1. A map f between spaces is *proper* if the pre-image of any compact set is compact. Let X and Y be locally compact Hausdorff spaces. Show that a continuous map $f : X \to Y$ extends to a continuous map $f^* : X^* \to Y^*$ between the 1-point compactifications if and only if it is proper.

Proof. ($\Rightarrow$) Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces, and suppose that it extends to a continuous map $f^* : X^* \to Y^*$ between the 1-point compactifications. Let $U \subseteq Y$ be a compact set. By the definition of the 1-point compactification we know that U^c is open in Y^*. Since f^* is continuous we find that $f^{*-1}(U^c) = f^{*-1}(U)^c$ is open in X^* and therefore $f^{*-1}(U)^c$ is either open in X or $f^{*-1}(U)$ is compact in X. Since $U \subseteq Y$ and since f^* is an extension of f it must be the case that $f^{*-1}(U^c) \subseteq X^*$ and hence $f^{*-1}(U)$ is compact in X. Since $U \subseteq Y$ we know that $f^{*-1}(U) = f^{-1}(U)$ is compact in X. $\square$

Proof. ($\Leftarrow$) Let $f : X \to Y$ be a continuous and proper map between locally compact Hausdorff spaces. Consider an extension of f, $f^* : X^* \to Y^*$ which takes $\infty_x \to \infty_y$. We will proceed to show that this map is continuous. Let $U \subseteq Y^*$ be open. By definition of the one-point compactification topology, U is either open in Y, in which case it follows that $f^{*-1}(U)$ is open in X^* since f is continuous, and f^* is an extension. Let U^c be compact in Y, since f is proper we know that $f^{-1}(U^c)$ is compact in X. Therefore it follows that $f^{-1}(U^c)^c$ is open in X^*. Note that since f^* is an extension of f and $U^c \subseteq Y$ it follows that $f^{-1}(U^c)^c = f^{*-1}(U^c)^c = f^{*-1}(U)$, hence f^* is a continuous map. $\square$

2. The Möbius band is the quotient of $[0, 1] \times \mathbb{R}$ where $(0, y) \sim (1, -y)$.

(a) Show that the Möbius band is a 2-manifold.

Proof. Let M be the quotient of $[0, 1] \times \mathbb{R}$ where $(0, y) \sim (1, -y)$. First we will shoe that M is locally euclidean dimension-2. $\square$

(b) Show that the Möbius band is homotopy equivalent to a circle.

(c) No rigor please, just a picture or two: what familiar space is the 1-point compactification of the Möbius band?

3. Let G be a group acting by homeomorphism on a topological space X, and let $O \subseteq X \times X$ be the subset defined by,

$$O = \{(x_1, x_2) : x_1 = g \cdot x_2 \quad \text{for some} \quad g \in G\}$$

it is called the orbit relation since $(x_1, x_2) \in O$ if and only if x_1 and x_2 are in the same orbit.

1. **(a)** Conclude that X/G is Hausdorff if and only if O is closed in $X \times X$.

Proof. ($\Rightarrow$) Suppose X/G is Hausdorff. We will proceed to show that O is closed by showing O^c is open. Consider $(x_1, x_2) \notin O$, so clearly $q(x_1) \neq q(x_2)$ and since X/G is Hausdorff there exists disjoint $q(x_1) \in U$ and $q(x_2) \in V$. Since q is a quotient map, by definition $q^{-1}(U)$ and $q^{-1}(V)$ are open in X with the property that $(x_1, x_2) \in q^{-1}(U) \times q^{-1}(V) \subseteq O^c$ $\square$

Proof. ($\Leftarrow$) Suppose O is closed in $X \times X$ and let $y_1, y_2 \in X/G$ be distinct. Since q is surjective there exists $q(x_1) = y_1$ and $q(x_2) = y_2$. Since y_1 and y_2 are distinct by definition $(x_1, x_2) \in O^c$. Since O^c is open, there exists a basic open set such that $(x_1, x_2) \in U \times V \subseteq O^c$. In a previous homework we showed that q is an open map, and therefore there exists open sets $y_1 \in q(U)$ and $y_2 \in q(V)$ and since $U \times V \subseteq O^c$ these sets are disjoint in X/G □

2. **(b)** Show that $\mathbb{RP}^n$ is Hausdorff.

Proof. Recall that by definition $\mathbb{RP}^n$ is the set of all lines through the origin in $\mathbb{R}^{n+1}$. Consider the equivalence relation on $R^{n+1,*}$ define by $x \sim y$ if $y = \lambda x$ for some $\lambda \in \mathbb{R}^*$. Then it is clear that $\mathbb{RP}^n$ is homeomorphic to $R^{n+1,*}/\sim$, this relation however can be described as R^* under multiplication acting by homeomorphism on $R^{n+1,*}$ so we get the following, $\mathbb{RP}^n \sim R^{n+1,*}/R^*$. By the previous result it is now be sufficient to show that O, the orbit relation of R^* acting on $R^{n+1,*}$ is closed in $R^{n+1,*} \times R^{n+1,*}$. Let $\langle (x_1, x_2)_\alpha \rangle_{\alpha \in A}$ be a convergent net whose terms are contained in O, with limit (a, b). For this net to converge in the product it must be the case that $\langle x_{1\alpha} \rangle_{a \in A} \to a$ and $\langle x_{2\alpha} \rangle_{a \in A} \to b$ in $R^{n+1,*}$. However clearly since $\langle (x_1, x_2)_\alpha \rangle_{\alpha \in A}$ is contained in O we know that $\langle x_{2\alpha} \rangle_{a \in A} = \langle \lambda x_{1\alpha} \rangle_{a \in A}$. Therefore it follows that $\langle x_{2\alpha} \rangle_{a \in A} = \lambda \langle x_{1\alpha} \rangle_{a \in A}$ which converges to λa. Since $R^{n+1,*}$ is Hausdorff we know convergent nets have unique limits and therefore $b = \lambda a$ and thus by definition $(a, b) \in O$. We conclude that O is closed in $R^{n+1,*} \times R^{n+1,*}$. □

4. Consider the metric on $\mathbb{R}$ given by $\overline{d}(x, y) = min(|x - y|, 1)$. For $z, w \in \mathbb{R}^\omega$,define

$$d(z, w) = \sum_{k=1}^{\infty} 2^{-k} \overline{d}(z_k, w_k).$$

it can be easily show that d is a metric. Prove that the topology this metric induces on $\mathbb{R}^\omega$ is the product topology.

Proof. Consider the following topological spaces $(\mathbb{R}^\omega)_d$ with topology induced by the metric d and $(\mathbb{R}^\omega)_p$, with the product topology. Consider an open ball $\mathcal{B}_r(x) \subseteq (\mathbb{R}^\omega)_d$. We wish to show that this open ball is also open in $(\mathbb{R}^\omega)_p$. Let $g \in \mathcal{B}_r(x)$ and note that

$$d(g, x) = \sum_{k=1}^{\infty} 2^{-k} \min(|g_k - x_k|, 1) < r$$

□

5.

a Show that the upper half sphere $S^{n,+}$ with antipodal points on $\delta S^{n,+}$ identified is homeomorphic to $\mathbb{RP}^n$.

Proof. Note that points on the open upper half sphere $S^{n,+}$ are uniquely identified by lines through the origin. By definition, antipodal points on $\delta S^{n,+}$ are identified by a single line. We call this map $f : S^{n,+} \to \mathbb{RP}^n$ We wish to show is that this map f, descends to the quotient $S^{n,+}/\sim$ as a bijection where $x \sim y$ if they are antipodal points in $\delta S^{n,+}$ Let $x, y \in S^{n,+}$ such that $q(x) = q(y)$, so by definition $\sim$ we know that $x, y \in \delta S^{n,+}$ and x and y

are antipodal. Therefore x, y are colinear with the origin and by definition f, we know that $f(x) = f(y)$. So f descends to the quotient as a continuous function $\hat{f} : S^{n,+}/\sim \to \mathbb{RP}^n$.

Now note that the upper half sphere is a closed and bounded subset of $\mathbb{R}^3$, hence by Heine-Borel it is also compact. Since quotient maps are surjective it follows that $S^{n,+}/\sim$ is compact. By the previous problem we know that $\mathbb{RP}^n$ is Hausdorff. Therefore by the Closed Map Theorem it is sufficient to show that $\hat{f}$ is a bijection to get a homeomorphism from $S^{n,+}/\sim$ and $\mathbb{RP}^n$.

This map is very clearly a bijection since f is a bijection when restricted to points other than the boundary, and it makes the same identifications as q when on the boundary. □

b Consider $X = \{(xy, yz, zx, x^2, y^2, z^2) \in \mathbb{R}^6 : x^2 + y^2 + z^2 = 1\}$. Prove that this set is homeomorphic to $\mathbb{RP}^n$.

Proof. □

6. Show that for each continuous map $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$, there is a 2×2 integer matrix $D(\varphi)$, with the following properties:

(a) Two continuous maps φ and ψ are homotopic if and only if $D(\varphi) = D(\psi)$.

Solution:
Note that $\mathbb{T}^2 \cong S^1 \times S^1$. Therefore we will define the map ι_i as an embedding of the product of S^1 and $\{1\}$ which takes S^1 into i^{th} factor of $S^1 \times S^1$. Note that the choice of factor $\{1\}$ is arbitrary as the result is still homeomorphic to S^1 so we consider ι_i as a map from S^1. By composing with the factor embeddings ι_i and projections π_i we get maps through φ and ψ which map $S^1 \to S^1$. Define them as follows,

$$\varphi_{i,j} = \pi_j \circ \varphi \circ \iota_i$$
$$\psi_{i,j} = \pi_j \circ \psi \circ \iota_i$$

These maps are continuous as they are compositions of continuous functions, and therefore we can define the entries of $D(f)$ by the $D(f)_{i,j} = \deg(f_{i,j})$.

With this definition to show is that to prove ($\Rightarrow$) all that is left to show is that $\varphi_{i,j}$ is homotopic to $\psi_{i,j}$, since that would imply $\deg(\varphi_{i,j}) = \deg(\psi_{i,j})$.

Proof. ($\Rightarrow$) Suppose φ and ψ are homotopic, then there exists a homotopy $H : I \times (S^1 \times S^1) \to S^1 \times S^1$ with the property that $H(0, (x, y)) = \psi(x, y)$ and $H(1, (x, y)) = \varphi(x, y)$. Consider the following $\tilde{H}_{i,j} : I \times S^1 \to S^1$ defined by $\tilde{H}(t, z) = \pi_j(H(t, \iota_i(z)))$. Note this map is continuous since it is again a composition of continuous functions and it has the desired property that $\tilde{H}_{i,j}(0, z) = \pi_j(H(0, \iota_i(z))) = \pi_j(\psi(\iota_i(z))) = \psi_{i,j}$ and similarly $\tilde{H}_{i,j}(1, z) = \varphi_{i,j}$. Therefore we conclude that $\varphi_{i,j}$ are homotopic $\psi_{i,j}$. □

Proof. ($\Leftarrow$) Suppose $D(\varphi) = D(\psi)$, where D is defined

□

(b) $D(\psi \circ \varphi)$ is equal to the matrix product $D(\psi)D(\varphi)$.

Proof. □

7. For each of the following spaces give a presentation of the fundamental group together with a specific loop representing each generator.

(a) A closed disk with two interior points removed.

(b) The projective plane with two points removed.

(c) a connected sum of n tori with one point removed.

(d) a connected sum of n tori with two points removed.

8. Let S be a set, let R and R' be subsets of the free group $F(S)$, and let $\pi : F(S) \to \langle S|R \rangle$ be the projection onto the quotient group. Prove that $\langle S|R \cup R' \rangle$ is a presentation of the quotient group $\langle S|R \rangle / \overline{\pi(R')}$.

Proof. □

9. Let $\pi : X \to Y$ be a quotient map. Suppose Y is connected and each fiber $\pi^{-1}(y)$ is connected. Show that X is connected.

Proof. Let $\pi : X \to Y$ be a quotient map, and let Y be connected with each fiber $\pi^{-1}(y)$ also connected. Suppose to the contrary that X is not connected. By definition there exists sets U and V such that that $U \cap V = \emptyset$ and $U \cup V = X$. Since quotient maps are surjective, if $\pi(U) \cap \pi(V) = \emptyset$ it follows that Y is not connected. Suppose $\pi(U) \cap \pi(V) \neq \emptyset$, then there exists some $y \in \pi(U) \cap \pi(V)$. Note that since $y \in \pi(U) \cap \pi(V)$, the fiber $\pi^{-1}(y)$ must have elements in both U and V, we find that $(\pi^{-1}(y) \cap U) \cup (\pi^{-1}(y) \cap V) = \emptyset$ so $\pi^{-1}(y)$ is not connected.

□

10. Recall that a set $A \subset X$ is a retract of X if there is a continuous $f : X \to A$ such that $f(a) = a$ for all $a \in A$.

(a) Show that if X is Hausdorff and A is a retract of X then A is closed.

Proof. Suppose X is Hausdorff and A is a retract of X. By definition there exists a continuous function $r : X \to A$ such that $r \circ \iota_A = Id_A$ where ι_A is the inclusion map.

We will proceed to show that A is closed by showing that for every convergent net $\langle x_\alpha \rangle \to x$ where $\langle x_\alpha \rangle \subseteq A$ then $x \in A$. Since r is continuous we know that $\langle r(x_\alpha) \rangle \to r(x)$. Since $r \circ \iota_A = Id_A$ it follows that $\langle r(x_\alpha) \rangle = \langle x_\alpha \rangle$ and by definition we know that $r(x) \in A$. Recall that convergent nets in Hausdorff spaces converge to a single limit, and therefore $r(x) = x$, and therefore $x \in A$. Hence A is closed. □

(b) Let A be a two point subset of $\mathbb{R}^2$. Show that it is not a retract of $\mathbb{R}^2$

Proof. Let A be a two point subset of $\mathbb{R}^2$ and suppose to the contrary that A is a retract of $\mathbb{R}^2$. By definition there exists a continuous function $f : \mathbb{R}^2 \to A$ such that $f(a) = a$ for all $a \in A$. However for f to be a continuous function from $\mathbb{R}^2$ into a discrete space like A it must be a constant map, but clearly it isn't since $f(a_1) = a_1 \neq a_2 = f(a_2)$, a contradiction. □

(c) Show that the closed ball $\mathbb{B}^2$ is a retract of $\mathbb{R}^2$.

Proof. Consider the function $f : \mathbb{R}^2 \to \mathbb{B}^2$ defined by,

$$f(x) = \begin{cases} x & ||x|| \leq 1, \\ \dfrac{x}{|||x||} & ||x|| > 1. \end{cases}$$

Clearly this function has the desired property that $f(\mathbb{B}^2) = \mathbb{B}^2$. We can quickly see that it's continuous as $f_1 : \mathbb{B}^2 \to \mathbb{B}^2$ defined by $f_1(x) = x$ and $f_2 : \text{int}(\mathbb{B}^2)^2 \to \mathbb{B}^2$ defined by $f_2(x) = \frac{x}{||x||}$ are continuous function whose domains form a closed cover which partitions $\mathbb{R}^2$ which agree on $\partial\mathbb{B}^2$. So by glueing lemma f is continuous. □

(d) Show that S^1 is not a retract of $\mathbb{R}^2$.

Proof. Suppose to the contrary that s^1 is a retract of $\mathbb{R}^2$. The there exists a continuous map $f : \mathbb{R}^2 \to S^1$ such that $f(x) = x$ for all $x \in S^1$. Also consider $\iota : S^1 \to \mathbb{R}^2$ the embedding map. Since f is a retraction when we compose $f \circ \iota = Id_{S^1}$. Let $p \in S^1$ and note that since these maps are continuous and the idenity on S^1, they descend to maps on the fundamental groups, $f^* : \pi_1(\mathbb{R}^2, p) \to \pi_1(S^1, p)$ and $\iota^* : \pi_1(S^2, p) \to \pi_1(\mathbb{R}^2, p)$. Note that $f^* \circ \iota^* = Id^*_{S^1}$ Since $Id^*_{S^1}$ is a bijection we know that f^* must be an surjection and ι^* must be an injection. However since $\mathbb{R}^2$ is contractable we know that $\pi_1(\mathbb{R}^2, p)$ is trivial and since $\pi_1(S^1, p) \cong \mathbb{Z}$ we know that f be a surjection must be a contradiction. □