1. The point of this exercise is to settle some details from the proof of the Brower fixed point theorem. We suppose $f : \mathbb{B}^2 \to \mathbb{B}^2$ is continuous and that f does not have a fixed point.

a) Prove that for all $x \in \mathbb{B}^2$ there exists a unique $t \in [1, \infty)$ such that $f(x) + t(x)(x - f(x)) \in S^1$.

Proof. Let $x \in \mathbb{B}^2$ and suppose $f : \mathbb{B}^2 \to \mathbb{B}^2$ is continuous and that f does not have a fixed point. Note that for $f(x) + t(x - f(x)) \in S^1$ it must be the case that $|f(x) + t(x - f(x))| = 1$. Consider when $t = 1$, it follows that $|f(x) + t(x - f(x))| = |x| \leq 1$ since $x \in \mathbb{B}^2$. Also note that $|f(x) + t(x - f(x))| = |(1 - t)f(x) + t(x)|$ and since $\lim_{t\to\infty} |(1 - t)f(x) + t(x)| = \infty$ since $|f(x) + t(x - f(x))|$ is continuous by the intermediate value theorem we know that there exists a $t \in [1, \infty)$ such that $|f(x) + t(x - f(x))| = 1$.

For notation's sake, let $u = f(x)$ and $v = (x - f(x))$, then written as an inner product we know that,

$$\begin{aligned} 1 = \langle u + tv, u + tv\rangle &= \langle u + tv, u\rangle + \langle u + tv, tv\rangle \\ &= \langle u, u\rangle + \langle tv, u\rangle + \langle u, tv\rangle + \langle tv, tv\rangle \\ &= \langle u, u\rangle + 2\langle u, v\rangle t + \langle v, v\rangle t^2 \end{aligned}$$

In standard form we have,

$$\langle v, v\rangle t^2 + 2\langle u, v\rangle t + \langle u, u\rangle - 1 = 0.$$

Thus we have expressed $|f(x) + t(x - f(x))| - 1$ as a positive quadratic in t, so it has two roots. Note that when $t = 0$ the quadratic takes on a value of $\langle u, u\rangle - 1$, and since $u = f(x) \in \mathbb{B}^2$ we know that $\langle u, u\rangle - 1 \leq 0$. Therefore by properties of positive quadratic functions there exists a single solution in $(0, \infty)$. Thus the t we proved existed via IVT, is unique. □

b) Define

$$r(x) = f(x) + t(x)(x - f(x)),$$

so $r : \mathbb{B} \to S^1$. The graph of r is a subset of $\mathbb{B} \times S^1$. We wish to show that r is continuous, and since S^1 is compact and Hausdorff it is enough to show that the graph of r is closed. Do so. Hint: Suppose $(x_n, r(x_n)) \to (x, z) \in \mathbb{B} \times S^1$. Now show that $z = r(x)$. We'll discuss in the problem session what a boon the closed graph theorem is here.

Proof. Suppose $f : \mathbb{B}^2 \to \mathbb{B}^2$ is continuous, f does not have a fixed point and $(x_n, r(x_n)) \to (x, z) \in \mathbb{B} \times S^1$. Note that $x - f(x) \neq 0$ so without loss of generality suppose that the first coordinate $(x - f(x))_1 \neq 0$. Now consider $(r(x_n))_1 = (f(x_n))_1 + t(x_n)(x - f(x))_1$. Solving for $t(x_n)$ we get that,

$$t(x_n) = \frac{(r(x_n) - f(x_n))_1}{(x_n - f(x_n))_1}$$

Since f is continuous it follows that $(x_n - f(x_n))_1 \to (x - f(x))_1$, and since $(x - f(x))_1 \neq 0$ there exists an N such that for all $n \geq N$, $(x_n - f(x_n))_1 \neq 0$. Therefore since $x_n \to x$ we know that $t(x_n)$ converges to t such that,

$$t(x_n) = \frac{(r(x_n) - f(x_n))_1}{(x - f(x))_1} \to \frac{(z - f(x))_1}{(x - f(x))_1} = t$$

Finally, since each of our terms converge since $x_n \to x$ it follows that,

$$r(x_n) = f(x_n) + t(x_n)(x_n - f(x_n)) \to f(x) + t(x - f(x)) = r(x).$$

Since S^1 is Hausdorff limits are unique so $r(x) = z$ and therefore the graph of r is closed in $\mathbb{B} \times S^1$. By the Closed Graph Theorem it follows that $r(x)$ is a continuous function.

□

2. If $f, g : S^1 \to S^1$ are two continuous maps, express $\deg(f \circ g)$ in terms of $\deg(f)$ and $\deg(g)$. Use this to show that $f \circ g$ is homotopic to $g \circ f$.

Proof. Suppose $f, g : S^1 \to S^1$ are two continuous maps. Let $w_i : S^1 \to S^1$ be defined by $w_i(z) = x^i$ and note that $\deg(w_i) = i$. Let $n = \deg(f)$ and $m = \deg(g)$ and note that $\deg(w_n) = n$ and $\deg(w_m) = m$. Recall that $f \sim g$ if and only if $\deg(f) = \deg(g)$ and therefore $w_n \sim f$ and $w_m \sim g$ and it follows that,

$$\begin{aligned}
\deg(f \circ g) &= \deg([f \circ g]) \\
&= \deg([f] \circ [g]) \\
&= \deg([w_n] \circ [w_m]) \\
&= \deg([w_n \circ w_m]) \\
&= \deg(w_n \circ w_m) \\
&= \deg(w_{nm}) \\
&= nm \\
&= \deg(f)\deg(g)
\end{aligned}$$

Clearly it follows that, $\deg(f \circ g) = \deg(f)\deg(g) = \deg(g)\deg(f) = \deg(g \circ f)$ and therefore $f \circ g$ is homotopic to $g \circ f$. □

3. Let X be a locally compact Hausdorff space. The one-point compactification of X is the topological space X^* defined as follows. Let ∞ be some object not in X, and let $X^* = X \coprod \{\infty\}$ with the following topology:

$$\begin{aligned}
\mathcal{T} &= \{\text{ open subsets of } X\} \cup \{U \subseteq X^* : X^* \setminus U \text{ is a compact subset of } X\} \\
&= \mathcal{T}_1 \cup \mathcal{T}_2
\end{aligned}$$

(a) Show that $\mathcal{T}$ is a topology.

Proof. Suppose X is a locally compact Hausdorff space. Clearly $\mathcal{T}_1$ is closed with respect to arbitrary unions and finite intersections of open subsets $U \subseteq X$ since X it a topological space. Let $U_i \in \mathcal{T}_2$ and note for an arbitrary union,

$$\left(\bigcup_{i \in I} U_i \right)^c = \bigcap_{i \in I} (U_i)^c.$$

Since X is Hausdorff we know that the compact sets $(U_i)^c$ are closed. Let x_α be a net contained in $\bigcap_{i \in I}(U_i)^c$, clearly $x_\alpha \subseteq (U_i)^c$ for all i. Since $(U_i)^c$ are compact there exists a convergent subnet $x_{\alpha_\beta} \subseteq (U_i)^c$ and by definition $x_{\alpha_\beta} \subseteq \bigcap_{i \in I}(U_i)^c$. Therefore it follows that $\bigcap_{i \in I}(U_i)^c$ is compact in X and by definition $\bigcup_{i \in I} U_i \in \mathcal{T}_2$. Now consider,

$$\left(\bigcap_{i=1}^{n} U_i \right)^c = \bigcup_{i=1}^{n} (U_i)^c$$

Note we have finite union of compact sets in X, which is also compact in X and therefore $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_2$.

Since we have shown that $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies all that is left to show is that for any pair $U \in \mathcal{T}_1$ and $V \in \mathcal{T}_2$, $U \cap V \in \mathcal{T}$ and $U \cup V \in \mathcal{T}$. Note that

$$(U \cap V)^c = U^c \cup V^c.$$

Since it $(U \cap V)^c$ is a finite union of closed sets in X, $(U \cap V)^c$ is closed in X. Therefore $U \cap V$ is open in X and hence $U \cap V \in \mathcal{T}_1 \subseteq \mathcal{T}$. Note that

$$(U \cup V)^c = U^c \cap V^c.$$

Since V^c is compact in X and U^c is closed in X we know that $(U \cup V)^c$ is a compact in X since it is a closed subset of a compact set, therefore $U \cup V \in \mathcal{T}_2 \subseteq \mathcal{T}$.

□

(b) Show that X^* is a compact Hausdorff space.

Proof. Suppose $\{U_i\}$ is an open cover of X^*. There exists some U_j such that $\infty \in U_j$ and therefore $U_j \in \mathcal{T}_2$. By definition U_j^c is a compact subset of X. Note that since $\{U_i\}$ is an open cover of X^* and therefore a cover of X it follows that $\{U_i : U_i \subseteq X\}$ must be an open cover of U_j^c. Since U_j^c, there exists a finite subcover $\{U_{i_\alpha} : U_{i_\alpha} \subseteq X\}$. Therefore $\{U_i\}$ admits a finite subcover, $X^* = U_{i_\alpha} \cup U_j$.

Suppose X is locally compact Hausdorff. By our hypothesis, since $X^* = X \cup \{\infty\}$, to show that X^* is Hausdorff, it is sufficient to show that for any $x \in X$ there exists open sets $x \in U$ and $\infty \in V$ such that $U \cap V = \emptyset$. Let $x \in X$ and since X is locally compact Hausdorff, there exists an open set U and compact set K such that $x \in U \subseteq K$. Since K is compact in X, by definition K^c is open in X^* such that $\infty \in K^c$. Note that $U \cap K^c = \emptyset$ since $U \subseteq K$. Thus X^* is Hausdorff.

□

4. Prove that every nonconstant polynomial in one complex variable has a zero. [Hint: if $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, write $p_\epsilon(z) = \epsilon^n p(z/\epsilon)$ and show that there exists $\epsilon > 0$ such that $|p_\epsilon(z) - z^n| < 1$ when $z \in S^1$. Suppose that if p has no zeroes, then $p_\epsilon|_{S^1}$ is homotopic to $p_n(z) = z^n$ and use degree theory to derive a contradiction.]

Proof. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and suppose p has no zeroes. Let $\epsilon > 0$ and consider $p_\epsilon(z) = \epsilon^n p(z/\epsilon)$. Consider $f(z,t) : S^1 \times (0,1) \to S^1$ defined by $f(z,t) = \frac{p_{-\ln(t)}(z)}{|p_{-\ln(t)}(z)|}$. Since p has no zeroes and $-ln(t) > 0$ we know that $p_{-\ln(t)}(z)$ also has no zeros and therefore for all $t \in (0,1)$, $|p_{-\ln(t)}(z)| > 0$. Thus f is continuous. Note that,

$$\lim_{t\to 1} f(z,t) = \lim_{t\to 1} \frac{z^n + (-\ln(t))a_{n-1}z^{n-1} + \cdots + (-\ln(t))^n a_0}{|z^n + (-\ln(t))a_{n-1}z^{n-1} + \cdots + (-\ln(t))^n a_0|} = z^n.$$

$$\lim_{t\to 0} f(z,t) = \lim_{t\to 0} \frac{z^n + (-\ln(t))a_{n-1}z^{n-1} + \cdots + (-\ln(t))^n a_0}{|z^n + (-\ln(t))a_{n-1}z^{n-1} + \cdots + (-\ln(t))^n a_0|} = \frac{a_0}{|a_0|} = 1.$$

Now we define the homotopy, $H(z,t) : S^1 \times I \to S^1$ by

$$H(z,t) = \begin{cases} 1, & t = 0 \\ f(z,t), & t \in (0,1) \\ z^n, & t = 1 \end{cases}$$

Thus we have shown that z^n is homotopic to a constant which is a contradiction since $\deg(z^n) = n$ and $\deg(1) = 0$.

□

5. Suppose X is a topological space, and g is any path in X from p to q. Let $\phi_g : \pi_1(X,p) \to \pi_1(X,q)$ denote the group isomorphism defined in Thoerem 7.13.

a Show that if h is another path in X starting at q, then $\phi_{g\cdot h} = \phi_h \circ \phi_g$.

Proof. Suppose h is another path in X from q to r. Let $\phi_{g\cdot h} : \pi_1(X,p) \to \pi_1(X,r)$ be the isomorphism defined by,

$$\phi_{g\cdot h}[f] = [\overline{g\cdot h}] \cdot [f] \cdot [g \cdot h].$$

Now note that the path $\overline{g\cdot h}$ goes from r to p, first via $\overline{h}$ and then via $\overline{g}$, and therefore $\overline{g\cdot h} = \overline{h}\cdot\overline{g}$. Applying this substitution, and by properties of the product of path classes we get,

$$\begin{aligned} \phi_{g\cdot h}[f] &= [\overline{g\cdot h}] \cdot [f] \cdot [g\cdot h] \\ &= [\overline{h}\cdot\overline{g}] \cdot [f] \cdot [g\cdot h] \\ &= [\overline{h}] \cdot [\overline{g}] \cdot [f] \cdot [g] \cdot [h] \\ &= \phi_h \circ \phi_g \end{aligned}$$

□

b Suppose $\psi : X \to Y$ is continuous, and show that the following diagram commutes:

Proof. To show that the diagram commutes, we must show that $\psi_* \circ \phi_g = \phi_{\psi \circ g} \circ \psi_*$. Let $[f] \in \pi_1(X, p)$ and note that by definition of the change of base point, $\psi_* \circ \phi_g([f]) = \psi_*([\overline{g}] \cdot [f] \cdot [g])$. By properties of path classes we know that,

$$\psi_* \circ \phi_g([f]) = \psi_*([\overline{g}] \cdot [f] \cdot [g]) = \psi_*([\overline{g} \cdot f \cdot g])$$

By definition of the induced homomorphism induced by q we know that,

$$\psi_* \circ \phi_g([f]) = [\psi(\overline{g} \cdot f \cdot g)].$$

By path multiplication we know that,

$$\psi_* \circ \phi_g([f]) = [\psi(\overline{g}) \cdot \psi(f) \cdot \psi(g)] = [\psi(\overline{g})] \cdot [\psi(f)] \cdot [\psi(g)].$$

Note that by definition $[g] \cdot [\overline{g}] = id_X$ apply our group homomorphism on both sides we get, $\psi_*([g] \cdot [\overline{g}]) = id_Y$. Simplifying the left hand side we get, $\psi_*([g] \cdot [\overline{g}]) = [\psi(g)] \cdot [\psi(\overline{g})] = id_Y$, and therefore $\overline{[\psi(g)]} = [\psi(\overline{g})]$. By substitution we arrive at the desired identity,

$$\psi_* \circ \phi_g([f]) = \overline{[\psi(g)]} \cdot [\psi(f)] \cdot [\psi(g)] = \overline{[\psi(g)]} \cdot \psi_*([f]) \cdot [\psi(g)] = \phi_{\psi \circ g} \circ \psi_*([f])$$

□

6. Let X be a path-connected topological space, and let $p, q \in X$. Show that all paths from p to q give the same isomorphism of $\pi_1(X, p)$ with $\pi_1(X, q)$ is and only if $\pi_1(X, p)$ is abelian.

Proof. ($\Rightarrow$) Suppose all paths from p to q give the same isomorphism of $\pi_1(X, p)$ with $\pi_1(X, q)$. Let $[f], [g] \in \pi_1(X, p)$ such that $[f] \neq [g]$ and h is a path from p to q. Note $f \cdot h$ is also a path from p to q. By our hypothesis it follows that $\phi_{f \cdot h}([g]) = \phi_h([g])$. By definition of the change of base point it follows that.

$$\begin{aligned}
\phi_{f \cdot h}([g]) &= \phi_h([g]), \\
[\overline{f \cdot h}] \cdot [g] \cdot [f \cdot h] &= [\overline{h}] \cdot [g] \cdot [h], \\
[\overline{h} \cdot \overline{f}] \cdot [g] \cdot [f \cdot h] &= [\overline{h}] \cdot [g] \cdot [h], \\
[\overline{h}] \cdot [\overline{f}] \cdot [g] \cdot [f] \cdot [h] &= [\overline{h}] \cdot [g] \cdot [h], \\
[\overline{f}] \cdot [g] \cdot [f] &= [g], \\
[g] \cdot [f] &= [f] \cdot [g].
\end{aligned}$$

Hence $\pi_1(X, p)$ is abelian. □

Proof. ($\Leftarrow$) Suppose $\pi_1(X, p)$ is abelian and let g, f be distinct paths from p to q and let $[h] \in \pi_1(X, q)$. Note $f \cdot h \cdot \overline{g}$ is a loop of p. Since g, f are distinct we know that $g \cdot \overline{f}$ is also a loop of p. Since $\pi_1(X, p)$ is abelian it follows that,

$$\begin{aligned}
[f \cdot h \cdot \overline{g}] \cdot [g \cdot \overline{f}] &= [g \cdot \overline{f}] \cdot [f \cdot h \cdot \overline{g}] \\
[f \cdot h \cdot \overline{g} \cdot g \cdot \overline{f}] &= [g \cdot \overline{f} \cdot f \cdot h \cdot \overline{g}] \\
[f \cdot h \cdot \overline{f}] &= [g \cdot h \cdot \overline{g}] \\
[f] \cdot [h] \cdot [\overline{f}] &= [g] \cdot [h] \cdot [\overline{g}] \\
\phi_{\overline{f}}([h]) &= \phi_{\overline{g}}([h])
\end{aligned}$$

Since these are isomorphisms it follows that their inverses are also equivalent so we conclude that $\phi_f([j]) = \phi_g([j])$. $\square$