Problem P34: Consider the heat equation $u_t = Du_{xx}$ for $D > 0$ constant, $x \in [0, 1]$ and Dirichlet boundary conditions $u(t, 0) = 0$ and $u(t, 1) = 0$. Suppose we have initial condition $u(0, x) = \sin(5\pi x)$.

a Confirm that,

$$u(t, x) = e^{-25\pi^2 Dt} \sin(5\pi x)$$

is an exact solution.

Solution:
Note that the second derivative with respect to x gives,

$$u_{xx} = -(5\pi)^2 e^{-25\pi^2 Dt} \sin(5\pi x).$$

The first derivative with respect to t gives,

$$u_t = -25\pi^2 De^{-25\pi^2 Dt} \sin(5\pi x) = D(-(5\pi)^2 e^{-25\pi^2 Dt} \sin(5\pi x)) = Du_{xx}.$$

And clearly this solution satisfies the Dirichlet conditions since $\sin(0) = \sin(5\pi) = 0$, as well as the initial condition since $e^{-25\pi^2 D(0)} = e^0 = 1$. Therefore this is an exact solution to the problem.

b Implement the backward Euler method, as applied to MOL ODE system, to solve this heat equation problem. Specifically, use diffusivity $D = 1/20$ and final time $t_f = .1$. Note that you do not need to use Newton's method to solve the implicit equation, a linear system, but you should use sparse storage and a linear solver.

Solution:

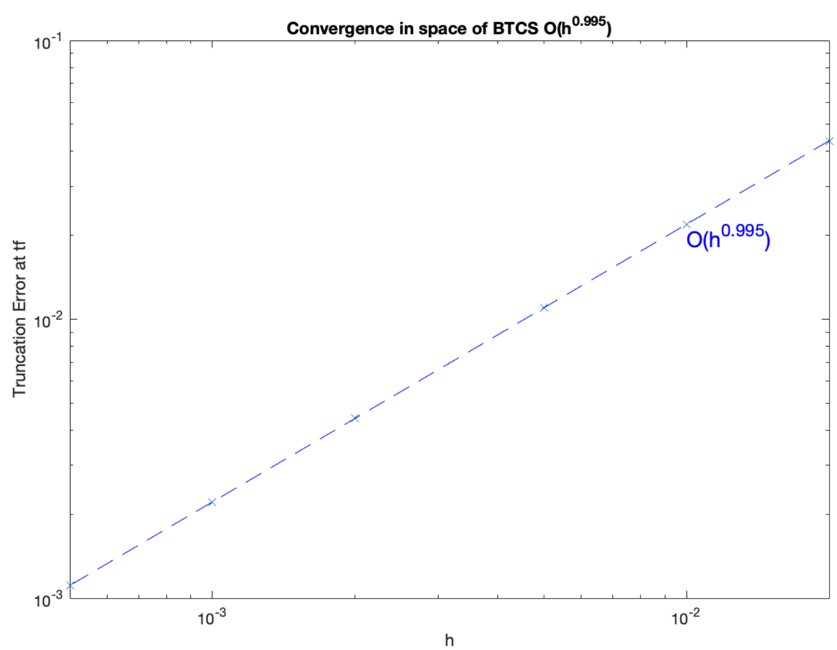
Code:

```
function [tt,zz, xx] = HEATbeulerD(m,feta,t0,tf,N, D)
% HEATBACKWARDEULER  Solve
%    u_t = Du_xx , u(0, x) = eta, u(t,0) = u(t,1) = 0
% for u(t, x) on the interval [t0,tf] with N steps in time
% and m+2 steps in space. Backward Euler in time via method of lines.
%
% Usage: [tt,zz] = HEATbeulerED(m,feta,t0,tf,N,D)

% Compute step size in space
h = 1/(m + 1);

% Generate matrix for system of IVPs in time
A = (D/h^2) * spdiags([ones(m,1), ...
        -2*ones(m,1), ones(m,1)], ...
        [-1, 0, 1], m, m);

% Define step in time.
dt = (tf - t0) / N;
% Sparse representation of
```

```
% backward euler system,
% U^n+1 = U^n + kAU^n+1
% (1 - kA)U^n+1 = U^n
fU = speye(m,m) - dt*A;

tt = t0:dt:tf;          % row vector of times
xx = linspace(0, 1, m+2); % row vector of spatial location
eta = feta(xx(2:end-1)');   % Solving for needed initial condition
zz = zeros(m, N+1);     % jth column is U at t_{j-1}
zz(:,1) = eta;
for j = 1:N          % Backward Euler Solve
    zz(:,j+1) = (fU)\zz(:,j);
end

% Putting together dirichlet conditions.
zz = [feta(0)*ones(N+1, 1)'; zz; feta(1)*ones(N+1, 1)'];
```

c Suppose we set $k = h$ for the 'refinement path'. What do you expect for the convergence $O(h^p)$? Then measure it by suing the exact solution from **a)**, at the final time, and the infinity norm $\|\cdot\|_\infty$ and $h = .02, .01, .005, .002, .001, .0005$. Make a log-log convergence plot of h versus error.

Solution:



Code:

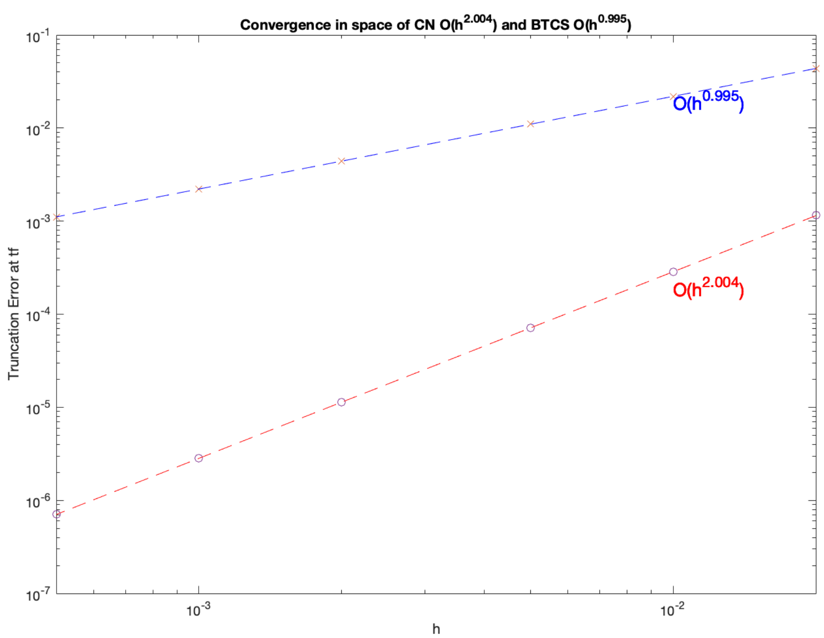
```
feta = @(x) sin(5*pi*x);
t0 = 0;
tf = .1;
D = 1/20;

uexact = @(t,x) exp(-25*pi^2*(1/20)*t)*sin(5*pi.*x);
Nlist = [.02, .01, .005, .002, .001, .0005];
err = [];


for k = 1:length(Nlist)
    [tt,zz,xx] = HEATbeulerD((1/Nlist(k))-1,feta,0,.1,(.1/Nlist(k)),D);
    err = [err max(abs(zz(:,end) - uexact(tf, xx)'))];
end

p = polyfit(log(Nlist),log(err),1);
fprintf('convergence at rate O(h^k) with k = %f\n',p(1))
loglog(Nlist,err,'o',Nlist,exp(p(2) + p(1)*log(Nlist)),'r--')
xlabel k, ylabel('Truncation Error at tf')
text(0.01,0.0002,sprintf('O(h^{%.3f})',p(1)),'Color','r','FontSize',14)
title(sprintf('Convergence in space of BTCS O(h^{%.3f})',p(1)))
axis tight
```

d Repeat parts b and c but with the trapezoidal rule. Use the same refinement path. Add the result to the same plot.

Solution:
The following is the code for trapezoid rule applied to the MOL ODE system as before,
Code:

```
function [tt,zz, xx] = HEATtrapezoidD(m,feta,t0,tf,N, D)
% HEATBACKWARDEULER Solve
%    u_t = Du_xx , u(0, x) = eta, u(t,0) = u(t,1) = 0
% for u(t, x) on the interval [t0,tf] with N steps in time
% and m+2 steps in space. Backward Euler in time via method of lines.
%
% Usage: [tt,zz] = HEATtrapezoidD(m,feta,t0,tf,N,D)

% Compute step size in space
h = 1/(m + 1);

% Define step in time.
k = (tf - t0) / N;

r = ((D*k)/(2*h^2));

% Generate matrix for system of IVPs in time
% Centered space trapezoid rule in time
A = -r*spdiags([ones(m,1), ...
        -2*ones(m,1), ones(m,1)], ...
```

```
        [-1, 0, 1], m, m);

fU = speye(m,m) + A;

tt = t0:k:tf;           % row vector of times
xx = linspace(0, 1, m+2); % row vector of spatial location
eta = feta(xx(2:end-1)');    % Solving for initial conditions


% Setting up solution matrix
zz = zeros(m, N+1);     % jth column is U at t_{j-1}
zz(:,1) = eta;


for j = 1:N
    %Constructing RHS
    z = zeros(m, 1);
    z(1) = (1 - 2*r)*zz(1,j) + r*zz(2, j);
    for i = 2:m-1
        z(i) = r*zz(i-1, j)+ (1 - 2*r)*zz(i,j) + r*zz(i+1, j);
    end
    z(m) = (1 - 2*r)*zz(m,j) + r*zz(m-1, j);
    %Solving
    zz(:,j+1) = (fU)\z;
end

% Putting together dirichlet conditions.
zz = [feta(0)*ones(N+1, 1)'; zz; feta(1)*ones(N+1, 1)'];
```



Problem P35: Consider the following scheme, which applies centered differences to both sides of the heat equation $u_t = u_{xx}$:

$$U_j^{n+2} = U_j^n + \frac{2k}{h^2}(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1})$$

a Compute the truncation error to determine the order of accuracy of this method, in space and time. The answer will be in the form $\tau(t,x) = O(k^p + h^q)$; determine p, q.

Solution:
The truncation error for the richardson scheme is based on the form,

$$\frac{U_j^{n+2} - U_j^n}{2k} = \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{h^2}.$$

Note the following,

$$\tau(x, t+k) = \frac{u(x, t+2k) - u(x,t)}{2k} - \frac{u(x-h, t+k) - 2u(x, t+k) + u(x+h, t+k)}{h^2}.$$

Let $u^{+k} = u(x, t+k)$. We will proceed by expanding all terms about u^{+1} with Taylor's Theorem. Doing so we get that,

$$\begin{aligned}
\tau(x, t+k) = \frac{1}{2k}\Big(&(u^{+k} - ku_t^{+k} + \frac{1}{2}k^2u_{tt}^{+k} - \frac{1}{6}k^3u_{ttt}^{+k} + \frac{1}{24}k^4u_{tttt}^{+k} + O(k^5)) \\
&- (u^{+k} + ku_t^{+k} + \frac{1}{2}k^2u_{tt}^{+k} + \frac{1}{6}k^3u_{ttt}^{+k} + \frac{1}{24}k^4u_{tttt}^{+k} + O(k^5))\Big) \\
-\frac{1}{h^2}\Big(&(u^{+k} + hu_x^{+k} + \frac{1}{2}h^2u_{xx}^{+k} + \frac{1}{6}h^3u_{xxx}^{+k} + \frac{1}{24}h^4u_{xxxx}^{+k} + O(h^5)) \\
&- 2u^{+k} \\
&+ (u^{+k} - hu_x^{+k} + \frac{1}{2}h^2u_{xx}^{+k} - \frac{1}{6}h^3u_{xxx}^{+k} + \frac{1}{24}h^4u_{xxxx}^{+k} + O(h^5))\Big)
\end{aligned}$$

$$\begin{aligned}
\tau(x, t+k) = &\frac{1}{2k}\Big((-2ku_t^{+k} - \frac{1}{3}k^3u_{ttt}^{+k} + O(k^6))\Big) \\
&-\frac{1}{h^2}\Big((h^2u_{xx}^{+k} + \frac{1}{12}h^4u_{xxxx}^{+k} + O(h^6))\Big)
\end{aligned}$$

$$\begin{aligned}
\tau(x, t+k) &= u_t^{+k} - \frac{1}{6}k^2u_{ttt}^{+k} + O(k^5) - u_{xx}^{+k} - \frac{1}{12}h^2u_{xxxx}^{+k} + O(h^4) \\
&= -\frac{1}{6}k^2u_{ttt}^{+k} - \frac{1}{12}h^2u_{xxxx}^{+k} + O(h^4) + O(k^5)
\end{aligned}$$

Therefore we have shown that the truncation error for this scheme is order $O(k^2 + h^2)$.

b Derive the method by applying the midpoint ODE method, to the MOL ODE system (9.10). Also, find the region of absolute stability of the midpoint method; it is in the textbook. Is the method likely to generate reasonable results? Why or why not?

Solution:
Recall the midpoint ODE method,

$$\frac{U^{n+1} - U^{n-1}}{2k} = f(U^n).$$

Also recall the MOL ODE system from 9.10,

$$U_i'(t) = \frac{1}{h^2}\left(U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)\right) \qquad \text{i = 1,2, } \ldots m$$

Which can be written in matrix vector form, where A is the typical tridiagonal matrix,

$$U'(t) = AU(t).$$

Applying Midpoint ODE method where $f(U^n) = AU(t)$, a matrix multiplication we get a new system,

$$\frac{U^{n+1} - U^{n-1}}{2k} = AU^n.$$

Considering the i^{th} row of this system we get the desired richardson scheme,

$$\frac{U_i^{n+1} - U_i^{n-1}}{2k} = \frac{1}{h^2}\left(U_{i-1}^n - 2U_i^n + U_{i+1}.^n\right)$$

To find the region of absolute stability of the midpoint method, we apply the scheme to the test equation $u' = \lambda u$. Doing so we get,

$$\frac{U^{n+2} - U^n}{2k} = \lambda U^{n+1}$$

Solving for the latest step we get,

$$U^{n+2} = U^n + 2k\lambda U^{n+1} = U^n + 2zU^{n+1}.$$

Solving this linear recurrence relation we substitute $U^n = \zeta^n$ (n is a power on the the the RHS), with $\zeta \neq 0$ and we get,

$$\begin{aligned} \zeta^{n+2} &= \zeta^n + 2z\zeta^{n+1}, \\ \zeta^{n+2} - 2z\zeta^{n+1} - \zeta^n &= 0, \\ \zeta^2 - 2z\zeta^1 - 1 &= 0. \end{aligned}$$

So we have our *stability polynomial*,

$$\pi(\zeta; z) = \zeta^2 - 2z\zeta - 1$$

With roots $\zeta_{1,2} = z \pm \sqrt{z^2 + 1}$. From our text we know that the region of absolute stability is given by all z such that $|\zeta_i| \leq 1$, with a strict inequality whenever ζ_i is a repeated root.

Problem P36: Consider the Jacobi iteration for the linear system $Au = b$ arising from a centered FD approximation fo the boundary value problem $u''(x) = f(x)$. Show that this iteration can be interpreted as forward Euler time-stepping applied to a heat equation MOL system (9.10) with time step $k = \frac{1}{2}h^2$.

Solution:

Recall the forward Euler scheme and the MOL system described in 9.10. Note that $f_i = f(x_i)$,

$$\frac{U^{n+1} - U^n}{k} = f(U^n).$$

$$U_i'(t) = \frac{1}{h^2}\left(U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)\right) + f_i \qquad \text{i} = 1,2,\dots m$$

Just like before written in matrix vector form, where A is the typical tridiagonal matrix and f is the source described in $u_t(t,x) = u_{xx}(t,x) - f(x)$,

$$U'(t) = AU(t) - f.$$

Applying forward Euler ODE method where $f(U^n) = AU(t) + f$, we get a new system,

$$\frac{U^{n+1} - U^n}{k} = AU^n - f.$$

We recall that $k = \frac{1}{2}h^2$, so by substitution we get,

$$\frac{2U^{n+1}}{h^2} - \frac{2U^n}{h^2} = AU^n + f.$$

Adding the $\frac{2U^n}{h^2}$ term we remove the diagonal entries of A, so if $A = D - L - U$ we get $A - D = -(L + U)$ and therefore by substitution we get the desired Jacobi iteration,

$$\begin{aligned}
\frac{2U^{n+1}}{h^2} &= -(L + U)U^n - f, \\
-\frac{2}{h^2}U^{n+1} &= f + (L + U)U^n, \\
DU^{n+1} &= f + (L + U)U^n.
\end{aligned}$$