Lab 4.7

Exercise 1: Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. find the change of coordinates matrix from $B \to C$ and from $C \to B$

$$b_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, b_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, c_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, c_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix},$$

(1) Set up the matrix M whose columns are $[c_1, c_2, b_1.b_2]$
Answer: Consider the following matrix,

$$M = \begin{bmatrix} -2 & 4 & 1 & -2 \\ 3 & 4 & 4 & 3 \end{bmatrix},$$

(2) Row reduce matrix M.
Answer: Row reducing matrix M,

$$\begin{bmatrix} -2 & 4 & 1 & -2 \\ 3 & 4 & 4 & 3 \end{bmatrix} \approx \begin{bmatrix} 3 & 4 & 4 & 3 \\ 0 & \frac{20}{3} & \frac{11}{3} & 0 \end{bmatrix} \text{ Row swap and row replacement at the same time.}$$
$$\approx \begin{bmatrix} 3 & 4 & 4 & 3 \\ 0 & 1 & \frac{11}{20} & 0 \end{bmatrix}$$
$$\approx \begin{bmatrix} 3 & 0 & \frac{9}{5} & 3 \\ 0 & 1 & \frac{11}{20} & 0 \end{bmatrix}$$
$$\approx \begin{bmatrix} 1 & 0 & \frac{3}{5} & 1 \\ 0 & 1 & \frac{11}{20} & 0 \end{bmatrix}$$

(3) Calculate ${P_c}^{-1}$, where $P_c = [c_1, c_2]$.
Answer: Consider,

$$P_c = \begin{bmatrix} -2 & 4 \\ 3 & 4 \end{bmatrix}$$

We know that probably the fastest way to calculate the inverse of a 2x2 matrix is to just multiply the whole thing by the reciprocal of the determinant. So,

$${P_c}^{-1} = \frac{1}{-20} \begin{bmatrix} -2 & 4 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{bmatrix}$$

(4) Calculate $N = {P_c}^{-1} P_b$.
Answer: Note that $P_b = [b_1, b_2]$ now through some matrix algebra we see that,

$$N = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{bmatrix} * \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} \left(-\frac{1}{5}\right) \cdot 1 + \frac{1}{5} \cdot 4 & \left(-\frac{1}{5}\right)(-2) + \frac{1}{5} \cdot 3 \\ \frac{3}{20} \cdot 1 + \frac{1}{10} \cdot 4 & \frac{3}{20}(-2) + \frac{1}{10} \cdot 3 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{5} & 1 \\ \frac{11}{20} & 0 \end{bmatrix}$$

(5) What do you notice about the last two columns of the reduced echelon form of M and the matrix N
Answer: They are the same, this makes sense because when we row reduce matrix M we are finding the change of coordinates matrix from $C \to B$ and it should be clear that when when we multiply P_b by the $P_c{}^{-1}$ we are essentially doing the same thing, for example take a vector written in terms of Basis C when we multiply by $P_c{}^{-1}$ we are converting it to the standard basis, and finally we multiply it by P_b to get it in terms of Basis B.

Exercise 2: Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. find the change of coordinates matrix from $B \to C$ and from $C \to B$

$$b_1 = \begin{bmatrix} 9 \\ 9 \end{bmatrix}, b_2 = \begin{bmatrix} -12 \\ 6 \end{bmatrix}, c_1 = \begin{bmatrix} -5 \\ 13 \end{bmatrix}, c_2 = \begin{bmatrix} 14 \\ 2 \end{bmatrix},$$

(1) Set up the matrix M whose columns are $[c_1, c_2, b_1.b_2]$
Answer:

$$\begin{aligned} M &= \begin{bmatrix} -5 & 14 & 9 & -12 \\ 13 & 2 & 9 & 6 \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & 0 & \frac{9}{16} & \frac{9}{16} \\ 0 & 1 & \frac{27}{32} & -\frac{21}{32} \end{bmatrix} \end{aligned}$$

(2) Calculate $P_c{}^{-1}$, where $P_c = [c_1, c_2]$.
Answer: Note that,

$$P_c = \begin{bmatrix} -5 & 14 \\ 13 & 2 \end{bmatrix}$$

Same as before, since we have a 2x2 matrix we can just multiply by the reciprocal of the determinant.

$$\begin{aligned} P_c{}^{-1} &= \frac{1}{-192} * \begin{bmatrix} -5 & 14 \\ 13 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{96} & \frac{7}{96} \\ \frac{13}{192} & \frac{5}{192} \end{bmatrix} \end{aligned}$$

Calculate $N = P_c{}^{-1} P_b$.
Answer: Note that $P_b = [b_1, b_2]$ now through some matrix algebra we see that,

$$\begin{aligned} N &= \begin{bmatrix} -\frac{1}{96} & \frac{7}{96} \\ \frac{13}{192} & \frac{5}{192} \end{bmatrix} * \begin{bmatrix} 9 & -12 \\ 9 & 6 \end{bmatrix} \\ &= \begin{bmatrix} \left(-\frac{1}{96}\right) \cdot 9 + \frac{7}{96} \cdot 9 & \left(-\frac{1}{96}\right)(-12) + \frac{7}{96} \cdot 6 \\ \frac{13}{192} \cdot 9 + \frac{5}{192} \cdot 9 & \frac{13}{192}(-12) + \frac{5}{192} \cdot 6 \end{bmatrix} \\ &= \begin{bmatrix} \frac{9}{16} & \frac{9}{16} \\ \frac{27}{32} & -\frac{21}{32} \end{bmatrix} \end{aligned}$$

Exercise 3: What is the pattern that we have established in exercises 1 and 2?
Answer: I believe I have already addressed this question in exercise 1.5, and quite thoroughly as well.

Exercise 4: Suppose we are given bases $B = \{b_1, b_2, ...b_n\}$ and $C = \{c_1, c_2, ..., c_n\}$ for a finite dimensional vector space $V = R_n$. Consider now the reduced row echelon form M of the matrix $M = [c_1, c_2, ..., c_n, b_1, b_2, ...b_n]$. Denote the columns of M by m_1. Thus for $i = 1, 2, ..., n$, $m_i = e_i$. What then are the vectors m_i for $i = n + 1, n + 2, ..., 2n$?

Answer: The vectors m_i for $i = n+1, n+2, ..., 2n$ form the change of coordinate matrix from $C \to B$ In other words they are the C- coordinate vectors in the basis B.

Exercise 5: For each Matrix M obtained in exercises 1 and 2 compare ${P_c}^{-1}M$ and ${P_c}^{-1}P_b$
Answer: Consider the following matrices from exercise 1,

$$ {P_c}^{-1}M = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{bmatrix} * \begin{bmatrix} 1 & 0 & \frac{3}{5} & 1 \\ 0 & 1 & \frac{11}{20} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & -\frac{1}{100} & -\frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} & \frac{29}{200} & \frac{3}{20} \end{bmatrix} $$

$$ {P_c}^{-1}P_b = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{bmatrix} * \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & 1 \\ \frac{11}{20} & 0 \end{bmatrix} $$

Consider the following matrices from exercise 2,

$$ {P_c}^{-1}M = \begin{bmatrix} -\frac{1}{96} & \frac{7}{96} \\ \frac{13}{192} & \frac{5}{192} \end{bmatrix} * \begin{bmatrix} 1 & 0 & \frac{9}{16} & \frac{9}{16} \\ 0 & 1 & \frac{27}{32} & -\frac{21}{32} \end{bmatrix} = \begin{bmatrix} -\frac{1}{96} & \frac{7}{96} & \frac{57}{1024} & -\frac{55}{1024} \\ \frac{13}{192} & \frac{5}{192} & \frac{123}{2048} & \frac{43}{2048} \end{bmatrix} $$

$$ {P_c}^{-1}P_b = \begin{bmatrix} -\frac{1}{96} & \frac{7}{96} \\ \frac{13}{192} & \frac{5}{192} \end{bmatrix} * \begin{bmatrix} 9 & -12 \\ 9 & 6 \end{bmatrix} = \begin{bmatrix} \frac{9}{16} & \frac{9}{16} \\ \frac{27}{32} & -\frac{21}{32} \end{bmatrix} $$

They both essentially represent the same operation but because of the notation M doesn't give us a useful answer when we change coordinates.

Exercise 6: What us the result of ${P_c}^{-1}P_b[x_b]$
Answer: Well we can see that the following statement simplifies,

$$ {P_c}^{-1}P_b[x_b] = {P_c}^{-1}x = [x]_c $$

and this is due to the fact that any P_z change-of-coordinates matrix will convert z-coordinate vectors, ie $[x_z]$ into standard ones. It goes the opposite way for the inverse of a change-of-coordinates matrix, goes from standard to z-coordinate vectors.

Exercise 7: Explain why the change-of-coordinates matrix from $C \to B$ is equal to the inverse of the change-of-coordinates matrix from $B \to C$
Answer: This is discussed in the previous question. except instead of going in between the standard basis and the a z basis its now a c basis and a b basis.