1. Let G be a topological group,

(a) Prove that up to isomorphism $\pi_1(G, g)$ is independent of the choice of basepoint $g \in G$.

Proof. Suppose G is a topological group and let $g, h \in G$ and consider $\pi_1(G, g)$ and $\pi_1(G, h)$. Recall since g is a topological group, group multiplication m, group inverse i, and $f : G \times G \to G$ defined by $f(a, b) = ab^{-1}$ are continuous functions.

The component functions $f_i : G \to G$ defined by $f_i(x) = ix^{-1}$ and $m_i : G \to G$ defined by $m_i(x, i) = xi$ are continuous.

Note that $\phi : G \to G$ defined by $\phi(x) = m_g(f_h(x)) = hx^{-1}g$ is continuous. It also follows that $\varphi : G \to G$ defined by $\varphi(x) = m_h(f_g(x)) = gx^{-1}h$ is continuous.

Since $\phi(g) = h$ and $\varphi(h) = g$ are continuous functions they induce homomorphisms $\phi^* : \pi_1(G, g) \to \pi_1(G, h)$ and $\varphi^* : \pi_1(G, h) \to \pi_1(G, g)$.

Let $[x] \in \pi_1(G, h)$ and note that $\varphi^*([x]) \in \pi_1(G, g)$ such that $\phi^*(\varphi^*[x]) = [\phi(\varphi(x))] = [h(gx^{-1}h)^{-1}g] = [x]$. Hence ϕ^* is a surjection.

Note ϕ^* has a trivial kernel since the only loop in $\pi_1(G, g)$ which maps to a constant loop in $\pi_1(G, h)$ would have to be a constant loop in $\pi_1(G, g)$. Hence ϕ^* is an injection and therefore an isomorphism. □

(b) Prove that $\pi_1(G, g)$ is abelian.

Proof. Suppose G is a topological group. By the previous result it is sufficiency to show that $\pi_1(G, 1)$ is abelian, as it is isomorphic to $\pi_1(G, g)$. Let $[a], [b] \in \pi_1(G, 1)$ and consider the map $F : I \times I \to G$ given by $F(s, t) = a(s)b(t)$. Clearly this map is continuous and it has the property that,

$$\begin{aligned} F(s, 0) &= a(s)b(0) = a(s)1 = a(s), \\ F(1, t) &= a(1)b(t) = 1b(t) = b(t), \\ F(0, t) &= a(0)b(t) = 1b(t) = b(t), \\ F(s, 1) &= a(s)b(1) = a(s)1 = a(t). \end{aligned}$$

Thus by the Square Lemma we know that $a \cdot b \sim b \cdot a$. So it follows that $[a][b] = [a \cdot b] = [b \cdot a] = [b][a]$. □

2. Prove that a retract of a Hausdorff space is a closed subset.

Proof. Suppose X is Hausdorff and A is a retract of X. By definition there exists a continuous function $r : X \to A$ such that $r \circ \iota_A = Id_A$ where ι_A is the inclusion map.

We will proceed to show that A is closed by showing that for every convergent net $\langle x_\alpha \rangle \to x$ where $\langle x_\alpha \rangle \subseteq A$ then $x \in A$. Since r is continuous we know that $\langle r(x_\alpha) \rangle \to r(x)$. Since $r \circ \iota_A = Id_A$ it follows that $\langle r(x_\alpha) \rangle = \langle x_\alpha \rangle$ and by definition we know that $r(x) \in A$. Recall that convergent nets in Hausdorff spaces converge to a single limit, and therefore $r(x) = x$, and therefore $x \in A$. Hence A is closed. □

3.

(a) Suppose $U \subseteq \mathbb{R}^2$ is an open subset and $x \in U$. Show that $U^* = U \setminus \{x\}$ is not a simply connected space.

Show that U^* has nontrivial fundamental groups.

Lemma 1: The closed unit ball with the origin removed is a retract of $\mathbb{R}^2$ with the origin removed. Consider the function $r : \mathbb{R}^{2^*} \to \overline{\mathbb{B}}^*$ defined by,

$$r(x) = \begin{cases} x, & |x| <\leq 1 \\ \frac{x}{|x|}, & |x| > 1 \end{cases}$$

This function is clearly continuous and has the property that $\iota_{\overline{\mathbb{B}}^*} \circ r(x) = Id_{\overline{\mathbb{B}}^*}$, as desired.

Proof. Let $U \subseteq \mathbb{R}^2$ be an open subset with $x \in U$ and suppose to the contrary that U^* is simply connected, and therefore it has trivial fundamental groups.

We will proceed to show that this implies the fundamental groups on S^1 are trivial.

Since $U \subseteq \mathbb{R}^2$ we can define $V = \overline{\mathbb{B}_r(x)} \subseteq U$ and $V^* = V \setminus \{x\}$ and consider $p \in V^*$. Note that $\iota_{V^*} : V^* \to U^*$, the embedding of V^* into U^* is continuous, and therefore induces a homomorphism $\iota_V^* : \pi_1(V^*, p) \to \pi_1(U^*, p)$.

Now note that via Lemma 1, up to a translation of the origin, V is a retraction of $\mathbb{R}^2 \setminus \{x\}$ and therefore we can construct a continuous function $r' : \mathbb{R}^2 \setminus \{x\} \to V^*$ such that $\iota_{V^*} \circ r' = Id_V$. Note that since $U^* \subseteq \mathbb{R}^*$ and $V^* \subseteq U^*$ we can construct a retraction from $r : U^* \to V^*$ by $r = r' \circ \iota_{U^*}$. This function is continuous, and therefore induces a homomorphism $r^* : \pi_1(U^*, p) \to \pi_1(V^*, p)$.

Let $[j] \in \pi_1(V^*, p)$ and note that composing these homomorphisms, we get the identity map on $\pi_1(V^*, p)$,

$$r^* \circ \iota_V^*([j]) = r^*([\iota_{V^*}(j)]) = r^*([j]) = [r(j)] = [r'(\iota_{U^*}(j))] = [j]$$

However since U^* is simply connected we know that $\pi_1(U^*, p)$ is trivial. Since $r^* \circ \iota_V^*$ is the identity map on $\pi_1(V^*, p)$ it follows that $\pi_1(V^*, p)$ must have been trivial as well, which is a contradiction since $\pi_1(U^*, p)$ is isomorphic to $\pi_1(S^1, x)$. □

(b) Show that if $n > 2$, then $\mathbb{R}^n$ is not homeomorphic to any open subset of $\mathbb{R}^2$.

Proof. Let $n > 2$ and suppose to the contrary that $\mathbb{R}^n$ is homeomorphic to any open subset of $\mathbb{R}^2$. Let $U \subseteq \mathbb{R}^2$ be open with $x \in U$ and note that by our hypothesis there exists an $f : U \to \mathbb{R}^n$ that is a homeomorphism. It follows that $f' : U^* \to \mathbb{R}^n \setminus \{f(x)\}$ is a homeomorphism, which is a contradiction since U^* is not simply connected and $\mathbb{R}^n \setminus \{f(x)\}$ for $n > 2$ is. □

4. Prove that a non-empty topological space cannot be both a 2-manifold and an n-manifold for some $n > 2$.

Proof. Let X be a topological space, and suppose to the contrary that it is both a 2-manifold and an n-manifold for some $n > 2$. Let $x \in X$ and by definition of locally euclidean we know that there exists open sets $x \in U$ and $x \in V$ such that U is homeomorphic to $\mathbb{R}^2$ via f and V is homeomorphic to $\mathbb{R}^n$ via g. Let $W = U \cap V$ and note that $f(W)$ is homeomorphic to $g(W)$ via $g \circ f^{-1}$.

Since W is an open subset of V we can can construct an open ball $\mathbb{B} \subseteq \mathbb{R}^n$ such that $g(x) \in \mathbb{B}$ and $\mathbb{B} \subseteq g(W)$. It follows that $\mathbb{B}$ is homeomorphic to $f \circ g^{-1}(\mathbb{B})$. It then follows that $\mathbb{B} \setminus g(x)$ is homeomorphic to $f \circ g^{-1}(\mathbb{B} \setminus \{g(x)\})$. Note that $\mathbb{B} \setminus g(x)$ is a ball in $\mathbb{R}^n$, $n > 2$ with one point removed so it is simply connected. However $f \circ g^{-1}(\mathbb{B} \setminus \{g(x)\})$ since is an open set of $\mathbb{R}^2$ with one point removed by the previous exercises it is not simply connected, hence a contradiction.

□

5. Show that the continuous map $\varphi : S^1 \to S^1$ has an extension to a continuous map $\phi : \mathbb{B}^2 \to S^1$ if and only if it has degree zero.

Proof. ($\Leftarrow$) Suppose $\varphi : S^1 \to S^1$ is a map with degree zero. Since φ has degree zero, we know that it is homotopic to a constant. Suppose φ is homotopic to a constant $\varphi(0)$, then we define the homotopy by $H(x, t) : S^1 \times I \to S^1$ where $H(x, 1) = \varphi(x)$ and $H(x, 0) = \varphi(0)$. Note that $\mathbb{B}^2$ is a reparameterization of $S^1 \times I$ in rectangular coordinates. Since $H(x, 1) = \varphi(x)$, applying this reparameterization to the homotopy gives an extension of φ defined by $\phi : \mathbb{B}^2 \to S^1$. □

Proof. ($\Rightarrow$) Suppose the continuous map $\varphi : S^1 \to S^1$ has an extension to a continuous map $\phi : \mathbb{B}^2 \to S^1$. Again $S^1 \times I$ is a reparameterization of $\mathbb{B}^2$ into polar coordinates. We apply this reparameterization and we get a function $H : S^1 \times I \to S^1$ where $H(x, 1) = \varphi(x)$ since ϕ was an extension and $H(x, 0) = 0$ by polar coordinates. Hence φ is homotopic to a constant and therefore has degree zero. □

6. Suppose $\varphi, \psi : S^1 \to S^1$ are continuous maps of different degrees. Show that there is a point $z \in S^1$ where $\varphi(z) = -\psi(z)$.

Proof. We will proceed by the contrapositive. Let $\varphi, \psi : S^1 \to S^1$ be continuous maps, and suppose that for all $z \in S^1$ we know that $\varphi(z) = -\psi(z)$. By problem 6 of Homework 9 it follows that φ is not homotopic to ψ and therefore they cannot have the same degree. □