Section 2.2

Exercise 7: Decide whether each of the following sequences converges, and if so, find its limit.

d. $z_n = \dfrac{n(2+i)}{n+1}$

Solution:
Consider the sequence, after we divide both terms in fraction by n,

$$\begin{aligned} z_n &= \frac{n(2+i)}{n+1}, \\ &= \frac{\frac{n(2+i)}{n}}{\frac{n+1}{n}}, \\ &= \frac{(2+i)}{\frac{n+1}{n}}. \end{aligned}$$

This gives us that the limit as $n \to \infty$ is

$$\lim_{n\to\infty} z_n = \frac{(2+i)}{\frac{n+1}{n}} = 2 + i.$$

e. $z_n = \left(\dfrac{1-i}{4}\right)^n$

Solution:
Consider the moduli of the sequence,

$$\begin{aligned} |z_n| &= \left|\frac{1-i}{4}\right|^n, \\ &= \left(\frac{|1-i|}{|4|}\right)^n, \\ &= \left(\frac{\sqrt{2}}{4}\right)^n, \\ &= 0. \end{aligned}$$

Note that if a sequence $|z_n|$ converges to zero it follows that z_n must also converge to zero. Let $\epsilon > 0$ and note that $||z_n| - 0| < \epsilon$ for some $N \in \mathbb{N}$ where $n \geq N$. Note the following,

$$||z_n| - 0| = ||z_n|| = |z_n| = |z_n - 0| < \epsilon.$$

Exercise 11: Find each of the following limits,

d $\lim_{z\to i} \dfrac{z^2+i}{z^4-1}$

Solution:
Note that the limit is undefined at $z = i$, since $(i)^4 - 1 = 1 - 1 = 0$. For $z \neq i$ consider the following,

$$\begin{aligned}\lim_{z\to i} \frac{z^2+i}{z^4-1} &= \lim_{z\to i} \frac{z^2+i}{(z^2+1)(z^2-1)},\\ &= \lim_{z\to i} \frac{1}{z^2-1},\\ &= \frac{1}{-2}.\end{aligned}$$

f $\lim_{z\to 1+2i} |z^2-1|$

Solution:
Simply substituting the limit value we get,

$$\begin{aligned}\lim_{z\to 1+2i} |z^2-1| &= |(1+2i)^2-1|,\\ &= |-4+4i|,\\ &= \sqrt{32},\\ &= 4\sqrt{2}.\end{aligned}$$

Exercise 25: Find each of the following limits involving infinity,

a $\lim_{z\to 2i} \dfrac{z^2+9}{2z^2+8}$

Solution:
Simply plugging in the value of the limit we see that the, denominator approaches 0 while the numerator stays constant therefore the limit of the sequence approaches

infinity.

$$\lim_{z \to 2i} \frac{z^2+9}{2z^2+8} = \frac{(2i)^2+9}{2(2i)^2+8}$$
$$= \frac{-4+9}{0}$$
$$= \frac{5}{0} = \infty.$$

b $\lim_{z\to\infty} \dfrac{3z^2-2z}{z^2-iz+8}$

Solution:
Consider the following factorization,

$$\lim_{z\to\infty} \frac{3z^2-2z}{z^2-iz+8} = \lim_{z\to\infty} \frac{z^2(3-\frac{2}{z})}{z^2(1-\frac{i}{z}+\frac{8}{z^2})},$$
$$= \lim_{z\to\infty} \frac{3-\frac{2}{z}}{1-\frac{i}{z}+\frac{8}{z^2}}.$$

Applying Theorem 1 we can break apart the limit into the denominator and the numerator,

$$\lim_{z\to\infty} 3 - \frac{2}{z} = 3 - 0 = 3,$$
$$\lim_{z\to\infty} 1 - \frac{i}{z} + \frac{8}{z^2} = 1 - 0 + 0 = 1.$$

Therefore the final limit is,

$$\lim_{z\to\infty} \frac{3z^2-2z}{z^2-iz+8} = \frac{3}{1} = 3.$$

Section 2.3

Exercise 7: Use rules (5)-(9) to find the derivatives fo the following function.

b $f(z) = (z^2-3i)^{-6}$

Solution:
Applying the power rule and chain rule we get the following,

$$f'(z) = (-6)(z^2-3i)^{-7}2z,$$
$$= (-12)z(z^2-3i)^{-7}.$$

d $f(z) = \dfrac{(z+2)^3}{(z^2+iz+1)^4}$

Solution:
Applying the quotient rule, power rule, and chain rule we get the following,

$$\begin{aligned}
f'(z) &= \frac{((z^2+iz+1)^4)(3(z+2)^2) - ((z+2)^3)(4(2z+i)(z^2+iz+1)^3)}{((z^2+iz+1)^4)^2},\\
&= \frac{(z^2+iz+1)(3(z+2)^2) - (z+2)^3(4(2z+i))}{(z^2+iz+1)^5},\\
&= \frac{(z+2)^2(3(z^2+iz+1) - 4(z+2)(2z+i))}{(z^2+iz+1)^5},\\
&= \frac{(z+2)^2(3z^2+3iz+3-8z^2-4iz-16z-8i)}{(z^2+iz+1)^5},\\
&= \frac{(z+2)^2(-5z^2-iz+3-16z-8i)}{(z^2+iz+1)^5}.
\end{aligned}$$

Exercise 9: For each of the following expressions determine the points at which the function is not analytic.

b $\dfrac{iz^3+2z}{z^2+1}$

Solution:
This expression is an example of a complex rational expression. We know that it is undefined, and therefore not differentiable when the denominator is zero. Solving for when the denominator is zero,

$$\begin{aligned}
z^2+1 &= 0,\\
z^2 &= -1,\\
z &= \sqrt{-1} = i.
\end{aligned}$$

d $z^2(2z^2-3z+1)^{-1}$

Solution:
Again this expression is a complex rational expression. Solving for when the denom-

inator is equal to zero using the quadratic formula we get the following,

$$\begin{aligned} z &= \frac{3 \pm \sqrt{(-3)^2 - 4(2)(1)}}{2(2)}, \\ &= \frac{3 \pm \sqrt{9-8}}{4}, \\ &= \frac{3 \pm 1}{4}. \end{aligned}$$

Exercise 11: Discuss the analyticity of each of the following functions,

d $x^2 - y^2 + 2xyi$.

Solution:
This function is a polynomial and is therefore differentiable on $\mathbb{C}$. Thus the function is analytic and entire.

f $(x + \frac{x}{x^2+y^2}) + i(y - \frac{y}{x^2+y^2})$

Solution:
First consider simplifying the function,

$$\begin{aligned} (x + \frac{x}{x^2 + y^2}) + i(y - \frac{y}{x^2 + y^2}) &= x + iy + \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2} \\ &= z + \frac{\bar{z}}{|z|^2}. \end{aligned}$$

Therefore the function is undefined and not analytic when $|z|^2 = 0$. Thus the function is analytic everywhere except $z = 0$.

2.4

Exercise 1: Use the Cauchy-Riemann equations to show that the following function are nowhere differentiable.

1. b $w = Rez$

Solution:
Let $z = x + iy$ and note that $w = Rez = x$. Checking the Cauchy-Riemann equations we get that,

$$\frac{du}{dx} = 1,$$
$$\frac{dv}{dy} = 0.$$

Since $\frac{du}{dx} \neq \frac{dv}{dy}$ we know that function is nowhere differentiable.

2. c $w = 2y + ix$

Solution:
Checking the Cauchy-Riemann equations we get that,

$$\frac{du}{dx} = 0,$$

$$\frac{dv}{dy} = 0.$$

Considering the next pair of partial derivatives,

$$\frac{du}{dy} = 2,$$

$$-\frac{dv}{dx} = -1.$$

Since $\frac{du}{dy} \neq -\frac{dv}{dx}$ we know that the function is nowhere differentiable.

Exercise 2: Show that $h(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable on the coordinate axes but is nowhere analytic.

Solution:
Consider the Cauchy-Riemann equations,

$$\frac{du}{dx} = 3x^2 + 3y^2 - 3,$$

$$\frac{dv}{dy} = 3x^2 + 3y^2 - 3.$$

Considering the next pair of partial derivatives,

$$\frac{du}{dy} = 6xy,$$

$$-\frac{dv}{dx} = -6xy.$$

Note that $\frac{du}{dy} \neq -\frac{dv}{dx}$ accept in the case when either $x = 0$ or $y = 0$. The function satisfies the Cauchy-Riemann equations are satisfied when $x = 0$ or $y = 0$, thus the function is differentiable on the coordinate axes.

Exercise 3: Use Theorem 5 to show that $g(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$ is entire. Write the function in terms of z.

Solution:
Consider the Cauchy-Riemann equation,

$$\frac{du}{dx} = 6x + 2,$$

$$\frac{dv}{dy} = 6x + 2.$$

Considering the next pair of partial derivatives,

$$\frac{du}{dy} = -6y,$$

$$-\frac{dv}{dx} = -6y.$$

Note that the first partial derivatives are continuous and satisfy the Cauchy-Riemann equations at every points in the plane. Hence by Theorem 5 $g(z)$ is entire. Simplifying the function to get it in terms of z,

$$\begin{aligned}
g(z) &= 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y),\\
&= 3x^2 + 2x - 3y^2 - 1 + i6xy + i2y,\\
&= 3x^2 + i6xy - 3y^2 + 2x + i2y - 1,\\
&= 3(x^2 + i2xy - y^2) + 2(x + iy) - 1,\\
&= 3(x + iy)^2 + 2(x + iy) - 1,\\
&= 3(z)^2 + 2(z) - 1.
\end{aligned}$$

Exercise 5: Show that $f(z) = e^{x^2-y^2}[cos(2xy) + isin(2xy)]$ is entire and find its derivative.

Solution:
Consider the Cauchy-Riemann equation,

$$\frac{du}{dx} = e^{x^2-y^2}\sin(2xy)(-2y) + \cos(2xy)e^{x^2-y^2}(2x),$$

$$\frac{dv}{dy} = e^{x^2-y^2}\cos(2xy)(2x) + \sin(2xy)e^{x^2-y^2}(-2y).$$

Considering the next pair of partial derivatives,

$$\frac{du}{dy} = e^{x^2-y^2}\sin(2xy)(-2x) + \cos(2xy)e^{x^2-y^2}(-2y),$$

$$-\frac{dv}{dx} = -(e^{x^2-y^2}\cos(2xy)(2y) + \sin(2xy)e^{x^2-y^2}(2x)).$$

Note that the first partial derivatives are continuous and satisfy the Cauchy-Riemann equations at every points in the plane. Hence by Theorem 5 $g(z)$ is entire. Computing the derivative we get,

$$\begin{aligned} f'(z) &= \frac{du}{dx} + \frac{dv}{dx}i \\ &= e^{x^2-y^2}\sin(2xy)(-2y) + \cos(2xy)e^{x^2-y^2}(2x) + e^{x^2-y^2}\cos(2xy)(2y) + \sin(2xy)e^{x^2-y^2}(2x)i \end{aligned}$$