Section 6.3:

Exercise 6.3.14: Find the best approximation to z by the vectors of the form $c_1v_1 + c_2v_2$.

$$z = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}, v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$$

Solution: We are looking for the orthogonal projection of z onto the span of v_1, v_2. So using the orthogonal projection theorem we get,

$$\begin{aligned}
\hat{z} &= \frac{z \cdot v_1}{v_1 \cdot v_1} \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} + \frac{z \cdot v_2}{v_2 \cdot v_2} \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} \\
&= \frac{7}{14} \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} + \frac{0}{49} \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ \frac{-1}{2} \\ \frac{-3}{2} \end{bmatrix} + 0
\end{aligned}$$

So we can see that the best approximation of z in the form of $c_1v_1 + c_2v_2$ is ,

$$\hat{z} = \begin{bmatrix} 1 \\ 0 \\ \frac{-1}{2} \\ \frac{-3}{2} \end{bmatrix}$$

Exercise 6.3.22: All vectors and subspaces are in R^n. Mark each statement True or False.

(1) If W is a subspace of R^n and of v is in both W and $W^\perp$ then v must be the zero vector.
Solution: True. The only vector that can be doted with itself and still result in zero is the zero vector. Consider the definition of orthogonal and definition of orthogonal sets.

(2) In the Orthogonal Decomposition Theorem, each term in formula (2) for $\hat{y}$ is itself an orthogonal projection of y onto subspace W
Solution: True. Consider this excerpt from page 351,

"When W is a one-dimensional subspace, the formula (2) for $proj_W y$ contains just one term. Thus, when dim $W > 1$, each term in (2) is itself an orthogonal projection of y onto a one-dimensional subspace spanned by one of the u's in the basis for W "

(3) If $y = z_1 + z_2$, where z_1 is in the subspace of W and z_2 is in $W^{\perp}$, then z_1 must be the orthogonal projection of y onto W
Solution: True. If y is composed of two orthogonal vectors, z_1 and z_2 then those vectors are the orthogonal projections of y onto their respective subspaces. For clarity reference figure 3 on page 351.

(4) The best approximation to y by elements of a subspace W is given by the vector $y - proj_W y$
Solution: False.The best approximation of y by the elements in the subspace W is given by just the $proj_W y$, also consider Theorem 9.

(5) If an nxp matrix U has orthonormal columns, then $UU^T x = x$ for all x in R^n
Solution: False. Statement is only true when x is an element of the column space of U. Consider Theorem 10 that gives the equality "$proj_W y = UU^T y$, s.t U is a orthonormal basis for W".

Exercise 6.3.24: Let W be a subspace of R^N with an orthogonal basis $\{w_1, ..., w_p\}$ and let $\{v_1, ...v_q\}$ be an orthogonal basis for $W^{\perp}$

(1) Explain why $\{w_1, ..., w_p, v_1, ...v_q\}$ is an orthogonal set.
Solution: Note, each vector in the set $\{w_1, ..., w_p\}$ is pair-wise orthogonal, by the definition of an orthogonal basis, and likewise for the set $\{v_1, ...v_q\}$. By our definition of orthogonal compliments we know that any pair of vectors taken from $W^{\perp}$ and W will be orthogonal, it must be true that the union of both $\{w_1, ..., w_p\}$ and $\{v_1, ...v_q\}$ creates an orthogonal set as well.

(2) Explain why the set in part (a) spans R^n
Solution: Consider the Orthogonal Decomposition Theorem that states any y in R^n can be written as $y = \hat{y} + z$ such that $\hat{y}$ is from W and z is from $W^\perp$. Since $\hat{y}$ and z can be written as linear combinations of the vectors in $W^\perp$ and W respectively, it must follow that y can be written as a linear combination of the vectors in W union $W^\perp$.

(3) Show that $dimW^\perp + dimW = n$
Solution: Since the set $\{w_1, ..., w_p, v_1, ...v_q\}$ is an orthogonal set, it has linearly independent vectors. We know that $\{w_1, ..., w_p, v_1, ...v_q\}$ spans R^n, therefore we know that $dimW^\perp + dimW = q + p = n$.

Section 6.4:

Exercise 6.4.4: the given set is a basis for the subspace W. Use the Gram-Schmidt process to produce the orthogonal basis W,

$$x_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, x_2 = \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$$

Solution: First we let,

$$v_1 = x_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$$

Then by the Gram-Schmidt Process we calculate v_2,

$$\begin{aligned} v_2 &= x_1 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\ &= \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} - \frac{-100}{50} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \end{aligned}$$

So then the orthogonal basis W is,

$$\left\{ \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \right\}$$

Exercise 6.4.12: Find the orthogonal basis for the column space of each matrix,

$$\begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$$

Solution: First let,

$$v_1 = x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Then calculating v_2,

$$\begin{aligned} v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\ &= \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - \frac{16}{4} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Now calculating v_3,

$$\begin{aligned} v_3 &= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \\ &= \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \frac{14}{4} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{12}{2} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix} \end{aligned}$$

Finally we can see the the orthogonal basis for the column space is,

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix} \right\}$$

Exercise 6.4.18: All vectors and subspaces are in R^n. Mark Each statement True, or False

(a) If $W = Span\{x_1, x_2, x_3\}$ with $\{x_1, x_2, x_3\}$ linearly independent and if $\{v_1, v_2, v_3\}$ is an orthogonal set in W, then $\{v_1, v_2, v_3\}$ is a basis for W
Solution: True. By our definition being $W = Span\{x_1, x_2, x_3\}$ we know that any set of three linearly independent vectors that span W must also be a basis for W. An Orthogonal set is also defined as a set of linearly independent vectors, where pair - wise vectors are orthogonal. Thus $\{v_1, v_2, v_3\}$ spans W and therefore is also a basis for W.

(b) If x is not in a subspace W, then $x - proj_W x$ is not zero.
Solution: True. If x is not in the subspace W then it can never have the same magnitude as $proj_W x$ and therefore $x - proj_W x \neq 0$.

(c) In a QR factorization, say $A = QR$ (when A has linearly independent columns), the columns of Q form an orthonormal basis for the column space of A.
Solution: True. Consider Theorem 12 Chapter 6.4.

Section 6.5:

Exercise 6.5.14: Let,

$$A = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}, u = \begin{bmatrix} 4 \\ -5 \end{bmatrix} v = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

Compute Au and Av, and compare them with b. Is it possible that at least one of u or v could be a least-squares solution of $Ax = b$? (Do not compute least-squares solution)

Solution: Computing Au and Av,

$$Au = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$$

$$Av = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix}$$

Now we calculate $b - Au$ and $b - Av$,

$$b - Au = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$$

$$b - Av = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$$

Finally we want to calculate the magnitude of $b - Av$ and $b - Au$ to see if either u or v could be the least squares solution.

$$||b - Au|| = \sqrt{2^2 + (-4)^2 + 2^2} = \sqrt{24}$$

$$||b - Av|| = \sqrt{(-2)^2 + 2^2 + (-4)^2} = \sqrt{24}$$

Since the least squares solution is unique neither u or v is the least squares solution.

Exercise 6.5.18: A is a an m x n matrix and b is in R^m. Mark each statement True or False.

(a) If b is in the column space of A, then every solution of $Ax = b$ is a least-squares solution.
Solution: True. Consider this excerpt from page 362,
"So we seek an x that makes Ax the closest point in $ColA$ to b. See Figure 1. (Of course, if b happens to be in $ColA$, then b is Ax for some x, and such an x is a "least-squares solution.")". the solution is also a vector not a point.

(b) The least - squares solution of $Ax = b$ is the point in the columns space of A that is closest to b.
Solution: True. By definition of Least Squares solution, normally the solution though is written as $A\hat{x} = b$, but it's the same thing the least squared solution is the vector in the span of A that is closest to b.

(c) A least - squares solution of $Ax = b$ is a list of weights that, when applied to the columns of A produces the orthogonal projections of be onto $ColA$
Solution: True. Consider the following excerpt from page 363,
"Since $\hat{b}$ is the closest point in $ColAtob$, a vector $\hat{x}$ is a least-squares solution of $Ax = b$ if and only if $\hat{x}$ satisfies (1). Such an $\hat{x}$ in R^n is a list of weights that will build $\hat{b}$ out of the columns of A. See Figure 2."

(d) If $\hat{x}$ is a least-squares solution of $Ax = b$, then $\hat{x} = (A^T A)^{-1} A^T b$ **Solution:** False. Columns of A must be linearly independent for this Theorem to work.

(e) The normal equations always provide a reliable method for computing least - squares solutions.
Solution: False. Consider the excerpt form page 363
"In some cases, the normal equations for a least-squares problem can be ill- conditioned; that is, small errors in the calculations of the entries of $A^T A$ can sometimes cause relatively large errors in the solution $\hat{x}$."

(f) If A has a QR factorization, say $A = QR$, then the best way to compute the least-square solution of $Ax = b$ is to compute $\hat{x} = R^{-1}Q^T b$ **Solution:** False. The Numerical Note on page 367 says that the best way to compute the least-square solution is to use $Rx = Q^T b$.

Exercise 6.5.25: Describe all least-squares solutions of the system,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Solution: the least - squares solution can be found by solving,

$$A^T Ax = A^T b$$

So,

$$A^T b = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

and,

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Thus we solve the following,

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 6 \\ 2 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

So our solution is,

$$x = 3 - y$$

$$y = y$$

Where y is free. So the Least-Square Solution is,

$$\begin{bmatrix} 3 - y \\ y \end{bmatrix} \text{ such that } y \in R$$