Arabic Mathematics

Al-Kashi's Method for $\sin(1^\circ)$

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Jamshid al-Kashani

- Born around 1380 in Kashan, Iran.
- Was a prominent member, and later the leader of the Samarkand Observatory.
- A prodigious calculator, his work largely involved producing more precise trigonometric tables.
- First methodical approach to our modern decimal notation. (vertical line)
- Law of Cosines

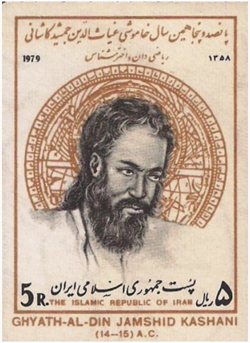

Figure 1: Depiction of Al-Kashi from MathStamps

Motivation for sin(1°)

- Recall that trigonometric tables were made incrementally.
- Ptolemy's Construction
  - Pythagorean special right triangles: 30,45,60
  - Elements Book 4 Proposition 10: 36,72,54,18
  - Half Angle 54: 27
  - Angle Difference 30-27: 3
  - Half Angle 3 : 1.5 , 0.75
  - Finally sin(1) is approximated
- Precise incremental unit $\rightarrow$ precise table
- Astronomy requires precise trig tables.

Al-Kashi's Method: The Setup

- First he discovered the triple angle identity (Mixture of construction and trig identity).

$$sin(3\theta) = 3sin(\theta) - 4sin^3(\theta)$$

- Substitute $\theta = 1$ and $sin(1) = x$ and we get a cubic,

$$sin(3) = 3x - 4x^3.$$

- Solve for x,

$$x = \frac{4}{3}x^3 + \frac{1}{3}sin(3).$$

- Now we have a function to iterate over.
- Arbitrary precision.

Al-Kashi's Method: Iteration Step

- Al-Kashi knew that $sin(1) \approx \frac{1}{3}sin(3) \approx .01$
  - Let $x_0 = .01$
- Suppose that $x = .01d_1d_2d_3d_4d_5\dots$ where $0 \leq d_i \leq 9$.
- Substituting into our function we get,

$$.01d_1d_2d_3d_4d_5\dots = \frac{4}{3}(.01d_1d_2d_3d_4d_5\dots)^3 + \frac{1}{3}sin(3).$$

- Subtract x_0 from both sides,

$$.00d_1d_2d_3d_4d_5\dots = \frac{4}{3}(.01d_1d_2d_3d_4d_5\dots)^3 + (\frac{1}{3}sin(3) - .01).$$

- Note that,

$$\frac{1}{3}sin(3) - .01 = .007445$$

$$\frac{4}{3}(.01d_1d_2d_3d_4d_5\dots)^3 = \frac{4}{3}(.000001d_1d_2d_3d_4d_5\dots)$$

- Thus $d_1 = 7$, set $x_1 = .017$ and repeat.

Al-Kashi's Method: Results

- With this method Al-Kashi calculated,

$$sin(1) = 0.017452406437283510$$

- 18 digits vs 16 digits with IEEE Double Precision
- Phone calculator demo

Comparison to Modern Methods: Power Series

- Recall the power series of *sin*,

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

- Matlab Demo

Comparison to Modern Methods: Counting FLOPs

- Al-Kashi
  - $\sim 7n$ operations $= O(n)$
- Power Series
  - $\sim 4n^2 - 2n - 1$ operations $= O(n^2)$