Section 1.7:

Exercise 1.7.14: In Exercises 11–14, find the value(s) of h for which the vectors are linearly dependent. Justify each answer.

$$\begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}-5\\7\\8\end{bmatrix}, \begin{bmatrix}1\\1\\h\end{bmatrix} \tag{1}$$

Solution: By the dependence relation, to make a set of vectors linearly dependent we need to make sure that the homogeneous matrix equation has a nontrivial solution. Consider Ax=0

$$\begin{bmatrix}1 & -5 & 1 & 0\\-1 & 7 & 1 & 0\\3 & 8 & h & 0\end{bmatrix} \tag{2}$$

From here we want to do row operations until we get a free variable, that is sufficient enough to show that there is a non trivial solution to the homogeneous matrix equation.

$$\begin{bmatrix}1 & -5 & 1 & 0\\0 & 2 & 2 & 0\\3 & 8 & h & 0\end{bmatrix} r_1 + r_2 \tag{3}$$

$$\begin{bmatrix}1 & -5 & 1 & 0\\0 & 1 & 1 & 0\\3 & 8 & h & 0\end{bmatrix} \frac{1}{2}r_2 \tag{4}$$

$$\begin{bmatrix}1 & 0 & 6 & 0\\0 & 1 & 1 & 0\\3 & 8 & h & 0\end{bmatrix} -3r_1 + r_2 \tag{5}$$

$$\begin{bmatrix}1 & 0 & 6 & 0\\0 & 1 & 1 & 0\\0 & 8 & (h-18) & 0\end{bmatrix} -3r_1 + r_3 \tag{6}$$

$$\begin{bmatrix}1 & 0 & 6 & 0\\0 & 1 & 1 & 0\\0 & 0 & (h-26) & 0\end{bmatrix} -8r_2 + r_3 \tag{7}$$

Now we can see that in order to have a non trivial solution to the homogeneous matrix equation we must have $h = 26$.

Exercise 1.7.22: In Exercises 21 and 22, mark each statement True or False. Justify each answer on the basis of a careful reading of the text.

(a) Two vectors are linearly dependent if and only if they lie on a line through the origin.
Answer: True. By the dependence relation, if two vectors are linearly dependent then one must be a multiple of the other, therefore they both lie on the same line through the origin.

(b) If a set contains fewer vectors than there are entries in the vectors then the set is linearly independent

Answer: False. Consider the set of vectors {x,y} such that, $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. They are obviously linearly dependent, there are two vectors in the set and each vector has three entries.

(c) If x and y are linearly independent, and if z is in $Span\{x, y\}$, then $\{x, y, z\}$ is linearly dependent.
Answer: True. We know by the definition of Span that every vector in $Span\{x, y\}$, can be written as a linear combination of x and y. Furthermore we know from Theorem 7 that if at-least one vector in a set is a linear combination of the others that the whole set is then linearly dependent. Thus if z is in $Span\{x, y\}$, then $\{x, y, z\}$ must be linearly dependent.

(d) If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more vectors than there are entries for each vector.
Answer: False. It is not entirely necessary for there to be more vectors in the set than there are entries in each vector. Consider the example in part b, there are two vectors in $\mathbb{R}^3$ and they are linearly dependent.

Exercise 1.7.28: How many pivot columns must a 5x7 matrix have if its columns span $\mathbb{R}^5$? Why?
Solution: We know from Theorem 4 that for the columns of the matrix to span $\mathbb{R}^5$ there must be a pivot in every row. In order to have the columns span $\mathbb{R}^5$ we need to have linear independence in in at least 5 dimensions, thus the pivot in each row.

Section 1.8:

Exercise 1.8.22: In Exercises 21 and 22, mark each statement True or False. Justify each answer.

(a) Every matrix transformation is a linear transformation.
Answer: True. Linear transformations preserve vector addition and scalar multiplication. Every

matrix transformation does as well.

Proof: Suppose that T is a matrix transformation such that $T(x) = A\vec{x}$ for some matrix A,

$$T(c\vec{x} + d\vec{y}) = A(c\vec{x} + d\vec{y}) \tag{8}$$
$$= A(c\vec{x}) + A(d\vec{y}) \tag{9}$$
$$= cA\vec{x} + dA\vec{y} \tag{10}$$
$$= cT(\vec{x}) + dT(\vec{y}) \tag{11}$$

Since $T(c\vec{x} + d\vec{y}) = cT(\vec{x}) + dT(\vec{y})$ vector addition and scalar multiplication have been preserved and thus T must be a linear transformation.

$\square$

(b) The codomain of a transformation $x \to Ax$ is the set of all linear combinations of columns of A.
Answer: False. Suppose Ax is an mxn matrix, then the codomain would be $\mathbb{R}^m$ and the range would be set of all linear combinations of columns of A. There are some circumstances where the range is equal of the codomain, ie the Span of the columns of A is equal to $\mathbb{R}^m$ but that is not always true.

(c) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and if c is in $\mathbb{R}^m$, then a uniqueness question is "Is c in the range of T?"
Answer: False. Is c in the range of T is an existence question not a uniqueness question. Is T one-to-one, would be an example of a uniqueness question.

(d) A linear transformation preserves the operation of vector addition and scalar multiplication.
Answer: True. By the definition found on page 65.

(e) The superposition principle is a physical description of a linear transformation.
Answer: True. The book describes the superposition principle by thinking of vectors as signals that go into a system and the output is a linear transformation of the input signal. "The system satisfies the superposition principle if whenever an input is expressed as a linear combination of such signals, the system's response is the same linear combination of the responses to the individual signals." p(67)

$$T(c_1v_1 + ... + c_1v_p) = c_1T(v_1) + ... + c_pT(v_p) \tag{12}$$

Exercise 1.8.24: Suppose vectors $v_1, ..., v_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose $T(v_i) = 0$ for $i = 1, ..., p$. Show that T is the zero transformation. That is, show that if x is any vector in $\mathbb{R}^n$, then $T(x) = 0$.

Proof: Suppose the set of vectors $v_1, ..., v_p$ span $\mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, and $T(v_i) = 0$ for $i = 1, ..., p$. Let $\vec{x} \in \mathbb{R}^n$. Since we know that the set of vectors$v_1, ..., v_p$ span $\mathbb{R}^n$ we can write $\vec{x}$ as a linear combination of $v_1, ..., v_p$

$$\vec{x} = c_1v_1 + ... + c_pv_p. \tag{13}$$

When we try to apply the transformation T to $\vec{x}$ we can see that by the superposition principle,

$$T(\vec{x}) = c_1T(v_1) + ... + c_pT(v_p). \tag{14}$$

By substitution it is also true that,

$$T(\vec{x}) = c_1(0) + ... + c_p(0) \tag{15}$$

$$T(\vec{x}) = 0 \tag{16}$$

Thus T is the zero transformation.

$\square$

Exercise 1.8.25: Given $v \neq 0$ and p in $\mathbb{R}^n$, the line through p in the direction of v has the parametric equation $x = p + tv$. Show that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ maps this line onto another line or onto a single point (a degenerate line).

Proof: Suppose $v \neq 0$ and p in $\mathbb{R}^n$, the parametric equation $x = p + tv$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. Since T is a linear transformation, when we apply it to the parametric equation $x = p + tv$ we get,

$$T(x) = T(p + tv) \tag{17}$$

$$T(x) = T(p) + tT(v) \tag{18}$$

From here there are two cases.

Case 1: Suppose $T(v) = 0$. Here we can see by substitution that the T transformation of the line will result in a single point,

$$T(x) = T(p) + t(0) \tag{19}$$

where $T(x) = T(p)$ for all t.

Case 2: Suppose $T(v) \neq 0$. From here we can see that the T transformation of the line will result in another line where the parametric equation is just,

$$T(x) = T(p) + tT(v) \tag{20}$$

Thus T maps the line $x = p + tv$ into a line or single point.

□

Exercise 1.8.29: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$
(a) Show that f is a linear transformation when $b = 0$

Proof: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = mx + b$ and $b = 0$. Consider $f(cx + dy)$ such that $x, y \in \mathbb{R}$ and c, d are constants.

$$\begin{align} f(cx + dy) &= m(cx + dy) \tag{21} \\ &= mcx + mdy \tag{22} \\ &= c(mx) + d(my) \tag{23} \\ &= cf(x) + df(y) \tag{24} \end{align}$$

Thus we have shown that when $b = 0$, f respects vector addition and scalar multiplication and therefore is a linear transformation.

(b) Find the property of linear transformation that is violated when $b \neq 0$

Proof: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = mx + b$ and $b \neq 0$. let c be any constant and $x, y \in \mathbb{R}$. By substitution,

$$\begin{align} f(x) + f(y) &= (mx + b) + (my + b) \tag{25} \\ &= m(x + y) + b + b \tag{26} \\ &= f(x + y) + b \tag{27} \\ \tag{28} \end{align}$$

Thus when $b \neq 0$ f does not respect vector addition. Consider,

(c) Why is f called a linear function
Answer: f called a linear function because it is a polynomial function whose degree is at most one.

Linear Transformations and Geometry

1. Introduction

Today you are going to work in groups of 3 or 4 to answer the questions below. Please include your solutions (written up neatly) *with your homework due on Sept. 16.* If you need help, I'm happy to answer questions, but you should always ask one of your peers first!

The goal of today's activity is to explore some of the basic properties of linear transformations.

2. Some Definitions

The first key point is what the heck exactly *is* a *linear transformation*? Well, to start with, transformation is a word mathematicians use interchangeably with map or function. Namely, a transformation is a rule assigning to each element of a domain (the "source space") some element of the codomain (the "target space"). The elements that get hit by elements in the domain via the transformation are collectively known as the *range.*

Linear means "line-like." Transformations "change something." But what does that mean when we put these two words together? The key fact is that a transformation T taking points in $\mathbb{R}^n$ and sending them to points in $\mathbb{R}^m$, and it is doing it according to the following simple extra constraints:

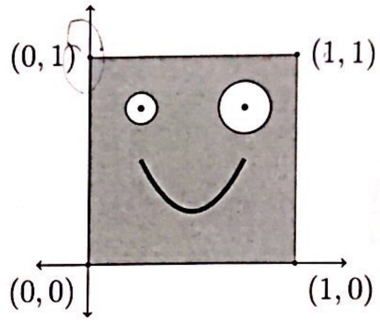
- $T(\vec{0}) = \vec{0}$;
- $T(a\vec{v}) = aT(\vec{v})$ if a is a scalar; and
- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$.

Exercise 1. Do these rules work for the functions $T_1(x) = 3x$ and for $T_2(x) = 3x + 2$? Which are linear transformations? (Note that here $x \in \mathbb{R}$.) What kind of graph does each of these functions have?

$T_1(\vec{0}) = 0$, $T_2(0) = 2$

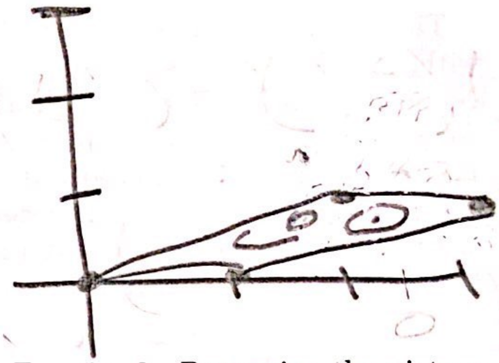
T_1 is linear $\quad$ T_2 is Not linear

Exercise 2. To the right is a picture drawn in the Cartesian plane. What happens to the picture under the transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$? (You can guess on the smiley face part... but the square you need to be sure about.)



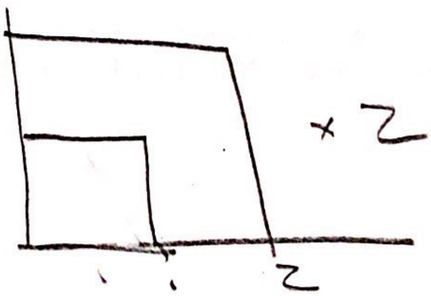
$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{matrix} 3 \\ 1 \end{matrix}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{matrix} 1 \\ 0 \end{matrix}$

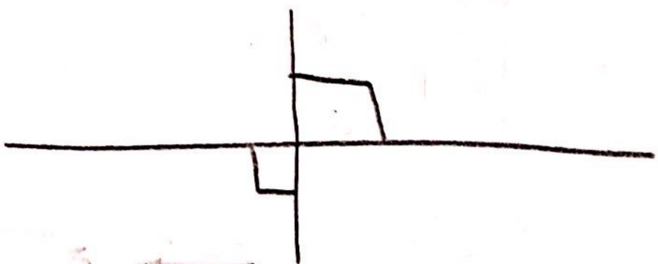


The following questions are all about the picture in Exercise 2. Determine the picture after applying the given transformation.

Exercise 3. $T(\vec{\mathbf{x}}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\vec{\mathbf{x}}$



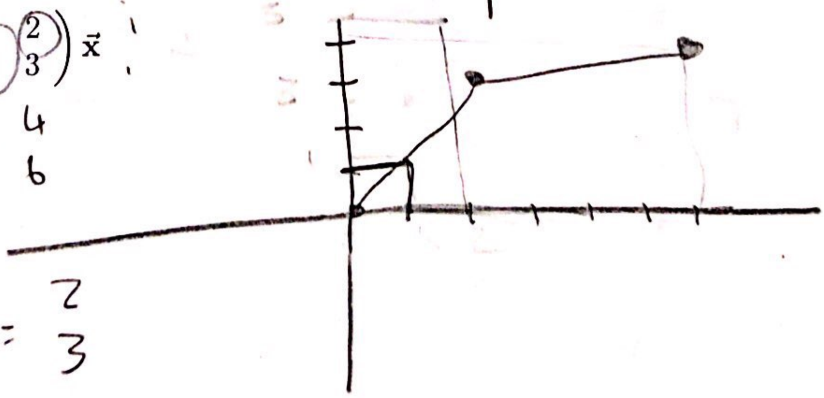
Exercise 4. $T(\vec{\mathbf{x}}) = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}\vec{\mathbf{x}}$



Exercise 5. $T(\vec{\mathbf{x}}) = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\vec{\mathbf{x}}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{matrix} 4 \\ 6 \end{matrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{matrix} 2 \\ 3 \end{matrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{matrix} 2 \\ 3 \end{matrix}$



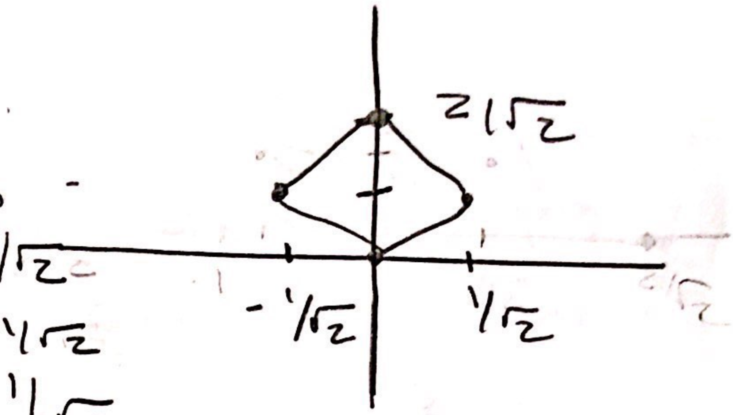
Exercise 6. $T(\vec{\mathbf{x}}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \vec{\mathbf{x}}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{matrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{matrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\sqrt{2} \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{matrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{matrix}$



Exercise 7. The linear transformation T such that $T(\vec{0}) = \vec{0}, T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$. (Hint: Think about where each of the earlier examples sent these three vectors.)

$(1)x_{11} + (0)x_{12} = 2$ $\quad (0)x_{11} + (1)x_{12} = -1$

$(1)x_{21} + (0)x_{22} = 3$ $\quad (0)x_{21} + (1)x_{22} = 4$

$T(\vec{x}) = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \vec{x}$

4. Back to the Definitions

Exercise 8. This one is harder. Consider the transformation given by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$, where A is some matrix with as many columns as there are entries in $\vec{\mathbf{x}}$. Other than the name of the course, can you come up with some reasons why we should believe that T is a linear transformation? As a starting point, consider the first rule above: Does it always hold, regardless of our choice of matrix A?

We know that every Matrix transform-ation is a linear transformation.

$T(\vec{0}) = \vec{0}$ always. (quick check).