1. Suppose $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y respectively. Show that $\mathcal{B} = \{U \times V : U \in B_X, V \in B_Y\}$ is a basis for $X \times Y$.

Proof. Suppose $X \times Y$ has the product topology and let $\mathcal{B} = \{U \times V : U \in B_X, V \in B_Y\}$. Let $O \subseteq X \times Y$ be an open subset with $(p, q) \in O$. Since $X \times Y$ has the product topology, there exists a basic open set $\pi_x^{-1}(\hat{U}) \cap \pi_y^{-1}(\hat{V})$ with $\hat{U}$ is open in X and $\hat{V}$ is open in Y such that $(p, q) \in \pi_x^{-1}(\hat{U}) \cap \pi_y^{-1}(\hat{V}) \subseteq O$. Note that since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y respectively we know that there exists $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ such that $p \in U \subseteq \hat{U}$ and $q \in V \subseteq \hat{V}$. Finally note that $(p, q) \in U \times V \subseteq \pi_x^{-1}(\hat{U}) \cap \pi_y^{-1}(\hat{V}) \subseteq O$.

□

2. Suppose $A \subset X$ and $B \subset Y$. Use the fact that the Characteristic Property of the Product Topology is characteristic to show that the subspace topology on $A \times B$ is the the same as its topology as a product of subspaces.

Proof. Suppose $A \subset X$ and $B \subset Y$ and let τ_s be the topology induced on $A \times B$ by $X \times Y$. By the characteristic property of the subspace topology we know that for any Z, a function f is continuous if and only if it's composite maps are continuous, which leads to the following commutative diagrams.

$$\begin{tikzcd} & X \\ Z \arrow[ur, "\iota_A \circ f"] \arrow[r, "f"'] & A \arrow[u, hook, "\iota_A"'] \end{tikzcd}$$

(1)

$$\begin{tikzcd} & Y \\ Z \arrow[ur, "\iota_B \circ f"] \arrow[r, "f"'] & B \arrow[u, hook, "\iota_B"'] \end{tikzcd}$$

(2)

$$\begin{tikzcd} & X \times Y \\ Z \arrow[ur, "\iota_{A \times B} \circ f"] \arrow[r, "f"'] & (A \times B)_s \arrow[u, hook, "\iota_{A \times B}"'] \end{tikzcd}$$

(3)

Note that $(A \times B)_s$ in (3) indicates open under τ_s and ι_I denotes the inclusion map of I into the ambient space. Since $X \times Y$ has the product topology, by the Characteristic Property of the Product Topology we know that for any Z, a function f is continuous into $X \times Y$ if and only if it is continuous into it's components, which yields the following commutative diagram,

$$\begin{tikzcd} & X \times Y \arrow[d, "\pi_I"] \\ Z \arrow[ur, "f"] \arrow[r, "f_I"'] & I \end{tikzcd} \qquad I \in \{X, Y\}$$

(4)

Note that π_I is the canonical projection. We will proceed to show that τ_s is equivalent to the product topology by demonstrating that τ_s satisfies the CPPT. Consider a space Z and suppose a function $f : Z \to (A \times B)_s$ is continuous. By (3) it follows that $\iota_{A\times B} \circ f$ is continuous into $X \times Y$. By (4) it follows that each of $\pi_X \circ \iota_{A\times B} \circ f$ and $\pi_Y \circ \iota_{A\times B} \circ f$ are continuous into X and Y respectively. Finally by (1) and (2) we know that the restrictions $(\pi_X \circ \iota_{A\times B} \circ f)|_A$ and $(\pi_Y \circ \iota_{A\times B} \circ f)|_B$ are continuous into A and B respectively.

Now suppose the component maps $f_A : Z \to A$ and $f_B : Z \to B$ are continuous. By (1) and (2) we know that $\iota_A \circ f_A$ and $\iota_B \circ f_B$ are continuous into X and Y respectively. Since these are simply component maps into X and Y, by (4) we know that f is continuous from Z into $X \times Y$, and finally by (3) we know that $f|_{(A\times B)}$ is continuous into $(A \times B)_s$.

Therefore τ_s satisfies the Characteristic Property of the Product Topology.

□

3. Show that $(X_1 \times X_2) \times X_3$ is homeomorphic to $X_1 \times X_2 \times X_3$. You may not use the words "open" or "closed" at any point in your proof. (*Hint:* Use the Characteristic Property, Luke!)

Proof. Let $f : (X_1 \times X_2) \times X_3 \to X_1 \times X_2 \times X_3$ be defined by $f(((x, y), z)) = (x, y, z)$. This function is a bijection. To show that $(X_1 \times X_2) \times X_3$ is homeomorphic to $X_1 \times X_2 \times X_3$ we must prove that f and f^{-1} are continuous. Let π_i, π_i' and π_i'' denote the canonical projections from $X_1 \times X_2 \times X_3$, $(X_1 \times X_2) \times X_3$ and $X_1 \times X_2$ respectively. First, we will show that f is continuous via the Characteristic Property of the Product Topology. Note that $\pi''_{X_1} \circ \pi'_{X_1\times X_2} = \pi_1 \circ f$, $\pi''_{X_2} \circ \pi'_{X_1\times X_2} = \pi_2 \circ f$, and $\pi'_{X_3} = \pi_{x_3} \circ f$ are all continuous, and thus by CPPT it follows that f is continuous.

Now we will show that f^{-1} is continuous. Note that $\pi'_{X_1\times X_2} \circ f^{-1}$ is a map from $X_1 \times X_2 \times X_3$ to $X_1 \times X_2$ whose component maps are π_1 and π_2 so by CPPT $\pi'_{X_1\times X_2} \circ f^{-1}$ is continuous. Also note that since $\pi_3 = \pi'_{X_3} \circ f^{-1}$ is continuous it follows by CPPT that f^{-1} is continuous.

□

4. Prove the following.

a) A projection map from an arbitrary product space is an open map.

Proof. Let $X^* = \prod_{\alpha\in A} X_\alpha$ and consider a projection map $\pi_i : X^* \to X_i$. Note that X^* has the product topology, and therefore a basic open set U is of the form $U = \bigcap_{j\in J} \pi_j^{-1}(U_j)$ where $J \subset A$ is a countable index set and U_j is open in X_j. Note that if $i \in J$ then $\pi_i(U) = U_i$ an open set, otherwise $\pi_i(U) = X_i$. In any case the basic open sets of X^* map to open sets in X_i, hence π_i is an open map. □

b) An arbitrary product of Hausdorff spaces is Hausdorff

Proof. Let $X^* = \prod_{\alpha\in A} X_\alpha$ such that each X_α is Hausdorff. Suppose $p, q \in X^*$ such that $p \neq q$. Since $p \neq q$ there exists some i such that $\pi_i(p) \neq \pi_i(q)$ where $\pi_i(p), \pi_i(q) \in X_i$. Since X_i is Hausdorff and $\pi_i(p) \neq \pi_i(q)$ there exist U_p, U_q open in X_i such that $\pi_i(p) \in U_p$, $\pi_i(q) \in U_q$ and $U_p \cap U_q = \emptyset$. Consider the

$\pi_i^{-1}(U_p), \pi_i^{-1}(U_q) \in X^*$ which are open by definition of the product topology. Note that $p \in \pi_i^{-1}(U_p)$ and $q \in \pi_i^{-1}(U_q)$ and clearly $\pi_i^{-1}(U_p) \cap \pi_i^{-1}(U_q) = \emptyset$ since $U_p \cap U_q = \emptyset$. □

Lemma 1: The set of all finite subsets of $\mathbb{N}$ is countable.

Proof. Let $X = \{A \subseteq \mathbb{N} : \text{A is finite}\}$ and consider $X_n = \{A \subseteq N : max(X) = n\}$. Note that $X_n = \mathcal{P}(\{1, 2, \dots, n\})$ so $|X_n| = 2^n$. Finally we see that $X = \cup_{i \in \mathbb{N}} X_n$ and therefore X must be countable because it is a countable union of finite sets. □

c) A countable product of second countable spaces is second countable.

Proof. Let $X^* = \prod_{\alpha \in A} X_\alpha$ such that each X_α is second countable. By definition of the product topology the following set is a basis for X^*,

$$\mathcal{B}^* = \left\{ \bigcap_{j \in J} \pi_j^{-1}(U_j) : \begin{array}{c} J \subseteq A \text{ is finite,} \\ U_j \subseteq X_j \text{ is open} \end{array} \right\}.$$

Since each X_j is second countable, for each X_j there exists a countable basis $\mathcal{B}_j$. Now consider the set of open sets in X^*,

$$\mathcal{B}^{**} = \left\{ \bigcap_{j \in J} \pi_j^{-1}(B_j) : \begin{array}{c} J \subseteq A \text{ is finite,} \\ B_j \in \mathcal{B}_j \end{array} \right\}.$$

Note that for each $p \in B^*$ where $B^* \in \mathcal{B}^*$ there exists a $B^{**} \in \mathcal{B}^{**}$ with $p \in B^{**} \subseteq B^*$ since for all $j \in J$ we know that $\pi_j^{-1}(p) \in \pi_j^{-1}(B_j) \subseteq \pi_j^{-1}(U_j)$. Thus $\mathcal{B}^{**}$ is a basis for X^*.

Counting $\mathcal{B}^{**}$ we find that by using Lemma 1, our choice of indexing set J has countably many options and recall that there are countably many options for each choice of $B_j \in \mathcal{B}_j$. So it follows that $|\mathcal{B}^{**}| = |\mathbb{N} \times \mathbb{N}|$ which is countable. Hence X^* is second countable.

□

5. Problem 3-8 Let X denote the cartesian product of countably infinitely copies of $\mathbb{R}$ endowed with the box topology. Define a map $f : \mathbb{R} \to X$ by $f(x) = (x, x, x, \dots)$. Show that f is not continuous even though each of its component functions is.

Proof. Let $X = \prod_{i=1}^{\infty} \mathbb{R}_i$ with the box topology. Suppose a map $f : \mathbb{R} \to X$ by $f(x) = (x, x, x, \dots)$. Note that under the box topology the following set is open,

$$A = \bigcap_{i=1}^{\infty} \pi_i^{-1}\left(\left(-\frac{1}{i}, \frac{1}{i}\right)\right)$$

Note that $0 \in \left(-\frac{1}{i}, \frac{1}{i}\right)$ for all $i \in \mathbb{N}$, and therefore $(0, 0, 0, \dots) \in A$. Suppose for the sake of contradiction that f is continuous, then $f^{-1}(A)$ is open. Since $(0, 0, 0, \dots) \in A$, we know that $0 \in f^{-1}(A)$ and there exists $\epsilon > 0$ such that $B_\epsilon(0) \subseteq f^{-1}(A)$, so $f(\epsilon/2) \in A$. However there exists an N such that all $n \geq N$,

$$\pi_i^{-1}\left(\frac{\epsilon}{2}\right) \notin \bigcap_{i=n}^{\infty} \pi_i^{-1}\left(\left(-\frac{1}{i}, \frac{1}{i}\right)\right).$$

□

6. Problem 3-9 Let X be as in the preceding problem. Let $X^+ \subseteq X$ be the subset consisting of sequences of strictly positive real numbers, and let z denote the zero sequence, that is, the one whose terms are $x_i = 0$ for all i. Show that z is in the closure of X^+, but there is no sequence of elements of X^+ converging to z. Then use the sequence lemma to conclude that X is not first countable, and thus not metrizable.

Proof. Let $X = \prod_{i=1}^{\infty} \mathbb{R}_i$ and $X^+ = \prod_{i=1}^{\infty} \mathbb{R}_i^+$ with $X^+ \subseteq X$. Note that since X has the box topology, there exists a basic open set B with $z \in B \subseteq U \in \mathcal{V}(z)$ of the form,

$$B = \bigcap_{i=1}^{\infty} \pi_i^{-1}(U_i),$$

where U_i are open in $\mathbb{R}$. To show that z is a contact point of X^+ it is sufficient to show that $B \cap X^+ \neq \emptyset$. Note that since $z \in B$ we know that $0 \in U_i$ and since U_i are open in $\mathbb{R}$, for every U_i there exists an $n_i > 0$ such that the open set $B_{n_i}(0) \cap \mathbb{R}^+ \subseteq U_i$. Thus it must follow that,

$$\bigcap_{i=1}^{\infty} \pi_i^{-1}(B_{n_i}(0) \cap \mathbb{R}^+) \subseteq B \cap X^+.$$

Hence z is a contact point of X^+.

Consider a sequence $\{x_i\}_{i=1}^{\infty} \in X^+$. Note that each x_i is itself a sequence $(x_{(i,1)}, x_{(i,2)}, \dots)$. Now consider the open set $U \in X$ containing z,

$$U = \bigcap_{i=1}^{\infty} \pi_i^{-1}((-x_{(i,i)}, x_{(i,i)})).$$

Finally note that $z \in U \cap X^+$ is open in X^+ and there exists no $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \subseteq U \cap X^+$. If there were such an N, it would follow that $x_N \in U \cap X^+$. However we know $x_{(N,N)} \notin (-x_{(N,N)}, x_{(N,N)})$ which by our construction of U implies that $x_n \notin U$, a contradiction.

Recall Lemma 2.48 which was proved in class (This is the sequence lemma, and we proved it in both direction but we only need one side for now).

Lemma 2.48: Let X be a topological space with $A \subseteq X$. If X is first countable then for all $p \in \overline{A}$ there exists a sequence in A converging to p.

From the contrapositive of Lemma 2.48, since $X^+ \subseteq X$ and $z \in \overline{X^+}$ with the property that there exists no sequences in X^+ which converge to z we can conclude that X is not first countable. Since every metric space is first countable, X is not metrizable.

□

7. Problem 3-13 a Suppose X and Y are topological spaces and $f : X \to Y$ is a continuous map. Prove that if f admits a continuous left inverse, it is a topological embedding.

Proof. Let X and Y are topological spaces and $f : X \to Y$ is a continuous map. Suppose that f admits a continuous left inverse, $g : Y \to X$. To show that f is a topological embedding we must show that it is an injective continuous map that is a homeomorphism into it's image.

Let $p, q \in X$ such that $f(p) = f(q)$. Since g is a well defined function and $f(p), f(q) \in Y$, applying g to both sides we get $p = q$, hence f is injective. Now we will show that the map f' defined as f restricted to it's image, is a homeomorphism. Clearly f' is a continuous bijection, since f is a continuous injection. Note that f'^{-1} is the same map as $g|_{f(x)}$ and since g is a continuous function so is it restriction, so f'^{-1} is continuous. Thus f is a topological embedding. □

8. Exercise 3.61 Prove that a continuous surjective map $q : X \to Y$ is a quotient map if and only if it takes saturated open subsets to open subsets, or saturated closed subsets to closed subsets.

In class we showed that q is a quotient map if and only if q is surjective, continuous, and takes saturated open sets to open sets.

Proof. ($\Rightarrow$) Suppose the continuous surjective map, $q : X \to Y$ is a quotient map. By our proof in class, q takes saturated opens sets to open sets. Let $U \subseteq X$ be a saturated and closed. Since U is saturated there exists a $W \in Y$ such that $U = q^{-1}(W)$ and since q is surjective $q(U) = q(q^{-1}(W)) = W$. Note that $U^c = (q^{-1}(W))^c = q^{-1}(W^c)$ and since U^c is open we know that $q(U^c) = q(q^{-1}(W^c)) = W^c$ is also open, hence W is closed. Therefore q takes saturated closed sets to closed sets. □

Proof. ($\Leftarrow$) Suppose the function $q : X \to Y$ is a continuous surjective map which takes saturated closed sets to closed sets. Let $U \subseteq X$ be saturated and open. Since U is saturated there exists a $W \in Y$ such that $U = q^{-1}(W)$ and since q is surjective $q(U^c) = q(q^{-1}(W)) = W$. Note that $U^c = (q^{-1}(W))^c = q^{-1}(W^c)$ and by our supposition we know that since U^c is closed $q(U^c) = q(q^{-1}(W^c)) = W^c$ is also closed, hence W is open. Therefore q takes saturated open sets to open sets and by our proof in class q is a quotient map. □