Exercise P8: On page 12 of the textbook, equation (2.4) says $(AB)^* = B^*A^*$. Prove this by showing the matrix entries are equal.

Solution:
Suppose that A is an mxn matrix and B is a nxl matrix. Considering the following $(AB)^*$ using the entry definition of matrix-matrix multiplication defined in equation (1.5) we know the following,

$$ab_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}.$$

Taking the adjoint of AB to get AB^* just requires us to swap the indices i, j in the left hand side,

$$(ab)^*_{j,i} = \sum_{k=1}^{n} a_{i,k}b_{k,j}.$$

Now we can consider B^*A^* and apply the same formula,

$$(b)^*(a)^*_{i,j} = \sum_{k=1}^{n} (b)^*_{i,k}(a)^*_{k,j}.$$

Note that the adjoint operation can be described with the following,

$$b_{i,j} = (b)^*_{j,i},$$
$$a_{i,j} = (a)^*_{j,i}.$$

Thus by substitution the following is true,

$$\sum_{k=1}^{n} a_{i,k}b_{k,j} = \sum_{k=1}^{n} (b)^*_{k,j}(a)^*_{i,k}.$$

Therefore $(AB)^* = B^*A^*$.

Exercise P9: On page 21 of the textbook, equation (3.10) gives a formula of the ∞-norm of an mxn matrix. Let a_i denote the i^{th} row of A and prove it,

$$||A||_\infty = \max_{1\leq i\leq m} ||a_i||_1$$

Solution:
Suppose an mxn matrix A. Now consider the set $\{x \in \mathbb{C}^n : \max_{1\leq j\leq n} |x_j|\}$ and note that the set of vectors Ax satisfy,

$$||Ax||_\infty = ||a_i^* x||_\infty \leq \max_{1\leq i\leq n} ||a_i^* x||_\infty.$$

Choosing x such that all the entries have the property that $|x_i| = 1$ we can maximize each inner product $||a_i||x||$. Note that this turns the inner product into the same operation as the one-norm for each a_i. Therefore the product $||A||_\infty$ attains the upper bound and we get the following,

$$||A||_\infty = \max_{1\leq i\leq m} ||a_i||_1.$$

Exercise 2.6: If u and v are m-vectors, the matrix $A = I + uv^*$ is known as the rank-one perturbation of the identity. Show that if A is nonsingular, then it's inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is the *null*(A).

Solution:
Suppose that A is nonsingular and u and v are m-vectors such that $A = I + uv^*$. Consider the following equation for some scalar α,

$$(I + uv^*)(I + \alpha uv^*) = I.$$

Expanding the product,

$$\begin{aligned}(I + uv^*)(I + \alpha uv^*) &= I,\\ \alpha uv^* + uv^* + uv^*\alpha uv^* + I &= I,\\ \alpha uv^* + uv^* + \alpha uv^*uv^* &= 0.\end{aligned}$$

Note that v^*u is a scalar, so the following applies,

$$\begin{aligned}\alpha uv^* + uv^* + \alpha u(v^*u)v^* &= 0,\\ \alpha uv^* + uv^* + \alpha(v^*u)uv^* &= 0,\\ (\alpha + 1 + \alpha(v^*u))uv^* &= 0.\end{aligned}$$

When $uv^* = 0$ we get the trivial case where $A = I$. Solving the other factor for α,

$$\begin{aligned}\alpha + 1 + \alpha(v^*u) &= 0,\\ \alpha + \alpha(v^*u) &= -1,\\ \alpha(1 + (v^*u)) &= -1,\\ \alpha &= \frac{-1}{(1 + (v^*u))}.\end{aligned}$$

Thus A has an inverse of the form $I + \alpha uv^*$ when $\alpha = \frac{-1}{(1+(v^*u))}$ and $v^*u \neq -1$.

To show when A is singular, consider all u and v such that $u^*v = -1$. Implicit in this consideration is the fact that $u \neq 0$ and $v \neq 0$. Evaluating Au we get the following,

$$Au = (I + uv^*)u = Iu + uv^*u = u + (-1)u = 0.$$

Since $u \neq 0$, A must be singular. Furthermore substituting any vector in the *span*(u) we also get 0 so by definition *span*(u) = *Null*(A).

Exercise 3.2: Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{mxm}$. Show that $\rho(A) \leq \|A\|$, where $\rho(A)$ is the spectral radius of A, i.e., the largest absolute value

$|\lambda|$ of an eigenvalue λ of A.

Solution:
By definition $||A||_{mxm}$ is the supremum of the ratios, $||Ax||_m/||x||_m$. Now note that by the definition of the eigenvalue, for some eigenvector v and the corresponding eigenvalue λ we know the following.

$$Av = \lambda v.$$

Taking the norm of both sides we get,

$$||Av||_m = ||\lambda v||_m.$$

Applying the linearity of vector norms, and solving for $|\lambda|$,

$$||Av||_m = |\lambda|||v||_m,$$
$$\frac{||Av||_m}{||v||_m} = |\lambda|.$$

Consider some $\hat{v}$ such that $|\lambda|$ is maximized and we get that,

$$\rho(A) = \max|\lambda| = \frac{||A\hat{v}||_m}{||\hat{v}||_m}.$$

Thus by definition $\rho(A)$ is contained in the set of all ratios $||Ax||_m/||x||_m$, where $||A||_{mxm}$ is the supremum, therefore we know

$$\rho(A) \leq ||A||.$$

Exercise 3.3: Vector and matrix p-norms are related by various inequalities, often involving the dimensions m or n. For each of the following, verify the inequality and give an example of a nonzero vector or matrix (for general m,n) for which the equality is achieved. In this problem x is an m-vector and A is an mxn matrix.

1. $||x||_\infty \leq ||x||_2$

   Solution:
   Consider the definition of the vector ∞-norm,

   $$||x||_\infty = \max_{1\leq i\leq m} |x_i|.$$

   Clearly we can replace the absolute value operator by squaring then square-rooting the max term. Doing so we get the following,

   $$||x||_\infty = \max_{1\leq i\leq m} \sqrt{(x_i)^2}.$$

If we add the square of the remaining x_i terms it must be the case that we produce a sum larger than just the maximum x_i term, thus

$$||x||_\infty = \max_{1\leq i\leq m} \sqrt{(x_i)^2} \leq \sqrt{\sum_{i=1}^{m}(x_i)^2} = ||x||_2.$$

For an example consider the vector $\hat{x} = [2, 1]$ and note that $||\hat{x}||_\infty = 2 \leq \sqrt{5} = ||\hat{x}||_2$

2. $||x||_2 \leq \sqrt{m}||x||_\infty$

Solution:
Consider the definition of the vector 2-norm.

$$||x||_2 = \sqrt{\sum_{i=1}^{m}(x_i)^2}.$$

Replacing every x_i in the sum with the $\max_{1\leq i\leq m} x_i$ we get the following,

$$||x||_2 \leq \sqrt{\sum_{i=1}^{m}(\max_{1\leq i\leq m} x_i)^2} = \sqrt{m(\max_{1\leq i\leq m} x_i)^2} = \sqrt{m}\max_{1\leq i\leq m}|x_i| = \sqrt{m}||x||_\infty.$$

For an example we can once again consider $\hat{x} = [2, 1]$ and note that $||\hat{x}||_2 = \sqrt{5} \leq \sqrt{2} * 2 = \sqrt{m}||\hat{x}||_\infty$.

Exercise 4.3: Write a MATLAB program which, given a real $2x2$ matrix A, plots the right singular vectors v_1, and v_2 in the unit circle and also the left singular vectors u_1 and u_2 in the appropriate ellipse, as in Figure 4.1. Apply your program to the matrix (3.7) and also to the $2x2$ matrices in Exercise 4.1

Solution:
Code:

```
function [v,u, U] = vismat(A)
% This function takes a 2x2 matrix A,
% ans returns the svd visualization.


%%%%%%%% ERROR CHECKING
DimensionCheck = size(A);
    if (DimensionCheck(1) ~= DimensionCheck(2))
        ||(DimensionCheck(1) + DimensionCheck(2) ~= 4)
```

```
        error('A is Unexpected Size')
    end

[U, S, V] = svd(A);
Vstar = V';

%%%%%%% Plotting input space vectors v_1 and v_2
theta = 0:pi/50:2*pi;
xCircle = cos(theta);
yCircle = sin(theta);
plot(xCircle, yCircle, 'Color','red');
hold on
plot([Vstar(1,1) 0], [Vstar(2,1) 0], 'Color','red')
plot([Vstar(1,2), 0], [Vstar(2,2), 0], 'Color','red')


%%%%%%%% Plotting output space vectors sigma_1u_1 and sigma_2u_2
%%%% Computing the matrix rotation
if U(1,1) > 0
    alpha = -acos(dot(U(:,1), [0; 1])) + pi/2;
else
    alpha = pi/2 + acos(dot(U(:,1), [0; 1]));
end

%%%% Parameterizing the unit ellipse formed in the output space.
xEllipse = S(1,1)*cos(theta)*cos(alpha) - S(2,2)*sin(theta)*sin(alpha);
yEllipse = S(1,1)*cos(theta)*sin(alpha) + S(2,2)*sin(theta)*cos(alpha);
plot(xEllipse, yEllipse, 'Color','blue');
hold on
plot([S(1,1)*U(1,1) 0], [S(1,1)*U(2,1) 0], 'Color','blue')
plot([S(2,2)*U(1,2) 0], [S(2,2)*U(2,2) 0], 'Color','blue')


v = Vstar;
u = U*S;
```

Figure 1: Vismat() with matrix from 3.7 [1 2;0 2]

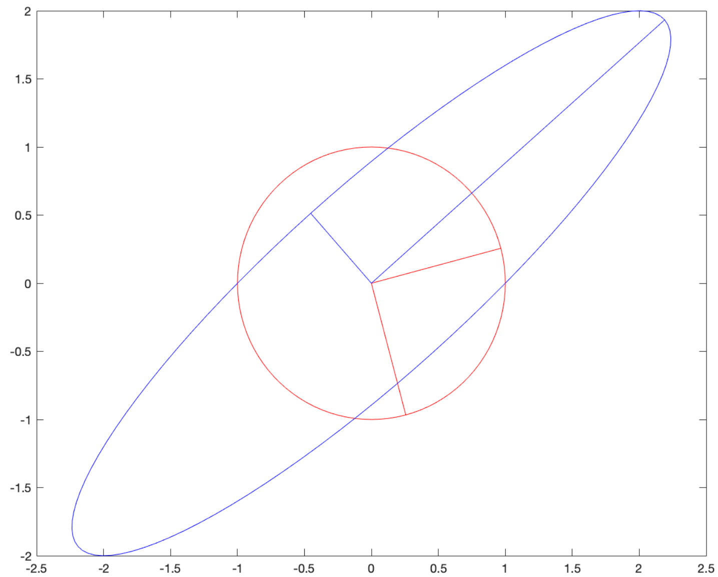

Figure 2: Vismat(A) with matrix from 4.1 [3 0;0 -2]

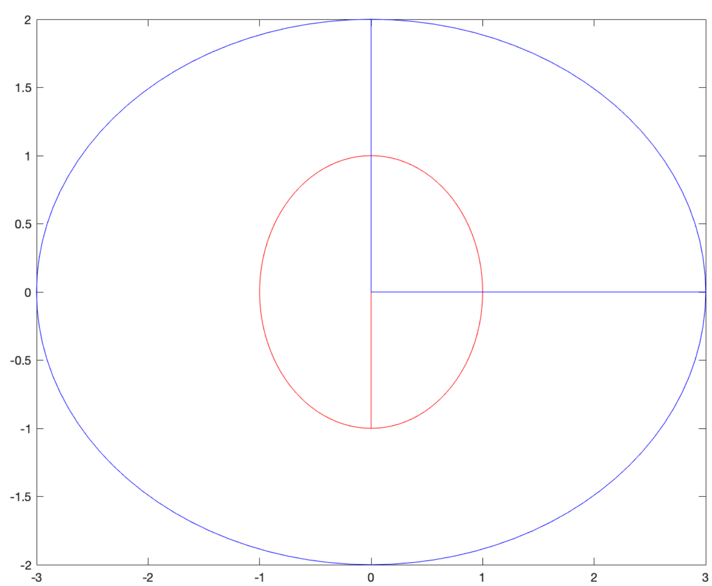

Figure 3: Vismat(A) with matrix from 4.1 [2 0;0 3]

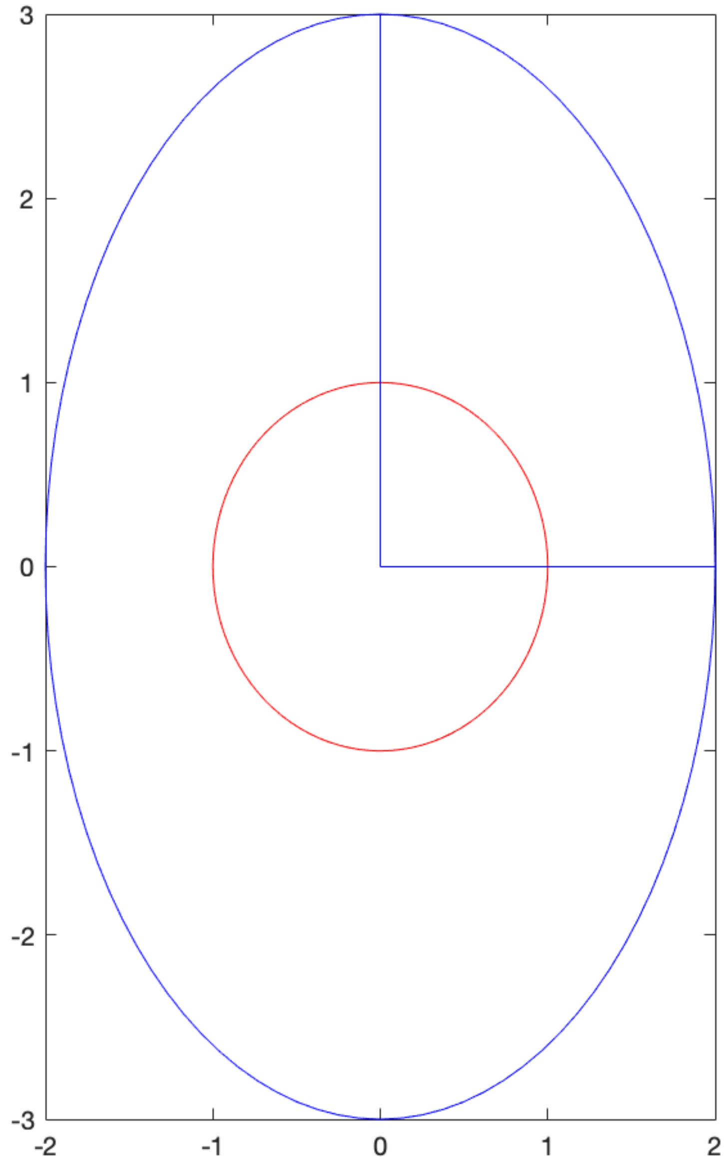

Figure 4: Vismat(A) with matrix from 4.1 [1 1;0 0]

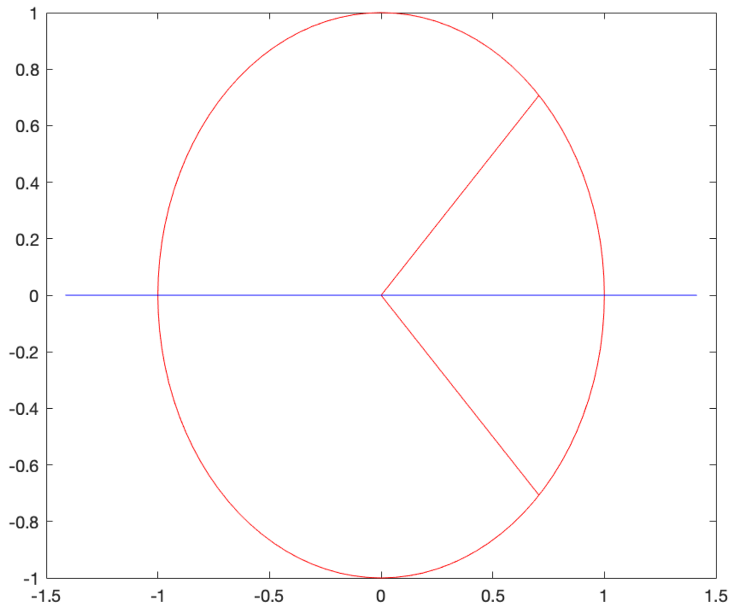

Figure 5: Vismat(A) with matrix from 4.1 [1 1;1 1]