Several of the problems on this assignment are repeats. Use nets!

1. Show that a topological space in X is Hausdorff if and only if every convergent net in X has exactly one limit.

Proof. ($\Rightarrow$) Suppose X is Hausdorff. Let $\langle x_\alpha \rangle_{\alpha \in A}$ is a net in X and suppose for the sake of contradiction that $\langle x_\alpha \rangle$ converges to both a and b. Since X is Hausdorff there exists $U_a \in \mathcal{V}(a)$ and $U_b \in \mathcal{V}(b)$ such that $U_a \cap U_b = \emptyset$. By definition there exists some $\alpha_b, \alpha_a \in A$ such that $\langle x_\alpha \rangle_{\alpha \in T(\alpha_a)} \subseteq U_a$ and $\langle x_\alpha \rangle_{\alpha \in T(\alpha_b)} \subseteq U_b$. Let $x_1 \in \langle x_\alpha \rangle_{\alpha \in T(\alpha_a)}$ and $x_2 \in \langle x_\alpha \rangle_{\alpha \in T(\alpha_b)}$ and note that $x_1, x_2 \in \langle x_\alpha \rangle_{\alpha \in A}$. Since $\langle x_\alpha \rangle_{\alpha \in A}$ is a net there exists some $x_3 \geq x_1, x_2$, and therefore $x_3 \in \langle x_\alpha \rangle_{\alpha \in T(\alpha_b)}$ and $x_3 \in \langle x_\alpha \rangle_{\alpha \in T(\alpha_a)}$ so therefore $x_3 \in U_a \cap U_a$, a contradiction. □

Proof. ($\Leftarrow$) Suppose X is a space which is not Hausdorff. Then there exists $a, b \in X$ such that for every $U_a \in \mathcal{V}(a)$ we know that $b \in U_a$. Let $A = \mathcal{V}(a)$ ordered by reverse inclusion and choose $x_U \in U_a \cap U_b$. Consider the net $\langle x_U \rangle_{U \in \mathcal{V}(a)}$, we will proceed by showing that this net converges to both a and b. Let $U_b \in \mathcal{V}(b)$, and note that $b \in U_b \cap W$ for some $W \in \mathcal{V}(a)$, therefore $\langle x_U \rangle_{U \in T(U_b \cap W)} \subseteq U_b$. Let $U_a \in \mathcal{V}(a)$, by definition $\langle x_U \rangle_{U \in T(U_a)} \subseteq U_a$. Hence $\langle x_U \rangle_{U \in \mathcal{V}(a)}$ has two limits. □

2. Consider the product space $X \times Y$. Find (and prove) a condition in terms of coordinate functions that characterizes convergence of nets in the product. Does your condition also work for an arbitrary product?

Let $X \times Y$ be a product space, a net $\langle (x, y)_\alpha \rangle_{\alpha \in A}$ converges to (x, y) if and only if $\pi_i(\langle (x, y)_\alpha \rangle_{\alpha \in A})$ converges to $\pi_i((x, y))$ for all canonical projections π_i.

Let f be the function associated to $\langle (x, y)_\alpha \rangle_{\alpha \in A}$. Define $\pi_i(\langle (x, y)_\alpha \rangle_{\alpha \in A})$ as the net $\langle \pi_i((x, y)_\alpha) \rangle_{\alpha \in A}$ defined by the function $\pi_i(f(\alpha))$.

Proof. ($\Rightarrow$) Let $X \times Y$ be a product space and suppose that $\langle (x, y)_\alpha \rangle_{\alpha \in A}$ converges to (x, y). Without loss of generality let $\pi = \pi_x$ the canonical projection into X. Let $U \in \mathcal{V}(\pi_i((x, y)))$. Note that $\pi^{-1}(U)$ is an open set in $X \times Y$ containing (x, y). By definition there exists some $\alpha' \in A$ such that $\langle (x, y)_\alpha \rangle_{\alpha \in T(\alpha')} \subseteq \pi^{-1}(U)$. Since π is a surjection it follows that $\pi(\langle (x, y)_\alpha \rangle_{\alpha \in T(\alpha')}) \subseteq U$. □

Proof. ($\Leftarrow$) Let $X \times Y$ be a product space with a net $\langle (x, y)_\alpha \rangle_{\alpha \in A}$ and suppose that $\pi_i(\langle (x, y)_\alpha \rangle_{\alpha \in A})$ converges to $\pi_i((x, y))$ for all canonical projections π_i. Let $U \in \mathcal{V}((x, y))$ and consider a basic open set B, such that $(x, y) \in B \subseteq U$. Note that $B = \pi_X^{-1}(V) \cap \pi_Y^{-1}(W)$ where V and W are open in X and Y respectively. Note that $x \in V$ and $y \in W$, by definition there exists $\alpha_x, \alpha_y \in A$ such that $\pi_X(\langle (x, y)_\alpha \rangle_{\alpha \in T(\alpha_x)}) \subseteq V$ and $\pi_Y(\langle (x, y)_\alpha \rangle_{\alpha \in T(\alpha_y)}) \subseteq W$. By properties of nets there exists an $\alpha' \geq \alpha_x, \alpha_y$ so therefore, $\pi_X(\langle (x, y)_\alpha \rangle_{\alpha \in T(\alpha')}) \subseteq V$ and $\pi_Y(\langle (x, y)_\alpha \rangle_{\alpha \in T(\alpha')}) \subseteq W$. Finally note that $\langle (x, y)_\alpha \rangle_{\alpha \in T(\alpha')} \subseteq \pi_X^{-1}(V), \pi_Y^{-1}(W)$ and therefore $\langle (x, y)_\alpha \rangle_{\alpha \in T(\alpha')} \subseteq B \subseteq U$. □

3. Show that a space X is Hausdorff if and only if the diagonal in $X \times X$ is closed.

Proof. ($\Rightarrow$) Suppose a space X is Hausdorff. Consider the subset $\Delta = \{(x,x) : x \in X\} \subseteq X \times X$. We will proceed to show that Δ is closed by showing that for every convergent net $\langle (x,x)_\alpha \rangle_{\alpha \in A} \subseteq \Delta$ with limit point (a,b), then $(a,b) \in \Delta$.

Let $\langle (x,x)_\alpha \rangle_{\alpha \in A} \subseteq \Delta$ which converges to (a,b). By the previous problem it follows that $\langle x_\alpha \rangle_{\alpha \in A}$ converges to both a and b. By Problem 1, since X is Hausdorff it follows that $a = b$ and therefore by definition $(a,b) \in \Delta$. □

Proof. ($\Leftarrow$) Suppose X is not Hausdorff. To show that Δ is not closed we will exhibit a net contained in Δ which has a limit point outside of Δ. Since X is not Hausdorff there exists distinct $a, b \in X$ such that for every $U_a \in \mathcal{V}(a)$ we know that $b \in U_a$. Let $A = \mathcal{V}(a)$ ordered by reverse inclusion and choose $x_U \in U_a \cap U_b$. Consider the net $\langle x_U \rangle_{U \in \mathcal{V}(a)}$ and recall that we showed this net converges to both a and b. By Problem 2 it follows that $\langle (x,x)_U \rangle_{U \in \mathcal{V}(a)}$ which is clearly contained in Δ converges to (a,b) which is not in Δ. □

4. Let G be a topological group and let H be a subgroup. Show that $\overline{H}$ is a subgroup.

Proof. Let $a, b \in \overline{H}$. We will proceed to show that $ab^{-1} \in \overline{H}$ by exhibiting a net contained in H which converges to ab^{-1}.

Recall that if $a, b \in \overline{H}$ then there exists nets $\langle a_\alpha \rangle_{\alpha \in A}$ and $\langle b_\beta \rangle_{\beta \in B}$ which converge to a and b respectively.

Note that $\langle a_\alpha \rangle$ and $\langle b_\beta \rangle$ are convergent nets in G and by Problem 2 we know that

$$\langle (a_\alpha, b_\beta) \rangle_{(\alpha,\beta) \in A \times B} = \langle (a,b)_\gamma \rangle_{\gamma \in C}$$

in $G \times G$ converges to (a,b). Recall that for a topological group G, the map $f : G \times G \to G$ defined by $f(a,b) = ab^{-1}$ is continuous. Since f is continuous and we know that $\langle (a,b)_\gamma \rangle \to (a,b)$ it follows that $\langle f((a,b)_\gamma) \rangle \to f((a,b)) = ab^{-1}$. What is left to show is that the net $f(\langle (a,b) \rangle_\gamma)$ is contained in H. Note that by definition $(a,b)_\gamma \in H \times H$ and since H is a subgroup $f(H \times H) \subseteq H$.

□

5. Suppose X is a space and Y is compact and Hausdorff. Show that a function $f : X \to Y$ is continuous if and only if the graph of f, G_f is closed.

Proof. ($\Rightarrow$) Let X be a space and Y be compact and Hausdorff. Suppose that function $f : X \to Y$ is continuous. We will proceed to show that G_f is closed by demonstrating that for every convergent net $\langle (x, f(x))_\alpha \rangle_{\alpha \in A}$ contained in G_f with limit (a,b), then $(a,b) \in G_f$.

By Problem 2 it follows that $\langle x_\alpha \rangle \to a$ and $\langle f(x_\alpha) \rangle \to b$. Since f is continuous we know that $\langle f(x_\alpha) \rangle = \langle f(x_\alpha) \rangle \to f(a)$. Since Y is Hausdorff, by Problem 1 $f(a) = b$ and therefore by definition $(a,b) \in G_f$. □

Proof. ($\Leftarrow$) Let X be a space and Y be compact and Hausdorff. Suppose that the function $f : X \to Y$ is not continuous. We will proceed to show that G_f is not closed. Since f is not continuous there must exists a convergent net $\langle x_\alpha \rangle_{\alpha \in A} \to x$ in X whose image does not converge to $f(x)$ in Y, i.e. $\langle f(x_\alpha) \rangle_{\alpha \in A} \not\to f(x)$. Therefore there exists a $U \in \mathcal{V}(f(x))$ and subnet $\langle f(x_{\alpha_\beta})\rangle_{\beta\in B}$ such that $f(x_{\alpha_\beta}) \notin U$ for all β. Since Y is compact however $\langle f(x_{\alpha_\beta})\rangle_{\beta\in B}$ itself must have a convergent subnet, $\langle f(x_{\alpha_{\beta_\gamma}})\rangle_{\gamma\in C} \to f(x)'$. Note that the corresponding net in the domain $\langle x_{\alpha_{\beta_\gamma}}\rangle_{\gamma\in C}$ is a subnet of a convergent net $\langle x_\alpha\rangle$ so therefore $\langle x_{\alpha_{\beta_\gamma}}\rangle_{\gamma\in C} \to x$. By Problem 2 it follows that $\langle (x_{\alpha_{\beta_\gamma}}, f(x_{\alpha_{\beta_\gamma}}))\rangle_{\gamma\in C}$ is a convergent net in $X \times Y$ contained in G_f which converges to $(x, f(x)')$. However since $f(x_{\alpha_\beta}) \notin U$ for all β, we know that $f(x)' \neq f(x)$. Therefore we have produced a convergent sequence contained in G_f whose limit point is not contained in G_f and hence G_f is not closed. □

6. Show that the homeomorphism group of a connected manifold acts transitively. In other words, show that if M is a connected manifold, then for any two points p and q in M there is a homeomorphism $\psi : M \to M$ such that $\psi(p) = q$.

Proof. Suppose M is a connected manifold and let $p, q \in M$. Since M is locally euclidean, for every point $x \in M$ there exists a $U \in \mathcal{V}(x)$ that is homeomorphic to $\mathbb{R}^n$, a connected set. These neighborhoods form a basis of path-connected open subsets and therefore M is locally path-connected. Since M is connected and locally path connected, it is also path connected. Since M is path connected there exists a path in M between p and q, i.e. a continuous $f : I \to M$ such that $f(0) = p$ and $f(1) = q$. Note that $f(I)$ is a compact set in M, and consider the collection $\{U_x \in \mathcal{V}(x) : x \in f(I), U \sim \mathcal{B}^n\}$ which covers $f(I)$ since $f(I)$ is compact, this collection admits a cardinality n finite refinement $\{U_i\}$.

Consider an ordering of $\{U_i\}$ such that $U_k \leq U_j$ if there exists a $b \in U_j$ such that $f^{-1}(b) \geq f^{-1}(a)$ for all $a \in U_k$ and $p \in U_1$ and $q \in U_n$. Note that $U_i \cap U_{i+1} \neq \emptyset$ since that would imply the collection does not cover $f(I)$.

Note that U_i, by construction is homeomorphic to the open $\mathcal{B}^n$, by some homeomorphism f. Recall that we can construct a homeomorphism $\phi_i : \mathcal{B}^n \to \mathcal{B}^n$ which maps $\phi_i(f(a)) = f(b)$ such that $a \in U_{i-1} \cap U_i$ and $b \in U_i \cap U_{i+1}$. We can define ϕ_i on the open $\mathcal{B}^n$ restriction and maintain our homeomorphism since $\phi_i(\partial\mathcal{B}^n) = \partial\mathcal{B}^n$ since the boundary was kept constant. Define ϕ_1 such that $\phi_1(f(p)) = f(k)$ where $k \in U_1 \cap U_2$ and ϕ_n such that $\phi_n(f(j)) = f(q)$ where $j \in U_{n-1} \cap U_n$.

Note that the map $f{-}1 \circ \phi_i \circ f$ is a homeomorphism from U_i to itself. To continue I need this map to be a homeomorphism from $\overline{U_i}$ to itself, it should follow pretty directly from ϕ_i being constant on the boundary but I'm not certain exactly how. To continue, let $f{-}1 \circ \phi_i \circ f$ be a homeomorphism from $\overline{U_i}$ to itself, such that all $x \in \partial\overline{U_i}$ get the identity map. Applying the glueing lemma with the identity function applied to $\overline{U_i}^c$ we get a homeomorphism from $g_i : M \to M$ which maps $g(a) = b$ for some $a \in U_{i-1} \cap U_n$ and $U_n \cap U_{n+1}$. Note that $\psi : M \to M$ defined by $\psi(x) = g_n \circ ... \circ g_1$ has the property that $\psi(p) = q$ and is a homeomorphism since it is defined as a composition of homeomorphisms.

□