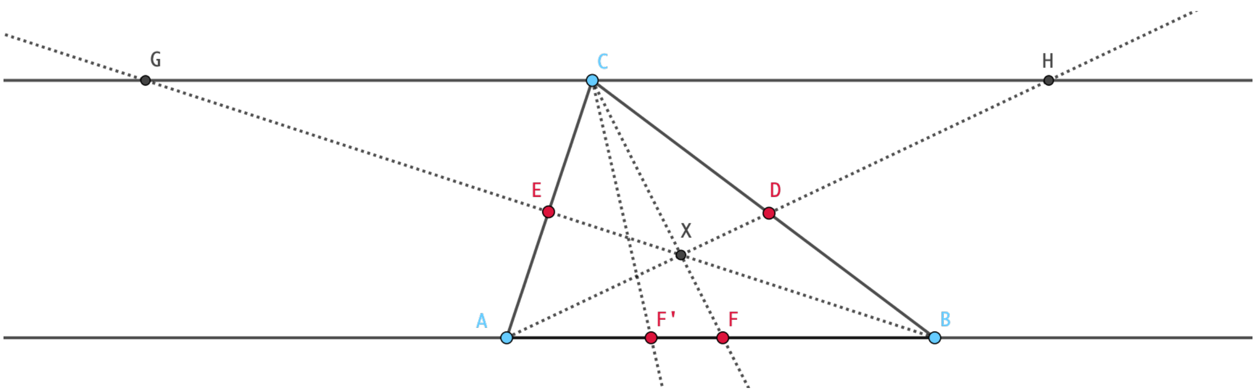
Exercise ($\square$ 4.2.1). Prove the converse of Ceva's Theorem. (HINT: Let $X = AD * BE$ and $F' = CX * AB$ Use Ceva's Theorem to say something about the situation involving F', then use that to sho $F = F'$

Proof. Let $\triangle ABC$ be a triangle, and let D, E, and F be on the lines BC, CA, and AB respectively.

FIGURE 1. Ceva's Converse



Suppose the following,

$$\frac{|AF'|}{|F'B|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1$$

We know that two lines that are not parallel must intersect at a point. Let AD and BE intersect at point X. Now Consider a third line that intersects point X, CF. By Ceva's Theorem we know that

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1$$

Yet we supposed that,

$$\frac{|AF'|}{|F'B|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1$$

Therefore by cancellation,

$$\frac{|AF|}{|FB|} = \frac{|AF'|}{|F'B|}.$$

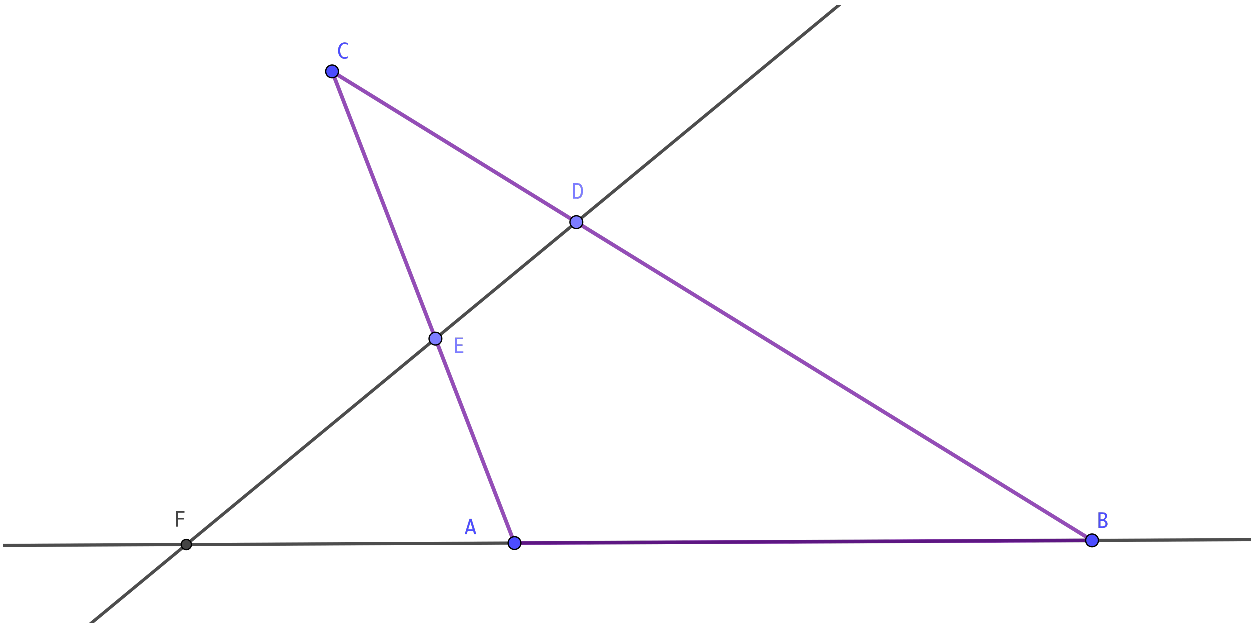
and thus $F' = F$ therefore lines AD, BE, and FC are concurrent.

$\square$

Exercise ($\square$ 4.2.2). Prove Menelaus's theorem, and while doing so, determine the constant that goes in the blank. (Hint: As in the proof of Ceva's theorem, draw a line through C parallel to AB and find similar triangles again. See Figure 4.6.)

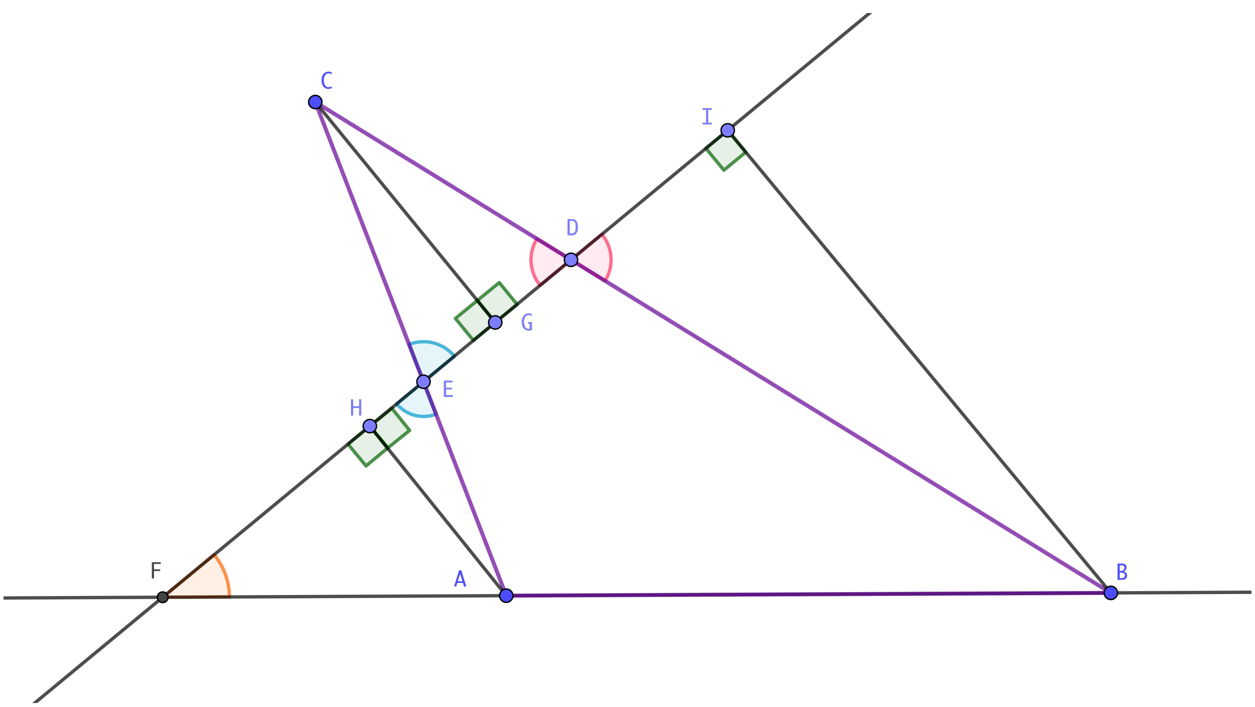
Answer: Let $\triangle ABC$ be a triangle, and let a transversal line intersect sides BC, CA, and AB at D, E, and F respectively such that D, E, and F are distinct from A, B, and C.

FIGURE 2. Setup For Menelaus's Theorem



Consider the triangles formed from perpendicular lines that go from the vertices of the triangle to the traversal.

FIGURE 3. Menelaus's Theorem with Right Triangles



So we can make a few observations about the triangles formed in the scene. Since they are all right triangles we know that $\triangle IBD \sim \triangle CGD$, $\triangle CGE \sim \triangle AHE$, and $\triangle IFB \sim \triangle HFA$, all by AA similarity (congruent angles are denoted by color, $\triangle IFB$ and $\triangle HFA$ share an angle). Then we can make the following statements about the ratio of edges,

$$\frac{BD}{CD} = \frac{BI}{CG}$$

$$\frac{CE}{AE} = \frac{CG}{AH}$$

$$\frac{FA}{FG} = \frac{AH}{BI}$$

Note that,

$$\frac{BI}{CG}\frac{CG}{AH}\frac{AH}{BI} = 1$$

Therefore by substitution,

$$\frac{BD}{CD}\frac{CE}{AE}\frac{FA}{FG} = 1$$

$\square$