Exercises 1.1

Exercise 5(b): Write the following in the form of $a + bi$

$$(8 + i) - (5 + i)$$

Solution:
By the addition and subtraction of complex numbers we get,

$$(8 + i) - (5 + i) = (8 - 5) + (1 - 1)i = 3 + 0i$$

Exercise 6(a): Write the following in the form of $a + bi$

$$(-1 + i)^2$$

Solution:
By the multiplication of complex numbers we get,

$$(-1 + i)^2 = (-1 + i)(-1 + i) = ((-1)(-1) - (1)(1)) + ((-1)(1) + (-1)(1))i = 0 - 2i$$

Exercise 7(b): Write the following in the form of $a + bi$

$$\frac{-1 + 5i}{2 + 3i}$$

Solution:
By the division of complex numbers we get,

$$\frac{-1 + 5i}{2 + 3i} = \frac{(-1)(2) + (5)(3)}{2^2 + 3^2} + \frac{(5)(2) - (-1)(3)}{2^2 + 3^2}i = 1 + 1i$$

Exercise 12: Write the following in the form of $a + bi$

$$(2 + i)(-1 - i)(3 + 2i)$$

Solution:
By the associative law of multiplication and the multiplication of complex numbers we get the following,

$$\begin{aligned}(2 + i)(-1 - i)(3 + 2i) &= ((2 + i)(-1 - i))3 + 2i),\\ &= (-1 + -3i)(3 + 2i),\\ &= 3 - 11i.\end{aligned}$$

Exercise 14: Show that $Re(iz) = -Im(z)$ for every complex number z.

Solution:
Suppose a complex number z. By definition z is of the form $z = a + bi$ where $a, b \in \mathbb{R}$. Now consider the following,

$$iz = i(a + bi) = ai + bi^2 = b(-1) + ai = -b + ai.$$

From the equality we know the following, $Re(iz) = -b$ and $-Im(z) = -b$. Thus, $Re(iz) = -Im(z)$.

Exercise 15: Let k be an integer. Show that,

1.
$$i^{4k} = 1$$

   Solution:
   Note that,
$$i^{4k} = i^{(2)(2)k} = \sqrt{-1}^{2^{2k}} = (-1)^{2k} = 1.$$

2.
$$i^{4k+1} = i$$

Solution:
Note that by substitution of the previous problem we get,

$$i^{4k+1} = i^{4k}i = 1(i) = i.$$

3.
$$i^{4k+2} = -1$$

Solution:
Note that by substitution of the previous problem we get,

$$i^{4k+1} = i^{4k}i^2 = 1\sqrt{-1}^2 = -1.$$

4.
$$i^{4k+3} = -i$$

Solution:
Note that by substitution of the previous problem we get,

$$i^{4k+3} = i^{4k+2}i = (-1)i = -i.$$

Exercise 18: Show that the complex number $z = -1 + i$ satisfies the following equation,

$$z^2 + 2z + 2 = 0.$$

Solution:
By substitution and algebra we get the following,

$$\begin{aligned} z^2 + 2z + 2 &= (-1 + i)^2 + 2(-1 + i) + 2, \\ &= -2i - 2 + 2i + 2, \\ &= 0. \end{aligned}$$

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Exercise 23: Let z be a complex number such that $Re(z) > 0$. Prove that $Re(1/z) > 0$

Solution:
Suppose a complex number z such that $Re(z) > 0$. By definition there z is of the form $z = a + bi$ where $a, b \in \mathbb{R}$. Note that,

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i.$$

Recall that $a = Re(z) > 0$ and therefore $Re(1/z) = \dfrac{a}{a^2+b^2} > 0$.

Exercise 24: Let z be a complex number such that $Im(z) > 0$. Prove that $Im(1/z) < 0$

Solution:
Suppose a complex number z such that $Re(z) > 0$. By definition there z is of the form $z = a + bi$ where $a, b \in \mathbb{R}$. Note that,

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i.$$

Recall that $b = Im(z) > 0$ and therefore $Im(1/z) = \dfrac{-b}{a^2+b^2} < 0$.

Exercises 1.2

Exercise 4: Let $z = 3 - 2i$. Plot the points $z, -z, \bar{z}, -\bar{z}$ in the complex plane. Do the same for $z = 2 + 3i$ and $z = -2i$.

Solution:

Figure 1: Plotting $z = 3 - 2i$

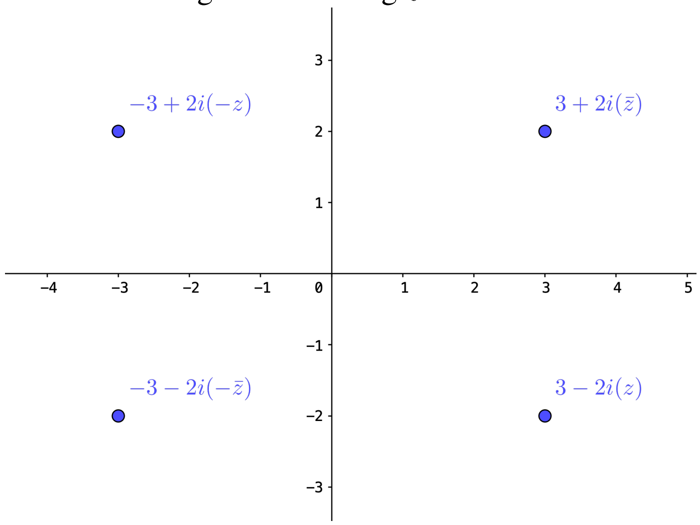


Figure 2: Plotting $z = 2 + 3i$

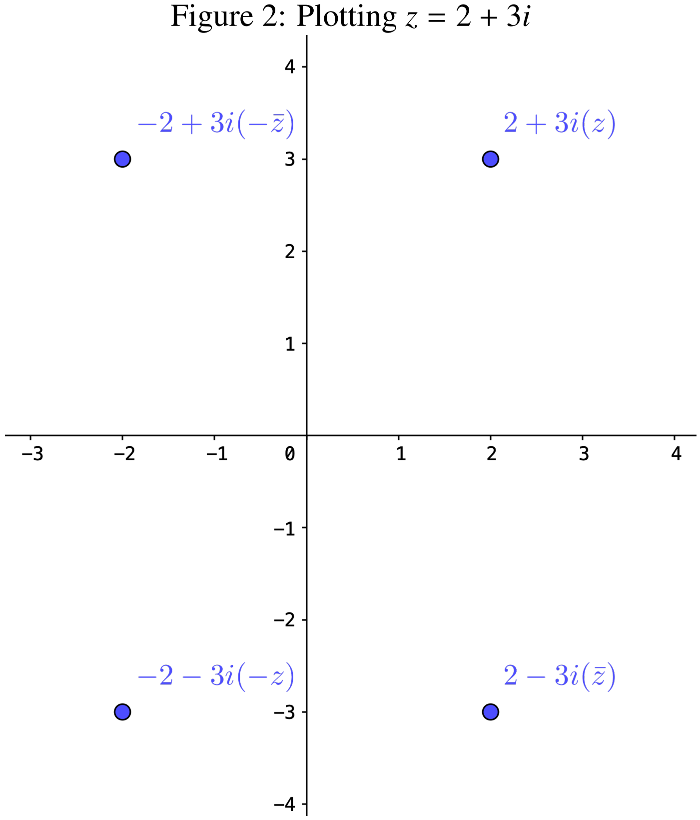
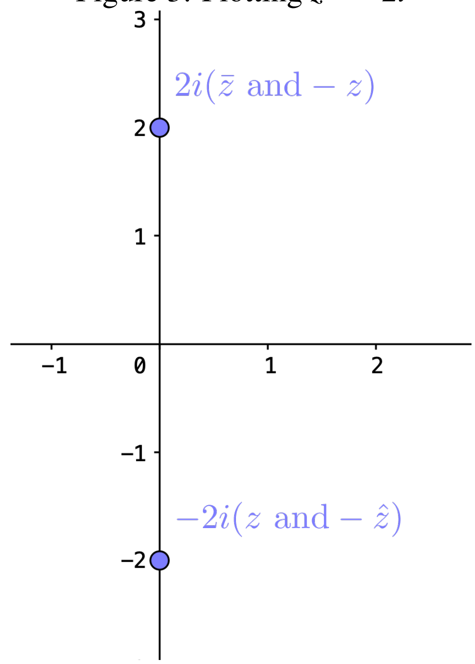


Figure 3: Plotting $z = -2i$



Exercise 7: Describe the set of points z in the complex plane that satisfies each of the following,

a $Im(z) = -2$

Solution:
This plots a horizontal line across the complex plane parallel to the real numbers at a height of -2.

c $|2z - i| = 4$

Solution:
Let $z = x + yi$ where $x, y \in \mathbb{R}$. By substitution and through algebra we get,

$$|2z - i| = 4,$$
$$|2(x + yi) - i| = 4,$$
$$|2x + (2y - 1)i)| = 4.$$

Applying the definition of modulus,

$$\begin{aligned} |2x + (2y-1)i)| &= 4, \\ \sqrt{(2x)^2 + (2y-1)^2} &= 4, \\ (2x)^2 + (2y-1)^2 &= 16. \end{aligned}$$

This final equation is a circle with radius 2 and center $(1, 0)$.

h $Rez \geq 4$

Solution:
This plots the inequality where everything to the right side of a vertical line at $x = 4$ is shaded in.

i $|z - i| < 2$

Solution:
Let $z = x + yi$ where $x, y \in \mathbb{R}$. By substitution and through algebra we get,

$$\begin{aligned} |z - i| &< 2, \\ |x + (y-1)i| &< 2. \end{aligned}$$

Applying the definition of modulus,

$$\begin{aligned} |x + (y-1)i| &< 2, \\ \sqrt{x^2 + (y-1)^2} &< 2, \\ x^2 + (y-1)^2 &< 4. \end{aligned}$$

This inequality describes all the points within the circle with radius 2 and center $(1, 0)$.

Exercise 8: Show both analytically and graphically, that $|z - 1| = |\bar{z} - 1|$.

Solution:
Let $z = a + bi$, with $a, b \in \mathbb{R}$. By definition $\bar{z} = a - bi$. By the definition of modulus,

$$|z - 1| = \sqrt{a^2 + b^2 - 1} = \sqrt{a^2 + (-b)^2 - 1} = |\bar{z} - 1|.$$

Graphically we know that the conjugate of a complex number z simply reflects it about the real axis. Note that $Re(z-1) = Re(\bar{z}-1)$, and $Im(z-1) = -Im(\bar{z}-1)$ so both numbers have the same distance from the origin, the only difference being direction.

Exercise 12: Verify properties (3), (4), and (5) for z_1 and z_2. First let $z_1 = a + bi$ and $z_2 = c + di$ for all $a, b, c, d \in \mathbb{R}$. Demonstrating (3), consider the division of complex numbers,

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{ac+bd}{c^2+d^2} - \frac{bc-ad}{c^2+d^2}i = \frac{ac+bd}{c^2+d^2} + \frac{(-b)c - a(-d)}{c^2+d^2}i = \frac{\bar{z}_1}{\bar{z}_2}$$

Demonstrating (4), let $z = a + bi$ and consider the following division,

$$\frac{z+\bar{z}}{2} = \frac{a+bi+a-bi}{2} = \frac{2a}{2} = a = Re(z).$$

Similarly we can demonstrate (5) by considering the following,

$$\frac{z-\bar{z}}{2i} = \frac{a+bi-a-bi}{2i} = \frac{2bi}{2i} = b = Im(z).$$

Exercise 13: Prove that if $(\bar{z})^2 = z^2$, then z is either real or pure imaginary.

Solution:
Let $z = a + bi$ for $a, b \in \mathbb{R}$ and suppose that, $(\bar{z})^2 = z^2$. Expanding we get,

$$\begin{aligned}(a-bi)^2 = (a+bi)^2, (a-bi)(a-bi) &= (a+bi)(a+bi),\\ a^2 - b^2 - 2abi = a^2 - b^2 + 2abi,&\\ 0 = 4abi.&\end{aligned}$$

By the last product we know that either $Re(z) = a = 0$ which concludes that z is pure imaginary or $Im(z) = b = 0$ which concludes that z is real, or the case where $z = 0$.

Exercise 14: Prove that $|z_1z_2| = |z_1||z_2|$ (Hint: Use equations (7) and (2)).

Solution:
Let z_1 and z_2 be complex numbers, and consider the square of the modulus of their product, $|z_1z_2|^2$. By equation (7) we know that the square fo the modulus of a complex number equals the number times its conjugate so we get the following,

$$|z_1z_2|^2 = z_1z_2\overline{z_1z_2}$$

By equation (2) we can distribute the conjugate among the right hand side product to get,

$$|z_1z_2|^2 = z_1z_2\overline{z_1}\,\overline{z_2}.$$

Since complex multiplication is commutative we can reorder the product and apply equation (7) to simplify,

$$\begin{aligned}
|z_1z_2|^2 &= z_1z_2\overline{z_1}\,\overline{z_2},\\
|z_1z_2|^2 &= z_1\overline{z_1}z_2\overline{z_2},\\
|z_1z_2|^2 &= |z_1|^2|z_2|^2,\\
|z_1z_2| &= |z_1||z_2|.
\end{aligned}$$

Exercise 15: Prove that $(\bar{z})^k = \overline{(z^k)}$ for every integer k (provided $z \neq 0$ when k is negative.)

Solution:
Suppose some complex number z let $k \in k \in \mathbb{Z}^+$. Note that the case where $k = 0$ is trivial. Consider the case where $k = 1$ and note that,

$$\begin{aligned}
(\bar{z})^1 &= \overline{(z^1)},\\
\bar{z} &= \bar{z}.
\end{aligned}$$

Suppose $(\bar{z})^k = \overline{(z^k)}$ for some k and we will proceed by induction on k. Consider the following.

$$(\bar{z})^{k+1} = (\bar{z})^k(\bar{z}).$$

By the induction hypothesis we know that,

$$(\bar{z})^{k+1} = \overline{(z^k)}(\bar{z}) = \overline{(z^k)(z)} = \overline{(z^{k+1})}.$$

For the case where $k \in \mathbb{Z}^-$, consider the previous result and with some algebra we get,

$$\begin{aligned}
(\bar{z})^k &= \overline{(z^k)},\\
(\bar{z})^{k^{-1}} &= \overline{(z^k)}^{-1},\\
(\bar{z})^{(-1)k} &= \overline{(z^{(-1)k})}.
\end{aligned}$$