1. **Exercise 2.22** Suppose $f : X \to Y$ is a homeomorphism and $U \subseteq X$ is an open subset. Show that $f(U)$ is open in Y and the restriction $f|_U$ is a homeomorphism from U to $f(U)$.

Proof. Suppose $f : X \to Y$ is a homeomorphism and $U \subseteq X$ is an open subset. Recall that since f is a homeomorphism we know that $f^{-1} : Y \to X$ is a continuous so $f(U)$, the pre-image of an open set U, must be open in Y.

Proving that $f|_U$ is a homeomorphism from U to $f(U)$ involves showing that $f|_U$ is a bijection and $f|_U$ and $f^{-1}|_U$ are continuous. Clearly since $U \subseteq X$, and $f : X \to Y$ is a bijection it must follow that any restriction $f|_U$ must also be a bijection (by contradiction this result is immediate).

Let $O \subseteq f(U)$ be an open set and note that $f^{-1}|_U(O) = U \cap f^{-1}(O)$. Since $f : X \to Y$ is continuous and O is also open in Y we know that $f^{-1}(O)$ must be open in X. Finally note that $U \cap f^{-1}(O)$ must be open in X and since $U \cap f^{-1}(O) \subseteq U$, $f^{-1}|_U(O) = U \cap f^{-1}(O)$ is open in U.

Let $O \subseteq U$ be an open set and note that $f|_U(O) = f(U) \cap f(O)$. Since $f^{-1} : Y \to X$ is continuous and O is also open in X we know that $f(O)$ must be open in Y. Therefore $U \cap f(O)$ must be open in Y and since $U \cap f(O) \subseteq U$, $f|_U(O) = U \cap f(O)$ is open in U.

$\square$

2. **Exercise 2.23** Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be topologies on the same set X. Show that the identity map of X is continuous as a map from $(X, \mathbb{T}_1)$ to $(X, \mathbb{T}_2)$ if and only if $\mathbb{T}_1$ is finer than $\mathbb{T}_2$, and is a homeomorphism is and only if and only if $\mathbb{T}_1 = \mathbb{T}_2$.

Proof. (is finer than) ($\Rightarrow$) Suppose the identity map f from $(X, \mathbb{T}_1)$ to $(X, \mathbb{T}_2)$ is continuous. Let $U \in \mathbb{T}_2$, and note that since f is continuous and the identity, it follows that $f^{-1}(U) = U$ must be open in $\mathbb{T}_1$. Thus $\mathbb{T}_2 \subseteq \mathbb{T}_1$.

($\Leftarrow$) Consider the identity map f from $(X, \mathbb{T}_1)$ to $(X, \mathbb{T}_2)$ and suppose that $\mathbb{T}_2 \subseteq \mathbb{T}_1$. Let $U \in \mathbb{T}_2$ and note that since f is the identity map $f^{-1}(U) = U$. Since $\mathbb{T}_2 \subseteq \mathbb{T}_1$ we conclude that $f^{-1}(U) \in \mathbb{T}_1$ and that f is continuous. $\square$

Proof. (Homeomorphism) ($\Rightarrow$) Suppose f is a homeomorphism from $(X, \mathbb{T}_1)$ to $(X, \mathbb{T}_2)$. By definition f is a bijection, and clearly since f is an identity map it must follow that $\mathbb{T}_1 = \mathbb{T}_2$.

($\Leftarrow$) Consider the identity map f from $(X, \mathbb{T}_1)$ to $(X, \mathbb{T}_2)$ and suppose that $\mathbb{T}_1 = \mathbb{T}_2$. By the previous result we can conclude that f and f^{-1} are continuous, and clearly since f is an identity map with $\mathbb{T}_1 = \mathbb{T}_2$ it is also a bijection. Thus f is a homeomorphism. $\square$

3. **Problem 2-4** Let X be a topological space and let $\mathcal{A}$ be a collection of subsets of X. Prove the following containments.

(a)

$$\overline{\bigcap_{A \in \mathcal{A}} A} \subseteq \bigcap_{A \in \mathcal{A}} \overline{A}$$

Proof. Note that the set $\bigcap_{A\in\mathcal{A}} \overline{A}$ is a closed set which must contain $\bigcap_{A\in\mathcal{A}} A$, since $\overline{A} \subseteq A$. Also recall that by definition $\overline{\bigcap_{A\in\mathcal{A}} A}$ is the intersection of all such closed subsets containing $\bigcap_{A\in\mathcal{A}} A$. Thus it follows that $\overline{\bigcap_{A\in\mathcal{A}} A} \subseteq \bigcap_{A\in\mathcal{A}} \overline{A}$. □

(b)

$$\overline{\bigcup_{A\in\mathcal{A}} A} \supseteq \bigcup_{A\in\mathcal{A}} \overline{A}$$

Proof. Let $x \in \bigcup_{A\in\mathcal{A}} \overline{A}$. Note that $x \in \overline{A}$ for some $A \in \mathcal{A}$. Note that $\overline{A}$ is the smallest closed set, which contains A, and $\overline{\bigcup_{A\in\mathcal{A}} A}$ is the smallest closed subset which contains $\bigcup_{A\in\mathcal{A}} A$, and since $A \subset \bigcup_{A\in\mathcal{A}} A$ it must follow that that $x \in \overline{A} \subseteq \overline{\bigcup_{A\in\mathcal{A}} A}$. □

(c)

$$\operatorname{Int}\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} \operatorname{Int}(A)$$

Proof. Note that $\operatorname{Int}(A)$ is the largest open subset contained in A, and since $\bigcap_{A\in\mathcal{A}} A \subseteq A$, for each $A \in \mathcal{A}$ it follows that, $\operatorname{Int}(\bigcap_{A\in\mathcal{A}} A) \subseteq \operatorname{Int}(A)$. Therefore we can conclude that $\operatorname{Int}\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} \operatorname{Int}(A)$. □

(d)

$$\operatorname{Int}\left(\bigcup_{A\in\mathcal{A}} A\right) \supseteq \bigcup_{A\in\mathcal{A}} \operatorname{Int}(A)$$

Proof. Again since $\operatorname{Int}(A)$ is the largest open subset contained in A, and since $\bigcup_{A\in\mathcal{A}} A \supseteq A$, for each $A \in \mathcal{A}$ it follows that, $\operatorname{Int}(\bigcup_{A\in\mathcal{A}} A) \supseteq \operatorname{Int}(A)$. Therefore we can conclude that $\operatorname{Int}\left(\bigcup_{A\in\mathcal{A}} A\right) \supseteq \bigcup_{A\in\mathcal{A}} \operatorname{Int}(A)$. □

(e) When $\mathcal{A}$ is a finite collection, show that equality holds in (b) and (c), but not necessarily in (a) or (d).

Proof. Note that $\overline{\cup_{A\in\mathcal{A}} A}$ is the smallest closed set containing $\cup_{A\in\mathcal{A}} A$ and $\cup_{A\in\mathcal{A}} \overline{A}$ contains $\cup_{A\in\mathcal{A}} A$. By our result from b and since $\cup_{A\in\mathcal{A}} \overline{A}$ is closed we get equality.

Similarly since $\operatorname{Int}(\cap_{A\in\mathcal{A}} A)$ is the largest open set contained in $\cap_{A\in\mathcal{A}} A$ and $\cap_{A\in\mathcal{A}} \operatorname{Int}(A)$ is contained in $\cap_{A\in\mathcal{A}} A$. By our result from c and since $\cap_{A\in\mathcal{A}} \operatorname{Int}(A)$ is now open we get equality.

For a counterexample for a consider $X = \mathbb{R}$ with the usual topology and $\mathcal{A} = \{(-1, 0), (0, 1)\}$. The closure of the intersection is empty, but the intersection of the closer is $\{0\}$.

For a counter example for d consider again $X = \mathbb{R}$ with the usual topology and $\mathcal{A} = \{[-1, 0], [0, 1]\}$. The interior of the intersection of the union is $(-1, 1)$, but the union of the interiors is $(-1,1)\backslash\{0\}$.

$\square$

4. Problem 2-5 (brief justifications only) For each of the following properties, give an example consisting of two subsets $X, Y \subseteq \mathbb{R}^2$, both considered as topological spaces with their Euclidean topologies, together with a map $f : X \to Y$ that has the indicated property.

For most examples I justified openness, or closedness of a function defined on $\mathbb{R}$ or subsets of $\mathbb{R}$ by looking at the basis of open intervals.

(a) f is open but neither closed nor continuous.

Proof. Let $f : (-\infty, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & x < 1 \\ 2 & x = 1 \end{cases}$$

This function is clearly discontinuous at $x = 1$ (construct a sequence $x_n = 1 - \frac{1}{n}$ which converges to 1 and note that $f(x_n) \not\to f(1)$). Let $(a, b) \subseteq (-\infty, 1]$ we find that $f((a, b)) = (a, b)$ an open interval. Consider the closed interval $[-1, 1]$ and note that $f([-1, 1]) = [-1, 1) \cup \{2\}$ which is not open in $\mathbb{R}$ since $f([-1, 1])^c = (-\infty, -1) \cup [1, 2) \cup (2, \infty)$ a not open set. $\square$

(b) f is closed but neither open nor continuous.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x < 0 \\ -1 & x \geq 0 \end{cases}$$

Clearly this function is discontinuous. It is not open since the only possible images are either $\{1\}, \{-1\}$ or $\{-1, 1\}$ which are closed sets in $\mathbb{R}$. Note that f must be closed for the same reason. $\square$

(c) f is continuous but neither open nor closed.

Proof. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |\arctan(x)|$. This function is continuous. Note that $f(\mathbb{R}) = [0, \frac{pi}{2})$ and since $[0, \frac{\pi}{2})$ is not closed and not open in $\mathbb{R}$, f is neither open nor closed. $\square$

(d) f is continuous and open but not closed.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$. This function is continuous. This function is open, if you take any open interval (a, b) we find that $f(a, b) = (e^a, e^b)$ an open interval. Note that $f(\mathbb{R}) = (0, \infty)$ an open set, thus f is not closed. $\square$

(e) f is continuous and closed but not open.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$. Clearly f is continuous. Note that any closed set on $\mathbb{R}$ will have a closed image of $\{1\}$ but so will any open set, hence f closed but not open. □

(f) f is open and closed but not continuous.

Proof. Let $f : [0, \infty) \to [0, \infty)$ defined by,

$$f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}.$$

Note f is not continuous at $x = 0$. Clearly the image of any open interval $f((a, b)) = (\frac{1}{b}, \frac{1}{a})$ is open. Note that closed intervals of the form $f([a, b]) = [\frac{1}{b}, \frac{1}{a}]$ where $a > 0$. Note that the image of closed intervals including 0 are given by $f([0, b]) = [\frac{1}{b}, \infty) \cup \{0\}$ which are also closed in $[0, \infty)$ since $f([0, b])^c = (0, \frac{1}{b})$. □

5. Problem 2-10 Suppose $f, g : X \to Y$ are continuous maps and Y is Hausdorff. Show that the set $A = \{x \in X : f(x) = g(x)\}$ is closed in X. Give a counterexample if Y is not Hausdorff.

Proof. Suppose $f, g : X \to Y$ are continuous maps and Y is Hausdorff. Consider $A^c = \{x \in X : f(x) \neq g(x)\}$ and let $x \in A^c$. By definition we know that $f(x) \neq g(x)$, and therefore since Y is Hausdorff there exists two open sets U_f and U_g such that $f(x) \in U_f$ and $g(x) \in U_g$ with $U_f \cap U_g = \emptyset$. Since f and g are continuous we know that $f^{-1}(U_f)$ and $g^{-1}(U_g)$ are open in X which both contain x. Now note that $f^{-1}(U_f) \cap g^{-1}(U_g) \subseteq A^c$, since $f(f^{-1}(U_f) \cap g^{-1}(U_g)) \subseteq U_f$ and $g(f^{-1}(U_f) \cap g^{-1}(U_g)) \subseteq U_g$ and $U_f \cap U_g = \emptyset$. Finally note that $x \in f^{-1}(U_f) \cap g^{-1}(U_g) \subseteq A^c$ so A^c is open and A is closed.

Consider $f, g : \mathbb{R} \to \mathbb{R}$ with both sets having the indiscrete topology where $f(x) = x$ and $g(x) = -x$. In this example $A = \{0\}$ and under $\mathbb{R}$ with the indiscrete topology this set is not closed, since $A^c \neq \emptyset, \mathbb{R}$. □

(You'll need the definition of a Hausdorff space, which we will see on Friday.)

6. Problem 2-15 Let X and Y be topological spaces. Suppose $f : X \to Y$ is continuous and $p_n \to p$ in X. Show that $f(p_n) \to f(p)$ in Y.

Proof. Let $U \in \mathcal{V}(f(p))$ and note that since f is continuous we know that $f^{-1}(U)$ is open in X. Since $p_n \to p$ in X and $f^{-1}(U) \in \mathcal{V}(p)$ there exists some $N \in \mathbb{N}$ such that $p_n \in f^{-1}(U)$ for all $n \geq N$. It then follows that $f(p_n) \in U$ for all $n \geq N$ and thus by definition $f(p_n) \to f(p)$.

□

7. (This is a modification of **Exercise 2.28**)

Consider the map $\exp : [0,1) \to S^1$ given by $\exp(x) = e^{2\pi i x} = \cos(2\pi x) + i\sin(2\pi x)$. This map is continuous (for example, it is sequentially continuous as a map between metric spaces). From familiar properties of trigonometric functions it is a bijection (though it would not be if we expanded the range to $[0,1]$ and it would not be if we shrunk the range!). Your job is to show that its inverse function is not continuous. Hint: Find a sequence $\{x_n\}$ in S^1 that converges to some point x, and yet $f^{-1}(x_n) \nrightarrow f^{-1}(x)$.

Proof. Let $\exp^{-1} : S^1 \to [0,1)$ be the inverse function and consider the sequence $x_n = e^{2\pi i \frac{-1}{n}}$. Note that,

$$\lim_{n\to\infty} e^{2\pi i \frac{-1}{n}} = \lim_{n\to\infty} e^{2\pi i} e^{\frac{-1}{n}} = 1$$

So $x_n \to 1$. Note that,

$$e^{2\pi i(1-\frac{1}{n})} = e^{2\pi i} e^{2\pi i \frac{-1}{n}} = (1) e^{2\pi i \frac{-1}{n}} = e^{2\pi i \frac{-1}{n}}$$

With $(1-\frac{1}{n}) \in [0,1)$ we know that $\exp^{-1}(x_n) = (1-\frac{1}{n})$ for all n. However clearly $\exp^{-1}(x_n) \to 1$, yet $\exp^{-1}(1) = 0$ and therefore $\exp^{-1}$ is not continuous at $1 \in S^1$.

□