. First let $(f_n) \subseteq X_K$ such that $f_n \to f$ with respect to the $C([0,1])$ infinity-norm, and since each (f_n) is a bounded continuous function we know that such convergence is equivalent to $(f_n) \to f$ uniformly. Since $(f_n) \to f$ uniformly $\int_0^1 f_n \to \int_0^1 f$ and since each $\int_0^1 f_n \leq 1$ it follows that $\int_0^1 f \leq 1$. Note that since $(f_n) \to f$ pointwise we also have that,

$$\frac{|f_n(x) - f_n(y)|}{|x-y|} \to \frac{|f(x) - f(y)|}{|x-y|}.$$

Since $\frac{|f_n(x) - f_n(y)|}{|x-y|} \leq K$ holds for all n we find that,

$$\frac{|f(x) - f(y)|}{|x-y|} \leq K.$$

Hence X_K is closed.

Now we will proceed to show that X_K is an equicontinuous set of functions. Let $\epsilon > 0$, and consider $\delta > 0$ such that $\delta < \frac{\epsilon}{K}$ and note that for all $x, y \in [0,1]$ and all $f \in X_K$ whenever $0 < |x-y| < \delta$ we find that,

$$|f(x) - f(y)| \leq K|x-y| \leq K\delta < \epsilon.$$

Hence X_K is equicontinuous.

Now we will proceed to show that X_K is uniformly bounded. Suppose for the sake of contradiction that for $M \in [0, \infty)$ there exist some $f \in X_K$ such that for some $x \in [0,1]$ $|f(x)| \geq M$. Now since f is lipschitz with constant K it follows that for all $y \in [0,1]$

$$|f(x) - f(y)| \leq K$$

Therefore it must follow that,

$$\int_0^1 |f| \geq M - K.$$

However M can be made arbitrarily large, say $(2+K)$ which would contradict $\int_0^1 |f| \leq 1$. Thus X_K is uniformly bounded. Therefore by Arzela-Ascoli we conclude that X_K is compact in $C([0,1])$.

$\square$

Proof. Note that X_K is sequentially compact in $C([0,1])$ so for any sequence $(f_n) \subseteq X_K$ we have a convergent subsequence in the $C([0,1])$ infinity norm, (uniformly convergent subsequence). Such a subsequence is necesarilly convergent in the $L^1([0,1])$ norm, since as $\left\|f_{n_k} - f\right\|_\infty \to 0$ we also have that,

$$\int_0^1 \left|f_{n_k} - f\right| \leq \int_0^1 \left\|f_{n_k} - f\right\|_\infty = \left\|f_{n_k} - f\right\|_\infty \to 0$$

Thus X_K is also sequentially compact in $L^1([0,1])$. □

5. (Riemann integrable functions are continuous a.e.)

(a) On a domain $I = [a, b]$, let (ψ_n) be an increasing sequence of step functions with $|\psi_n| \leq M$ for some M. Show that $\lim \psi_n$ is lower semicontinuous almost everywhere. That is, show that for almost every $x \in [a, b]$, if $x_n \to x$ then $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Proof. Let $I = [a, b]$ with (ψ_n) be an increasing sequence of step functions on I with $|\psi_n| \leq M$ for some M. Consider the set $E' = \cup_{i=1}^{\infty} \mathcal{P}_i$ where $\mathcal{P}_i$ is the step partition for ψ_i. Not that E' is comprised of a countable union of finite sets and is therefore countable, thus $m(E') = 0$. Now consider $E = [a, b] \setminus E'$, and by construction we know that for a fixed x the sequence $\psi_n(x)$ is well defined, increasing, and bounded above by M. Hence on E, the pointwise limit $\lim_{n\to\infty} \psi_n = \psi$ exists.

Let $x \in E$ and suppose $(x_n) \subset E$ such that $x_n \to x$. Let $\epsilon > 0$, and note that since the ψ is the pointwise limit of ψ_n we know that there exists a N_1 such that,

$$\psi(x) - \psi_{N_1}(x) < \epsilon,$$
$$\psi(x) < \epsilon + \psi_{N_1}(x).$$

However since ψ_{N_1} is a step function, and $x_n \to x$ we know that there exists an N_2 such that $\psi_{N_1}(x) = \psi_{N_1}(x_{N_2})$. Recall that the sequence of step functions is increasing, so we have that $\psi_{N_1}(x) = \psi_{N_1}(x_{N_2}) \leq \psi(x_{N_2})$. Therefore we find that,

$$\psi(x) < \epsilon + \psi(x_{N_2}),$$
$$\psi(x) - \epsilon < \psi(x_{N_2}).$$

Note that $\psi(x_{N_2})$ is an eventual lower bound, and therefore $\psi(x) \leq \liminf_{n\to\infty} \psi(x_n)$. Hence $\psi(x)$ is lower semicontinuous almost everywhere. □

(b) Suppose g, f and G are measurable functions on $[a, b]$, $g \leq f \leq G$, $g = G$ almost everywhere, g is lower semicontinuous, and G is upper semicontinuous. Show that f is continuous almost everywhere.

Proof. Let g, f and G are measurable functions on $[a, b]$, $g \leq f \leq G$, $g = G$ almost everywhere, g is lower semicontinuous, and G is upper semicontinuous. Let E be the set where $g = G$. Let $x \in E$ and consider $(x_n) \subset [a, b]$ such that $x_n \to x$. Since g is lower semicontinuous we know that $g(x) \leq \liminf_{n\to\infty} g(x_n)$ and similarly since G is upper semicontinuous we know that $G(x) \geq \limsup_{n\to\infty} G(x_n)$. The following chain of inequalities are produced by semicontinity of g, and G, $g \leq f \leq G$, the definition of lim sup, lim inf and the fact that $x \in E$ so $g(x) = f(x) = G(x)$,

$$\begin{aligned} f(x) = g(x) \leq \liminf_{n\to\infty} g(x_n) \leq \liminf_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} f(x_n) \\ \leq \limsup_{n\to\infty} f(x_n) \leq \limsup_{n\to\infty} G(x_n) \leq G(x) = f(x) \end{aligned}$$

□

(c) Show that Riemann integrable functions are continuous almost everywhere. [Hint: Find functions g and G with $g \leq f \leq G$ where $G = g$ a.e. and where g and G are continuous a.e.]

Proof. Suppose $f \in \mathcal{R}[a, b]$. Recall that by definition for any $\epsilon > 0$ there exists sequences of step function $g_n, G_n \in \text{Step}[a, b]$ with $g_n \leq f \leq G_n$ such that $g_n \to g$ and $G_n \to G$ where,

$$\lim_{n\to\infty} \int_a^b (G_n - g_n) \to 0 \qquad \text{alternatively,} \qquad \int_a^b (G - g) = 0.$$

Now, previously this definition of being Riemann integrable was defined using the Riemann integral over step functions. However we have shown that this integral is equivalent to the Lebesgue integral, so we get that that $G - g = 0$ almost everywhere, so $G = g$ almost everywhere. Note that by (a) we conclude that g is lower semicontinuous almost everywhere, and analogously with G being upper semicontinuous almost everywhere. Having satisfied the hypothesis of (b) (save for a few null sets) we can conclude that f is continuous almost everywhere. $\square$

6. Carothers 8.20 Let E be a noncompact subset of $\mathbb{R}$. Find a continuous function $f : E \to \mathbb{R}$ that is (*i*) not bounded; (*ii*) bounded but has no maximum value.

Proof. Let E be a noncompact subset of $\mathbb{R}$.

(i) Consider the function $f = \tan(x)$ on the noncompact set $(\frac{\pi}{2}, \frac{\pi}{2})$. This function is clearly unbounded.

(ii) Consider the function $f = \arctan(x)$ on the noncompact set $\mathbb{R}$. This function is bounded $|f| \leq \frac{\pi}{2}$ however has no maximum value.

$\square$

7. (Cauchy-Schwartz inequality for integrals.)

(a) Use the (ℓ_2) Cauchy-Schwartz inequality to prove that if f and g are simple and integrable, then

$$\int |f||g| \leq \left[\int |f|^2\right]^{1/2} \left[\int |g|^2\right]^{1/2}.$$

Proof. Suppose f and g be simple, integrable functions, Written in standard form we know that,

$$f = \sum_{i=1}^{n} a_i \chi_{A_i} \qquad g = \sum_{i=1}^{m} b_i \chi_{B_i}$$

Where $(a_n), (b_n)$ are distinct sequences of real numbers, and $\{A_i\}, \{B_i\}$ are pairwise disjoint collection of measurable sets. Now consider constructing a common refinement of the domain of measurable sets, $\{C_i\} = \{A_i\} \cup \{B_i\}$ with coefficients a_i' and b_i' so that

$$f = \sum_{i=1}^{j} a_i' \chi_{C_i} \qquad g = \sum_{i=1}^{j} b_i' \chi_{C_i}$$

On the common refinement it follows that,

$$\int |f||g| = \int \sum_{i=1}^{j} |a_i'||b_i'| \chi_{C_i} = \int \sum_{i=1}^{j} |(a_i' \chi_{C_i})(b_i' \chi_{C_i})|.$$

By the (ℓ_2) Cauchy-Schwartz inequality it follows that,

$$\int |f||g| \leq \int \left\|a_i' \chi_{C_i}\right\|_2 \left\|b_i' \chi_{C_i}\right\|_2 = \left[\int |f|^2\right]^{1/2} \left[\int |g|^2\right]^{1/2}.$$

□

(b) Suppose that f and g are measurable functions such that $|f|^2, |g|^2 \in L^1$. Show that $fg \in L^1$ and $\int |fg| \leq \left[\int |f|^2]\right]^{\frac{1}{2}} \left[\int |g|^2\right]^{\frac{1}{2}}$.

Proof. Suppose that f and g are measurable functions such that $|f|^2, |g|^2 \in L^1$. First since f and g are measurable consider the disjoint measurable sets $E = \{f \geq g\}$ and by definition its compliment $E^c = \{f < g\}$. Now since $|fg|$ is a nonnegative measurable function we know that (Corollary 18.9),

$$\int |fg| = \int_E |fg| + \int_{E^c} |fg| \leq \int_E \left|f^2\right| + \int_{E^c} \left|g^2\right| < \infty.$$

Hence $fg \in L^1$. Now appealing to the basic construction and the Monotone Convergence Theorem (Corollary 18.8) there exists sequences of nonnegative simple integral functions (f_n) (g_n) which are increasing and have the property that,

$$\int |f| = \lim_{n\to\infty} \int f_n \qquad \int |g| = \lim_{n\to\infty} \int g_n$$

Note that $\lim_{n\to\infty} |f_n||g_n| = |f||g|$ and $|f_n||g_n|$ is also an increasing sequence of nonnegative measurable functions we find, again by the Monotone Convergence Theorem that,

$$\lim_{n\to\infty} \int |f_n||g_n| = \int |f||g| = \int |fg|.$$

Now consider that by the previous problem, for all n we know that,

$$\int |f_n||g_n| \leq \left[\int |f_n|^2\right]^{1/2} \left[\int |g_n|^2\right]^{1/2}.$$

Considering the limit and noting that the terms on the right hand side converge independently we find that,

$$\lim_{n\to\infty} \int |f_n||g_n| \leq \lim_{n\to\infty} \left[\int |f_n|^2\right]^{1/2} \left[\int |g_n|^2\right]^{1/2},$$

$$\int |fg| \leq \left[\int |f|^2\right]^{\frac{1}{2}} \left[\int |g|^2\right]^{\frac{1}{2}}.$$

□

8. (The approximate with wild abandon problem.)

Suppose $f \in L^1[a,b]$ and $\int_a^b fg = 0$ for every polynomial g. Show that $f = 0$ almost everywhere.

[Hint: First show that $\int_I f = 0$ for every interval in $[a,b]$. Then show that $\int_E f = 0$ for every measurable set in $[a,b]$. You might find Exercise 18.35 (the even more is true part) to be handy, as well.]

Proof. Suppose $f \in L^1[a,b]$ and $\int_a^b fg = 0$ for every polynomial g. Let $c \in C[a,b]$ and note that by the Weierstrass Approximation Theorem there exists a sequence of polynomials $(p_n) \subset P[a,b]$ such that $p_n \to c$ uniformly. Note that $\chi_{[a,b]} f p_n \to \chi_{[a,b]fc}$ pointwise, also note that p_n are continuous function on a compact domain and hence each p_n is bounded, clearly a uniformly convergent sequence of bounded function is uniformly bounded and hence $|p_n| \leq M$ for some M. Note that

$$\int \left|\chi_{[a,b]} f p_n\right| = \int_a^b |f||p_n| \leq \int_a^b |f| M \in L^1$$

Having satisfied the hypothesis of the Dominating Convergence Theorem we conclude that,

$$\lim_{n\to\infty} \int \chi_{[a,b]} f p_n = \int \chi_{[a,b]fc}.$$

Since the left hand side is strictly a sequence of zeros we know that $\int_a^b fc = 0$ for any continuous function.

Now recall that by Theorem 18.27, which gives $C[a,b]$ is dense in $L^1[a,b]$ and since $\chi_I \in L^1[a,b]$ for any interval $I \subseteq [a,b]$ there exists a sequence of continuous functions $c_n \to \chi_I$ in the $L^1[a,b]$ norm. This implies that $fc_n \to f\chi_I$ and therefore

$$\lim_{n\to\infty} \int_a^b fc_n = \int_a^b f\chi_I,$$

and again since the left hand side is a sequence of zeros we know $\int_a^b \chi_I f = \int_I f = 0$.

Now consider a measurable set $E \subseteq [a,b]$, and let $\{I_k\}_n$ be a sequence of nested measuring covers, where each $I \in I_k$ is $I \subseteq [a,b]$, disjoint and,

$$\bigcap_n^\infty \bigcup_k^\infty I_{k_n} = E.$$

By continuity of the integral from above(for measurable set) and since inside each measuring cover, the intervals are disjoint we have (by Corollary 18.26) that,

$$\int_E f = \lim_{n\to\infty} \int_{\bigcup_k^\infty I_{k_n}} f = \lim_{n\to\infty} \sum_{k=1}^\infty \int_{I_{k_n}} f = 0.$$

Noting that each term in the sum is zero, and therefore each term in the sequence of integrals over measuring covers is also zero we get that $\int_E f = 0$. Finally let $E = \{f \geq 0\}$ and note that,

$$\int_a^b |f| = \int_E f + \int_{[a,b]\setminus E} -f = 0$$

Thus $f = 0$ almost everywhere. □

9. A sequence (f_n) is Cauchy in measure if for every $\epsilon > 0$ there is an index N such that if $n, m \geq N$ then $m(\{|f_n - f_m| > \epsilon\}) < \epsilon$. Show that if (f_n) is Cauchy in measure and has a subsequence that is convergent in measure, then the full sequence is convergent in measure.

Proof. Suppose (f_n) is Cauchy in measure and has a subsequence (f_{n_j}) that is convergent in measure.

Let $\epsilon > 0$, since (f_n) is Cauchy in measure we know that there exist an N_1 such that for all $n, m \geq N_1$ we have

$$m(\{|f_n - f_m| > \frac{\epsilon}{2}\}) < \frac{\epsilon}{2}.$$

Since (f_{n_j}) is convergent in measure we also have that there exists an N_2 such that for all $k \geq N_2$,

$$m(\{|f_{n_k} - f| > \frac{\epsilon}{2}\}) < \frac{\epsilon}{2}.$$

Now consider N_3 which is the index of N_2 in (f_n), and let $N = \max N_1, N_3$. Consider $n \geq N$, let $x \in \{|f_n - f| > \epsilon\}$ and note that by the triangle inequality for some $n_k > n$,

$$|f_n(x) - f(x)| \leq \left|f_n(x) - f_{n_k}(x)\right| + \left|f_{n_k} - f(x)\right|.$$

Since $\epsilon < |f_n(x) - f(x)|$ it must be the case that either,

$$\left|f_n(x) - f_{n_k}(x)\right| > \frac{\epsilon}{2} \qquad \text{or} \qquad \left|f_{n_k} - f(x)\right| > \frac{\epsilon}{2}.$$

Hence,

$$\{|f_n - f| > \epsilon\} \subseteq \{|f_n - f_{n_k}| > \frac{\epsilon}{2}\} \cup \{|f_{n_k} - f| > \frac{\epsilon}{2}\}.$$

Thus by monotonicity and subadditivity,

$$m(\{|f_n - f| > \epsilon\}) \leq m(\{|f_n - f_{n_k}| > \frac{\epsilon}{2}\}) + m(\{|f_{n_k} - f| > \frac{\epsilon}{2}\}) < \epsilon.$$

□

10. Consider the series $\sum_{k=1}^{\infty} a_k \sin(kx)$ on the domain $[0, 2\pi]$. Suppose that $\sum_{k=1}^{\infty}(a_k)^2$ converges. Prove that the series converges in $L^2([0, 2\pi])$.

Proof. Consider the series $\sum_{k=1}^{\infty} a_k \sin(kx)$ on the domain $[0, 2\pi]$ and suppose that $\sum_{k=1}^{\infty}(a_k)^2$ converges. Let (s_n) be the sequence of partial sum functions, and note that s_n is a continuous function on a compact interval, so it certainly is in $L^2([0, 2\pi])$.

Now we will show that (s_n) is a Cauchy sequence in the $L^2[0, 2\pi]$. Let $\epsilon > 0$ and consider N large enough so that for all $n, m \geq N$, without loss of generality $n \leq m$ we have that,

$$\sum_{i=n}^{m}(a_i)^2 < \frac{\epsilon}{\pi}.$$

Thus it follows that,

$$\int_0^{2\pi} |s_n - s_m|^2 = \int_0^{2\pi} \left|\sum_{k=1}^{n} a_k \sin(kx) - \sum_{k=1}^{m} a_k \sin(kx)\right|^2 = \int_0^{2\pi} \left|\sum_{k=n+1}^{m} a_k \sin(kx)\right|^2$$

We recall that for $m \neq n$ integrals of the form,

$$\int_0^{2\pi} \sin(mx)\sin(nx) = 0.$$

Therefore expanding the term on the right hand side we get that,

$$\int_0^{2\pi} \left|\sum_{k=n+1}^{m} a_k \sin(kx)\right|^2 = \int_0^{2\pi} \sum_{k=n+1}^{m} a_k^2 \sin^2(kx).$$

By linearity we get,

$$\int_0^{2\pi} \sum_{k=n+1}^{m} a_k^2 \sin^2(kx) = \sum_{k=n+1}^{m} \left(a_k^2 \int_0^{2\pi} \sin^2(kx)\right) = \sum_{k=n+1}^{m} a_k^2 \pi < \epsilon.$$

Therefore the sequence of (s_n) is Cauchy in $L^2([0, 2\pi])$ and since L^2 is complete (s_n) must converge so equivalently the series $\sum_{k=1}^{\infty} a_k \sin(kx)$ must converge. □