1. Suppose $\langle x\rangle_{\alpha\in A}$ is a net in X that does not converge to $x \in X$. Show that there is an open set U containing x and a subnet $\left\langle x_{\alpha_\beta}\right\rangle_{\beta\in B}$ such that $x_{\alpha_\beta} \notin U$ for all $\beta \in B$. Hint: For a particular 'bad' U, take B to be the entire subset of A such that $x_\beta \notin U$ and show that B is directed. Then show that there is a natural increasing cofinal map from B to A.

Proof. Suppose $\langle x_\alpha\rangle_{\alpha\in A}$ is a net in X that does not converge to $x \in X$. By definition there exists a $U \in \mathcal{V}(x)$ such that $\langle x_\alpha\rangle_{\alpha\in T(\alpha')} \nsubseteq U$ for all $\alpha' \in A$. Let $B = \{\alpha \in A : x_\alpha \notin U\}$. We will proceed by showing that B is directed with $\leq$ relation inherited from A. Clearly it's reflective, and transitive, by A. Let $a, b \in B$, and note that $\langle x_\alpha\rangle_{\alpha\in T(a)} \nsubseteq U$ and $\langle x_\alpha\rangle_{\alpha\in T(b)} \nsubseteq U$, since $x_a, x_b \notin U$. Since A is directed, there exist a $c \in A$ such that $c \geq a, b$. Since $c \in A$ we know that $\langle x'_c\rangle_{c'\in T(c)} \nsubseteq U$, and therefore there exists a $x^*_c \notin U$. Note that by definition $c^* \in B$ and since $c^* \in T(c)$ it follows $c^* \geq c \geq a, b$ and thus B is directed.

We will proceed to show that $f : B \to A$ defined by the identity is increasing and cofinal. Clearly this map is increasing, since our directness was inherited from A. Let $a \in A$, and note that $\langle x_\alpha\rangle_{\alpha\in T(a)} \nsubseteq U$ so there exists some $b \in T(a)$ such that $x_b \notin U$. Note that $b \in B$ and $f(b) = b \geq a$.

Therefore B defines a subnet $\langle x_{\alpha_\beta}\rangle$ which by construction has the property that $x_{\alpha_\beta} \notin U$ for all $\beta \in B$.

$\square$

2. Crossley 6.1 Show that the spaces $[0, 1]$ and $(0, 1)$ are homotopy equivalent by finding an explicit homotopy equivalence and its inverse between the two spaces.

Proof. Let $f : [0, 1] \to (0, 1)$ be defined by $f(x) = \frac{1+2(x)}{4}$. Let $g : (0, 1) \to [0, 1]$ be the identity map. Consider $g \circ f$ which is defined by $g(f(x)) = \frac{1+2(x)}{4}$. We will show this function is homotopy equivalent to the identity on $(0, 1)$ by exhibiting an explicit homotopy. Consider the function $H_1 : I \times I \to I$ defined by,

$$H_1(x, t) = \frac{1 + 2(x)}{4}(1 - t) + x(t).$$

This function is continuous and note that $H_1(x, 0) = \frac{1+2(x)}{4} = g(f(x))$ and $H_2(x, 1) = x$.

Consider $f \circ g$ which is defined by $f(g(x)) = \frac{1+2(x)}{4}$. We will show this function is homotopy equivalent to the identity on $(0, 1)$ by exhibiting an explicit homotopy. Consider the function $H_2 : (0, 1) \times I \to (0, 1)$ defined by,

$$H_2(x, t) = \frac{1 + 2(x)}{4}(1 - t) + x(t).$$

This function is also continuous and note that $H_2(x, 0) = \frac{1+2(x)}{4} = f(g(x))$ and $H_2(x, 1) = x$. $\square$

velcro

3. Crossley 6.4b

Suppose $f : X \to S^n$ is continuous and not surjective. Show that it is homotopic to a constant map.

Proof. Suppose $f : X \to S^n$ is continuous and not surjective. Let $p \notin f(X)$ and recall that $A = S^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$ via the stereographic projection map σ. Note that $\sigma(f(x))$ is a map from $X \to \mathbb{R}^n$, and recall that $\mathbb{R}^n$ is contractible and therefore there exists a homotopy, namely $H(x,t) = \sigma(f(x))(1-t) + q(t)$ with the property that $H(x,0) = \sigma(f(x))$ and $H(x,1) = q$ where $q = \sigma(i)$ for some $i \in A$. Note that $\sigma^{-1}(H(x,t))$ is a function defined from $X \times I \to A$ with the property that

$$\sigma^{-1}(H(x,t)) = \sigma^{-1}(\sigma(f(x))(1) + q(0)) = f(x)$$

and

$$\sigma^{-1}(H(x,t)) = \sigma^{-1}(\sigma(f(x))(0) + q(1)) = \sigma^{-1}(q) = \sigma^{-1}(\sigma(i)) = i,$$

a constant map in S^n since $i \in A \subseteq S^n$. Since $\sigma^{-1}(H(x,t))$ is a composition of continuous functions it is also continuous and thus we have constructed a homotopy from f to a constant map. □

4. Crossley 6.5

Show by means of an explicit homotopy that the map $f : S^1 \to S^1$ given by $f(x,y) = (-x,-y)$ is homotopic to the identity.

Proof. Suppose $f : S^1 \to S^1$ given by $f(x,y) = (-x,-y)$. Note that $(x,y) = z$ for some $z \in \mathbb{C}$ and then the function is equivalent to $f(z) = e^{i\pi}z$. Clearly our desired homotopy is given by $H(z,t) = e^{i\pi(t)}z$ since it is continuous and $H(z,0) = e^0 z = z$ and $H(z,1) = e^{i\pi(1)}z = f(z)$. □

5. Show that a space X is contractible if and only if $[X, X]$ consists of a single element.

Proof. ($\Rightarrow$) Suppose X is contractible. By definition X is homotopy equivalent to a one point space, and therefore for $Y = \{p\}$ there exists function $c_p : X \to Y$ and $c_0 : Y \to X$ such that $c_p \circ c_0$ is homotopic to i_Y and $c_0 \circ c_p$ is homotopic to i_X. Since Y is a one point space $[Y : X]$ has only one homotopy class and therefore,

$$\begin{aligned}
[f] &= [f \circ i_X] \\
&= [f] \circ [i_X] \\
&= [f] \circ [c_0 \circ c_p] \\
&= [f \circ c_0] \circ [c_p] \\
&= [c_0] \circ [c_p] \\
&= [g \circ c_0] \circ [c_p] \\
&= [g] \circ [c_0 \circ c_p] \\
&= [g] \circ [i_X] \\
&= [g \circ i_X] \\
&= [g]
\end{aligned}$$

□

Proof. ($\Leftarrow$) Suppose $[X, X]$ consists of a single element. Then the constant map for $p \in X$ is homotopic to the identity map in X. Consider the one point space $A = \{p\}$ and note that $c_p : X \to A$ and $i_X|_p$, the identity map restricted to A when composed as $c_p \circ i_X|_p$ give the identity in A and $i_X|_p \circ c_p$ give a constant map in X, which is homotopic to the identity in X and therefore $X \sim A$. □

6. Suppose that $f, g : S^n \to S^n$ are continuous maps such that $f(x) \neq -g(x)$ for any $x \in S^n$. Prove that f and g are homotopic.

Proof. Consider the function $H : S^n \times I \to S^n$,

$$H(x,t) = \frac{f(x)(1-t) + g(x)t}{|f(x)(1-t) + g(x)t|}.$$

Note that since $f, g : S^n \to S^n$ we know that $H(x, 0) = \frac{f(x)}{|f(x)|} = f(x)$ and $H(x, 1) = \frac{g(x)}{|g(x)|} = g(x)$. What is left to show is that $H(x,t)$ is continuous from $S^n \times I \to S^n$.

Note that $n(x) = \frac{x}{|x|}$ is a continuous function from $R^n \setminus \{0\} \to S^n$. Let $c(x,t) : S^1 \times I \to R^n$ be defined by $c(x,t) = f(x)(1-t) + g(x)t$ and suppose to the contrary that $f(x)(1-t) + g(x)t = 0$, that would imply that $g(x) = \frac{t-1}{t} f(x)$, and since f and g are maps into S^n, taking the absolute value of both sides gives,

$$\begin{aligned} |g(x)| &= \left|\frac{t-1}{t}\right| |f(x)|, \\ 1 &= \left|\frac{t-1}{t}\right|, \\ 1 &= \frac{|t-1|}{t}, \\ t &= |t-1|. \end{aligned}$$

so $t = 1/2$. Substituting back into our equation we get $f(x) = -g(x)$ a contradiction. Thus $f(x)(1-t) + g(x)t \neq 0$ for all $(x,t) \in S^n \times I$. Clearly $c(x,t)$ is continuous, and since $c(x,t) \neq 0$ we know that $n(c(x,t)) = H(x,t)$ is continuous, as it is a composition of continuous functions. □