Section 2.3:

Exercise 2.3.8: Determine if A is invertible.

$$A = \begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Solution: Matrix A is a 4x4 matrix with 4 pivot positions, therefore by the IMT we know that A must be invertible.

Exercise 2.3.10:
Solution: Determine if A is invertible.

$$A = \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & 5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix}$$

The fastest way to prove that A is an invertible matrix with the given information is to get it into row echelon form and count the number of pivots. Row reducing A we get,

$$: \begin{bmatrix} 9 & 6 & 4 & -9 & 5 \\ 0 & \frac{1}{3} & -\frac{1}{9} & 17 & \frac{46}{9} \\ 0 & 0 & -\frac{5}{3} & 36 & \frac{14}{3} \\ 0 & 0 & 0 & -\frac{2}{5} & -\frac{66}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since there is A is a 5x5 matrix with 5 pivot positions, by the IMT it must be invertible.

Exercise 2.3.12:

(1) If there is an nxn matrix D such that $AD = I$. then there is also an n x n matrix C such that such that $CA = I$

Answer: True. Consider axiom k of the IMT.

(2) If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^N$.(It is previously stated that A is an nxn matrix.)
Answer: True. If axiom e of the IMT then we know that h must also be true.

(3) If the equation $Ax = b$ has at least one solution for each $b. \in \mathbb{R}^n$, then the solution is unique for each b.
Answer: True. If If the equation $Ax = b$ has at least one solution for each $b. \in \mathbb{R}^n$ the using axiom g from the IMT we know that A must be invertible. Theorem 5 from chapter 2.2 states that if A is an invertible nxn matrix then the equation $Ax = b$ has a unique solution.

(4) If the linear transformation $x \to Ax$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then A has n pivots.
Answer: True. We know that in order for a matrix transformation to be onto, the associated matrix must have a pivot in every row. Since our associated matrix A is defined as being an nxn matrix it must also be true it has a pivot in every column, hence the bijection, $\mathbb{R}^n$ into $\mathbb{R}^n$.

(5) If there is a b in $\mathbb{R}^n$ such that the equation $Ax = b$ is inconsistent, the the transformation $x \to Ax$ is not one-to-one.
Answer: True. The only way to have the equation $Ax = b$ be inconsistent is if the associated matrix A has a row composed of zeroes when in echelon form. Since we know that A is defined as being nxn we also know that it must also have a free variable, therefore a non pivot column thus $x \to Ax$ is not one-to-one.

Exercise 2.3.14: An mxn lower triangular matrix is one whose entries above the main diagonal are 0's. When is a square lower triangular matrix invertible? Justify your answer.
Solution: If a lower triangular matrix with all zeroes above the main diagonal then, we know that there exists an inverse. When we take the transpose of A we get a matrix where the diagonal corresponds to pivot positions and since it has a pivot in every row by IMT it is invertible. We also know from Theorem 6 in Chapter 2.2 that $(A^T)^{-1} = (A^{-1})^T$. So A must be invertible.

Section 3.1:

Exercise 3.1.14: Compute the determinant by cofactor expansion. in each step choose the row that requires the least computation.

$$A = \begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$$

Solution: First let's begin by expanding along the fourth row.

$$det(A) = \begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} = -2 * \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ -5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{vmatrix}$$

The we want to expand by the last column because it has the most zeroes.

$$det(A) = -2 * \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ -5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{vmatrix} = -2 * -1 \begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

Then we want to expand by the first column,

$$det(A) = 2 * \begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix} = 2 * (3 * \begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix} + 2 * \begin{vmatrix} 2 & 4 \\ -4 & 1 \end{vmatrix})$$

Then we can easily calculate the determinants of the resulting 2x2 matrices .

$$det(A) = 2(3(-8-3) + 2(2+16)) = 6$$

Exercise 3.1.22: State the row operation and then describe how it affects the determinant.

$$\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 5+3k & 4+2k \end{bmatrix}$$

Solution: The row operation is row replacement, because we are replacing a row with another. We can do some quick algebra to compare the determinants.

$$\begin{aligned} 3*4-5*2 &= 12-10 \\ &= 2 \end{aligned}$$

Then with the other matrix,

$$
\begin{aligned}
3 * (4 + 2k) - 2 * (5 + 3k) &= 12 + 6k - 10 - 6k \\
&= 2
\end{aligned}
$$

Since both determinants are the same, we can conclude that in this case the row operation had no affect on the determinant.

Exercise 3.1.40:

(1) The cofactor expansion of the det(A) down a column is equal to the cofactor expansion along a row.
Answer: True. Consider Theorem 1 of Chapter 3 that states the determinant of an nxn matrix A can be computed by cofactor expansion along any row or down any column.

(2) The determinant of a triangular matrix is the is, of the entries along the diagonal.
Answer: False. The determinant of a triangular matrix is the product of the entries along the main diagonal. By Theorem 2 Chapter 3.

Section 3.2:

Exercise 3.2.28:

(1) If three row interchanges are made in succession then the new determinant equals the old determinant.
Answer: False. After the first row interchange the new determinant will be -det(A), the second det(A) and finally the third, -det(A). So after three row interchanges the new determinant will be -1 times the old determinant.

(2) The determinant of A is the product of the diagonal entries of A
Answer: False. In order for this to be true A must be in row echelon form ie. reduced to triangular form. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ here $det(A) = -3$ and the product along the diagonal is 1 since $1 = -3$ the statement is false.

(3) If the $det(A) = 0$, then two rows or two columns are the same, or a row or column is zero.

Answer: False. Consider the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$. Here we can see that $det(A) = 0$ even though there are no two rows or columns that are the same, and there is no zero column or row.

(4) $detA^{-1} = (-1)detA$.

Answer: False. The $detA^{-1} = 1/detA$. By Theorem 6 of Chapter 3 consider the equation $A^{-1} * A = I$, when we take the determinants of both sides of the equation. which we can by Theorem 6, we get $det(A^{-1}) * det(A) = 1$. So therefore it must be true that the $detA^{-1} = 1/detA$.

Exercise 3.2.34: Let A and P be square matrices, with P invertible. Show that $det(PAP^{-1}) = det(A)$

Solution: Consider $det(PAP^{-1})$ then by Theorem 6 Chapter 3 we can say that

$$det(PAP^{-1}) = det(P) * det(A) * det(P^{-1}).$$

Since determinants are just real numbers we can reorder them,

$$det(PAP^{-1}) = det(P) * det(P^{-1}) * det(A).$$

From the last problem we proved that for any invertible matrix A then the $detA^{-1} = 1/detA$, so we can say the same for $detP^{-1}$,

$$det(PAP^{-1}) = det(P) * \frac{1}{det(P)} * det(A).$$

Thus $det(PAP^{-1}) = det(A)$.

Exercise 3.2.36: Find a formula for $det(rA)$ when A is an nxn matrix

Solution: Suppose A is an nxn matrix. Since we know that $A = I * A$ where I is the nxn identity matrix, then it must follow that.

$$det(A) = det(IA)$$

Furthermore we can surmise, by substitution that,

$$det(rA) = det(rIA)$$

Where r is a scalar constant. Using Theorem 6 from Chapter 3 it is also true that,

$$det(rA) = det(rI) * det(A)$$

by Theorem 2 of Chapter 3 we also know that $det(rI) = r^n$. Thus by substitution,

$$det(rA) = r^n * det(A)$$

Section 3.3:

Exercise 3.3.18: Suppose that all the entries in A are integers and the $det(A) = 1$. Explain why all the entries in A^{-1} are integers.

Solution: We know from Theorem 8 in Chapter 3 that A^{-1} can be calculated by,

$$A^{-1} = \frac{1}{det(A)} * adj(A)$$

given that A is an invertible nxn matrix. From here we can see that all me need to do to prove that all the entries in A^{-1} are integers is to show that all the entries in $adj(A)$ are also integers. Recall from the textbook that the formula for each term in the $adj(A)$ matrix is given by,

$$(-1)^{(i+j)} * det(A_{ji}) = adj(A)_{ji}$$

Where A_{ji} denotes the submatrix of A formed by deleting row j and column i and $adj(A)_{ji}$ refers to the term in the j^{th} row and i^{th} column of $adj(A)$. Since A is composed of all integers it must follow that $det(A_{ji})$ is also an integer. Therefore each term in $adj(A)$ must be an integer. Thus we have shown that all the entries in A^{-1} are integers.

Exercise 3.3.30: Let R be a triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that

$$areaoftriangle = \frac{1}{2} * det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

Solution: First we want to translate the triangle to the origin to get a clear picture of the actual length of each side. To do so we subtract (x_1, y_1) from each vertex. so our new triangle is located on $(0, 0)$, $(x_2 - x_1, y_2 - y_1)$, and $(x_3 - x_1, y_3 - y_1)$. We can also find the area of this triangle by taking the area of the parallelogram formed by the two vectors $a_1 = [x_2 - x_1, y_2 - y_1]$ and $a_2 = [x_3 - x_1, y_3 - y_1]$ and then dividing it by two, so

$$areaoftriangle = \frac{1}{2} * det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$$

Doing some row operations and cofactor expansion to the given matrix,

$$\begin{aligned}
\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} &= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix}, \text{subtracting } R_1 \text{ from } R_2 \text{ and } R_3 \\
&= 1 * \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \text{cofactor expansion along third column} \\
&= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} \text{ equivalent by Theorem 5 Chapter 3}
\end{aligned}$$

Therefore by substitution,

$$areaoftriangle = \frac{1}{2} * det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

Exercise 3.3.32: Let S be the tetrahedron in $\mathbb{R}^3$ with the vertex at the vectors 0, e_1, e_2, and e_3, and let S' be the tetrahedron with vertices at vectors 0, v_1, v_2, and v_3.

(1) Describe a linear transformation that maps S onto S'.
Answer: The linear transformation that maps S onto S', maps $T(I) = [v_1 v_2 v_3]$ where I is the 3x3 identity matrix. Thus $T : Ax \to x$ such that $A = [v_1 v_2 v_3]$

the formula for the volume fo the tetrahedron S' using the fact that,

$$volume of S = \frac{1}{3} * area of base * height$$

Answer: First calculate the volume of S,

$$volume of S = \frac{1}{3} * \frac{1}{2}(1) * (1) = \frac{1}{6}$$

Thus since $T(S) = S'$ as defined by the previous problem we know that the volume of S',

$$volume of S' = \frac{1}{6}|det(A)|$$

Name: Stefano F.

Worksheet: Determinant Characteristics

Instructions: Work in your assigned groups. Write up your answers neatly on separate sheets of paper, *with your written homework.* This worksheet is due Monday October 7.

After class you will also want to read section 3.2 in the textbook (this worksheet replaces a lecture — don't read it right now).

Theorem 1. *The determinant of a upper-triangular, lower-triangular, or diagonal matrix is the product of the entries along the diagonal.*

Exercise 1: Consider the matrices A and B, where $\boxed{= 60}$

$= 3(4+2k)(5) - 215k$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 & 6 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

$\begin{matrix} 3 & 2 & 6 \\ 3k & 4+2k & 1+6k \\ 0 & 0 & 5 \end{matrix}$

$\begin{vmatrix} 1 & 2 \\ 3+k & 4+2k \end{vmatrix} = -2k$

(a) Compute $\det(A)$ and $\det(B)$. $\det(A) = 4 - 6 = -2$, $\det(B) = 60$

(b) For each of the following row operations applied to A and B, compute the determinant of the resulting matrix, called A' and B' respectively. Describe how the results relate to $\det(A)$ and $\det(B)$.

$4k - 6k$

$\det(A') = -2k$

$\det(B') = 60k$

i. A': replace R_2 by $3R_2$; B': replace R_3 by $R_3/3$ $\quad \begin{vmatrix} 1 & 2 \\ 9 & 12 \end{vmatrix}$ $\det(A') = -6$ $\quad \begin{vmatrix} 3 & 2 & 6 \\ 0 & 4 & 1 \\ 0 & 0 & 5/3 \end{vmatrix}$ $\det(B') = 20$

ii. A' and B': replace R_2 by kR_2

iii. A' and B': replace R_2 by $R_2 + kR_1$ $= 60$

iv. A' and B': swap R_1 and R_2 $= 2, -60$

(c) Make a guess: given any matrix M, what effect do the following elementary row operations have on $\det(M)$?

i. Replace R_i by kR_i $= k \det(M)$

ii. Replace R_i by $R_i + kR_j$ $= \det(M)$

iii. Swap R_i and R_j $= -\det(M)$

(d) i. If t is an arbitrary nonzero scalar, compute $\det(tA)$ for the matrix A given above. How does it compare to $\det(A)$? Then compute $\det(B)$ and $\det(tB)$. How do they compare? $\det(tA) = -2t^2$

ii. If an arbitrary matrix M is $n \times n$ and $\det(M) = q$, what is $\det(tM)$ for some scalar t? Explain how you know. Hint: think about what tM looks like, as a matrix, and think about what row operations you might have to use to get from M to tM. Then apply your guesses from 1(c).

$= qt^n$

$\det(tB) = 60t^3$

Exercise 2: Let $Q = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 2 & -7 \end{pmatrix}$.

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$$Q = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 2 & -7 \end{pmatrix}$$

$= 1(-28-2) - 2(-14-1) + 3(4-4)$

$= -30 + 30 = 0$

Name:

(a) Compute det(Q) using **cofactor expansion** along your choice of row or column. Det $(Q) = 0$

(b) What is det(I_n)? Det $(I_n) = 1$

(c) Suppose you begin with I_n and apply a sequence of elementary row operations to get to a matrix S (which implies that $RREF(S) = I_n$). Based on your answers to exercise 1(c), is there any way that det(S) can equal zero? Explain your reasoning. Yes, If we multiply I_n 0.

(d) Show that Q is not invertible (perhaps using one of the statements in the invertible matrix theorem and the previous results). Q is dependent so RREF $\neq I_n$ Not $AA^{-1} \neq I$

(e) Make a conjecture about matrices involving determinants and invertibility. (Note your answers for (b) - (d) actually prove your conjecture!)

If A is an $n \times n$ matrix whose Det$(A) = 0$ Then A is Not invertible.

Exercise 3: Give a counterexample to the following (false) claim:

$$\det(A + B) = \det(A) + \det(B)$$

Can you find a pair of matrices for which the claim is true?

Consider A, B = I then Det$(A+B) = 2^n$ and Det(A) + Det$(B) = 2$

Exercise 4: Two true facts:

Theorem 2. $\det(AB) = \det(A)\det(B)$

Theorem 3. $\det(A^T) = \det(A)$

Use the true facts to answer the following:

(a) Suppose A is invertible, and $\det(A) = q$. Using the fact that $AA^{-1} = I$, $= 1$ determine $\det(A^{-1})$. Det $(A^{-1}) = \frac{1}{q}$ $1/q$

Det A Det B

(b) Show that $\det(AB) = \det(BA)$. Find a counterexample to show that this does not imply that $AB = BA$. Sub, Swap, Sub Back

(c) A square matrix U is *orthogonal* if $U^T U = I$. Show that if U is orthogonal, $\det(U) = \pm 1$. $1 =$ Det $(U^T U) =$ Det(U^T) Det(U) $= 1$

Exercise 5: For each of the row operations below, describe how your guess in Exercise 1c can be justified in terms of the impact of the change on the elementary products used in calculating the determinant.

a.) $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

b.) $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

c.) $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

(a) Replace R_i by kR_i k Det (M), Do to further expansion on k row then pull k out

(b) Replace R_i by $R_i + kR_j$ Det(A) + kR(

(c) Swap R_i and R_j

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