1. Carothers 10.7 Let (f_n) and (g_n) be real-values functions on a set X, and suppose that (f_n) and (g_n) converge uniformly on X. Show that $(f_n + g_n)$ converges uniformly on X. Give an example showing that $(f_n g_n)$ need not converge uniformly on X.

Proof. Let $\epsilon > 0$. Since (f_n) and (g_n) are real-values functions which converge uniformly on X, there exists an N such that if $n \geq N$ then, for all $x \in X$,

$$|f_n(x) - f(x)| < \epsilon,$$

$$|g_n(x) - g(x)| < \epsilon.$$

Then by the triangle inequality it follows that for all $x \in X$,

$$\begin{aligned}|(f_n(x) + g_n(x)) - (f(x) + g(x))| &= |(f_n(x) - f(x)) + (g_n(x) - g(x))|,\\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)|,\\ &< 2\epsilon.\end{aligned}$$

Thus $(f_n + g_n)$ converges uniformly on X. For an example where $(f_n g_n)$ do not converge uniformly on X consider $g_n = 1/n$ a sequence of constant functions and $f_n = x^2$ a constant sequence of functions. Note both g_n and f_n are uniformly convergent on all of $\mathbb{R}$, g_n converges to the zero function and f_n converges to x^2. However the sequence $(g_n f_n) = (1/n)x^2$ is not uniformly convergent to its point wise limit, the zero function. Let $\epsilon_0 = 1/2$ and note that for each n we can take $x = \sqrt{n}$ and we find that,

$$|f_n(\sqrt{n}) - 0| = \left|\frac{1}{n}\sqrt{n}^2\right| = 1 > \frac{1}{2}$$

□

2. Carothers 10.9 For each of the following sequences, determine the point wise limit on the given interval (if it exists) and the interval on which the convergence is uniform (if any):

a. $f_n(x) = x^n$ on $(-1, 1]$:

Solution:
Note for a fixed $x \in (-1, 1)$, we get $f_n(x) \to 0$ and for $x = 1$ we find that $f_n(1) \to 1$ therefore the point wise limit of f_n on $(-1, 1]$ is,

$$f(x) = \begin{cases} 0, x \neq 1 \\ 1, x = 1 \end{cases} .$$

Let $\delta > 0$ and note that on $[-1 + \delta, 1 - \delta]$, a compact interval we know that f_n decreases point wise to the zero function, hence by Dini's Theorem f_n converges uniformly to $[-1 + \delta, 1 - \delta]$.

b. $f_n(x) = n^2x(1-x^2)^n$ on $[0,1]$:

Solution:
For a fixed $x \in (0,1)$ we find that, $0 < (1-x^2) < 1$ and therefore the sequence $f_n(x) = n^2x(1-x^2)^n$ is dominated by the $(1-x^2)^n$ term and converges to 0. Now when $x = 1$ or $x = 0$ we find that $f_n(x) = 0$. Hence f_n converges point wise to the zero function. This sequence is uniformly convergent to the zero function on the interval $[\delta, 1)$ for a $\delta > 0$. To see this note that there exists an N far enough out in the sequence of functions which such that if $n \geq N$ the 'hump' in $f_n(x)$ lies between $[0, \delta)$.

c. $f_n(x) = nx/(1+nx)$ on $[0,\infty)$:

Solution:
Fix $x \in (0,\infty)$ and note that $f_n(x) = nx/(1+nx) \to 1$ however clearly if $x = 0$, we get that $f_n(0) = n(0)/(1+n(0)) = 0$. So the point wise limit of f_n on $[0,\infty)$ is given by,

$$f(x) = \begin{cases} 0, x = 0 \\ 1, x \neq 0 \end{cases}$$

Again if we let $\delta > 0$ and we consider the interval $[\delta, \infty)$ we have uniform convergence.

d. $f_n(x) = nx/(1+n^2x^2)$ on $[0,\infty)$:

Solution:
Fix $x \in (0,\infty)$ we find that

$$f_n(x) = \frac{nx}{(1+n^2x^2)} < \frac{nx}{(nx)^2} = \frac{1}{nx} \to 0$$

Clearly when $x = 0$, $f_n(x) \to 0$ and therefore f_n on $[0,\infty)$ converges point wise to 0. Similarly to part b to get f_n to converge uniformly we must select an interval $[\delta, \infty)$ where $\delta > 0$, so we have the option to find a far enough N which pushes the 'hump' into the interval $[0, \delta)$.

e. $f_n(x) = xe^{-nx}$ on $[0,\infty)$:

Solution:
Fix $x \in (0,\infty)$ then it follows that $f_n(x) = xe^{-nx} \to 0$ and clearly $f_n(0) = 0$ so f_n converges point wise to the zero function. Note f_n is decreases point wise to zero on $[0,\infty)$, so considering a compact sub interval $[0,1]$ we get by Dini's Theorem that f_n converges uniformly on $[0,1]$.

f. $f_n(x) = nxe^{-nx}$ on $[0,\infty)$:

Solution:
Fix $x \in (0,\infty)$ then it follows that $f_n(x) = nxe^{-nx} \to 0$ since the sequence is dominated by the exponential decay in the e^{-nx}. It also follows tha that $f_n(0) = 0$

and therefore f_n is point wise convergent on $[0, \infty)$: to the zero function. Again similarly to part d and b we select interval $[\delta, \infty)$ where $\delta > 0$, so we have the option to find a far enough N which pushes the 'hump' into the interval $[0, \delta)$.

3. **Carothers 10.10** Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, and define $f_n(x) = f(x + 1/n))$. Show that f_n uniformly converges to f on $\mathbb{R}$.

Proof. Let $\epsilon > 0$. Since f is uniformly continuous there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Now choose N such that if $n \geq N$ then $|x - (x + 1/n)| = |1/n| < \delta$, then it follows that for all $x \in \mathbb{R}$

$$|f(x) - f_n(x)| = |f(x) - f(x + 1/n)| < \epsilon.$$

□

4. **Carothers 10.15** Let (X, d) and (Y, ρ) be metric spaces, and let $f, f_n : X \to Y$, with f_n uniformly converging to f on X. If each f_n is continuous at $x \in X$, and if $x_n \to x$ in X, prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

Proof. Let $\epsilon > 0$. Since f_n uniformly converges to f we can choose N_1 such that if $n \geq N_1$ then for all $x \in X$ it follows that $\rho(f_n(x), f(x)) < \epsilon$. Since each f_n is continuous at x and $x_n \to x$ then it follows that $f_n(x_n) \to f_n(x)$. Now choose N_2 such that if $i \geq N_2$ then $\rho(f_n(x_i), f_n(x)) < \epsilon$. Let $N = \max\{N_1, N_2\}$ and it follows that for all $n \geq N$,

$$\begin{aligned}
\rho(f_n(x_n), f(x)) &\leq \rho(f_n(x_n), f(x_n)) + \rho(f(x_n), f(x)),\\
&\leq \rho(f_n(x_n), f(x_n)) + \rho(f(x_n), f_n(x_n)) + \rho(f_n(x_n), f_n(x)) + \rho(f_n(x), f(x)),\\
&< 4\epsilon.
\end{aligned}$$

□

5. **Carothers 10.18** Here is a partial converse to Theorem 10.4, called Dini's Theorem. Let X be a compact metric space, and suppose that the sequence (f_n) in $C(X)$ increases pointwise to a continuous function $f \in C(X)$; that is, $f_n(x) \leq f_{n+1}(x)$ for each n and x, and $f_n(x) \to f(x)$ for each x. Prove that the convergence is actually uniform. The same is true if (f_n) decreases pointwise to f.

Proof. First we will reduce this problem to the case where (f_n) decreases pointwise to 0. Without loss of generality suppose (f_n) in $C(X)$ decreases pointwise to a continuous function $f \in C(X)$. Define a new sequence of function (g_n) where $g_n(x) = f_n(x) - f(x)$ and note that since $f_n(x) \to f(x)$ pointwise it follows that $g_n(x) \to 0$ pointwise. Since $f_n(x) \geq f_{n+1}(x)$ for each n and x, it follows by subtracting $f(x)$ to both sides that $g_n(x) \geq g_{n+1}(x)$ for each n and x. Therefore g_n decreases pointwise to g.

Let $\epsilon > 0$. Consider the open sets $U_n = \{x \in X : g_n(x) < \epsilon\}$, we will show that U_n covers X. Let $x \in X$, and note that since $g_n(x) \to 0$ pointwise there must exists an N such that $|g_N(x)| < \epsilon$, and therefore $x \in U_N$. Now since X is compact and the set of all U_n form an open cover, we know that there exists a finite subcover $\{U_i\}_{i \in I}$, for some finite index set I. Now let $x \in U_n$, then by definition $g_n(x) < \epsilon$ but clearly $g_{n+1}(x) \leq g_n(x) < \epsilon$, so it follows $x \in U_{n+1}$, and therefore $U_n \subseteq U_{n+1}$.

Now, since the set I is finite there exists a $U_M = \max_{n \in I} U_i$, and since $U_n \subseteq U_{n+1}$ and $\{U_i\}_{i \in I}$ is a cover of X, U_M covers X. Thus for all $n \geq M$ it follows that for all $x \in X$ (since U_M covers X) we have,

$$|g_n(x)| < \epsilon.$$

□

6. Carothers 10.19 Suppose (f_n) is a sequence of functions in $C[0,1]$ and that f_n converges uniformly to f on $[0,1]$. True or false $\int_0^{1-(1/n)} f_n \to \int_0^1 f$.

Proof. Note that since $(f_n) \subseteq C(0,1)$ and f_n converges uniformly to f, we know that $f \in C[0,1]$ and is therefore Riemann integrable. Let $\epsilon > 0$ and choose N to be the max of either $1/N < \epsilon$ or $\|f - f_n\|_\infty < \epsilon$ then if $n \geq N$,

$$\begin{aligned}
\left|\int_0^1 f(x)dx - \int_0^{1-(1/n)} f_n(x)dx\right| &= \left|\int_{1-(1/n)}^1 f(x)dx + \int_0^{1-(1/n)} f(x)dx - \int_0^{1-(1/n)} f_n(x)dx\right| \\
&\leq \left|\int_{1-(1/n)}^1 f(x)dx\right| + \left|\int_0^{1-(1/n)} f(x)dx - \int_0^{1-(1/n)} f_n(x)dx\right| \\
&= \left|\int_{1-(1/n)}^1 f(x)dx\right| + \left|\int_0^{1-(1/n)} f(x) - f_n(x)dx\right| \\
&\leq \int_{1-(1/n)}^1 |f(x)|\, dx + \int_0^{1-(1/n)} |f(x) - f_n(x)|\, dx \\
&\leq (1/n)\, \|f\|_\infty + (1 - (1/n))\, \|f - f_n\|_\infty \\
&< \epsilon\, \|f\|_\infty + (1 - \epsilon)\epsilon
\end{aligned}$$

□

7. Carothers 10.25 Show that $B[0,1]$ is not separable.

Proof. Consider the set of function $\delta_y(x) : [0,1] \to \mathbb{R}$ defined by,

$$\delta_y(x) = \begin{cases} 0, x \neq y \\ 1, x = y \end{cases}$$

Note that $\{\delta_y(x)\}_{y \in [0,1]} \subseteq B[0,1]$ since each $\delta_y(x)$ is bounded above by 1 and below by 0. Now consider the collection of sets $\{B_{1/2}(\delta_y(x))\}_{y \in [0,1]} \subseteq B[0,1]$. This collection is uncountable, and each set is disjoint, since if we fix $a \in [0, 1]$ and let $b \in [0, 1]$ such that $b \neq a$ we find that $\|\delta_a(x) - \delta_b(x)\|_\infty = 1 < 1/2$ since $\delta_a(x) - \delta_b(x)$ takes on a value of 1 at a, -1 at b and 0 elsewhere. Hence any countable dense subset would have to have a single element in each set of $\{B_{1/2}(\delta_y(x))\}_{y\in[0,1]}$ which is impossible since there are an uncountable number of them. □

8. Carothers 10.26 If $\sum_{n=1}^\infty |a_n| < \infty$, prove that $\sum_{n=1}^\infty a_n \sin(nx)$ and $\sum_{n=1}^\infty a_n \cos(nx)$ are uniformly convergent in $\mathbb{R}$.

Proof. First let $M_n = |a_n|$ and note that for a fixed n since $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$ we find that for all $x \in \mathbb{R}$,

$$|a_n \sin(nx)| = |a_n||\sin(nx)| \leq |a_n| = M_n,$$

$$|a_n \cos(nx)| = |a_n||\cos(nx)| \leq |a_n| = M_n.$$

By the Weirstrass M-test it follows that since $\sum_{n=1}^\infty M_n$ converges then $\sum_{n=1}^\infty a_n \sin(nx)$ and $\sum_{n=1}^\infty a_n \cos(nx)$ are uniformly convergent in $\mathbb{R}$. □

9. Carothers 10.27 Show that $\sum_{n=1}^\infty x^2/(1 + x^2)^n$ converges for all $|x| \leq 1$, but that convergence is not uniform.

Proof. First note that for all $0 < |x| \leq 1$, we get the following pointwise convergence,

$$\sum_{n=1}^\infty \frac{x^2}{(1 + x^2)^n} = x^2 \sum_{n=1}^\infty \left(\frac{1}{1 + x^2}\right)^n = \frac{x^2\left(\frac{1}{1 + x^2}\right)}{1 - \frac{1}{1 + x^2}} = \frac{\left(\frac{x^2}{1 + x^2}\right)}{\frac{x^2}{1 + x^2}} = 1.$$

However clearly if $x = 0$ we get,

$$\sum_{n=1}^\infty \frac{(0)^2}{(1 + (0)^2)^n} = 0$$

Therefore $\sum_{n=1}^\infty x^2/(1 + x^2)^n$ on $|x| \leq 1$ converges pointwise to f where,

$$f(x) = \begin{cases} 1, x \neq 0 \\ 0, x = 0 \end{cases}$$

Note that f is discontinuous on $|x| \leq 1$ and therefore $\sum_{n=1}^\infty x^2/(1+x^2)^n$ a series of continuous function, cannot converge uniformly.

□

10. Carothers 10.32

(a) if $\sum_{n=1}^{\infty} |a_n| < \infty$, show that $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on $[0, \infty)$.

Proof. Let $M_n = |a_n|$ and note that for all $x \in [0, \infty)$ and n, we know that $e^{-nx} \leq 1$. Therefore it follows that for all $x \in [0, \infty)$,

$$|a_n e^{-nx}| = |a_n| e^{-nx} \leq |a_n| = M_n$$

Since $\sum_{n=1}^{\infty} M_n < \infty$ by the Weirstrass M-test it follows that $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on $[0, \infty)$ □

(b) If we assume only that (a_n) is bounded, show that $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on $[\delta, \infty)$ for every $\delta > 0$.

Proof. Suppose (a_n) is bounded, and therefore there exists an $A \in \mathbb{R}$ such that $|a_n| \leq A$, for all n. Let $\delta > 0$, define $M_n = Ae^{-n\delta}$ and note that $M_n \geq 0$. Now note that for all $x \in [\delta, \infty)$ it follows that,

$$|a_n e^{-nx}| = |a_n| e^{-nx} \leq Ae^{-nx} = M_n.$$

Now to apply the Weirstrass M-test we must show that,$\sum_{n=1}^{\infty} M_n < \infty$. Consider the following,

$$\sum_{n=1}^{\infty} Ae^{-n\delta} = A \sum_{n=1}^{\infty} e^{-n\delta} = A \sum_{n=1}^{\infty} \left(\frac{1}{e^{\delta}} \right)^n.$$

Therefore $\sum_{n=1}^{\infty} M_n$ is a convergent geometric series since $\delta > 0$ forces $|1/e^{\delta}| < 1$. Thus by the Weirstrass M-test, $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on $[\delta, \infty)$ for every $\delta > 0$. □