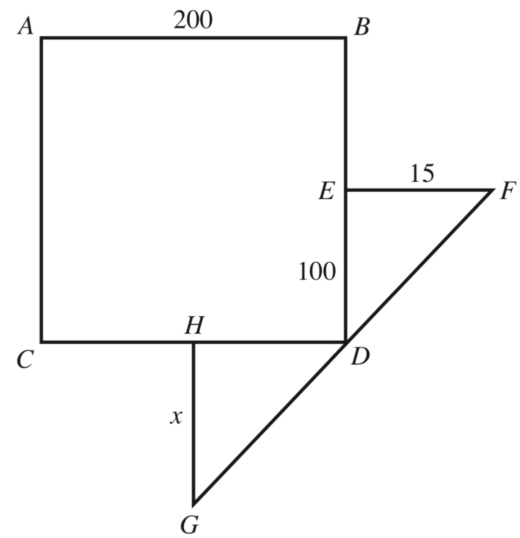
Section 5.5

Exercise 15: Solve the following problems from the Nine Chapters.

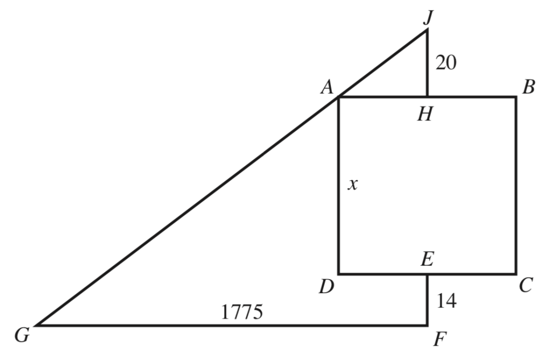
1. A square, walled city measures 200 paces on each side. Gates are located on the centers of each side. If there is a tree 15 paces form the east gate, how far must a man travel from the south gate in order to see the tree?



Solution:
First consider that by construction we know that $HG || ED$, and that $\angle GHD = \angle DEF = 90$. Now consider the transversal GD. By corresponding angles we know that, $\angle HGD = \angle EDF$. Thus by AA similarity we know that $\triangle HGD \sim \triangle EDF$. Now we can solve for x using the ratios from similar triangles,

$$\frac{x}{100} = \frac{100}{15}$$
$$x = \frac{100^2}{15}$$

2. A square walled city of unknown dimensions, had four gates, one at the center of each side. A tree stands 20 paces form the north gate. A man walks 14 paces southward orm the south gate, then turns west and walks 1775 paces before he can see the tree. What are the dimensions of the city.



Solution:
Consider that by construction we know $\angle JFG = \angle JHA = 90$. Since $\triangle JHA$ and $\triangle JFG$ share an angle we know that $\triangle JHA \sim \triangle JFG$ by AA similarity. Since the city is square, and the gates are at the center of each side we know that $AH = \frac{x}{2}$ and $JF = 20 + x + 14$. Solving for x,

$$\begin{aligned} \frac{20 + x + 14}{1775} &= \frac{(2)20}{x} \\ x^2 + 34x &= 40(1775) \\ x^2 + 34x - 71000 &= 0 \\ (x + 284)(x - 250) &= 0 \end{aligned}$$

Looking at just our positive solution we get that $x = 250$.

Exercise 16a: A certain number of people are purchasing chickens, jointly. If each person contributes 9 wen there is a surplus of 11 wen, and if each person contributes 6 wen there is deficiency of 16 wen. Find the number of people n, and the price of the chickens p.

Solution:
Converting this problem into our notation we get the following system of equations,

$$9n = p + 11,$$

$$6n = p - 16.$$

Consider the same equations rewritten,

$$9n - p = 11,$$

$$6n - p = -16.$$

Now consider our two guesses, $n_1 = 10$ and $n_2 = 2$. Solving for p_1 and p_2 using the first equation,

$$\begin{aligned} 9(10) &= p_1 + 11, \\ p_1 &= 79. \end{aligned}$$

$$\begin{aligned} 9(2) &= p_2 + 11, \\ p_2 &= 7. \end{aligned}$$

So our two guess pairs are $(10, 79)$ and $(2, 7)$. Substituting into the second equation we get,

$$6(10) - 79 = -19 = -16(-3),$$

$$6(2) - 7 = 5 = -16(+21).$$

Therefore our two failures are $f_1 = -3$ and $f_2 = 21$. We know from the proof in Chapter 2 that,

$$n = \frac{(f_1)(n_2) - (f_2)(n_1)}{f_1 - f_2}.$$

So finally we get that n is,

$$n = \frac{(-3)(2) - (21)(10)}{(-3) - 21} = 9.$$

Solving for p,

$$\begin{aligned} 9(9) &= p + 11, \\ p &= 70. \end{aligned}$$

Additional Problems

Exercise 1: Use the Chinese square root algorithm to find the square root of 142884.

Solution:
Given the order of 142884 we can say that it's root can be written in the form of,

$$\sqrt{142884} = 100a + 10b + c.$$

Consider $a = 3$ since $300^2 = 90000$ and $400^2 = 160000$. With $a = 3$ we calculate the remainder,

$$142884 - 90000 = 52884.$$

By the geometric argument in Chapter 5, b must satisfy the following inequality,

$$2(300)(10b) + (10b)^2 \leq 52884.$$

Finding a b that satisfies $6000b \leq 52884$ and then checking the inequality. Consider $b = 7$,

$$6000(7) + (10(7))^2 = 46900 \leq 52884.$$

Suppose we try 8,

$$6000(8) + (10(8))^2 = 54400 \nleq 52884.$$

With $b = 7$ we calculate our remainder.

$$52884 - 54400 = 5984$$

Now we need to find a c that satisfies the following,

$$2(300 + 70)c + c^2 \leq 5984.$$

Finding a c that satisfies $740c \leq 5985$ and then checking the inequality. Consider $c = 8$,

$$2(300 + 70)(8) + (8)^2 = 5984 \leq 5984.$$

Since our remainder is zero we have reached an exact answer and thus,

$$\sqrt{142884} = 100(3) + 10(7) + 8 = 378$$

Exercise 2: Solve problem 4 from Chapter 1 of Qin Jiushao's Mathematical Treatise, which is equivalent to solving this system,

$$\begin{aligned} N &\equiv 0 \mod 11 \\ N &\equiv 0 \mod 5 \\ N &\equiv 4 \mod 9 \\ N &\equiv 6 \mod 8 \\ N &\equiv 0 \mod 7 \end{aligned}$$

Solution:
To solve this problem we must first compute the product M, of the moduli m_i,

$$M = 11 * 5 * 9 * 8 * 7 = 27720.$$

The next step involves computing all M_i such that

$$M_i = \frac{M}{m_i}.$$

Therefore we get the following,

$$M_1 = \frac{27720}{11} = 2520$$
$$M_2 = \frac{27720}{5} = 5544$$
$$M_3 = \frac{27720}{9} = 3080$$
$$M_4 = \frac{27720}{8} = 3465$$
$$M_5 = \frac{27720}{7} = 3960$$

Now we reduce each M_i mod m_i, so find a P_i such that,

$$M_i = P_i \mod m_i.$$

Doing that we get,

$$\begin{aligned}2520 &\equiv 1 \mod 11\\ 5544 &\equiv 4 \mod 5\\ 3080 &\equiv 2 \mod 9\\ 3465 &\equiv 1 \mod 8\\ 3960 &\equiv 5 \mod 7\end{aligned}$$

Now we need to find one for each P_i so solving for some x_i that gives,

$$P_i x_i \equiv 1 \mod m_i.$$

Doing this we get,

$$\begin{aligned}(1)(1) &\equiv 1 \mod 11\\ (4)(4) &\equiv 1 \mod 5\\ (2)(5) &\equiv 1 \mod 9\\ (1)(1) &\equiv 1 \mod 8\\ (5)(3) &\equiv 1 \mod 7\end{aligned}$$

Finally we can compute N by the following formula,

$$N = \sum_i r_i M_i x_i \mod M.$$

We can see that for 3 of the equations $r_i = 0$ so our computation simplifies to,

$$N = 4(3080)(5) + 6(3465)(1) = 82390 = 26950 \mod 27720.$$

Exercise 3: Use the counting boards/algorithm for "finding one" to solve $88(x) \equiv 1 \mod 105$.

Solution:

The counting boards/algorithm for "finding one" begins with the following,

$$\begin{bmatrix} 1 & 88 \\ 0 & 105 \end{bmatrix}.$$

Subtracting 1 time we get,

$$\begin{bmatrix} 1 & 88 \\ 1 & 17 \end{bmatrix}.$$

Subtracting 5 times we get,

$$\begin{bmatrix} 6 & 3 \\ 1 & 17 \end{bmatrix}.$$

Subtracting 5 times we get,

$$\begin{bmatrix} 6 & 3 \\ 31 & 2 \end{bmatrix}.$$

Subtracting 1 time we get,

$$\begin{bmatrix} 37 & 1 \\ 31 & 2 \end{bmatrix}.$$

With our "one" found we get that $x = 37$,

$$88(37) = 3256 = 105(31) + 1.$$

Reflection

1. I definitely had to review all the algorithms that we went over last Thursday. Thankfully I was able to go back and watch the class recording.
2. I though it was interesting how badly the image in the textbook portrays problem 15. $\triangle EDF$ looks isosceles even though it most definitely is not.