Exercises 1.3

Exercise 2: Show that $|z_1 z_2 z_3| = |z_1||z_2||z_3|$.

Solution:
Consider the polar form for each complex number $z_1 = r_1(\cos(\theta_1)+i\sin(\theta_1))$, $z_2 = r_2(\cos(\theta_2)+i\sin(\theta_2))$ and $z_3 = r_3(\cos(\theta_3)+i\sin(\theta_3))$. Now consider the product of the three terms and applying equation (7) from the text, we get the following,

$$z_1 z_2 z_3 = r_1 r_2 r_3 \left(\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)\right).$$

Thus it follows that $|z_1 z_2 z_3| = r_1 r_2 r_3 = |z_1||z_2||z_3|$.

Exercise 4: Show that for any integer k. $|z^k| = |z|^k$ (provided that $z \neq 0$ when k is negative).

Solution:
Suppose some complex number z and let $k \in \mathbb{Z}^+$. Note that the case where $k = 0$ is trivial. Let $k = 1$ and note that,

$$|z^1| = |z| = |z|^1.$$

Suppose that $|z^k| = |z|^k$ for some k and we will proceed by induction on k. Consider the term $|z^{k+1}|$ applyting the induction hypothisis we get the following,

$$\begin{aligned}
|z^{k+1}| &= |z^k z|,\\
&= |z^k||z|,\\
&= |z|^k|z|,\\
&= |z|^{k+1}.
\end{aligned}$$

For the $k \in \mathbb{Z}^-$ consider the previous result and with some algebra we get the following,

$$\begin{aligned}
|z^{-k}| &= |\frac{1}{z^k}|,\\
&= \frac{1}{|z^k|},\\
&= \frac{1}{|z|^k},\\
&= |z|^{-k}.
\end{aligned}$$

Exercise 5: Find the following,

b.
$$|\overline{(1+i)}(2-3i)(-3+4i)|$$

Solution:
The result from problem (2) shows us that we can simplify with the following,

$$|\overline{(1+i)}(2-3i)(-3+4i)| = |\overline{(1+i)}||(2-3i)||(-3+4i)| = |(1-i)||(2-3i)||(-3+4i)|.$$

Computing the moduli for each term we get the following,

$$|\overline{(1+i)}(2-3i)(-3+4i)| = \sqrt{2}\sqrt{13}5 = 5\sqrt{26}.$$

c.
$$\left|\frac{i(2+i)^3}{(1-i)^2}\right|$$

Solution:
Recall equation (12) from the text ((2) from the lecture notes.) which allows us to distribute the moduli through the division. Furthermore our result from problem allows us to extract the exponents from the moduli therefore the problem is simplified to,

$$\left|\frac{i(2+i)^3}{(1-i)^2}\right| = \frac{|i(2+i)^3|}{|(1-i)^2|} = \frac{|i||(2+i)|^3}{|(1-i)|^2}.$$

Computing the moduli for each term we get,

$$\left|\frac{i(2+i)^3}{(1-i)^2}\right| = \frac{(1)(\sqrt{5})^3}{(\sqrt{2})^2)} = \frac{5\sqrt{5}}{2}.$$

d.
$$\left|\frac{(\pi+i)^{100}}{(\pi-i)^{100}}\right|$$

Solution:
Applying a similar technique to the previous problem we get the following simplified form,

$$\left|\frac{(\pi+i)^{100}}{(\pi-i)^{100}}\right| = \frac{|(\pi+i)|^{100}}{|(\pi-i)|^{100}}.$$

From here it's clear to see that the solution is one since for any complex number z we know that $|z| = |\overline{z}|$.

Exercise 7: Find the argument fo each of the following complex numbers and write each in polar form.

b. $z = -3 + 3i$

Solution:
Computing the moduli we get $r = |z| = \sqrt{-3^2 + 3^2} = 3\sqrt{2}$. Computing the argument with the inverse tan formula we get the following, $\theta = tan^{-1}(\frac{3}{-3}) = \frac{3\pi}{4}$. Therefore we get the following polar form $z = 3\sqrt{2}(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4}))$.

e. $z = (1 - i)(-\sqrt{3} + i)$

Solution:
Recall from equation (7) in the text that the polar form of a product of two complex numbers can be extracted by multiplying their moduli and summing their arguments. Computing the moduli by multiplying the moduli of each term in the product, $r = r_1 r_2 = \sqrt(2)2$. Applying the inverse tan formula to extract the argument for each term in the product we get the following,

$$\theta_1 = \tan^{-1}(\frac{-1}{1}) = \frac{3\pi}{4}.$$

$$\theta_2 = \tan^{-1}(\frac{1}{-\sqrt{3}}) = \tan^{-1}(\frac{1/2}{-\sqrt{3}/2}) = -\frac{\pi}{6}.$$

Thus we get the argument for z through addition,

$$\theta = \theta_1 + \theta_2 = \frac{3\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}.$$

Therefore our final polar form is,

$$z = 2\sqrt(2)(\cos(\frac{7\pi}{12}) + i\sin(\frac{7\pi}{12})).$$

h. $\dfrac{-\sqrt{7}(1 + i)}{\sqrt{3} + i}$.

Solution:
Recall from equations (10), (11), and (12) we can solve this problem in a similar way to the last one. We begin by computing the moduli for each term in quotient. $r_1 = |-\sqrt{7}||1 + i| = \sqrt{7}\sqrt{2}$ and $r_2 = |\sqrt{3} + i| = 2$. By (12) we know that moduli for z is the quotient of the moduli, thus $r = |z| = \frac{\sqrt{14}}{2}$. Now we compute the argument for each term in the quotient with the inverse tan formula,

$$\theta_1 = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4},$$

$$\theta_2 = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6},$$

By (11) we get the argument for z by subtraction,

$$\theta = \theta_1 - \theta_2 = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

Therefore our final polar form is,

$$z = \frac{\sqrt{14}}{2}(\cos(\frac{\pi}{12}) + i\sin(\frac{\pi}{12}).$$

Exercise 12: Find the following,

a. $Arg(-6-6i)$

Solution:
Applying the inverse tan() formula we get the following,

$$Arg(-6-6i) = \tan^{-1}(\frac{-6}{-6}) = \frac{5\pi}{4}.$$

c. $Arg(10i)$

Solution:
Since our complex number only point in the positive imaginary direction we know that it's argument is $\theta = \frac{\pi}{2}$.

Exercise 13: Decide which of the following statements are always true.

a. $Arg(z_1z_2) = Arg(z_1) + Arg(z_2)$ if $z_1, z_2 \neq 0$

Solution:
False. Let $z_1 = z_2 = -1 + 0i$ Note that $Arg(z_1z_2) = 0 \neq Arg(z_1) + Arg(z_2) = 2\pi$.

b. $Arg(\overline{z}) = -Arg(z)$ if z is not a real number.

Solution:
True. This is shown by equation (14) in the text.

c. $Arg(z_1/z_2) = Arg(z_1) - Arg(z_2)$ if $z_1, z_2 \neq 0$

Solution:
False. Let $z_1 = 0 + i$ and $z_2 = 0 - i$. Note that $z_1/z_2 = -1$ so clearly $Arg(z_1/z_2) = \pi$, however we get, $Arg(z_1) - Arg(z_2) = \frac{\pi}{2} - \frac{3\pi}{2} = -\pi$.

d. $arg(z) = Arg(z + 2\pi k)$ where $k \in \mathbb{Z}$ if $z \neq 0$.

Solution:
True. This definition is stated explicitly in the footnotes of section 1.3 after equation (5).

Exercises 1.4

Exercise 2: Write the given numbers in standard form.

b. $z = 2e^{3+i\pi/6}$.

Solution:
Simplifying z to pull the argument and moduli

$$z = 2e^{3+i\pi/6} = (2e^3)e^{i\pi/6}.$$

Since $\pi/6$ is the argument and the moduli is $2e^3$ we know that the standard form of z looks like, $z = 2e^3(\frac{\sqrt{3}}{2} + \frac{1}{2}i) = 3e^3 + e^3 i$.

c. $z = e^x$ where $x = 4e^{i\pi/3}$.

Solution:
First note that we can simplify the inner exponential(x) with the fact that the moduli is 4 and the argument is $\pi/3$ we get the following, $x = 4(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 2 + 2\sqrt{3}i$. No we simplify the outer exponential,

$$z = e^{2+2\sqrt{3}i} = e^2 e^{2\sqrt{3}i}.$$

Applying the applying equation (7) to extract the components of the standard form from an argument of $2\sqrt{3}$ we get,

$$z = e^2(\cos(2\sqrt{3}) + i\sin(2\sqrt{3}))$$

Exercise 4: Write each of the given numbers in polar form $re^{i\theta}$.

b. $z = \dfrac{2+2i}{-\sqrt{3}+i}$.

Solution:
Simplifying the numerator and denominator by factoring out the moduli then using the unit circle to pull the argument we get,

$$z = \frac{2+2i}{-\sqrt{3}+i} = \frac{2\sqrt{2}(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}})i}{2(-\frac{\sqrt{3}}{2} + \frac{1}{2})i} = \frac{2\sqrt{2}e^{i\pi/4}}{2e^{i5\pi/6}} = \sqrt{2}e^{(i(\pi/4-5\pi/6))} = \sqrt{2}e^{(i(-7\pi/12))}$$

c. $z = \dfrac{2i}{3e^{4+i}}$.

Solution:
Continuing in a similar fashion we get,

$$z = \frac{2i}{3e^{4+i}} = \frac{2e^{i(\pi/2)}}{3e^4e^i} = \frac{2}{3e^4}e^{i(2\pi-1)}.$$

Exercise 5: Show that $|e^{x+iy}| = e^x$ and $\arg e^{x+iy} = y + 2k\pi$ for $k \in \mathbb{Z}$.

Solution:
First consider that,

$$|e^{x+iy}| = |e^x e^{iy}| = |e^x||e^{iy}|$$

Applying equation (7) from the text we get.

$$|e^{x+iy}| = |e^x||\cos(y) + i\sin(y)| = |e^x|(1).$$

Showing the that the set $\arg e^{x+iy} = y + 2k\pi$ for $k \in \mathbb{Z}$ first consider that the following sets are equivalent,

$$arg(e^{x+iy}) = arg(e^{iy})$$

From equation (7) we get the following,

$$arg(e^{x+iy}) = arg(cos(y) + isin(y))$$

Computing the $arg(cos(y) + isin(y))$ using the inverse tangent formula we get $arg(e^{x+iy}) = \tan^{-1}(\tan(y)) = y + 2k\pi$ for $k \in \mathbb{Z}$.

Exercise 7: Show that $e^z = e^{z+2\pi i}$ for all z.

Solution:
By applying definition (5) to the right hand side we get the following,

$$e^{z+2\pi i} = e^z(\cos(2\pi) + i\sin(2\pi)) = e^z.$$

Exercise 8b: Show that $\overline{e^z} = e^{\bar{z}}$ for all z.

Solution:
Let $z = a + ib$ for some $a, b \in \mathbb{R}$. Applying definition (5) we get the following,

$$\overline{e^z} = \overline{e^a \cos(b) + ie^a \sin(b)} = e^a \cos(b) - ie^a \sin(b).$$

Recall the negative angle identities, $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$ and note that,

$$e^{\bar{z}} = e^a \cos(-b) + ie^a \sin(-b) = e^a \cos(b) - ie^a \sin(b).$$

Thus it follows that $\overline{e^z} = e^{\bar{z}}$ for all z.

Exercise 9: Show that $(e^z)^n = e^{nz}$ for any integer n.

Solution:
Suppose some complex number z and let $n \in \mathbb{Z}$. Note that the case where $n = 0$ is trivial. Let $n = 1$ and note that,

$$(e^z)^{(1)} = e^z = e^{(1)z}.$$

Suppose that $(e^z)^n = e^{nz}$ for some $n \in \mathbb{Z}^+$. We will proceed by induction on n. Consider $(e^z)^{n+1}$ and note that,

$$\begin{aligned}(e^z)^{n+1} &= (e^z)^n(e^z),\\ &= e^{nz}(e^z),\\ &= e^{(n+1)z}.\end{aligned}$$

For the $n \in \mathbb{Z}^-$ consider the previous result and with some algebra we get the following,

$$\begin{aligned}(e^z)^{(-1)}n &= \frac{1}{(e^z)^n},\\ &= \frac{1}{e^{nz}},\\ &= e^{(-1)nz}.\end{aligned}$$

Exercise 10: Show that $|e^z| \leq 1$ if $Rez \leq 0$.

Solution:
Suppose $z = a + bi$ is a complex number with $a, b \in \mathbb{R}$ such that $a \leq 0$. Consider the following,

$$|e^z| = |e^{a+bi}| = |e^a(\cos(b) + i\sin(b))| = e^a.$$

By the definition of the exponential when $Rez = a \leq 0$ we know that $|e^z| = e^a \leq 1$.

Exercise 11: Determine which of the following properties of the real exponential function remain true for the complex exponential function (that is, for x replaced by z).

a. e^x is never zero.

Solution:
Let $z = x + yi$, by the equation (8) we know that $e^z = e^x(\cos(y) + i\sin(y))$. Clearly $e^x > 0$, and we know that $(\cos(y) + i\sin(y)) \neq 0$ Since there is no solution to the system $\cos(y) = 0, \sin(y) = 0$. Thus this is true for the complex exponential function.

b. e^x is a one-to-one function.

Solution:
Consider equation (10) in the text and note that $e^{2\pi i} = e^{-2\pi i}$ thus this is false for the complex exponential function.

c. e^x is defined for all x.

Solution:
Recall again the decomposition of e^z via equation (8), $e^z = e^x(\cos(y) + i\sin(y))$. Clearly e^x is well defined and trig functions are also well defined for all $x, y \in \mathbb{R}$. The product of well defined functions is also well defined thus e^z is well defined.

d. $e^{-x} = 1/e^x$.

Solution:
Again recall the decomposition of e^z via equation (8), note that $e^{-z} = e^{(-1)x}(\cos((-1)y)+ i\sin((-1)y))$. by De Moivre's formula (16) we get the following,

$$e^{-z} = e^{(-1)x}(\cos((-1)y) + i\sin((-1)y)) = (e^x(\cos(y) + i\sin(y)))^{-1} = \frac{1}{e^z}.$$

Exercise 12a: Use De Moivre's formula together with the binomial formula to derive $\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)$.

Solution:
Note that the following is true by De Moivre's formula.

$$\cos(3\theta) + i\sin(3\theta) = (\cos(\theta) + i\sin(\theta))^3$$

Now applying the binomial formula to compute the expansion we get,

$$
\begin{aligned}
\cos(3\theta) + i\sin(3\theta) &= (\cos(\theta) + i\sin(\theta))^3, \\
&= \cos^3(\theta) + 3\cos^2(\theta)\sin(\theta)i - 3\cos(\theta)\sin^2(\theta) - \sin^3(\theta)i, \\
&= (\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)) + (3\cos^2(\theta)\sin(\theta) - \sin^3(\theta))i.
\end{aligned}
$$

Consider the imaginary component of both sides,

$$\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)$$

Exercise 13: Show how the following trigonometric identities follow from equations (11) and (12).

a. $\sin^2(\theta) + \cos^2(\theta) = 1$

Solution:
Applying equations (11) and (12) to the right hand side we get the following,

$$
\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 + \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 \\
&= \frac{(e^{i\theta} + e^{-i\theta})^2 - (e^{i\theta} - e^{-i\theta})^2}{4} \\
&= \frac{(e^{2i\theta} + 2 + e^{-2i\theta}) - (e^{2i\theta} - 2 + e^{-2i\theta})}{4} \\
&= \frac{4}{4} = 1.
\end{aligned}
$$

b. $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

Solution:
Applying equation (11) to the left hand side we get,

$$
\begin{aligned}
\cos(\theta_1 + \theta_2) &= \frac{e^{i(\theta_1+\theta_2)} + e^{-i(\theta_1+\theta_2)}}{2} \\
&= \frac{e^{i\theta_1+i\theta_2} + e^{-i\theta_1-i\theta_2}}{2} \\
&= \frac{e^{i\theta_1}e^{i\theta_2} + e^{-i\theta_1}e^{-i\theta_2}}{2}
\end{aligned}
$$

Applying equation (7) to the denominator we get the following,

$$
\begin{aligned}
\cos(\theta_1 + \theta_2) &= \frac{e^{i\theta_1}e^{i\theta_2} + e^{-i\theta_1}e^{-i\theta_2}}{2}, \\
&= \frac{((\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)))}{2}, \\
&+ \frac{((\cos(-\theta_1) + i\sin(-\theta_1))(\cos(-\theta_2) + i\sin(-\theta_2)))}{2}, \\
&= \frac{\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2)}{2}, \\
&+ \frac{\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) - i\sin(\theta_1)\cos(\theta_2) - i\cos(\theta_1)\sin(\theta_2)}{2}, \\
&= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2).
\end{aligned}
$$

Exercise 23b: Compute the following integral by using the representation (11) or (12) together with the binomial formula.

$$\int_0^{2\pi} sin^6(2\theta)d\theta$$

Solution:
Applying the complex representation,

$$\int_0^{2\pi} \left(\frac{e^{i2\theta} - e^{-i2\theta}}{2i}\right)^6 d\theta$$

Applying the binomial theorem we can simplify,

$$
\begin{aligned}
\int_0^{2\pi} \left(\frac{e^{i2\theta} - e^{-i2\theta}}{2i}\right)^6 d\theta &= \frac{1}{(2i)^6}\int_0^{2\pi} \left(e^{i2\theta} - e^{-i2\theta}\right)^6 d\theta \\
&= \frac{1}{(2i)^6}\int_0^{2\pi} \left(e^{i2\theta} - e^{-i2\theta}\right)^6 d\theta \\
&= \frac{1}{(2i)^6}\int_0^{2\pi} e^{12i\theta} - 6e^{8i\theta} + 15e^{4i\theta} - 20 + 15e^{-4i\theta} - 6e^{-8i\theta} + e^{-12i\theta}d\theta \\
&= \frac{1}{(2i)^6}\int_0^{2\pi} e^{12i\theta} - 6e^{8i\theta} + 15e^{4i\theta} - 20 + 15e^{-4i\theta} - 6e^{-8i\theta} + e^{-12i\theta}d\theta \\
&= \frac{1}{(2i)^6}\int_0^{2\pi} 2(1)\cos(12\theta) - 2(6)\cos(8\theta) + 2(15)\cos(4\theta) - 20 \\
&+ (i\sin(12\theta) - i\sin(12\theta) + 6i\sin(8\theta) - 6i\sin(8\theta) + 15i\sin(4\theta) - 15i\sin(4\theta))d\theta \\
&= -\frac{1}{64}\int_0^{2\pi} 2\cos(12\theta) - 12\cos(8\theta) + 30\cos(4\theta) - 20d\theta
\end{aligned}
$$

Integrating each term we get,

$$\begin{aligned}
\int_0^{2\pi} sin^6(2\theta)d\theta &= -\frac{1}{64}\int_0^{2\pi} 2\cos(12\theta) - 12\cos(8\theta) + 30\cos(4\theta) - 20 d\theta, \\
&= -\frac{1}{64}\left[2\left(\frac{\sin(12\theta)}{12}\right) - 12\left(\frac{\sin(8\theta)}{8}\right) + 30\left(\frac{\sin(4\theta)}{4}\right) - 20\theta\right]_0^{2\pi}, \\
&= -\frac{1}{64}(-20)(2\pi), \\
&= \frac{40\pi}{64}.
\end{aligned}$$