1. A subset A of a topological space X is said to be nowhere dense if $\text{Int}\,\overline{A} = \emptyset$.

a) Let U be an open subset of a topological space. Prove that ∂U is closed and nowhere dense.

Proof. Suppose U is an open subset of X. Clearly ∂U is closed as its defined as an intersection of closed sets, $\partial U = \overline{U} \cap \overline{U^c}$. Note that $\text{Int}\,\overline{\partial U} = \text{Int}\,\partial U$ since ∂U is closed. Suppose for the sake of contradiction that $x \in U' \subseteq \partial U$ is open. Since U is open and $\partial U \subseteq \overline{U^c} = U^c$ it follows that that $U \cap \partial U = \emptyset$. By definition $x \notin \overline{U}$ since $x \in U'$ such that $U' \cap U = \emptyset$ i.e x is not a contact point of U, a contradiction. □

b) Let V be a closed and nowhere dense set. Show that V is the boundary of an open set.

Proof. Suppose V is closed an $\text{Int}\ V = \emptyset$. Note that V^c is an open set with the property that $\partial V^c = (\overline{V^c} \cap \overline{(V^c)^c}) = (\overline{V^c} \cap \overline{V})$. Let $x \in V$ and note that since $\text{Int}\ V = \emptyset$ it follows that x is a contact point to V^c. So $x \in \overline{V^c}$ and since $x \in \overline{V}$ we conclude that $x \in \partial V^c$.

Let $x \in \partial V^c$. By definition $x \in \overline{(V^c)^c} = \overline{V} = V$. Thus $V = \partial V^c$. □

2. Let f and g be continuous maps from a topological space X to a Hausdorff space Y. Suppose $f = g$ on a dense subset of X. Prove that $f = g$.

Proof. Let f and g be continuous maps from a topological space X to a Hausdorff space Y. Suppose $f = g$ on a dense subset $X' \subseteq X$. For the sake of contradiction suppose that for some $x \in X$ $f(x) \neq g(x)$. Since Y is Hausdorff, there exists open sets $f(x) \in U_f$ and $g(x) \in U_g$ such that $U_f \cap U_g = \emptyset$. Since f and g are continuous we know that $f^{-1}(U_f)$ and $g^{-1}(U_g)$ are open in X. Note that $x \in f^{-1}(U_f) \cap g^{-1}(U_g) = U$, and since X' is dense in X there exists an $x' \in U$ such that $f(x') = g(x')$. Note that $x' \in f^{-1}(U_f)$ and $x' \in g^{-1}(U_g)$ so it follows that $f(x') \in U_f$ and $g(x') \in U_g$. Therefore $f(x') \in U_f \cap U_g$ a contradiction. □

3. Exercise 4.38 Let X be a compact space, and suppose $\{F_n\}$ is a countable collection of nonempty closed subsets of X what are nested, which means that $F_n \subseteq F_{n+1}$ for each n. Show that $\cap_n F_n$ is nonempty.

Proof. Let X be a compact space, and suppose $\{F_n\}$ is a countable collection of nonempty closed subsets of X what are nested, which means that $F_n \subseteq F_{n+1}$ for each n. For the sake of contradiction suppose $\cap_n F_n = \emptyset$ which is equivalent to showing $\cap_n (F_n)^c = X$. Note that the set $\{(F_n)^c\}$ is an open cover of X and therefore has a finite subcover $\mathcal{U} = \{F_\alpha\}_{\alpha \in A}$. Since $\mathcal{U}$ is finite there exists a largest element F_k^c, and by construction we know that

$$F_k^c \supseteq F_{k-1}^c \supseteq \dots \supseteq F_0^c$$

So it follows that, $F_k^c = \cup_{\alpha \in A} F_\alpha^c = X$ and therefore $F_k = \emptyset$ a contradiction. (this is a proof by contrapositive in disguise.)

□

4. Suppose X and Y are spaces and Y is compact. Show that the projection $\pi : X \times Y \to X$ is a closed map.

Proof. Suppose X and Y are spaces and Y is compact. Let U be a closed set in $X \times Y$. Note that to show $\pi(U)$ is closed it is sufficient to show that $\pi(U)^c$ is open. Note that $U^c \subseteq \pi^{-1}(\pi(U))^c = \pi^{-1}(\pi(U)^c)$. Let $x \in \pi(U)^c$, and note that U^c is an open set in $X \times Y$ containing $\{x\} \times Y$. By the tube lemma there exists an open V in X such that $V \times Y \subseteq U$, which implies that since $\{x\} \times Y \subseteq V \times Y$ then $V \times Y \subseteq \pi^{-1}(\pi(U)^c)$ which implies that $x \in V \subseteq \pi(U)^c$. Therefore $\pi(U)^c$ is an open set and thus $\pi(U)$ is closed. □

5. Let G be an algebraic group. We say that G is a **topological group** if in addition G is a topological space such that that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ defined by $m(g, h) = g \cdot h$ and $i(g) = g^{-1}$ are continuous.

a) Suppose G is an algebraic group and a T_1 topological space. Show that G is a topological group if and only if the map $f : G \times G \to G$ defined by $f(g, h) = gh^{-1}$ is continuous.

Proof. ($\Rightarrow$) Let G be an algebraic group and a topological space. Suppose G is a topological group. By definition $m : G \times G \to G$ and the inversion map $i : G \to G$ defined by $m(g, h) = g \cdot h$ and $i(g) = g^{-1}$ are continuous. Note that the map $f : G \times G \to G$ defined by $f(g, h) = gh^{-1}$, is equivalent to $f(g, h) = m(g, i(h))$ and since the components of $m(g, i(g))$ are continuous so is f.

($\Leftarrow$) Let G be an algebraic group and a topological space. Suppose the map $f : G \times G \to G$ defined by $f(g, h) = gh^{-1}$ is continuous. Note that the inversion map $i : G \to G$ defined $i(g) = g^{-1}$ is equivalent to $i(g) = f(f(g, g), g) = gg^{-1}g^{-1} = g^{-1}$. Since $i(g)$ is a composition of continuous functions, it is also continuous. Note that the multiplication map $m : G \times G \to G$ defined by $m(g, h) = g \cdot h$ is simply $m(g, h) = f(g, i(h)) = gh^{-1}$. Since $h(g, h)$ is a composition of continuous functions, it is also continuous. Thus G is a topological group. □

b) Let G be a topological group and let H be a subgroup. Show that $\overline{H}$ is a subgroup. Hint: that map f from the previous part is continuous.

Proof. Suppose G is a topological group and let H be a subgroup. Let $a, b \in \overline{H}$. Note that since H is a subgroup of G, $f(H \times H) \subseteq H$ and since $H \subseteq \overline{H}$ it follows that $H \times H \subseteq f^{-1}(\overline{H})$. Since $\overline{H}$ is closed and f is a continuous function, $f^{-1}(\overline{H})$ is closed. Therefore it follows that $\overline{H \times H} \subseteq f^{-1}(\overline{H})$. Note that $\overline{H} \times \overline{H} \subseteq \overline{H \times H}$ since for all $x = (x_1, x_2) \in \overline{H} \times \overline{H}$, we know that x is a contact point of $H \times H$ because x_1 and x_2 are contact points of H. Hence $f(\overline{H} \times \overline{H}) = f\overline{H}$ and therefore $\overline{H}$ is a subgroup.

□

6. Let $\{x_n\}_n$ be a sequence in an arbitrary product $\prod X_\alpha$. Show that $x_n \to x$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for every α. Then show that this result is false if we assume instead that $\prod X_\alpha$ is given the box topology.

Proof. ($\Rightarrow$) Let $\{x_n\}_n$ be a sequence in an arbitrary product $\prod_{\alpha \in A} X_\alpha$. Suppose that $x_n \to x$, and note that by definition we know that for all $U \in \mathcal{V}(x)$ there exists some N, such that for all $n \geq N$, $\{x_n\}_{n \geq N} \in U$. Consider the sequence $\pi_\alpha(x_n)$ for some factor α. Let $U' \in \mathcal{V}(\pi_\alpha(x))$, and note that $\pi_\alpha^{-1}(U')$ is an open set in $\prod_{\alpha \in A} X_\alpha$ under the product topology, so there exists some N such that for all $n \geq N$, $\{x_n\}_{n \geq N} \in \pi_\alpha^{-1}(U')$ so therefore $\pi_\alpha(\{x_n\}_{n \geq N}) = \{\pi_\alpha(x_n)\}_{n \geq N} \subseteq U'$. □

Proof. ($\Leftarrow$) Let $\{x_n\}_n$ be a sequence in an arbitrary product $\prod_{\alpha\in A} X_\alpha$. Suppose $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for every α. Suppose $U \in \mathcal{V}(x)$ and since $\prod_{\alpha\in A} X_\alpha$ has the product topology we know that there exists a basic open set U', with $x \in U' \subseteq U$ of the form $U' = \cap_{j\in J}\pi_j^{-1}(U_j)$ for some finite $J \subseteq A$ with U_j an open set in X_j. Note that for each U_j there exists an N_j such that for all $n \geq N_j$, $\{\pi_j(x_n)\}_{n\geq N_j} \subseteq U_j$. Since J is finite, there exists a maximal N_j, N where for all $j \in J$, $\{\pi_j(x_n)\}_{n\geq N} \subseteq U_j$. Therefore it follows that $\{x_n\}_{n\geq N} \subseteq \{\pi_j^{-1}(\pi_j(x_n))\}_{n\geq N} \subseteq \pi_j^{-1}(U_j)$, for all $j \in J$ hence $\{x_n\}_{n\geq N} \subseteq U' \subseteq U$. □

Clearly the ($\Leftarrow$) direction does not hold if $\prod_{\alpha\in A} X_\alpha$ is endowed with the box topology. Let $U \in \mathcal{V}(x)$ be the basic open set of $U = \cap_{j\in J}\pi_j^{-1}(U_j)$ for some infinite $J \subseteq A$. There would then exists no maximal N in the set of all N_j.

7. Show that the following topological spaces are not manifolds:

(a) The union of the x-axis and the y-axis in $\mathbb{R}^2$.

Proof. Let S be the union of the x-axis and the y-axis in $\mathbb{R}^2$. Consider the point $(0,0) \in S$ and note that for every $U' \in \mathcal{V}((0,0))$ there exists an $r > 0$ such that $U \subseteq U'$ where $U = B_r((0,0)) \cap S$ is open via the subspace topology. Suppose to the contrary that there exists a homeomorphism $f : U \to \mathbb{R}^n$ for all $n \geq 2$. Let $U^* = U \setminus \{(0,0)\}$ and note that $f|_{U^*} : U^* \to \mathbb{R}^n \setminus \{f((0,0))\}$ is a homeomorphism. Note that U^* is disconnected since $\{(x,y) \in \mathbb{R}^2 : y > x\} \cap S$ and $\{(x,y) \in \mathbb{R}^2 : y < x\} \cap S$ are open in S and partition U^*. Clearly for all $n \geq 2$, $\mathbb{R}^n \setminus \{f((0,0))\}$ is connected, thus a contradiction. Therefore for all $n \geq 2$, $U \not\sim \mathbb{R}^n$ and thus $U' \not\sim \mathbb{R}^n$ so S is not locally Euclidean dimension n.

We will conclude by showing that $U' \not\sim \mathbb{R}$, and therefore S is not locally Euclidean of dimension 1. Suppose there exist a homeomorphism $f : U \to \mathbb{R}$. Again it follows that $f|_{U^*} : U^* \to \mathbb{R}^n \setminus \{f((0,0))\}$ is a homeomorphism. Clearly U^* has 4 components, namely $(\mathbb{R}^- \times \{0\} \cap U^*)$, $(\mathbb{R}^+ \times \{0\} \cap U^*)$, $(\{0\} \times \mathbb{R}^+ \cap U^*)$, and $(\{0\} \times \mathbb{R}^- \cap U^*)$ whose images in $\mathbb{R} \setminus \{f((0,0))\}$ must be disjoint. Note that f is a closed and open map and there are only 2 disjoint closed and open sets in $\mathbb{R} \setminus \{f((0,0))\}$, namely $(-\infty, f((0,0)))$ and $(f((0,0)), \infty)$. Therefore f is a homeomorphism which maps a pair of disjoint sets to the same set, a contradiction. Thus $U \not\sim \mathbb{R}$ and therefore $U' \not\sim \mathbb{R}$. □

(b) The conical surface $C \subseteq \mathbb{R}^3$ defined by

$$C = \{(x,y,z) : z^2 = x^2 + y^2\}$$

Proof. Suppose for the sake of contradiction that C is an n-manifold. Consider the point $p^* = (0,0,0)$ and note that for every every $U' \in \mathcal{V}(p)$ there exists an $r > 0$ such that $U \subseteq U'$ where $U = B_r(p^*) \cap C$ is open via the subspace topology. Let $U^* = U \setminus \{p^*\}$. Since C is a n-manifold there exists a homeomorphism $f : U \to \mathbb{R}^n$, and therefore $f|_{U^*} : U^* \to \mathbb{R}^n \setminus \{f(p^*)\}$ is also a homeomorphism. Note that U^* is disconnected since $\{(x,y,z) \in \mathbb{R}^3 : z > 0\} \cap U^*$ and $\{(x,y,z) \in \mathbb{R}^3 : z < 0\} \cap U^*$ partition U^* and are open via the subspace topology. Therefore it follows that $\mathbb{R}^n \setminus \{f(p^*)\}$ is also disconnected, thus it must have been the case that $n = 1$ with $U \sim \mathbb{R}$. So C is a 1-manifold.

Now consider the set $C^+ = C \setminus \{(x,y,z) \in C : z > 0\}$. We will proceed by showing that the map $\pi : C^+ \to \mathbb{R}^2 \setminus \{(0,0)\}$ defined by $\pi((x,y,z)) = (x,y)$ is homeomorphism. First suppose $\pi((x,y,z)) = \pi((x_1,y_1,z_1))$ and by π we know that $x = x_1$ and $y = y_1$, and since our domain is C^+ $z = \sqrt{x^2+y^2} = \sqrt{x_1^2+y_1^2} = z_1$, so f is injective. Let $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and note that by definition of C^+ we know that $(x, y, \sqrt{x^2+y^2}) \in C^+$ and by π we get $\pi((x, y, \sqrt{x^2+y^2})) = (x, y)$. So f is a surjection. Note that the component maps of π are simply the identity so they are continuous and thus π is continuous. Define $\pi : \mathbb{R}^2\setminus\{(0, 0)\} \to C^+$ by $\pi^{-1}((x, y)) = (x, y, \sqrt{x^2+y^2})$. Again this map has two component maps which are the identity, and another defined by $\pi_z^{-1}(x, y) = \sqrt{x^2+y^2}$ which is also continuous.

So $\pi : C^+ \to \mathbb{R}^2 \setminus \{(0, 0)\}$ is a homeomorphism, and since $\mathbb{R}^2 \setminus \{(0, 0)\}$ is a 2-manifold, so is C^+. Finally we have that there exists a neighborhood U_2 of $p \in C^+$ which is homeomorphic to $\mathbb{R}^2$ and since $p \in C$, and C is a 1-manifold there exists another neighborhood of p, U_1 homeomorphic to $\mathbb{R}$. So $U_2 \cap U_1$ is homeomorphic to both $\mathbb{R}^2$ and $\mathbb{R}$. □

8. Let $M = \mathbb{S}^1 \times \mathbb{R}$, and let $A = \mathbb{S}^1 \times \{0\}$. Show that the space M/A obtained by collapsing A to a point is homeomorphic to the space C of the previous problem, and thus is Hausdorff and second countable but not locally Euclidean.

Proof. Let $M = \mathbb{S}^1 \times \mathbb{R}$, and let $A = \mathbb{S}^1 \times \{0\}$. Let $q : M \to M/A$ be the natural projection sending M to it's equivalence class. To show that M/A is homeomorphic to C we will proceed by first exhibiting a function $f : M \to C$ which is continuous and constant on the fibers of q. Consider f defined by $f(((x, y), z)) = (xz, yz, z)$. Clearly the components of this function are continuous from $\mathbb{R}^3$ and hence continuous from the restriction M. Since it's component maps are continuous, f is a continuous function. Suppose $q(x) = q(x')$ for $x \neq x'$, and note that this is only the case when $x, x' \in A$ and are therefore of the form $x = ((x, y), 0)$ and $x' = ((x', y'), 0)$ so clearly $f(x) = f(x')$. Thus f descends to the quotient as continuous function $\hat{f} : M/A \to C$ where $f(x) = \hat{f}(q(x))$.

Now let $g : C \to M \setminus \{S^1\} \cup \{(1, 1, 0)\}$ be defined by

$$g((x, y, z)) = \begin{cases} \left(\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right), z\right) & (x, y, z) \in C^* \\ ((1, 1), 0) & (x, y, z) \in S^1 \end{cases}$$

Note that as a map from $C^* \to M \setminus \{S^1\}$, g has continuous component maps since C^* excludes $(0, 0, 0)$ and thus g is continuous restricted onto C^*. Note that any open set $U \in M\setminus\{S^1\}\cup\{(1, 1, 0)\}$ has a pre-image, $g^{-1}(U) = U^* \cup \{(0, 0, 0)\}$ where U^* is open in C^* and therefore $g^{-1}(U)$ is open in C. Thus g is continuous and there exists a continuous map $\hat{g} : C \to M/A$ defined by $\hat{g}(x) = q(g(x))$.

Finally we must show that $\hat{f}$ and $\hat{g}$ are inverses, note that exhibiting an inverse is sufficient to show a bijection between $M/A \to C$. Note that $\hat{g}(\hat{f}(q(x))) = q(g(\hat{f}(x))) = q(g(f(x))) = q(x)$. □

Lemma 1: Let X be a topological space with $a, b, c \in X$. If there exists a path in X from a to c and from b to c, then there exists a path from a to b.

Proof. Suppose there exists a path in X from a to c and from b to c, so there are continuous function $f_a : I \to X$ and $f_b : I \to X$ with $f_a(0) = a$, $f_a(1) = c$, $f_b(0) = b$, and $f_b(1) = c$. Note that $[0, 1/2]$ and $[1/2, 1]$ are homeomorphic to I by g_a and g_b respectively where $g_a(x) = 2x$ and $g_b(x) = -2(x - 1)$. Note that $[0, 1/2]$ and $[1/2, 1]$ form a finite closed cover of I and since $f_a(g_a(1/2)) = f_a(1) = c = f_b(1) = f_b(g_b(1/2))$ by the glueing lemma there exists a continuous map

$f : I \to X$ whose restrictions on $[0, 1/2]$ and $[1/2, 1]$ is equal to $f_a(g_a(x))$ and $f_b(g_b(x))$. Note that f is a path from a to b in X. □

9. Let X be a topological space, and let CX be the cone on X.

1. (a) Show that CX is path-connected

   Proof. Recall that CX is defined as the quotient space $(X \times [0, 1])/(X \times \{0\})$ and let $q : X \to CX$ be the natural projection of X onto its equivalence classes. Note that to show that CX is path-connected, by Lemma 1 it is sufficient to show that there exists a path from an $x \in X$ to some fixed $x \in X$. Let $c \in CX$ be the point whose identified by $c = q(X \times \{0\})$. Let $x \in CX$, and note that since q is a surjection there exists some $(x', i) \in X \times [0, 1]$ with the property that $q((x', i)) = x$. we will proceed by exhibiting a continuous function $f : I \to X \times [0, 1]$ with the property that $f(0) = (x', i)$ and $f(1) = (x', 0)$, whose composition with q will define the path from x to c in CX. Let $f(z) = (x', (1 - z)i)$ and note that f is continuous since it's component maps are also continuous. Finally since q is continuous we know that $q \circ f : I \to CX$ is continuous with the property that $q(f(1)) = q((x', 0)) = c$ and $q(f(0)) = q((x', i)) = x$. □

   Lemma 2: The product of locally (path-)connected spaces is locally (path-)connected. Suppose X and Y are locally (path-)connected. Let $p \in X \times Y$ and consider $U \in \mathcal{V}(p)$. Note that by the product topology we know $U = \pi_x^{-1}(U_x) \cap \pi_y^{-1}(U_y)$ where U_x and U_y are open sets in X and Y respectively. Since U_x and U_y belong to locally (path-)connected spaces there exists (path-)connected sets $U'_x \subseteq U_x$ and $U'_y \subseteq U_y$ such that $\pi_x^{-1}(p) \in U'_x$ and $\pi_y^{-1}(p) \in U'_y$. Clearly $U' = U'_x \times U'_y$ is (path-)connected, since it's a product of (path-)connected sets and also by construction has the property that $p \in U' \subseteq U$. Hence $X \times Y$ is locally (path-)connected.

2. (b) Show that CX is locally connected if and only if X is, and locally path connected if and only if X is.

   Proof. ($\to$) Suppose CX is locally (path-)connected. To show that X is locally (path-)connected we will show that for any $p \in X$, $U \in \mathcal{V}(p)$ there exists an open (path-)connected subset U' such that $p \in U' \subseteq U$. Let $p \in X$ and $U \in \mathcal{V}(p)$. Note that by the product topology, $U \times [0, 1]$ is open in $X \times [0, 1]$, and since q is a quotient map it also must follow that $q(U \times [0, 1])$ is open in CX. Note that since CX is locally (path-)connected there exists a (path-)connected set $CU' \subseteq q(U \times [0, 1])$ whose pre-image $q^{-1}(CU')$ is open in $X \times [0, 1]$ since q is a quotient map and contains $p \times [0, 1]$. Let $U' = \pi_(q^{-1}(CU'))$ and note that by construction $p \in U' \subseteq U$. Thus X is locally (path-)connected.

   □

   Proof. ($\leftarrow$) Suppose X is locally path connected. By Lemma 2 we know that since X is locally (path-)connected, $X \times (0, 1]$ is also locally (path-)connected. Since $q : X \times [0, 1] \to CX$ is a continuous map, it follows that the image $q(X \times (0, 1])$ is locally (path-)connected in CX. Let $v = q(X \times \{0\})$, what remains to be shown is that for any $U \in \mathcal{V}(v)$ there exists a (path-)connected subset U' such that $v \in U' \subseteq U$. By the part (a) if this problem we know that CX is path connected, so there must exists such a U'. Therefore CX is locally (path-)connected. □

10. Let X be a topological space. The **suspension** of X, denoted by ΣX, is the quotient of $X \times [-1, 1]$ where all points of the form $(x, 1)$ are identified, and all points of the form $(x, -1)$ are identified. Determine, with proof, a familiar space that is homeomorphic to ΣS^n.

Proof. Consider ΣS^n, and note that by definition ΣS^n is the quotient of $S^n \times [-1, 1]$ where all points of the form $(x, 1)$ are identified, and all points of the form $(x, -1)$ are identified. Let $q : S^n \times [-1, 1] \to \Sigma S^n$ be the natural projection of $X \times [-1, 1]$ onto its equivalence classes. Note that S^n is compact, one can see that quickly by noting S^n is closed and bounded and applying Heine-Borel. By Tychonoff's Theorem the product $S^n \times [-1, 1]$ is compact. Now note that S^{n+1} is a Hausdorff space, since for any $a, b \in S^{n+1}$ we can find a disjoint $a \in U_a$ and $b \in U_b$ in $\mathbb{R}^{n+2}$ and by the subspace topology on S^{n+1}, $U_a \cap S^{n+1}$ and $U_b \cap S^{n+1}$ are open and disjoint. We will conclude by exhibiting a map surjective continuous map $f : S^n \times [-1, 1] \to S^{n+1}$ which is closed via the closed map lemma and therefore a quotient map, showing that it makes the same identifications as q and therefore by the uniqueness of quotient spaces $\Sigma S^n \sim S^{n+1}$. Let $f : S^n \times [-1, 1] \to S^{n+1}$ be defined by $f((x, t)) = (\sqrt{1 - t^2}x, t)$. The component maps for f are continuous from $\mathbb{R}^{n+2}$ since the first is a nonzero scaling of a vector, and the second is a projection. Therefore f is continuous on its domain and its image is S^{n+1} since all for $(x, t) \in S^n \times [-1, 1]$ we get $f((x, t)) = (\sqrt{1 - t^2}x, t)$ with,

$$
\begin{aligned}
|f((x,t))| = |(\sqrt{1-t^2}x, t)| &= \left(\left(\sum_{i=1}^{n+1}(\sqrt{1-t^2}x_i)^2\right) + t^2\right)^{1/2} \\
&= \left(\left(\sum_{i=1}^{n+1}(1-t^2)x_i^2\right) + t^2\right)^{1/2} \\
&= \left((1-t^2)\left(\sum_{i=1}^{n+1}x_i^2\right) + t^2\right)^{1/2} \\
&= \left(1 - t^2 + t^2\right)^{1/2} = 1.
\end{aligned}
$$

Let $(x_1, \ldots, x_{n+2}) \in S^{n+1}$. Note that $f\left(\frac{1}{\sqrt{1-x_{n+2}^2}}(x_1, \ldots x_{n+1}), x_{n+2}\right) = (x_1, \ldots, x_{n+2})$ and $\frac{1}{\sqrt{1-x_{n+2}^2}}(x_1, \ldots x_{n+1}) \in S^n$ since,

$$
\begin{aligned}
\left|\frac{1}{\sqrt{1-x_{n+2}^2}}(x_1, \ldots x_{n+1})\right| &= \sum_{i=1}^{n+1}\left(\frac{1}{\sqrt{1-x_{n+2}^2}}x_i\right)^2 \\
&= \frac{1}{1-x_{n+2}^2}\left(\sum_{i=1}^{n+1}x_i^2\right) \\
&= \frac{1}{\sum_{i=1}^{n+1}x_i^2}\left(\sum_{i=1}^{n+1}x_i^2\right) = 1.
\end{aligned}
$$

So f is surjective and continuous. Now we will conclude by showing that f makes the same identifications as q. Note that as defined q makes the same non identity identification when (x, t)

is of the form $(x, 1)$ or $(x, -1)$, which clearly $f((x, 1)) = (0, 1)$ and $f((x, -1)) = (0, -1)$ so f makes the same identifications $\square$