1. Although quotient maps must take saturated open sets to open sets and saturated closed sets to closed sets, a quotient map need not be open or closed. The point of this exercise is to see an example.

Let A be the set of points (x, y) in $\mathbb{R}^2$ with $y = 0$ or $x \geq 0$. Let $\pi((x, y)) = x$. Show that $\pi : A \to \mathbb{R}$ is a quotient map, but that it is neither open nor closed.

Proof. Let A be the set of points (x, y) is $\mathbb{R}^2$ with $y = 0$ or $x \geq 0$. Let $\pi((x, y)) = x$. To prove that π is a quotient map we must show that π is a continuous surjective map which takes saturated open sets to open sets. Clearly π is a continuous surjection, since π is simply $\pi_1|_A$ a restriction on the natural projection onto the first component of $\mathbb{R}^2$. Let $V \subseteq A$ be an closed saturated set, by definition there exists some $W \subseteq \mathbb{R}$ such that $V = \pi^{-1}(W)$. Let $\tilde{\mathbb{R}} \subseteq A$ be the set of points (x, y) with $y = 0$. Note $\mathbb{R}$ is homeomorphic to $\tilde{\mathbb{R}}$ and since $V \cap \tilde{\mathbb{R}}$ is closed in $\tilde{\mathbb{R}}$ by the subspace topology induced by A, it follows that it's homeomorphic image $\pi(V)$ is closed in $\mathbb{R}$. □

2. Let $\pi : X \to Y$ be a quotient map and let $A \subseteq X$ be a saturated closed set or a saturated open set. Show that $\pi|_A : A \to \pi(A)$ is a quotient map.

Proof. Let $\pi : X \to Y$ be a quotient map and let $A \subseteq X$ be a saturated closed set or a saturated open set. Now consider $\pi|_A : A \to \pi(A)$. Note that it is a continuous surjection, since π was continuous and we've restricted it's codomain to $\pi(A)$. Now suppose $V \subseteq \pi(A)$ such that $W = \pi|_A^{-1}(V)$ is open in A. Note that since A has the subspace topology W must be open in X and therefore since π is a quotient map $\pi(W)$ is open in Y. Since $W \subseteq A$ it follows that $\pi|_A(W) = \pi|_A(\pi|_A^{-1}(V)) = V$ is open in $\pi(A)$. □

3. Problem 3-14 Show that the real projective space $\mathbb{P}^n$ is an n-manifold. [Hint: Consider the subsets $U_i \subseteq \mathbb{R}^{n+1}$ where $x_i = 1$.]

4. Problem 3-16 Let X be the subset $(\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \subset \mathbb{R}^2$. Define an equivalence relation on X by declaring $(x, 0) \sim (x, 1)$ if $x \neq 0$. Show that the quotient space $X/\sim$ is locally euclidean and second countable, but not Hausdorff.

Your solution should not be longer than a page. Extra credit for the shortest correct solution.

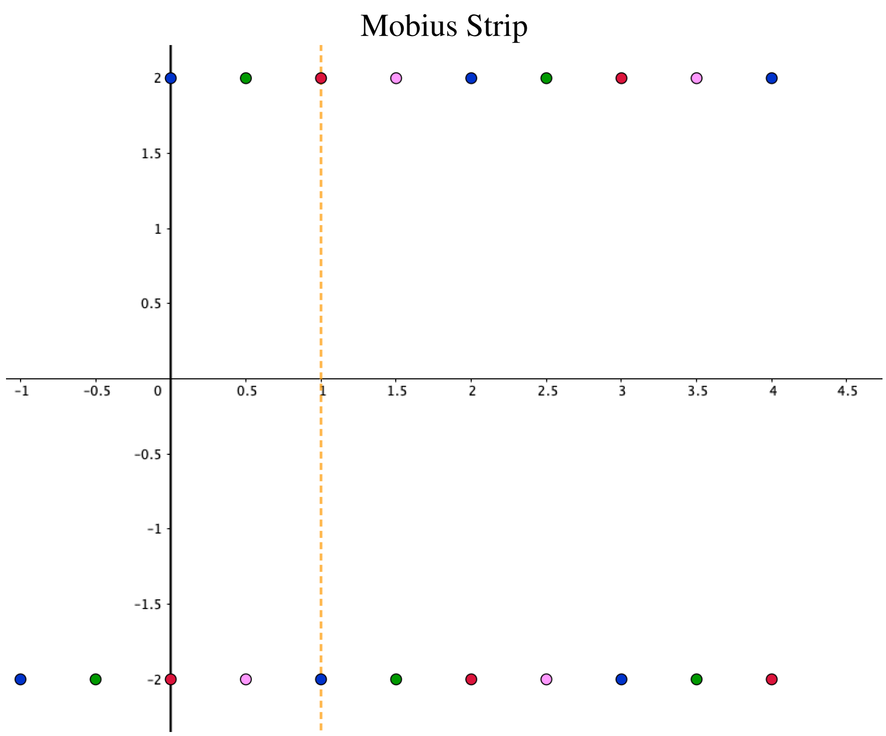
5. No rigor please on this problem. Just coherent explanations, and maybe a picture or two.

a) Define an equivalence class on $\mathbb{C}$ where $z \sim w$ if there is $u \in S^1$ with $z = wu$. The quotient space $\mathbb{C}/\sim$ is a familiar topological space. Name it. ("is" means "is homeomorphic to, of course").

Proof. We can see that the equivalence classes formed in the quotient space consist of $x \in \mathbb{C}$ with the same magnitude (or norm), so naturally we can thing of $\mathbb{C}/\sim$ as being homeomorphic to the interval $[0, \infty)$. To show this It would be appropriate to use the Uniqueness of Quotient Spaces, let $q : \mathbb{C} \to \mathbb{C}/\sim$ be the natural quotient map sending each element to its equivalence class. We would like to show that $f : \mathbb{C} \to [0, \infty)$ defined by $f(a + bi) = \sqrt{a^2 + b^2}$ is a quotient map and makes the same identifications as q. $\square$

b) Define an equivalence class on $\mathbb{R}^2$ where $(x, y) \sim (x + 1, -y)$ (along with all the relations then implied by transitivity). The resulting quotient space is a familiar one. Name it.

Proof. Note that the quotient space is homeomorphic to $M = [0, 1) \times \mathbb{R}$, where each point in M represents an equivalence class of points related by $(x, y) \sim (x + 1, -y)$. Some of these classes are illustrated below,



$\square$

6. Let $X = \mathbb{R} \times \{0, 1\}$ and define an equivalence relation on X by $(0, 0) \sim (0, 1)$. Rigorously show that $X/\sim$ is homeomorphic to the union of the x- and y-axes in the plane.

Proof. Let $\mathbb{R}^{xy} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$ and consider the $q : X \to X/\sim$ be the natural projection sending each element of X to its equivalence class. We will use the Uniqueness of Quotient spaces to show that $\mathbb{R}^{xy}$ is homeomorphic to $X/\sim$, by constructing a quotient map $q^* : X \to \mathbb{R}^{XY}$ which makes the same identifications as q. Let $q^*((x, y))$ be defined by,

$$q^*((x, y)) = \left\{ \begin{array}{ll} (x, 0), & y = 0 \\ (0, x), & y = 1 \end{array} \right\}.$$

Let $(x, y) \in \mathbb{R}^{xy}$ and note that by definition either $x = 0$ or $y = 0$, without loss of generality let $x = 0$ now note that for $(y, 1) \in X$ we know that $q^*((y, 1)) = (0, y) = (x, y)$. Hence q^* is surjective.

Now note that $\{\mathbb{R} \times \{0\}, \mathbb{R} \times \{1\}\}$ form a finite closed cover for X. Let $q_0^* : \mathbb{R} \times \{0\} \to \mathbb{R} \times \{0\}$ be defined by q^* restricting it's domain. Clearly q_0^* is continuous as it becomes the identity map. Let $q_1^* : \mathbb{R} \times \{1\} \to \{0\} \times \mathbb{R}$ be defined by restricting the domain of q^*. Note that $q_1^* = \iota \circ \pi_1$ where π_1 is the projection into the first component, and ι is the natural embedding into $\{0\} \times \mathbb{R}$. Since q_1^* is a composition of continuous functions, q_1^* is continuous. Since $q_1^*((0, 0)) = (0, 0) = q_0^*((0, 0))$ it follows by the glueing lemma that q^* is a continuous map.

Now we will show that q^* is a closed map. Let $C \subseteq X$ be a closed set note that $C = C_0 \cup C_1$ where C_0 and C_1 are closed in $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ respectively. So it follows that that $q^*(C) = q_0^*(C_0) \cup q_1^*(C_1)$. Clearly q_0^* and q_0^* are homeomorphisms, so they are closed maps, whose images in $\mathbb{R}^{xy}$ are closed, and thus $q_0^*(C_0)$ and $q_1^*(C_1)$ are closed in $\mathbb{R}^{xy}$. Finally we can conclude that $q^*(C)$ is closed in $\mathbb{R}^{xy}$ and that q^* is a closed map.

Therefore q^* is a quotient map. Note that $q((0, 0)) = q((0, 1))$ and evidently $q^*((0, 1)) = (0, 0) = q^*((0, 0))$ so q and q^* make the same identifications and thus by the Uniqueness of the Quotient Spaces we know that there exists homeomorphism $f : X/\sim \to \mathbb{R}^{xy}$.

□

7. Problem 4-1 Show that for $n > 1$, $\mathbb{R}^n$ is not homeomorphic to any open subset of $\mathbb{R}$.

Proof. Let $n > 1$ and suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a homeomorphism. Let $y \in \mathbb{R}$ and $x \in \mathbb{R}^n$ such that $f(x) = y$. Since f is a homeomorphism it must them follow that $f^* : \mathbb{R}^n - \{x\} \to \mathbb{R} - \{y\}$ defined by $f^*(x) = f|_{\mathbb{R}^n - \{x\}}$ is a homeomorphism as well. However note that $\mathbb{R} - \{y\}$ is a disconnected space while $\mathbb{R}^n - \{x\}$, for $n > 1$ is still connected. □

8. **Exercise 4-4** Prove that a topological space X is disconnected if an only if there exists a nonconstant continuous function from X to the discrete space $\{0, 1\}$.

Proof. ($\Rightarrow$) Suppose a topological space X is disconnected. Then by definition there exists nonempty, open $U, V \subseteq X$ such $U \cup V = X$ and $U \cap V = \emptyset$. Define $f : X \to \{0, 1\}$ such that $f(x) = 1$ when $x \in U$ and $f(x) = 0$ when $x \in V$. This function is continuous. □

Proof. ($\Leftarrow$) Suppose there exists a continuous function $f : X \to \{0, 1\}$. Note that $f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = f^{-1}(\{0\} \cap \{1\}) = \emptyset$ and that $f^{-1}(\{0\}) \cup f^{-1}(\{1\}) = f^{-1}(\{0\} \cap \{1\}) = \emptyset$ and that $f^{-1}(\{0\} \cup \{1\}) = X$. So therefore X is disconnected. □