1. Problem 2-23 Show that every manifold has a basis of coordinate balls.

Proof. Let M be a manifold. Since M is locally euclidean of dimension n we know that for every $x \in M$ every $U \in \mathcal{V}(x)$ is homeomorphic to a ball in $\mathbb{R}^n$. Choose a collection of open sets $\{U_x\}$ such that $U_x \in \mathcal{V}(x)$ for all $x \in M$. Clearly $\cup U_x = M$ and since each U_x is homeomorphic to an open ball in $\mathbb{R}^n$ they are by definition coordinate balls. □

2. Problem 3-2 Suppose X is a topological space and $A \subseteq B \subseteq X$. Show that A is dense in X if and only if A is dense in B and B is dense in X.

Proof. ($\Rightarrow$) Suppose X is a topological space with $A \subseteq B \subseteq X$ where A is dense in X. By definition, every $x \in X$ is a contact point of the set A. Since $B \subseteq X$ it follows A must also be dense in B. Since $A \subseteq B$ it follows directly that every point in X is also a contact point of B, hence B is dense in X.

($\Leftarrow$) Suppose X is a topological space with $A \subseteq B \subseteq X$ where A is dense in B and B is dense in X. Let $x \in X$ and consider some $U_x \in \mathcal{V}(x)$. Since B is dense in X there must exists some $b \in B$ such that $b \in U_x$. Now consider $U_b \in \mathcal{V}(b)$ and consider the open set $U_x \cap U_b$. Note that since A is dense in B there exists some $a \in A$ such that $a \in U_x \cap U_b$. Therefore there exists some $a \in U_x$ and thus A is dense in X. □

3. Problem 3-3 Show by giving a counterexample that the conclusion of glueing lemma need not hold if $\{A_i\}$ is an infinite closed cover.

Proof. Let $X = [-1, 1]$ be a topological space inheriting the subspace topology on $\mathbb{R}$. Consider the infinite closed cover, $\{A_n\} \cup \{-1, 1\}$ where $A_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$ for $n \in \mathbb{N}$. Define the following maps $f_0 : \{-1, 1\} \to \mathbb{R}$ with $f_0(x) = -x$ and $f_n : A_n \to \mathbb{R}$ with $f_n(x) = x$. The glueing lemma would have us believe that the function $f : [-1, 1] \to \mathbb{R}$ defined by,

$$f(x) = \left\{ \begin{array}{ll} x, & x \in (-1, 1) \\ -x, & x \in \{-1, 1\} \end{array} \right\}$$

is continuous, which clearly it is not. □

4. Exercise 3.7 Suppose X is a topological space and $U \subseteq S \subseteq X$.

(a) Show that the closure of U in S is equal to $\overline{U} \cap S$.

Proof. By definition the closure of U in S denoted $\overline{U_S}$ is the intersection of all subsets closed with respect to S which contain U. Note that $\overline{U} = \cap U_i$ where $U \subseteq U_i$ and U_i is closed in X. It follows that $\overline{U} \cap S = (\cap U_i) = \cap(U_i \cap S)$. Note that $(U_i \cap S)$ are closed with respect to S via the subspace topology and contain U, therefore $\overline{U} \cap S = \cap(U_i \cap S) = \overline{U_s}$. □

(b) Show that the interior of U in S contains $\operatorname{Int} U \cap S$; Give an example to show that they might not be equal.

Proof. Note that the interior of U in S denoted $\operatorname{Int}_S U$ is simply the largest subset which is open with respect to S contained in U. Let $X = \mathbb{R}$, $S = \mathbb{Z}$, and $U = \mathbb{Z}$. Note that $\operatorname{Int} U \cap S = \emptyset$ since $\operatorname{Int} \mathbb{Z}$ in $\mathbb{R}$ is empty. But $\operatorname{Int}_S U = \mathbb{Z}$ since S as a subspace must be open. □

5. Give a rock solid proof that the cylinder $M\{(x, y, z) \in \mathbb{R} : x^2 + y^2 = 1\}$ is a 2-manifold.

Proof. Let $S^* = S^2/\{(0, 0, 1), (0, 0, -1)\}$ and consider the function $f : M \to S^*$ defined by $f(x, y, z) = \frac{(x,y,z)}{\sqrt{1+z^2}}$. Recall that S^2 is a 2-manifold, and clearly S^*, an open subset of S^2 is also a 2-manifold. Therefore to prove that M is a 2-manifold we will proceed by showing that f is a homeomorphism.

First we will show that f is a bijection. Let $P, Q \in M$ such that $f(P) \neq f(Q)$. Applying f we find that

$$\begin{aligned}
\frac{z_p}{\sqrt{1+z_p^2}} &\neq \frac{z_q}{\sqrt{1+z_q^2}} \\
z_p\sqrt{1+z_q^2} &\neq z_q\sqrt{1+z_p^2} \\
z_p^2(1+z_q^2) &\neq z_q^2(1+z_p^2) \\
z_p^2 &\neq z_q^2 \\
z_p &\neq z_q
\end{aligned}$$

So clearly $P \neq Q$, and thus f is an injection.

Let $P \in S^*$, and note that by definition $x_p^2 + y_p^2 + z_p^2 = 1$. Since S^* removes the poles we can choose Q such that $Q = \frac{1}{\sqrt{x_p^2+y_p^2}}(x_p, y_p, z_p)$. Note that $Q \in M$, since

$$\left(\frac{x_p}{\sqrt{x_p^2+y_p^2}}\right)^2 + \left(\frac{y_p}{\sqrt{x_p^2+y_p^2}}\right)^2 = 1.$$

Finally we can see that applying f to Q gives P,

$$\begin{aligned}
f\left(\frac{1}{\sqrt{x_p^2+y_p^2}}(x_p, y_p, z_p)\right) &= \frac{1}{\sqrt{1+\left(\frac{z_p}{\sqrt{x_p^2+y_p^2}}\right)^2}}\frac{1}{\sqrt{x_p^2+y_p^2}}(x_p, y_p, z_p) \\
&= \frac{1}{\sqrt{\frac{x_p^2+y_p^2+z_p^2}{x_p^2+y_p^2}}}\frac{1}{\sqrt{x_p^2+y_p^2}}(x_p, y_p, z_p) \\
&= \frac{1}{\frac{1}{\sqrt{x_p^2+y_p^2}}}\frac{1}{\sqrt{x_p^2+y_p^2}}(x_p, y_p, z_p) \\
&= (x_p, y_p, z_p).
\end{aligned}$$

Hence f is a surjection, and we can conclude that f is a bijection.

Finally we will show that f and f^{-1} are continuous functions. Note that the component maps from $\mathbb{R}^3 \to \mathbb{R}^3$ of f are continuous. Consider the first component map $f_x(x, y, z) = \frac{x}{\sqrt{1+z^2}}$ is clearly a continuous function for any value of z and x. Similarly we know that f_y is continuous. We also know that $f_z(x, y, z) = \frac{z}{\sqrt{1+z^2}}$ is continuous, and since each component map of f is continuous f is also continuous.

We can show that $f^{-1} : S^* \to M$ defined by $f^{-1}(x, y, z) = \frac{1}{\sqrt{x^2+y^2}}$ is continuous. Here note that the component maps from $\mathbb{R}^3/\{x, y, z : x, y = 0\} \to \mathbb{R}^3/\{x, y, z : x, y = 0\}$ of f^{-1} are continuous. Consider $f_x^{-1}(x, y, z) = \frac{x}{\sqrt{x^2+y^2}}$ is continuous on $\mathbb{R}^3/\{x, y, z : x, y = 0\}$. Similarly with f_y^{-1} and f_z^{-1}.

Thus we have shown that M and S^* are homeomorphic.

□

6. Using metric space arguments only, show that a sequence $\{x_n\}$ in $\mathbb{R}^k$ converges to a limit x if and only if each projection sequence $\{\pi_j(x_n)\}$ converges to $\pi_j(x)$, $1 \le j \le k$.

Proof. ($\Rightarrow$) Suppose that a sequence $\{x_n\}$ in $\mathbb{R}^k$ converges to a limit x. By definition for all $\epsilon > 0$ there exists an N, such that for all $n \ge N$ it follows that,

$$d(x_n, x) = \sqrt{\sum_{i=1}^{k}(x_{i_n} - x_i)^2} < \epsilon.$$

Clearly it follows that for all $\pi_j(x)$, $1 \le j \le k$,

$$d(\pi_j(x_n), \pi_j(x)) = \sqrt{(x_{j_n} - x_j)^2} \le \sqrt{\sum_{i=1}^{k}(x_{i_n} - x_i)^2} < \epsilon.$$

□

Proof. ($\Leftarrow$) Let $\{x_n\}$ in $\mathbb{R}^k$ and suppose each projection sequence $\{\pi_j(x_n)\}$ converges to $\pi_j(x)$, $1 \le j \le k$. By definition we know that for each k there exists a N_k such that $n \ge N_k$ $\sqrt{(x_{k_n} - x_k)^2} < \frac{\epsilon}{k}$. Let $\epsilon > 0$ and note that for $N = max\{N_k\}$ it follows that for all $n \ge N$,

$$d(x_n, x) = \sqrt{\sum_{i=1}^{k}(x_{i_n} - x_i)^2} \le \sum_{i=1}^{k} \sqrt{(x_{i_n} - x_i)^2} < \sum_{i=1}^{k} \frac{\epsilon}{k} = \epsilon.$$

□

7. Let X be a topological space. The ***diagonal*** of $X \times X$ is the subset $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$ Show that X is Hausdorff if and only if Δ is closed in $X \times X$.

Proof. ($\Rightarrow$) Let X be a topological space and suppose X is Hausdorff. Consider the subset $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$. We will proceed to show that Δ is closed by showing that Δ^c is open. Let $(p, q) \in X \times X$. Since X is Hausdorff there exists open sets $U_p, U_q \subseteq X$ such that $p \in U_p$, $q \in U_q$, and $U_p \cap U_q = \emptyset$. Note that $(p, q) \in U_p \times U_q \subseteq \Delta^c$ since $U_p \cap U_q = \emptyset$. Therefore Δ^c is open and thus Δ is closed. $\square$

Proof. ($\Leftarrow$) Let X be a topological space and suppose Δ is closed in $X \times X$. Consider $p, q \in X$ such that $p \neq q$. Note that by definition $(p, q) \in \Delta^c$, and since Δ^c is open there exists a basic open set $p \in \pi^{-1}(U_p) \cap \pi^{-1}(U_q) \subseteq \Delta^c$ where U_p and U_q are open in X. Therefore it follows that $p \in U_p$ and $q \in U_q$, with $U_p \cap U_q = \emptyset$, and thus X is Hausdorff. $\square$

8. Let X and Y be topological spaces such that every $f : X \to Y$ is continuous. Show that either X is discrete or Y is indiscrete.

Proof. Let X and Y be topological spaces such that every $f : X \to Y$ is continuous. Suppose Y is not the indiscrete topology, which means that there exists some open set $U \subset Y$ such that $U \neq, Y$. Let $V \subseteq X$, we can construct f such that $V = f^{-1}(U)$ and since all f are continuous V must be open in X. Since V was chosen arbitrarily, every subset of X is open and thus X has the discrete topology. $\square$