Problem P22: **(a)** Compute *by hand* the eigenvalues and eigenvectors of,

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Solution:
Forming the characteristic equation we get the following. First note that,

$$(A - \lambda I) = \begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & -\lambda & 1 \\ -1 & 1 & -\lambda \end{bmatrix}$$

Solving for λ when the determinant is zero we get,

$$\begin{aligned} \det(A - \lambda) &= (2-\lambda)(\lambda^2 - 1) + (\lambda + 1) - (-1 - \lambda), \\ &= -\lambda^3 + 2\lambda^2 + 3\lambda, \\ &= (\lambda)(-\lambda^2 + 2\lambda + 3), \\ &= (\lambda)(-\lambda + 3)(\lambda + 1). \end{aligned}$$

So our matrix has eigenvalues $\lambda = 0, 3, -1$. Solving for our corresponding eigenvectors,for $\lambda = 0$ we get

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So we get an eigenspace of,

$$v_0 = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 3$ we get the following,

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So we get an eigenspace of,

$$v_3 = x \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = -1$ we get the following,

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} v_{-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} v_{-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So we get an eigenspace of,

$$v_{-1} = x \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

So our eigenvectors are,

$$v_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad v_{-1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

(b) Continuing with the same matrix A, do the following using Matlab, and show the command-line session or code: Choose a vector $u \in \mathbb{R}^3$ at random. Apply A to it 50 times, call it w. Now compute $\|Aw\|_2/\|w\|_2$. You will get 3.0000. Why? Explain in several sentences, using equations to make it clear.

Solution:
Consider the following Matlab output,

Console:

```
>> A = [2 -1 -1; -1 0 1; -1 1 0];
>> u = randn(3, 1)

   0.8622
   0.3188
  -1.3077

>> w = A^(50)*u

  1.0e+23 *

   6.4928
  -3.2464
  -3.2464

>> norm(A*w, 2)/norm(w, 2)

    3
```

Recall that A has 3 linearly independent eigenvectors they form a basis in $\mathbb{R}^3$. Therefore we can express u as some linear combination of our eigenvectors,

$$u = x_1 v_3 + x_2 v_0 + x_3 v_{-1}.$$

Note that multiplying by A we get the following,

$$\begin{aligned} Au &= A(x_1 v_3 + x_2 v_0 + x_3 v_{-1}) \\ &= x_1(Av_3) + x_2 A(v_0) + x_3(Av_{-1}) \\ &= x_1(3)v_3 + x_2(0)v_0 + x_3(-1)v_{-1}. \end{aligned}$$

A simple induction show that for any n,

$$A^n u = x_1(3)^n v_3 + x_2(0)^n v_0 + x_3(-1)^n v_{-1}.$$

ome For large enough n we can see that the v_3 term will dominate, and so $A^n u$ will get closer and closer in the direction of the eigenvector associated with the largest eigenvalue by magnitude. Thus for a sufficiently large enough n,

$$\frac{||Aw||_2}{||w||_2} = \frac{||3w||_2}{||w||_2} = 3.$$

(c) Note that $w = A^{50}u$ from part **(b)** has a very large norm. Why? For a random u, give an estimate of the norm fo the vector A^k for large k.

Solution:
Like before consider our eigenvectors scaled to be unit vectors, and we can write u as a linear combination,

$$u = x_1 v_3 + x_2 v_0 + x_3 v_{-1}.$$

As discussed previously, $A^k u$ we can be written as,

$$||A^k u|| = ||x_1(3)^k v_3 + x_2(0)^k v_0 + x_3(-1)^k v_{-1}|| = O(3^k).$$

Problem P23: **(a)** Consider this matrix-valued function of x,

$$M(x) = \begin{bmatrix} 2 & x & x \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Use Matlab to generate a single figure showing all eigenvalues of all matrices $M(x)$ for $x \in [-1, 5]$. Label this figure in an attempt to clarify how the eigenvalues depend on x.

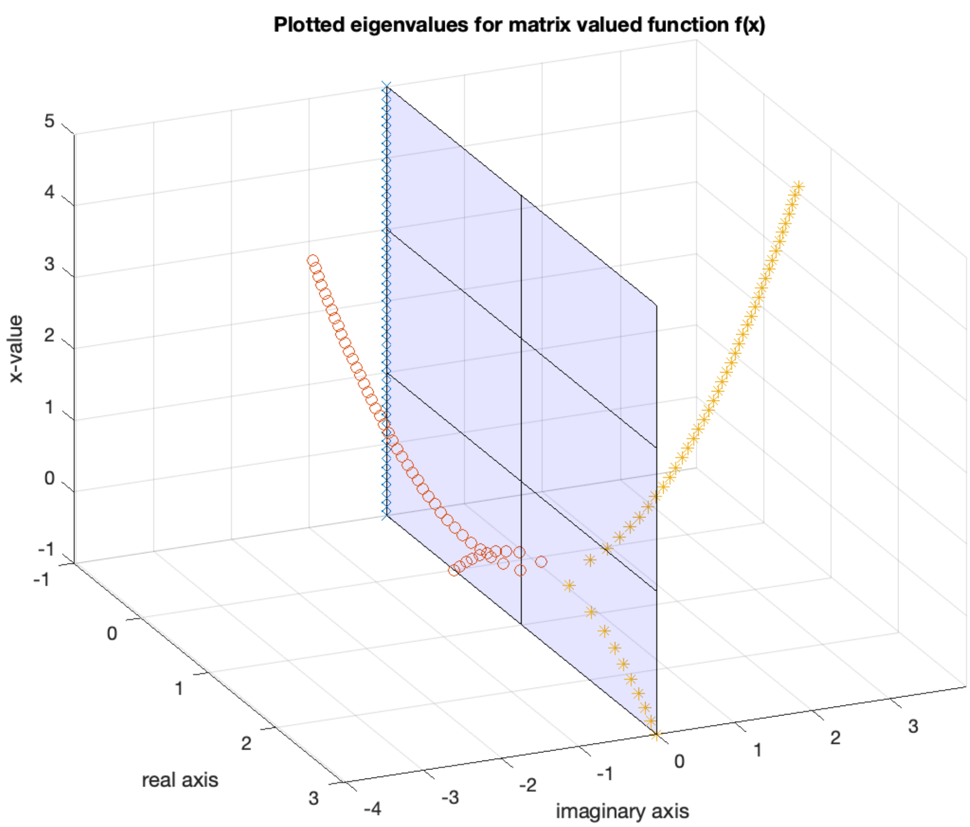
Solution:
The following Matlab code produces the desired figure,
Code:

```
f = @(x) [2 x x; -1 0 1; -1 1 0];
n = 50;
x = linspace(-1, 5, n);
Hist = [];
for i = 1:n
    Hist = [Hist sort(eig(f(x(i))))];
end

plot(x,Hist(1, :), 'x', x , Hist(2, :), 'O', x,Hist(3, :), '*')
plot3(real(Hist(1, :)), imag(Hist(1, :)),x, 'x', ...
    real(Hist(2, :)), imag(Hist(2, :)),x, ...
    'O',real(Hist(3, :)), imag(Hist(3, :)), x,'*')

xlabel('real axis')
ylabel('imaginary axis')
zlabel('x-value')
title(['Plotted eigenvalues for ' ...
    'matrix values function f(x)'])
grid on
```

Figure 1: Plane of 'real' eigenvalues in purple.



(b) By doing a by-hand calculation, at what x value in the interval $[-1, 5]$ do non-real eigenvalues first appear?

Solution:
Forming the characteristic equation like before we get that,

$$(A - \lambda I) = \begin{bmatrix} 2-\lambda & x & x \\ -1 & -\lambda & 1 \\ -1 & 1 & -\lambda \end{bmatrix}$$

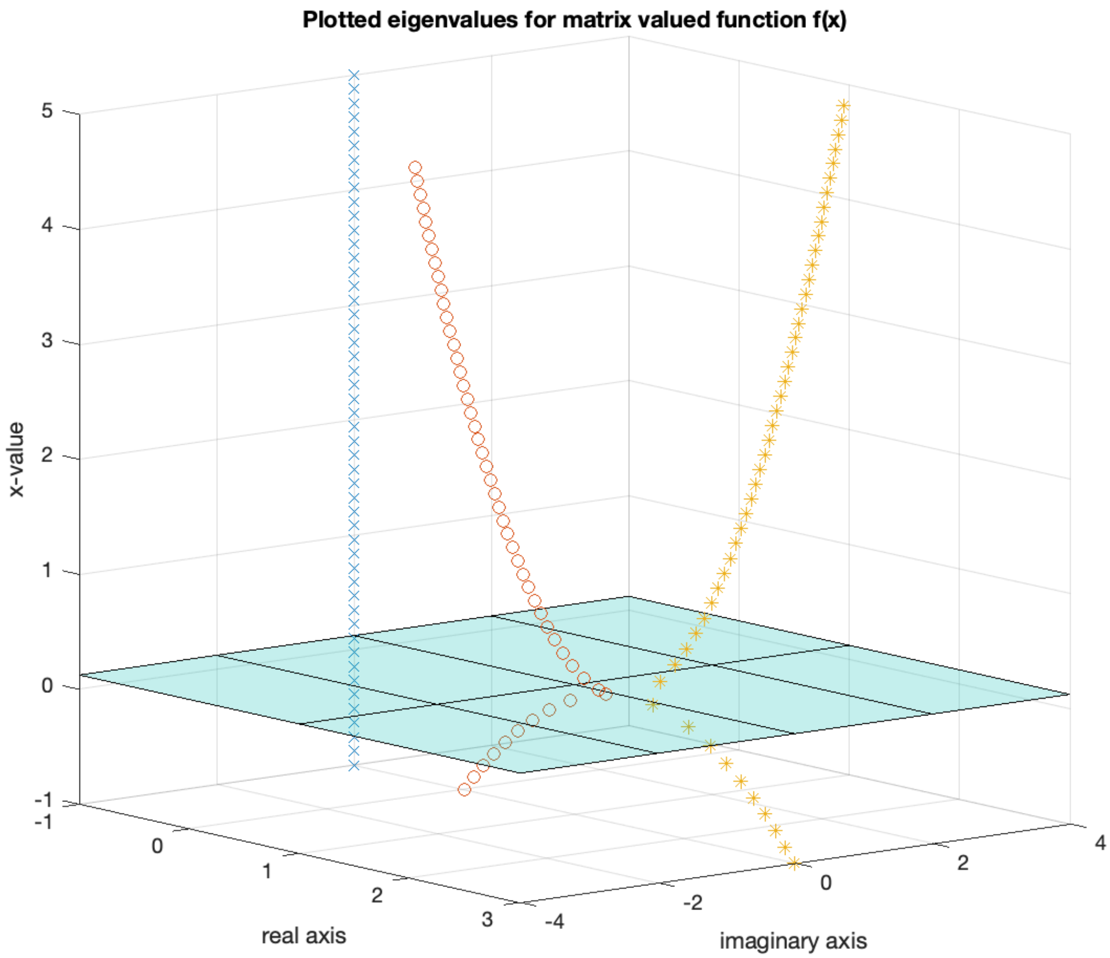
Solving for λ when the determinant is zero we get,

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(\lambda^2 - 1) - x(\lambda + 1) + x(-1-\lambda), \\ &= (2-\lambda)(\lambda^2 - 1) - 2x(\lambda + 1), \\ &= (2-\lambda)(\lambda - 1)(\lambda + 1) - 2x(\lambda + 1), \\ &= (\lambda + 1)((2-\lambda)(\lambda - 1) - 2x), \\ &= (\lambda + 1)(-\lambda^2 + 3\lambda - 2(x+1)). \end{aligned}$$

Applying the quadratic formula we get that in order to have complex eigenvalues the following expression must hold,

$$\begin{aligned} 9 - 4(-1)(-2(x+1)) &< 0, \\ 9 - 8x - 8 &< 0, \\ x &> \frac{1}{8}. \end{aligned}$$

Figure 2: The teal plane depicts $x = \frac{1}{8}$.



Problem P24: Check that the solution $u(t)$ given by Duhamel's principle, equation (5.8) in the textbook, satisfies ODE(5.6) and the initial condition $u(t_0) = \eta$.

Solution:

Recall that ODE(5.6) is a constant coefficient linear system where $A \in \mathbb{R}^{S\times S}$ is constant,

$$u'(t) = Au(t) + g(t).$$

Also recall that the solution given by Duhamel's principle is given by,

$$u(t) = e^{A(t-t_0)}\eta + \int_{t_0}^{t} e^{A(t-\tau)}g(\tau)d\tau.$$

To verify this solution we must first solve for $u'(t)$. Clearly the first part of the sum differentiates to,

$$u'(t) = Ae^{A(t-t_0)}\eta + \frac{\delta}{\delta t}\left(\int_{t_0}^{t} e^{A(t-\tau)}g(\tau)d\tau\right)$$

One can see this by considering the Taylor series definition of the matrix exponential.

Applying a special case of the Leibniz Integral rule,

$$\frac{\delta}{\delta x}\left(\int_a^x f(x,t)dt\right) = f(x,x) + \int_a^x \frac{\delta}{\delta x} f(x,t)dt, \tag{1}$$

we get the following,

$$\begin{aligned}
\frac{\delta}{\delta t}\left(\int_{t_0}^{t} e^{A(t-\tau)}g(\tau)d\tau\right) &= e^{A(t-t)}g(t) + \int_{t_0}^{t} \frac{\delta}{\delta t}\left(e^{A(t-\tau)}g(\tau)\right)d\tau \\
&= (1)g(t) + \int_{t_0}^{t} Ae^{A(t-\tau)}g(\tau)d\tau \\
&= g(t) + A\int_{t_0}^{t} e^{A(t-\tau)}g(\tau)d\tau
\end{aligned}$$

Finally by substitution we get,

$$\begin{aligned}
u'(t) &= Ae^{A(t-t_0)}\eta + g(t) + A\int_{t_0}^{t} e^{A(t-\tau)}g(\tau)d\tau, \\
&= A\left(e^{A(t-t_0)}\eta + \int_{t_0}^{t} e^{A(t-\tau)}g(\tau)d\tau\right) + g(t), \\
&= Au(t) + g(t).
\end{aligned}$$

Problem P25: Consider the ODE system

$$u_1' = 2u_1,$$

$$u_2' = 3u_1 - 2u_2$$

with some initial conditions at $t = 0$: $u_1(0) = a$, $u_2(0) = b$.

Solve this system in two ways:

(a) Solve the first equation. Then insert this into the second equation to get a non-homogenous linear ODE for u_2. Solve using Duhamel's principle.

Solution:
Clearly the first equation's solution is $u_1 = ae^{2t}$. Substitution into the second equation gives,

$$u_2' = -2u_2 + 3ae^{2t}.$$

Applying Duhamel's principle we get,

$$\begin{aligned}
u_2 &= e^{-2t}b + \int_0^t e^{-2(t-\tau)}(3ae^{2\tau})d\tau, \\
&= be^{-2t} + 3a\int_0^t e^{-2t}e^{2\tau}e^{2\tau}d\tau, \\
&= be^{-2t} + 3ae^{-2t}\int_0^t e^{4\tau}d\tau, \\
&= be^{-2t} + 3ae^{-2t}\left(\frac{1}{4}e^{4t} - \frac{1}{4}\right), \\
&= be^{-2t} + \frac{3}{4}ae^{2t} - \frac{3}{4}ae^{-2t},
\end{aligned}$$

(b) Write the system as $u' = Au$, compute the matrix exponential, and get the solution in the form of equation (D.30) in Appendix D.

Solution:
Written as a system, we get the following,

$$u' = \begin{bmatrix} 2 & 0 \\ 3 & -2 \end{bmatrix} u,$$

$$u(0) = \eta = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Computing the eigendecomposition of A we first note that it's eigenvalues are $\lambda = 2, -2$ since it forms the following characteristic equation, $0 = (2-\lambda)(-2-\lambda)$. Solving for the associated eigenvectors we get,

$$\begin{aligned}
\begin{bmatrix} 0 & 0 \\ 3 & -4 \end{bmatrix} v_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
v_2 &= \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} v_{-2} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
v_{-2} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}$$

Forming R and Λ we get

$$R = \begin{bmatrix} 1 & 0 \\ \frac{3}{4} & 1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Solving for R^{-1} we get,

$$R^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{3}{4} & 1 \end{bmatrix}$$

From (D.30) in Appendix D we can now form the matrix exponential and solve our ODE system,

$$u(t) = e^{At}\eta = (Re^{\Lambda r}R^{-1})\eta = \begin{bmatrix} 1 & 0 \\ \frac{3}{4} & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-2r} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{3}{4} & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}.$$

Simplifying we get the same solution as part (a),

$$\begin{aligned} u(t) &= \begin{bmatrix} 1 & 0 \\ \frac{3}{4} & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-2r} \end{bmatrix}\begin{bmatrix} a \\ -\frac{3}{4}a + b \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 0 \\ \frac{3}{4} & 1 \end{bmatrix}\begin{bmatrix} ae^{2t} \\ -\frac{3}{4}ae^{-2t} + be^{-2t} \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 0 \\ \frac{3}{4} & 1 \end{bmatrix}\begin{bmatrix} ae^{2t} \\ -\frac{3}{4}ae^{-2t} + be^{-2t} \end{bmatrix}, \\ &= \begin{bmatrix} ae^{2t} \\ \frac{3}{4}ae^{2t} - \frac{3}{4}ae^{-2t} + be^{-2t} \end{bmatrix}. \end{aligned}$$

Problem P26: The ODE IVP

$$v'' = -9v, \qquad v(0) = v_0, \qquad v'(0) = w_0$$

has solution,

$$v(t) = v_0 \cos(3t) + \frac{w_0}{3}\sin(3t).$$

Verify this.

Construct this solution by first rewriting the ODE as a first order system $u' = Av$. Then compute the solution $u(t) = e^{At}u(0)$ by using equation (D.30) in Appendix D

Solution:

We begin by first converting this second order ODE IVP into a system of first order ODE IVP. Consider the following substitution, and note that $u_1(0) = v_0$ and $u_2(0) = w_0$

$$u_1 = v$$

$$u_2 = v'$$

Differentiating and by substitution we get the following system of differential equations,

$$u_1' = v' = u_2$$

$$u_2' = v'' = -9v = -9u_1$$

Written out as a system and reordering the rows we get,

$$\begin{bmatrix} u_2' \\ u_1' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} u$$

$$\eta = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}.$$

Note that we get eigenvalues $\lambda = 3i, -3i$ since the characteristic equation is of the form $(\lambda^2 - (-3)^2)$. Consider $\lambda = 3i$ and solving for the corresponding eigenvector we get,

$$\begin{bmatrix} -3i & 1 \\ -9 & -3i \end{bmatrix} v_{3i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -3i & 1 \\ 0 & 0 \end{bmatrix} v_{3i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$v_{3i} = \begin{bmatrix} -\frac{i}{3} \\ 1 \end{bmatrix}.$$

Now with $\lambda = -3i$ we get,

$$\begin{bmatrix} 3i & 1 \\ -9 & 3i \end{bmatrix} v_{-3i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3i & 1 \\ 0 & 0 \end{bmatrix} v_{-3i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$v_{-3i} = \begin{bmatrix} \frac{i}{3} \\ 1 \end{bmatrix}.$$

Note we can now construct the following,

$$R = \begin{bmatrix} -\frac{i}{3} & \frac{i}{3} \\ 1 & 1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 3i & 0 \\ 0 & -3i \end{bmatrix}.$$

Solving for R^{-1} we get,

$$R^{-1} = \frac{1}{-2i/3} \begin{bmatrix} 1 & -\frac{i}{3} \\ -1 & -\frac{i}{3} \end{bmatrix} = \frac{3i}{2} \begin{bmatrix} 1 & -\frac{i}{3} \\ -1 & -\frac{i}{3} \end{bmatrix} = \begin{bmatrix} \frac{3i}{2} & \frac{1}{2} \\ -\frac{3i}{2} & \frac{1}{2} \end{bmatrix}$$

Having diagonalized our matrix we can proceed by forming the solution with equation (D.30) in Appendix D we get,

$$u = e^{At}\eta = (Re^{\Lambda r}R^{-1})\eta = \begin{bmatrix} -\frac{i}{3} & \frac{i}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3i} & 0 \\ 0 & e^{-3i} \end{bmatrix} \begin{bmatrix} \frac{3i}{2} & \frac{1}{2} \\ -\frac{3i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$$

Expanding to check the given solution we get,

$$\begin{aligned}
u &= \begin{bmatrix} -\frac{i}{3} & \frac{i}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(3i)t} & 0 \\ 0 & e^{(-3i)t} \end{bmatrix} \begin{bmatrix} \frac{(3i)t}{2} v_0 + \frac{1}{2} w_0 \\ -\frac{(3i)t}{2} v_0 + \frac{1}{2} w_0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{i}{3} & \frac{i}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(3i)t}(\frac{3i}{2} v_0 + \frac{1}{2} w_0) \\ e^{(-3i)t}(-\frac{3i}{2} v_0 + \frac{1}{2} w_0) \end{bmatrix} \\
&= \begin{bmatrix} -\frac{i}{3} & \frac{i}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(3i)t}(\frac{3i}{2} v_0 + \frac{1}{2} w_0) \\ e^{(-3i)t}(-\frac{3i}{2} v_0 + \frac{1}{2} w_0) \end{bmatrix}
\end{aligned}$$

Which gives our solution as,

$$u_1 = v(t) = \left(-\frac{i}{3}\right) e^{(3i)t} \left(\frac{3i}{2} v_0 + \frac{1}{2} w_0\right) + \left(\frac{i}{3}\right) e^{(-3i)t} \left(-\frac{3i}{2} v_0 + \frac{1}{2} w_0\right)$$

Applying Euler's formula and simplifying we get the desired solution,

$$\begin{aligned}
v(t) &= \left(\frac{i}{3}\right)\left(-e^{(3i)t}\left(\frac{3i}{2} v_0 + \frac{1}{2} w_0\right) + e^{(-3i)t}\left(-\frac{3i}{2} v_0 + \frac{1}{2} w_0\right)\right) \\
&= \left(\frac{i}{3}\right)\left(-(\cos(3t) + i\sin(3t))\left(\frac{3i}{2} v_0 + \frac{1}{2} w_0\right) + (\cos(-3t) + i\sin(-3t))\left(-\frac{3i}{2} v_0 + \frac{1}{2} w_0\right)\right) \\
&= \left(\frac{i}{3}\right)\left((-\cos(3t) - i\sin(3t))\left(\frac{3i}{2} v_0 + \frac{1}{2} w_0\right) + (\cos(3t) - i\sin(3t))\left(-\frac{3i}{2} v_0 + \frac{1}{2} w_0\right)\right) \\
&= \left(\frac{i}{3}\right)\left(-2\frac{3i}{2} v_0 \cos(3t) - i w_0 \sin(3t)\right) \\
&= v_0 \cos(3t) + \frac{w_0}{3} \sin(3t).
\end{aligned}$$