1. Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for topologies τ_1 and τ_2. Show that $\tau_1 \subseteq \tau_2$ if and only if for every $B_1 \in \mathcal{B}_1$ and every $x \in B_1$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.

Proof. ($\Rightarrow$) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for topologies τ_1 and τ_2 and suppose $\tau_1 \subseteq \tau_2$. Let $B_1 \in \mathcal{B}_1$, and note that since $\mathcal{B}_1$ is a basis for τ_1 we know that $\mathcal{B}_1 \in \tau_1, \tau_2$. Since $\mathcal{B}_1$ is in τ_2 and $\mathcal{B}_2$ is a basis for τ_2 it follows that for every $x \in B_1$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.

($\Leftarrow$) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for topologies τ_1 and τ_2 and suppose that for every $B_1 \in \mathcal{B}_1$ and every $x \in B_1$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$. Let $U \in \tau_1$, and note that since $\mathcal{B}_1$ is a basis for τ_1 we know that for some index set I_1 of $\mathcal{B}_1$,

$$U = \bigcup_{i \in I_1} B_1^i.$$

Note that by our supposition for every $B_1^i \in I_1$ there exists an index set I_i of $\mathcal{B}_2$ such that,

$$B_1^i = \bigcup_{j \in I_i} B_2^j.$$

Therefore we know that,

$$U = \bigcup_{i \in I_1} \bigcup_{j \in I_i} B_2^j.$$

Having expressed U as a union of sets in $\mathcal{B}_2$ we know that $u \in \tau_2$ and thus $\tau_1 \subseteq \tau_2$. □

2. Given a family $\{\tau_\alpha\}_{\alpha \in I}$ of topologies in X, show that there is a unique smallest topology containing each τ_α. Show also that there is a unique largest topology contained in each τ_α. Take advantage of past work!

Proof. Let $\{\tau_\alpha\}_{\alpha \in I}$ be a family of topologies in X. Let $B = \bigcup_{\alpha \in I} \tau_\alpha$. Now consider the pre-basis $\mathcal{B}$, a set consisting of all unions of finite intersections of elements of B. Let τ' be the topology generated by $\mathcal{B}$. Suppose that $\tau \subseteq \tau'$ such that τ contains each τ_α. Therefore τ must contain all elements of B, and since it's a topology it must be closed with respect to unions and finite intersections therefore it must also contain $\mathcal{B}$. Hence $\tau = \tau'$.

Let $A = \bigcap_{\alpha \in I} \tau_\alpha$ and note that it is a topology contained in each τ_α. Showing A is a topology, simply note that any union or finite intersection of elements in A must have also been in every τ_α since they are also topologies and therefore A is closed with respect to unions and finite intersections. Suppose there exists some τ contained in each τ_α such that $A \subseteq \tau$. Let $x \in \tau$ and note that $x \in \tau_\alpha$ for all α, therefore by definition $x \in A$.

□

3. Let $\mathcal{B} = \{[a, b) : a, b \in \mathbb{Q}\}$. Show that $\mathcal{B}$ is a pre-basis and hence generates a topology $\tau_{\mathcal{B}}$. Compare this topology to the lower-limit topology τ_ℓ. In particular, determine if it is finer or coarser or neither or both.

Proof. Let $\mathcal{B} = \{[a, b) : a, b \in \mathbb{Q}\}$. Clearly we can see that $\cup_{B\in\mathcal{B}} B = \mathbb{R}$. Let $[a, b), [c, d) \in \mathcal{B}$ and consider some $x \in [a, b) \cap [c, d)$. If the intersection is non-empty either the clopen intervals overlap or one is contained in the other. In either case the resulted intersection is another clopen interval $[y, z) \in \mathcal{B}$, so finally $x \in [y, z) \subseteq [a, b) \cap [c, d)$ thus $\mathcal{B}$ is a pre-basis.

Let $\tau_{\mathcal{B}}$ be the topology generated by $\mathcal{B}$. Note that the interval $[\pi, 1) \in \tau_\ell$ is not open with respect to $\tau_{\mathcal{B}}$ since there is no $[a, b) \in \mathcal{B}$ such that $\pi \in [a, b) \subseteq [\pi, 1)$. Thus $\tau_{\mathcal{B}}$ is the coarser topology.

□

4. **Problem 2-12** Suppose X is a set, and $\mathcal{A} \subseteq \mathcal{P}(X)$ is any collection of subsets of X. Let $\tau \subseteq \mathcal{P}(X)$ be the collection of subsets consisting of X, $\emptyset$, and all unions of finite intersection of elements of $\mathcal{A}$.

   (a) Show that τ is a topology.

   Proof. Suppose a set X, and $\mathcal{A} \subseteq \mathcal{P}(X)$ with $\tau \subseteq \mathcal{P}(X)$ be the collection of subsets consisting of X, $\emptyset$, and all unions of finite intersection of elements of $\mathcal{A}$. By definition $X, \emptyset \in \tau$. Let $\{U_i\}_I \subseteq \tau$ and note that for each U_i there exists a collection $\{A_{i,j}\}_J$ where each $A_{i,j}$ is some finite intersection of the elements of $\mathcal{A}$ such that $U_i = \cup_{j\in J} A_{i,j}$.

   By substitution we get,
$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} \bigcup_{j\in J} A_{i,j}$$
   Note that this union is itself a union of finite intersections of elements of $\mathcal{A}$, thus τ is closed with respect to unions. Let $\{U_i\}_I \subseteq \tau$ be a finite subset. Note that by substitution and associativity of intersection we get,
$$\bigcap_{i\in I} U_i = \bigcap_{i\in I} \bigcup_{j\in J} A_{i,j} = \bigcup \bigcap_{\substack{i\in I\\ j\in J}} A_{i,j}.$$
   Again we've managed to write our finite intersection as a union of finite intersections of elements of $\mathcal{A}$, thus τ is closed with respect to finite intersections. □

   (b) Show that τ is the coarsest topology for which all the sets in $\mathcal{A}$ are open.

   Proof. Suppose there exists some topology τ' such that $\tau' \subseteq \tau$ and $\mathcal{A} \subseteq \tau'$. Let $U \in \tau$ and by definition we know that U equal to the union of some finite intersection of the elements in $\mathcal{A}$. Well since τ' is a topology with $\mathcal{A} \subseteq \tau'$ we must have that $x \in \tau'$. Thus τ is the coarsest topology. □

   (c) Let Y be any topological space. Show that a map $f : Y \to X$ is continuous if and only if $f^{-1}(U)$ is open in Y for every $U \in \mathcal{A}$.

   Proof. ($\Rightarrow$) Let Y be any topological space and suppose the map $f : Y \to X$ is continuous. Note that by definition $U \in \mathcal{A}$ is open in X and by continuity we know that $f^{-1}(U)$ must be open in Y. □

Proof. ($\Leftarrow$) Let Y be any topological space, consider the map $f : Y \to X$ and suppose that for every $A \in \mathcal{A}$, $f^{-1}(A)$ is open in Y. Let $U \subseteq X$, and recall that by definition there exists a collection $\{\hat{A}_j\}_J$ where each A_j is some finite intersection of the elements of $\mathcal{A}$ such that,

$$U = \bigcup_{j\in J} \hat{A}_j = \bigcup_{j\in J} \bigcap_{i\in I}^{n} A_{j,i}.$$

Considering the pre-image we find that,

$$f^{-1}(U) = f^{-1}\left(\bigcup_{j\in J} \bigcap_{i\in I}^{n} A_{j,i}\right) = \bigcup_{j\in J} \bigcap_{i\in I}^{n} f^{-1}\left(A_{j,i}\right)$$

Since Y is a topological space and $f^{-1}(A)$ is open for every $A \in \mathcal{A}$ we can conclude that $f^{-1}(U)$ is open in Y and thus f is continuous. □

(d) Conclude that the topology generated by a pre-basis $\mathcal{B}$ is the smallest topology in which every set from $\mathcal{B}$ is open.

Proof. This conclusion comes directly from parts (a) and (b) of this problem. Note that τ from (a) is the topology generated by a pre-basis. In (b) we showed that τ is the coarsest (or smallest) topology for which all of the sub-basis elements are included. □

5. Problem 2-15 Let X and Y be topological spaces.

(a) Suppose $f : X \to Y$ is continuous and $p_n \to p$ in X. Show that $f(p_n) \to f(p)$ in Y. (This was proved in last weeks homework.)

Proof. Let $U \in \mathcal{V}(f(p))$ and note that since f is continuous we know that $f^{-1}(U)$ is open in X. Since $p_n \to p$ in X and $f^{-1}(U) \in \mathcal{V}(p)$ there exists some $N \in \mathbb{N}$ such that $p_n \in f^{-1}(U)$ for all $n \geq N$. It then follows that $f(p_n) \in U$ for all $n \geq N$ and thus by definition $f(p_n) \to f(p)$. □

(b) Prove that if X is first countable then the converse is true: if $f : X \to Y$ is a map such that $p_n \to p$ in X implies $f(p_n) \to f(p)$ in Y, then f is continuous.

Proof. We will proceed by proving the contrapositive. Suppose that $f : X \to Y$ is not continuous. Then there exists some $U \subseteq Y$ such that $f^{-1}(U)$ is not open in X. Therefore there exists some $x \in f^{-1}(U)$ such that for every $U' \in \mathcal{V}(x)$, $U' \nsubseteq f^{-1}(U)$. Choose one such U' and since X is first countable there exists a nested neighborhood basis $\{U'_k\}$ about x such that for all k, $x \in U'_k \subseteq U'$. Choose $p_k \in U'_k$ such that $p_k \notin f^{-1}(U)$. By construction we know that $p_k \to x$, yet clearly $f(p_k) \notin f(x)$ since $f(x) \in U$ but $f(p_k) \notin U$ for all k.

Thus we have constructed a sequence which converges in X, whose image does not converge in Y. □

6. Let A be a subset of a topological space X, and let $\mathcal{B}$ be a basis for the topology.

(a) Show that $x \in \overline{A}$ if and only if for every $B \in \mathcal{B}$ with $x \in B$, $B \cap A \neq \emptyset$.

Proof. ($\Rightarrow$) Suppose $x \in \overline{A}$. Observe that by definition x is a contact point of A and since all $B \in \mathcal{B}$ are open, $B \cap A \neq \emptyset$ is immediate. □

Proof. ($\Leftarrow$) Suppose that for every $B \in \mathcal{B}$ with $x \in B$, $B \cap A \neq \emptyset$. Let $U \subseteq X$ such that $x \in U$. Note that since $\mathcal{B}$ is a basis there exists a collection of $\{B_i\}_{i \in I} \subseteq \mathcal{B}$ such that $U = \cup_{i \in I} B_i$. By our supposition we see that,

$$U \bigcap A = \left(\bigcup_{i \in I} B_i \right) \bigcap A = \bigcup_{i \in I} \left(B_i \bigcap A \right) \neq \emptyset$$

□

(b) Show that $x \in \partial A$ if and only if for every $B \in \mathcal{B}$ with $x \in B$, $B \cap A \neq \emptyset$ and $B \cap A^c \neq \emptyset$

Proof. ($\Rightarrow$) Suppose $x \in \partial A$. By our definition of $\partial A = \overline{A} \cap \overline{A^c}$ we know that $x \in \overline{A}, \overline{A^c}$. Therefore by ($a$) we can conclude for every $B \in \mathcal{B}$ with $x \in B$, $B \cap A \neq \emptyset$ and $B \cap A^c \neq \emptyset$. □

Proof. ($\Leftarrow$) Suppose for every $B \in \mathcal{B}$ with $x \in B$, $B \cap A \neq \emptyset$ and $B \cap A^c \neq \emptyset$. Again by ($a$) we can conclude that $x \in \overline{A}, \overline{A^c}$ and therefore $x \in \partial A$. □

(c) Show that $\text{Int}(A) \cap \partial A = \emptyset$ and $\overline{A} = \text{Int}(A) \cup \partial A$.

Proof. Suppose $x \in \partial A$. Let $U \in \mathcal{V}(x)$ and note that since $x \in \partial A$ we know that $U \cap A^c \neq \emptyset$ and since $\text{Int}(A)$ is open we conclude that $x \notin \text{Int}(A)$. Thus $\text{Int}(A) \cap \partial A = \emptyset$. □

Proof. Let $x \in \overline{A}$. By definition x is a contact point of A and therefore for all $U \in \mathcal{V}(x)$ we know that $U \cap A \neq \emptyset$. If there exists a U such that $U \subseteq A$ then $x \in \text{Int}(A)$ otherwise for all $U \in \mathcal{V}(x)$ we know that $U \cap A^c \neq \emptyset$ which implies $x \in \partial A$. Showing containment in the other direction comes from the fact that all points in $\text{Int}(A) \cap \partial A$ are contact points of A. □

7. Problem 2-20 Show that second countability, separability, and the Lindelöf property are all equivalent for metric space.

Proof. (2^{nd} countable $\Rightarrow$ Lindelöf) *This proof was done in class.*

Suppose X is a second countable metric space. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for X. Since X is second countable we know that there exists $\{W_k\}$, a countable basis for X. Since $\{W_k\}$ is a basis we can consider the index set K, with the property that for all $k \in K$, $W_k \subseteq U_\alpha$ for some $\alpha \in I$. Therefore, for each $k \in K$ there exists some α_k such that $W_k \subseteq U_{\alpha_k}$. Clearly $U_{\alpha_k k \in K}$ is a countable refinement of $\{U_\alpha\}_{\alpha \in I}$. Note that since $\{W_k\}$ is a basis, for every $x \in X$ we know that $x \in W_k \subseteq U_{\alpha_k}$ for some k and therefore $\{U_\alpha\}_{\alpha \in I}$ is still an open cover of x. □

Proof. (2^{nd} countable $\Rightarrow$ separable) *This proof was outlined in class.*

Suppose X is a second countable metric space. Since X is second countable we know that there exists $\{W_k\}$, a countable basis for X. Consider the set $\{p_k\}$ such that $p_k \in W_k$. Clearly $\{p_k\}$ is countable so we will proceed to show that $\{p_k\}$ is dense. Let $U \subseteq X$ be open and choose $x \in U$. Since $\{W_k\}$ is a basis we know that for some k, $x \in W_k \subseteq U$. Immediately it follows that $p_k \in W_k \subseteq U$ and thus $\{p_k\}$ is dense in X. □

Proof. (Lindelöf $\Rightarrow$ separable) *Glen and I worked on this together with your help.*

Let X be a Lindelöf metric space. For all $n \in \mathbb{N}$ consider the sets $\mathcal{U}_n = \{B_{1/n}(x) : x \in X\}$. Clearly each $\mathcal{U}_n$ is an open cover of X and since X is Lindelöf there exists countable subcovers $\mathcal{U}'_n$. Let $\{x_i^n\}_{i\in\mathbb{N}}$ be the set of centers of balls in $\mathcal{U}'_n$. Note that since $\mathcal{U}'_n$ was a countable subcover, $\{x_i^n\}_{i\in\mathbb{N}}$ is also countable. Now consider the following set,

$$A = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} x_i^n.$$

Note that A is a countable set. We will proceed by showing that A is dense in X by showing that for all $x \in X$, for every $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$. Let $x \in X$ and $\epsilon > 0$, consider the subset of A such that $n \geq 2/\epsilon$ and note that since points in this subset are at most $\epsilon/2$ distance apart there must exists some $x_i^n \in B_\epsilon(x)$. Hence A is a countable dense subset of X. □

Proof. (seperable $\Rightarrow$ 2^{nd} countable)

Suppose X is a seperable metric space. Since X is seperable there exists a set countable dense subset, A. Now consider the countable set of open balls,

$$\mathcal{A} = \left\{ B_{\frac{1}{n}}(x) \subseteq X : x \in A, n \in \mathbb{N} \right\}.$$

We will proceed by showing that this set is a basis for X. Let $\epsilon > 0$ and consider $B_\epsilon(x)$ for some $x \in X$. Since A is dense there exists some $x' \in A$ such that $x' \in B_{\epsilon/4}(x)$. Note that by construction of $d(x, x') < \epsilon/4$ it follows that for all $B_n(x') \in \mathcal{A}$ with $n \geq 2/\epsilon$ we know $B_n(x') \subseteq B_\epsilon(x)$ since for all $x'' \in B_n(x')$,

$$d(x, x'') \leq d(x, x') + d(x', x'') = \epsilon/4 + \epsilon/2 < \epsilon.$$

Since the open balls about x' have at most radius $\epsilon/2$ there must exists some $B_n(x')$ such that $x \in B_n(x') \subseteq B_\epsilon(x)$. □