1. Prove that every ball $B_r(x)$ in a metric space (X, d) is an open set.

Solution:

Proof. Let $a \in B_r(x)$, and choose $\delta = r - d(x, a)$. Now consider $B_\delta(a)$ and note that to show $B_r(x)$ is open we must prove that $B_\delta(a) \subseteq B_r(x)$. Let $y \in B_\delta(x)$ and note that since (X, d) is a metric space the following expression comes from the triangle inequality,

$$d(x, y) \leq d(x, a) + d(a, y).$$

By the definition of $B_\delta(a)$ we know that $d(a, y) < r - d(x, a)$ and by substitution we get that $d(x, y) < r$. Thus by definition $y \in B_r(x)$ and therefore we get that $B_\delta(a) \subseteq B_r(x)$. Therefore $B_r(x)$ is open.

□

2. Let V be a subset of a metric space (X, d). The set of limit points of V are those points x that can be written as the limit of a sequence of points in V. Show that a set $V \subseteq X$ is closed if and only if it contains its limit points.

Solution:

Proof. Suppose that $V \subseteq X$ is a closed set. By definition we know that V^c is open. For the sake of contradiction suppose V does not contain its limit points, therefore there exists a limit point of the set V, x such that $x \in V^c$. By the previous result there exists an open ball $B_r(x)$ such that $B_r(x) \subseteq V^c$. Note that since $x_n \in V$ and $x_n \to x$, we can choose n such that $d(x_n, x) < r$ and therefore $x_n \in B_r(x)$. Thus $x_n \in V^c$ a contradiction.

We will prove the reverse direction via contrapositive. Suppose V is not closed and therefore V^c is not open. Thus there exists some point $x \in V^c$ such that for all $r > 0$, $B_r(x) \nsubseteq V^c$. Choose the set $\{B_i(x) : i = \frac{1}{n}\}$ and by the previous result we know that inside each $B_i(x)$ there exists some $x_i \in V$. We will proceed by showing that $x_i \to x$. Let $\epsilon > 0$, and choose N such that $\frac{1}{N} < \epsilon$. Then for all $n \geq N$ we get $d(x_i, x) \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Therefore the sequence $\{x_i\} \in V$ converges to a limit point $x \in V^c$, so V does not contain all it's limit points. □

3. Let d_1 and d_2 be two metrics on a set X. Show that the following conditions are equivalent.

a) For every sequence $\{p_i\}_{i=1}^{\infty}$, if $p_i \underset{d_2}{\longrightarrow} p$ then $p_i \underset{d_1}{\longrightarrow} p$.

b) For every function $f : X \to \mathbb{R}$, if f is continuous with respect to d_1 then f is continuous with respect to d_2.

c) For every set V, if V is closed with respect to d_1 then V is closed with respect to d_2.

d) For every set U, if U is open with respect to d_1 then U is open with respect to d_2.

Solution:

1. ($a \iff b$) Suppose that for every sequence $\{p_i\}_{i=1}^{\infty}$, if $p_i \underset{d_2}{\longrightarrow} p$ then $p_i \underset{d_1}{\longrightarrow} p$. Let $f : X \to \mathbb{R}$ be a continuous function with respect to d_1. Consider a sequence $\{p_i\}_{i=1}^{\infty}$ such that $p_i \underset{d_2}{\longrightarrow} p$. By our supposition since $p_i \underset{d_2}{\longrightarrow} p$ we know that $p_i \underset{d_1}{\longrightarrow} p$. Since f is continuous with respect to d_1, $p_i \underset{d_1}{\longrightarrow} p$ implies that $f(p_i) \to f(p)$ in $\mathbb{R}$. Thus f is continuous with respect to d_2.

   For the converse we will make use of the following lemmas.

   Lemma 1 (Reverse Triangle Inequality): Let X be a metric space, then for all $a, b, c \in X$
   $$|d(a,c) - d(c,b)| \leq d(a,b).$$

   Proof. Since X is a metric space, by the triangle inequality we know that
   $$d(a,c) \leq d(a,b) + d(c,b),$$
   $$d(a,c) - d(c,b) \leq d(a,b).$$

   Case 1: Suppose $d(a,c) - d(c,b) \geq 0$. Then it follows that $|d(a,c) - d(c,b)| \leq d(a,b)$.

   Case 2: Suppose $d(a,c) - d(c,b) < 0$. By triangle inequality we also get that,
   $$d(c,b) \leq d(a,b) + d(a,c),$$
   $$d(c,b) - d(a,c) \leq d(a,b).$$

   Since $d(a,c) - d(c,b) < 0$ we know that $d(c,b) - d(a,c) \geq 0$ and therefore it follows that $|d(c,b) - d(a,c)| \leq d(a,b)$ or equivalently, $|d(a,c) - d(c,b)| \leq d(a,b)$. □

   Lemma 2: Let X be a metric space with a fixed point x^*. Then the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, x^*)$ is continuous.

   Proof. Suppose $a \in X$ and let $\epsilon > 0$. Choose $\delta = \epsilon$ such that $d(x,a) < \delta$ implies that,
   $$\begin{aligned} |f(x) - f(a)| &= |d(x,x^*) - d(a,x^*)|, \\ &\leq d(x,a), \\ &< \delta = \epsilon. \end{aligned}$$
   □

Suppose that for every function $f : X \to \mathbb{R}$, if f is continuous with respect to d_1 then f is continuous with respect to d_2. Let $\{p_i\}_{i=1}^{\infty}$ be a sequence such that $p_i \underset{d_2}{\longrightarrow} p$. Consider the function $f : X \to \mathbb{R}$ defined by $f(x) = d_1(x, p)$. By Lemma 2 we know that f is a continuous function with respect to d_1 and therefore by assumption f is continuous with respect to d_2. Therefore it follows that since $p_i \underset{d_2}{\longrightarrow} p$ we know that $d_1(x, p_i) \to 0$ which implies $p_i \underset{d_1}{\longrightarrow} p$.

2. ($a \implies c$) Suppose that for every sequence $\{p_i\}_{i=1}^{\infty}$, if $p_i \underset{d_2}{\longrightarrow} p$ then $p_i \underset{d_1}{\longrightarrow} p$ and let V be a set that is not closed with respect to d_2. Since V is not closed with respect to d_2 there exists a contact point $p \notin V$ where the sequence $\{p_i\}_{i=1}^{\infty} \in V$ $p_i \underset{d_2}{\longrightarrow} p$. By assumption if $p_i \underset{d_2}{\longrightarrow} p$ then $p_i \underset{d_1}{\longrightarrow} p$ and since $\{p_i\}_{i=1}^{\infty} \in V$ and $p \notin V$, V is not closed with respect to d_1.

3. ($c \implies d$) Suppose that for every set V, if V is closed with respect to d_1 then V is closed with respect to d_2, and U is an open set with respect to d_1. Note that if U is open with respect to d_1 then U^c is closed with respect to d_1. It then follows that U^c is closed with respect to d_2 and thus U is open with respect to d_2.

4. ($d \implies a$) Suppose that for every set U, if U is open with respect to d_1 then U is open with respect to d_2 and let $\{p_i\}_{i=1}^{\infty} \in X$ such that $p_i \underset{d_2}{\longrightarrow} p$. Let $r > 0$ and consider an open ball $B_r^1(p)$. Since $B_r^1(p)$ is open with respect to d_1, then by assumption $B_r^1(p)$ is open with respect to d_2. It then follows that there exists some $\hat{r}$ such that $B_{\hat{r}}^2(p) \subseteq B_r^1(p)$. Note that since $p_i \underset{d_2}{\longrightarrow} p$ we can choose an N such that for all $n \geq N$, $p_n \in B_{\hat{r}}^2(x) \subseteq B_r^1(p)$. Thus $p_i \underset{d_1}{\longrightarrow} p$.

Proof. □

4. Let X be an infinite set.

a) Show that
$$\mathcal{T}_1 = \{U \subseteq X : U = \emptyset \text{ or } U^c \text{ is finite}\}$$
is a topology on X, called the finite complement topology.

Solution:

Proof. Note that by definition $\emptyset \in \mathcal{T}_1$ and since $X^c = \emptyset$ a finite set we get that $X \in \mathcal{T}_1$. Let $\{U_i\}_I$ be a set of open subsets of X. By definition we know that $\{U_i{}^c\}_I$ are all finite and therefore their intersection $\cap_{i \in I} U_i{}^c$ is also finite. By De Morgan's Law we conclude that,
$$\bigcap_{i \in I} U_i{}^c = \left(\bigcup_{i \in I} U_i \right)^c.$$

Thus $\cup_{i\in I} U_i$ is open. Let $\{U_i\}_I$ be a finite set of open subsets of X. Note that $\cup_{i\in I} {U_i}^c$ is a finite union of finite sets and is therefore also a finite set. By De Morgan's law we conclude that,

$$\bigcup_{i\in I} {U_i}^c = \left(\bigcap_{i\in I} U_i\right)^c .$$

Thus $\cup_{i\in I} {U_i}^c$ is open. □

b) Show that

$$\mathcal{T}_2 = \{U \subseteq X : U = \emptyset \text{ or } U^c \text{ is countable}\}$$

is a topology on X, called the countable complement topology.

Solution:

Proof. Note that by definition $\emptyset \in \mathcal{T}_2$. Since $X^c = \emptyset$ is a countable set we get that $X \in \mathcal{T}_2$. Let $\{U_i\}_I$ be a set of open subsets of X. By definition we know that $\{{U_i}^c\}_I$ are all countable and therefore their intersection $\cap_{i\in I} {U_i}^c$ is also countable. By De Morgan's Law we conclude that,

$$\bigcap_{i\in I} {U_i}^c = \left(\bigcup_{i\in I} U_i\right)^c .$$

Thus $\cup_{i\in I} U_i$ is open. Let $\{U_i\}_I$ be a finite set of open subsets of X. Note that $\cup_{i\in I} {U_i}^c$ is a finite union of countable sets which is also countable. Thus $\cup_{i\in I} {U_i}^c$ and by De Morgans law

$$\bigcup_{i\in I} {U_i}^c = \left(\bigcap_{i\in I} U_i\right)^c .$$

Thus $\cup_{i\in I} {U_i}^c$ is open. □

c) Let p be a fixed point in X, show that

$$\mathcal{T}_3 = \{U \subseteq X : U = \emptyset \text{ or } p \in U\}$$

is a topology on X, called the particular point topology.

Solution:

Proof. Note that by definition $\emptyset \in \mathcal{T}_3$. Since $p \in X$ we get that $X \in \mathcal{T}_3$. Let $\{U_i\}_I$ be a set of open subsets of X. By definition we know that each set in $\{{U_i}^c\}_I$ must be nonempty and exclude p and therefore either $\cap_{i\in I} {U_i}^c$ is empty or nonempty and excluding p. By De Morgans law we know that $(\cup_{i\in I} U_i)^c$ also excludes p and thus $p \in \cup_{i\in I} U_i$, an open set.

Let $\{U_i\}_I$ be a finite set of open subsets of X. Note that $\cup_{i\in I} {U_i}^c$ again is either empty or nonempty and excluding p. By De Morgans law we know that $\cup_{i\in I} {U_i}^c = (\cap_{i\in I} U_i)^c$ and therefore $p \in \cap_{i\in I} U_i$ an open set. □

d) Let p be a fixed point in X, show that

$$\mathcal{T}_4 = \{U \subseteq X : U = X \text{ or } p \notin U\}$$

is a topology on X, called the excluded point topology.

Solution:

Proof. Note that by definition $X \in \mathcal{T}_4$. Since $p \notin \emptyset$ we get that $\emptyset \in \mathcal{T}_4$. Let $\{U_i\}_I$ be a set of open subsets of X. By definition each set in $\{{U_i}^c\}_I$ must be nonempty and must include p, therefore either $\cap_{i\in I}{U_i}^c$ is empty, or nonempty and including p. By De Morgans law we know that $(\cup_{i\in I}U_i)^c$ also includes p and thus $p \notin \cup_{i\in I}U_i$, an open set.

Let $\{U_i\}_I$ be a finite set of open subsets of X. Again each $\{{U_i}^c\}_I$ must be nonempty and must include p, therefore either $\cup_{i\in I}{U_i}^c$ is empty, or nonempty and including p. By De Morgans law we know that $(\cap_{i\in I}U_i)^c$ also includes p and thus $p \notin \cap_{i\in I}U_i$, an open set. □

e) Determine whether,

$$\mathcal{T}_5 = \{U \subseteq X : U = X \text{ or } U^c \text{ is infinite}\}$$

is a topology on X.

Solution:
For a counter example suppose $\mathcal{T}_5$ is a topology on $\mathbb{R}$. By definition the intervals $(-\infty, 0)$ and $(0, \infty)$ are open and therefore $A = (-\infty, 0) \cap (0, \infty)$ is open, yet $A \neq X$ and $A^c = 0$ is a finite set.

5. Let X be a set, and suppose $\{\mathcal{T}_\alpha\}_{\alpha\in A}$ is a collection of topologies on X. Show that the intersection $\mathcal{T} = \cap_{\alpha\in A}\mathcal{T}_\alpha$ is a topology on X.

Solution:

Proof. Let X be a set, and suppose $\{\mathcal{T}_\alpha\}_{\alpha\in A}$ is a collection of topologies on X. Note that for all $\alpha \in A$, $X, \emptyset \in \mathcal{T}_\alpha$ since each $\mathcal{T}_\alpha$ is a topology on X. So it follows $X, \emptyset \in \mathcal{T}$.

Now let $\{U_i\}_I$ be a set of subsets from $\mathcal{T}$. Note that by definition, for all $\alpha \in A$, $\{U_i\}_I \subseteq \mathcal{T}_\alpha$. Since $\mathcal{T}_\alpha$ are topologies on X it follows that $\cup_{i\in I}U_i \in \mathcal{T}_\alpha$. Thus $\cup_{i\in I}U_i \in \mathcal{T}$.

Now let $\{U_i\}_I$ be a finite set of subsets from $\mathcal{T}$. Note that by definition, for all $\alpha \in A$, $\{U_i\}_I \subseteq \mathcal{T}_\alpha$. Since $\mathcal{T}_\alpha$ are topologies on X it follows that $\cap_{i\in I}U_i \in \mathcal{T}_\alpha$. Thus $\cap_{i\in I}U_i \in \mathcal{T}$. □