1. Lee 9-2 The center of a group G is the set Z of elements of G that commute with every element of G: thus $Z = \{g \in G : gh = hg \text{ for all } h \in G\}$. Show that a free group on two or more generators has center consisting of the identity alone.

Proof. Suppose $|S| = n$ such that $n \geq 2$, we want to show that the center of $F(S)$ consists of only the identity. Suppose to the contrary that there exists some non identity $w \in F(S)$ such that $w \in Z$. Let

$$w = \sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\dots\sigma_k^{\alpha_k}$$

where $\sigma_i \in S$, $\alpha_i \neq 0$ and $\sigma_i \neq \sigma_{i+1}$. Now suppose the case where $k \geq 3$, and consider the element $h = \sigma_{k-1}^{\alpha_{k-1}}\sigma_1^{-\alpha_1}$ and note that since $w \in Z$ we get the following,

$$\begin{aligned} hw &= wh, \\ (\sigma_{k-1}^{\alpha_{k-1}}\sigma_1^{-\alpha_1})(\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\dots\sigma_k^{\alpha_k}) &= (\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\dots\sigma_k^{\alpha_k})(\sigma_2^{\alpha_2}\sigma_1^{-\alpha_1}), \\ \sigma_{k-1}^{\alpha_{k-1}}\sigma_2^{\alpha_2}\dots\sigma_k^{\alpha_k} &= (\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\dots\sigma_k^{\alpha_k})(\sigma_{k-1}^{\alpha_{k-1}}\sigma_1^{-\alpha_1}). \end{aligned}$$

Note that the left-hand side cannot reduce since $\sigma_i \neq \sigma_{i+1}$ and therefore since these words are clearly not equivalent this is a contradiction. Clearly a word with one element cannot be in the center, so now we consider the case when $k = 2$. Note that with element $h = \sigma_2^{\alpha_2}\sigma_1^{-\alpha_1}$ we get the following,

$$\begin{aligned} hw &= wh, \\ (\sigma_2^{\alpha_2}\sigma_1^{-\alpha_1})(\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}) &= (\sigma_1^{\alpha_1}\sigma_2^{\alpha_2})(\sigma_2^{\alpha_2}\sigma_1^{-\alpha_1}), \\ \sigma_2^{\alpha_2+\alpha_2} &= \sigma_1^{\alpha_1}\sigma_2^{\alpha_2+\alpha_2}\sigma_1^{-\alpha_1}. \end{aligned}$$

□

2. Lee 9-4 (Read "Presentations of Groups", pages 241–243 first) Let G_1, G_2, H_1, H_2 be groups and let $f_i : G_i \to H_i$ be a group homomorphism from $i = 1, 2$.

(a) Show that there exists a unique homomorphism $f_1 * f_2 : G_1 * G_2 \to H_1 * H_2$ such that the following diagram commutes for $i = 1, 2$:

$$\begin{tikzcd} G_1 * G_2 \arrow[r, "f_1 * f_2"] & H_1 * H_2 \\ G_i \arrow[u, "\iota_i"] \arrow[r, "f_i"'] & H_i \arrow[u, "\iota_i'"'] \end{tikzcd}$$

where $\iota_i : G_i \to G_1 * G_2$ and $\iota_i' : H_i \to H_1 * H_2$ are the canonical injections.

Proof. Recall that in order to apply the characteristic property of the free product to our collection $\{G_1, G_2\}$, the free product $G_1 * G_2$, and the group $H_1 * H_2$ we must show is that for each $G_i \in \{G_1, G_2\}$ there exists a homomorphism into $H_1 * H_2$. By the characteristic property of the free product these homomorphisms extend into the desired unique homomorphism from $G_1 * G_2$ to $H_1 * H_2$.

From our hypothesis we know that there exists a homomorphism $f_i : G_i \to H_i$ and and a canonical projection $\iota'_i : H_i \to H_1 * H_2$. Let $\phi_i = \iota'_i \circ f_i$ and note that $\phi_i : G_i \to H_1 * H_2$. We will conclude by showing that this map is a homomorphism. First note that since ι'_i is the canonical injection, $\iota'_i(h) = h$ for all $h \in H_i$. Let $a, b \in G_i$ it then follows that,

$$\begin{aligned}
\phi_i(ab) &= \iota'_i(f_i(ab)), \\
&= \iota'_i(f_i(a)f_i(b)), \\
&= f_i(a)f_i(b), \\
&= \iota'_i(f_i(a))\iota'_i(f_i(b)), \\
&= \phi_i(a)\phi_i(b).
\end{aligned}$$

□

(b) Let S_1 and S_2 be disjoint sets, and let R_i be a subset of the free group $F(S_i)$ for $i = 1, 2$. Prove that $\langle S_1 \cup S_2 | R_1 \cup R_2 \rangle$ is a presentation of the free product group $\langle S_1 | R_1 \rangle * \langle S_2 | R_2 \rangle$

Proof. □

3. Lee 10-1 (Wait until Monday to start) Use the Seifert-Van Kampen Theorem to give another proof that S^n is simply connected when $n \geq 2$.

Proof. Consider S^n with $n \geq 2$ and recall that in order to show that if S^n is simply connected we must show that $\pi_1(S^n)$ is trivial. Let $U = S^n \setminus \{p\}$ and $V = S^n \setminus \{q\}$ with $p \neq q$. Clearly these two sets are open, path-connected, and cover S^n. Note that $U \cap V = S^n \setminus \{p, q\}$ is clearly path-connected.

Consider the fundamental groups $\pi_1(U)$ and $\pi_1(V)$ and note that since the spaces are S^n with one point removed we know that they are homeomorphic to $\mathbb{R}^n$ and since $n \geq 2$, $\mathbb{R}^n$ is always homotopic to a constant. Hence $\pi_1(U)$ and $\pi_1(V)$ are trivial groups.

Apply Seifert-Van Campen, we know that $\pi_1(S^n) \cong {\pi_1(U) * \pi_1(V)}/{\overline{C}}$. Since $\pi_1(U)$ and $\pi_1(V)$ are both trivial the free product is also trivial, and so is a quotient of the free product. Hence $\pi_1(S^n)$ is trivial and S^n is simply connected.

□

4. Lee 10-5 (You may be a little informal in your proof)