Section 6.6:

Exercise 6.6.4: Find the equation $y = b_0 + b_1 x$ of the last squares line that best fits the given data points. $(2,3),(3,2),(5,1),(6,0)$.

Solution: First we want to construct the Design matrix, and observation vector given our data. Consider the following,

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}, y = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Now that we have the parameters for our linear model, $y = Xb + e$ where b is our parameter vector and e is our residual vector. Our goal is to minimize e, which means we need to find the least squares solution to $Xb = y$. From Theorem 14 the least squares solution b is given by $X^T X b = X^T y$. Through some matrix algebra,

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$$

Now we have the following equation,

$$\begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$$

The fastest way to solve this system is to find $X^T X^{-1}$ and then multiplying both sides. Using the 2 x 2 inverse formula,

$$b = \{X^T X\}^{-1} X^T y$$

$$\{X^T X\}^{-1} = \frac{1}{40} \begin{bmatrix} 74 & -16 \\ -16 & 4 \end{bmatrix}$$

Therefore, by substitution,

$$\begin{aligned} b &= \frac{1}{40} \begin{bmatrix} 74 & -16 \\ -16 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 17 \end{bmatrix} \\ &= \begin{bmatrix} \frac{43}{10} \\ -\frac{7}{10} \end{bmatrix} \end{aligned}$$

Therefore we know that the line $y = \frac{43}{10} - \frac{7}{10} x$ is the best fit for our data.

Exercise 6.6.8: A simple curve that often makes a good model for variable costs of a company, as a function of the sales level x, has the form $y = b_1 x + b_2 x^2 + b_3 x^3$. There is not constant term because fixed costs are not included.

(1) Give the design matrix and the parameter vector for the linear model that leads to a least - square fit of the equation above, with data $(x_1, y_1), ..., (x_n, y_n)$

Solution: The design matrix for the given data and the desired model is,

$$X = \begin{bmatrix} x_1 & x_1^2 & x_1^3 \\ \vdots & \vdots & \vdots \\ x_n & x_n^2 & x_n^3 \end{bmatrix}$$

We can see that the parameter vector is gonna be,

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The next step would be to find the least - squares solution for $y = Xb + e$ where y is the observation vector or,

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

and e is the residual vector (we want to minimize this).

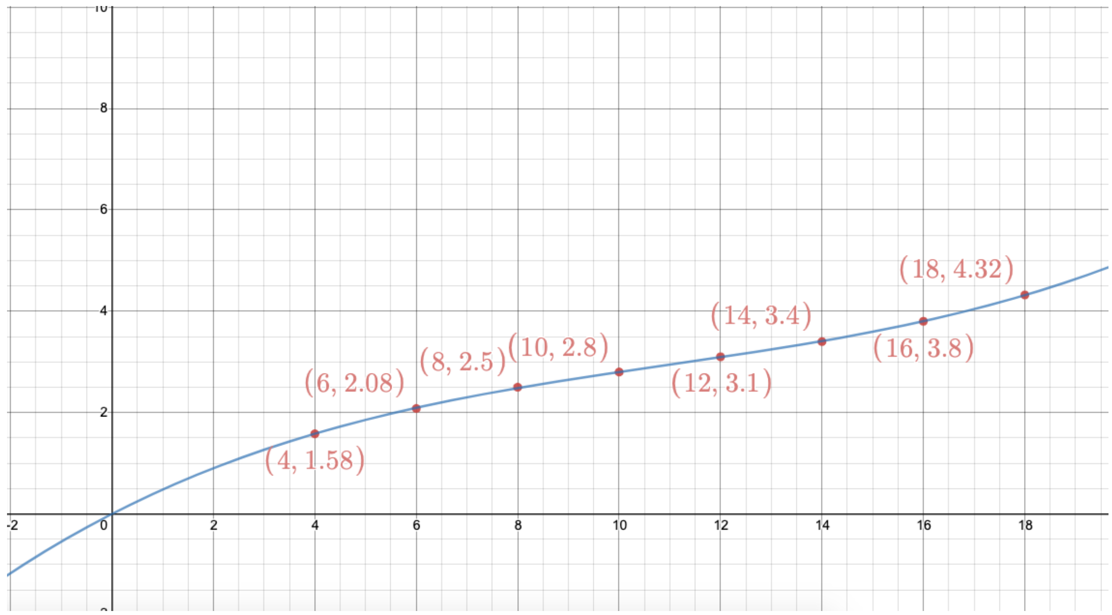
(2) {MATLAB} Find the least - squares curve of the form above to fit the data $(4, 1.58)$, $(6, 2.08)$, $(8, 2.5)$, $(10, 2.8)$, $(12, 3.1)$, $(14, 3.4)$, $(16, 3.8)$ and, $(18, 4.32)$ with the values in the thousands. If possible, produce a graph that shows and the graph of the cubic approximation.

Solution: First we want to set up the design matrix X, and the observation vector y. Plugging into the matrix in part a,

$$X = \begin{bmatrix} 4 & 16 & 64 \\ 6 & 36 & 216 \\ 8 & 64 & 512 \\ 10 & 100 & 1000 \\ 12 & 144 & 1728 \\ 14 & 196 & 2744 \\ 16 & 256 & 4096 \\ 18 & 324 & 5832 \end{bmatrix}, y = \begin{bmatrix} 1.58 \\ 2.08 \\ 2.5 \\ 2.8 \\ 3.1 \\ 3.4 \\ 3.8 \\ 4.32 \end{bmatrix}$$

From Matlab we get,

$$b = \{X^T X\}^{-1} X^T y = \begin{bmatrix} 0.5132 \\ -0.03348 \\ 0.001016 \end{bmatrix}$$



Section 6.7:

Exercise 6.7.14: Let T be a one-to-one linear transformation from a vector space V into R^n. Show that for u, v in V, the formula $< u, v >= T(u)T(v)$ defines an inner product on V

Solution:

(1) Suppose u, v in V then,

$$\begin{aligned} < u, v > &= T(u)T(v) \text{ (by definition of inner product)} \\ &= T(v)T(u) \text{ (because } T(v), T(u) \in R^n \text{ and is commutative)} \\ &=< v, u > \end{aligned}$$

(2) Suppose u, v and w in V then,

$$\begin{aligned} < u + w, v > &= T(u + w)T(v) \text{ (by definition of inner product)} \\ &= \{T(u) + T(w)\}T(v) \text{ (linear transformations respect vector addition)} \\ &= T(u)T(v) + T(w)T(v) \\ &=< u, v > + < w, v > \end{aligned}$$

(3) Suppose u, v in V and c in R then,

$$\begin{aligned} < cu, v > &= T(cu)T(v) \text{ (by definition of inner product)} \\ &= cT(u)T(v) \text{ (linear transformations respect multiplication by scalars)} \\ &= c < u, v > \end{aligned}$$

(4) Suppose u in V then,

$$\begin{aligned} < u, u > &= T(u)T(u) \text{ (by definition of inner product)} \\ &= T(u)^2 \end{aligned}$$

From here we can see that any vector in R^n when its squared will be larger than or equal to the 0 vector.

Exercise 6.7.18: Use the inner product axioms to verify the statement,

$$||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$$

Solution: Suppose,

$$\begin{aligned} ||u+v||^2 + ||u-v||^2 &=< u+v, u+v > + < u-v, u-v > \text{ (by definition of a norm)} \\ &=< u, u > + < v, v > + < u, v > + < v+u > + < u, u > + < v, v > - < u, v > - < v, u > \\ &\text{(expanding by inner product axiom 2)} \\ &= 2 < u, u > +2 < v, v > \\ &= 2||u||^2 + 2||v||^2 \end{aligned}$$

Exercise 6.7.22: Refer to $V = C[0, 1]$, with the inner product given by an integral. Compute $< f, g >$, where $f(t) = 5t - 3$ and $g(t) = t^3 - t^2$

Solution: By simply computing the inner product, by the given definitions we get,

$$\begin{aligned} < f, g > &= \int_0^1 (5t-3)(t^3-t^2)dt \\ &= \int_0^1 5t^4 - 8t^3 + 3t^2 dt \\ &= t^5 - 2t^4 + t^3|_0^1 \\ &= 1 - 2 + 1 = 0 \end{aligned}$$