1. Problem 4-11 b) (Just the connected part; the path connected is similar)

(Sorry this is the same as the midterm. Don't waste your time if you've already seen it.)

Proof. ($\rightarrow$) Suppose CX is locally (path-)connected. To show that X is locally (path-)connected we will show that for any $p \in X$, $U \in \mathcal{V}(p)$ there exists an open (path-)connected subset U' such that $p \in U' \subseteq U$. Let $p \in X$ and $U \in \mathcal{V}(p)$. Note that by the product topology, $U \times [0,1]$ is open in $X \times [0,1]$, and since q is a quotient map it also must follow that $q(U \times [0,1])$ is open in CX. Note that since CX is locally (path-)connected there exists a (path-)connected set $CU' \subseteq q(U \times [0,1])$ whose pre-image $q^{-1}(CU')$ is open in $X \times [0,1]$ since q is a quotient map and contains $p \times [0,1]$. Let $U' = \pi_X(q^{-1}(CU'))$ and note that by construction $p \in U' \subseteq U$. Thus X is locally (path-)connected.

□

Proof. ($\leftarrow$) Suppose X is locally path connected. By Lemma 2 we know that since X is locally (path-)connected, $X \times (0,1]$ is also locally (path-)connected. Since $q : X \times [0,1] \rightarrow CX$ is a continuous map, it follows that the image $q(X \times (0,1])$ is locally (path-)connected in CX. Let $v = q(X \times \{0\})$, what remains to be shown is that for any $U \in \mathcal{V}(v)$ there exists a (path-)connected subset U' such that $v \in U' \subseteq U$. By the part (a) of this problem we know that CX is path connected, so there must exists such a U'. Therefore CX is locally (path-)connected. □

2. Problem 4-9 [Modified] Let M be an n-manifold.

a) Show that each component of M is a (connected) manifold.

Proof. Let M be an n manifold. Suppose S is a component of M. By definition S is a maximal nonempty connected subset of M. Let $x \in S$ and note that since M is a manifold there exists a $U \in \mathcal{V}(x)$ which is open in M and homeomorphic to $\mathbb{R}^n$. Note that U is a connected subset of M containing x, and by definition S is the maximal connected subset containing x is follows that $U \subseteq S$.

Clearly S is Hausdorff under the subspace topology. Let $\{U_i\}$ be a countable basis for M, and note that by the subspace topology $\{U_i \cap S\}$ must also be a countable basis for S. Let $x \in S$ and $U \in \mathcal{V}(x)$ open in S. By the subspace topology there exists some U' open in M such that $U' \cap S = U$. Note there exists some U_i such that $x \in U_i \subseteq U'$ and clearly it follows that $x \in U_i \cap S \subseteq U' \cap S = U$. Hence S is second countable. □

b) Show that there are at most countably many components.

Proof. Suppose there are uncountably many components of M, $\{S_\alpha\}_{\alpha \in A}$. Note this collection is an open cover of M and partition M. Therefore $\{S_\alpha\}_{\alpha \in A}$ is a open cover with no finite subcover, as removing an S_α would not cover M. Hence M is not Lindeloff and therefore not second countable, a contradiction. □

c) Suppose $f : M \rightarrow Z$ is a map into a topological space Z. Show that f is continuous if and only if its restriction to each component is.

Proof. ($\Leftarrow$) Suppose $f : M \to Z$ is a continuous map into a topological space Z. Let S be a component of M and let $U \subseteq Z$ be open. Consider $f|_A^{-1}(U)$ and note that since f is continuous we know that $f^{-1}(U)$ is open in M. By the subspace topology $f^{-1}(U) \cap S = f|_A^{-1}(U)$ is open in S. □

Proof. ($\Rightarrow$) Let $\{S_i\}$ is the set of components of M and suppose each $f|_{S_i}$ is continuous into Z. Note that $\{S_i\}$ form an open cover and partition M so therefore each restriction vacuously agrees on their overlapping domains, since there are none. By the Glueing Lemma $f : M \to Z$ is continuous. □

d) Read Theorem 3.41. Then conclude that an n-manifold is homeomorphic to a disjoint union of countably many connected n-manifolds.

Solution:
Let M be an n-manifold with $\{S_i\}$ the collection of components. Note that clearly the identity map $f : M \to M$ is continuous. By the previous result we know that each $f|_{S_i}$ is continuous into M, and by Theorem 3.41 we conclude that $f : \coprod S_i \to M$ is continuous. Similarly we know that the identity map $g : \coprod S_i \to \coprod S_i$ is continuous. By Theorem 3.41 each $g|_{S_i}$ is continuous into $\coprod S_i$ and by the previous result we get that $g : M \to \coprod S_i$ is continuous. Since $\{S_i\}$ is a partition, the bijectivity of the identity maps carry through.

3. Let $f : X \to Y$ where X is a space and Y is compact and Hausdorff. Show that f is continuous if and only if the graph of f is closed in $X \times Y$. The graph of f is $G_f = \{(x, f(x)) : x \in X\}$.

Proof. ($\Rightarrow$) Let $f : X \to Y$ where X is a space and Y is compact and Hausdorff. Suppose f is continuous. We will proceed by showing that G_f^c is open. Let $(x, y) \in G_f^c$. By definition of G_f it follows that $y \neq f(x)$ and since Y is Hausdorff, there exists open sets $y \in U$ and $f(x) \in V$ such that $U \cap V = \emptyset$. Note that since f is continuous $x \in f^{-1}(V)$ is open in X. Let $(a, b) \in f^{-1}(V) \times U$, since $b \in U$, $b \notin V$ and since $a \in f^{-1}(V)$, $f(a) \in V$ therefore $b \neq f(a)$ and $(a, b) \notin G_f$. Finally note that $f^{-1}(V) \times U$ is open in $X \times Y$ and is contained in G_f^c. □

Proof. ($\Leftarrow$) Let $f : X \to Y$ where X is a space and Y is compact and Hausdorff. Suppose that G_f is closed in $X \times Y$. Let $A \subseteq Y$ be closed closed and therefore it's preimage under projection into Y which is $X \times A$ must also be closed in $X \times Y$ because projections are continuous. Note that since G_f, is closed $G_f \cap X \times A$ is also closed. Recall, we have shown that if Y is compact, then the projection $\pi : X \times Y \to X$ is a closed map, and therefore $A^* = \pi(G_f \cap X \times A)$ is closed in X. We will proceed by showing that $A^* = f^{-1}(A)$. Let $x \in A^*$, and therefore we know that by definition $(x, f(x)) \in G_f \cap X \times A$ and therefore $f(x) \in A$ and thus $x \in f^{-1}(A)$. Let $x \in f^{-1}(A)$ and then it follows that $f(x) \in A$. By definition we know that $(x, f(x)) \in G_f \cap X \times A$ so $x \in A^*$. Thus we have shown that the pre-image of a closed set under f is closed and therefore f is continuous. □

4. If (X, d) is a metric space, a function $f : X \to X$ is an isometry if for all $x, y \in X$, $d(f(x), f(y)) = d(x, y)$. Show that every isometry is continuous and injective. Then show that if X is compact and f is an isometry then f is surjective as well and quickly conclude that f is a homeomorphism. Hint: Show that a is not in the image of f, then for some $\epsilon > 0$, $B_\epsilon(a)$ is also not in the image of f. Then show that if $x_0 = a$, $x_1 = f(x_0)$, etc, then $d(x_n, x_m) > \epsilon$ for $n \neq m$.

Proof. Let (X, d) is a metric space, and suppose $f : X \to X$ is an isometry. Let $p \in X$ and $\epsilon > 0$. Note that for $\delta = \epsilon$, so then when $d(p, x) < \delta$ it follows that $d(f(p), f(x)) = d(p, x) < \delta = \epsilon$. Thus f is a continuous function.

Let $x, y \in X$ such that $f(x) = f(y)$. Note that since $f(x) = f(y)$ we know that $d(x, y) = d(f(x), f(y)) = 0$ and since X is a metric space it follows that $x = y$. □

Proof. Let (X, d) is a metric space and $f : X \to X$ is an isometry. Suppose f is not surjective. Then it follows that there exists some $a \in X$ such that $a \notin f(X)$.

Not sure how this works
Since f is continuous injection for some $\epsilon > 0$ there exists $B_\epsilon(a) \not\subseteq f(X)$. Construct a sequence, $\{x_i\}$ where $x_0 = a$ and $x_n = f(x_{n-1})$. Note since a is not in the image of f, we know that $d(x_0, x_1) = d(a, f(a)) > \epsilon$. Let $f^n(x)$ denote n compositions of f consider x_n, x_m, such that $n \neq m$ and without loss of generality let $n > m$. By definition of isometry we know that $d(x_m, x_n) = d(f^m(a), f^n(a)) = d(a, f^{n-m}(a)) > \epsilon$. Therefore any subsequence will also not be cauchy and thus there are no convergent subsequences in $\{x_i\}$. Having constructed a sequence with no convergent subsequence we know X is not sequentially compact, and since X is a metric space X is also not compact. □

5. Show that if p and q are elements of the interior of the closed unit ball

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \leq 1\},$$

then there is a homeomorphism $\phi : \mathbb{B}^n \to \mathbb{B}^n$ such that $\phi(p) = q$ and such that $\phi(x) = x$ for all x with $|x| = 1$. Be as rigorous as you can, but avoid writing a tome.

Proof. Consider the function $\phi_q : \mathbb{B}^n \to \mathbb{B}^n$ defined via the convex combination,

$$\phi_q(x) = x + (1 - |x|)q.$$

Note that this function has the property that $\phi_q(0) = q$ and any point s along the boundary of B^n has the property that $\phi_q(s) = s$. Note that showing ϕ_q is a homeomorphism, is sufficient in obtaining the property that $\phi(p) = q$, as $\phi_q(\phi_p^{-1}(p))$ would also be, such a homeomorphism. Also note that $\mathbb{B}^n$ is a closed and bounded set in $\mathbb{R}^n$ and thus by Heine-Borel it is also compact. We also note that as a subset of $\mathbb{R}^n$, $\mathbb{B}^n$ is also Hausdorff. Therefore showing that ϕ_q is a continuous bijection will be sufficient for to show it is a homeomorphism. Let $a, b \in \mathbb{B}^n$ and suppose $\phi_q(a) = \phi_q(b)$. By definition we know that

$$\begin{aligned} a + (1 - |a|)q &= b + (1 - |b|)q \\ a - b &= (1 - |b|)q - (1 - |a|)q \\ a - b &= (|a| - |b|)q \\ |a - b| &= ||a| - |b||\,|q| \\ |a| - |b| \leq |a - b| &= ||a| - |b||\,|q|. \end{aligned}$$

Now suppose $|a| = |b|$ and we get that

$$|a - b| = ||a| - |b||\,|q| = |0||q| = 0$$

Which implies $a = b$. Otherwise $|a| \neq |b|$ without loss of generality let $|a| \geq |b|$, which implies

$$\begin{aligned} |a| - |b| &\leq ||a| - |b||\,|q| \\ |a| - |b| &\leq (|a| - |b|)|q| \\ 0 &\leq (|a| - |b|)|q| \\ 0 &\leq (q - 1)|a - b| \\ 0 &\leq (q - 1) \\ 1 &\leq q. \end{aligned}$$

However $q < 1$ and therefore $|a| \neq |b|$ implies a contradiction. Thus $a = b$ and $\phi_q(x)$ is an injection. Let $b \in \mathbb{B}^n$. Note that b can be written as a convex combination between some point on the boundary $\frac{x}{|x|}$ and q,

$$b = |x|\frac{x}{|x|} + (1 - |x|)q.$$

The ratio of this convex combination is, $\frac{|x|}{1-|x|}$. Applying this ratio between $\frac{x}{|x|}$ and 0 we get,

$$w = \frac{0 + \frac{|x|}{1-|x|}\frac{x}{|x|}}{1 + \frac{|x|}{1-|x|}} = x.$$

So therefore $x \in\in \mathbb{B}^n$ and clearly $f(x) = x + (1 - |x|)q = |x|\frac{x}{|x|} + (1 - |x|)q = b$, so ϕ_q is surjective. Finally note that the component function of $\phi_{q_i}(x) = x_i + (1 - |x|)q_i$ is continuous from $\mathbb{B}^n$ to $\mathbb{R}^n$, and since we've just shown the function itself is a bijection, it's the component functions are continuous into their image, $\mathbb{B}^n$. Hence ϕ_q is continuous.

□

6. Let G be a group acting by homeomorphism on a topological space X. Let $O \subseteq X \times X$ be the subset defined by

$$O = \{(x_1, x_2) : x_1 = g \cdot x_2 \text{ for some } g \in G\}.$$

Show that the quotient map $X \to X/G$ is an open map.

Proof. Let $\pi : X \to X/G$ be the the natural quotient map, and let $U \subseteq X$ be open. By the quotient topology, we know that $\pi(U)$ is open in X/G if and only if $\pi^{-1}(\pi(U))$ is open in X. Note that $\pi^{-1}(\pi(U))$ can be expressed as the union of the orbits of the elements of U under

G. So we know that

$$\begin{aligned}
\pi^{-1}(\pi(U)) &= \bigcup_{u\in U}\{g\cdot u : g \in G\} \\
&= \bigcup_{u\in U}\bigcup_{g\in G} g\cdot u \\
&= \bigcup_{g\in G}\bigcup_{u\in U} g\cdot u \\
&= \bigcup_{g\in G} gU.
\end{aligned}$$

Since G acts by homeomorphism on X we know that each gU is open in X and thus we've expressed $\pi^{-1}(\pi(U))$ as a union of open sets in X, so $\pi^{-1}(\pi(U))$ is open. Therefore π is an open map. $\square$