Section 4.5:

Exercise 4.5.19:

(1) The number of pivot columns of a matrix equals the dimension of it's column space.
Answer: True. By Theorem 6 which says that the pivot columns of a matrix A for a basis for the $Col(A)$. Therefore it must be true that the number of pivot columns is equal to the number of pivots.

(2) A plane in $\mathbb{R}^3$ is a two-dimensional subspace pf $\mathbb{R}^3$.
Answer: False. Consider a plane that does not go through the origin, It cannot be a subspace.

(3) The dimension of vector space $\mathbb{P}_4$ is 4.
Answer: False. $\mathbb{P}_4$ is all fourth degree polynomials, which have 5 coefficients. Therefore the dimension of $\mathbb{P}_4$ is 5.

(4) If the $DimV = n$ and S is linearly independent set in V , then S is a basis for V.
Answer: False. It is possible that the set S does not span V therefore it is not always a basis. Consider any $V = \mathbb{R}^2$ and $S = [1, 1]$, S is a linearly independent set that contains one vector, but It doesn't span $\mathbb{R}^2$ therefore it is not a basis.

(5) If a set $\{v_1, ...v_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.
Answer: True. By spanning set theorem.

Exercise 4.5.22: The first four Laguerre polynomials are $1, 1-t, 2-4t+t^2$, and $6-18t+9t^2-t^3$ show that these polynomials for a basis for $\mathbb{P}_3$
Solution: Consider the Laguerre polynomials in the form of coordinate vectors,

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{1}$$

It is clear that we have 4 linearly independent vectors, which means that the given vector span $\mathbb{P}_3$. Through the spanning set theorem we can say the form a basis for. $\mathbb{P}_3$ because the set is linearly independent.

Exercise 4.5.24: Let B be a basis for $\mathbb{P}_2$ consisting of the first three Laguerre polynomials, and let $p(t) = 7 - 8t + 3t^2$. Find the coordinate vector of p relative to B.

Solution: Consider the following matrix equation,

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix} [P]_b = \begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix} \tag{2}$$

All we have to do now is solve the system. Consider the following matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \tag{3}$$

So we can see that the solution i.e, $[P]_b = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

Section 4.6:

Exercise 4.6.17: A is an n x m matrix.

(1) The row space of A is the same as the column space of A^T.
Answer: True. The rows of A are the columns of A^T, and pivot positions stay the same through transpose.

(2) If B is any echelon form of A, and if B has three nonzero rows, then the first three rows of A form a basis for Row A.
Answer: False. The non zero rows of B form the basis for Row A.

(3) The dimensions of the row space and the columns space of A are the same, even if A is not a square.
Answer: True. Pivot positions determine the dimensions of the column and row spaces. Since a pivot column is also a pivot row the have the same dimension.

(4) The sum fo the dimensions of the row space and the null space A equals the number of rows in A.
Answer: False. By the rank-nullity that the dimension of the row space is equal to $n - dimNulA$. Let $n = 5$ and $dimNulA = 2$ then $dimrowA = 3$ and $3 \neq 2$.

(5) On a computer, row operation can change the apparent rank of a matrix.
Answer: True. Consider the numerical note in page 238.

Exercise 4.6.20: Suppose a non homogeneous system of six linear equations in eight unknown has a solution, with two free variables. Is it possible to change some constants on the equations right side to make the new system inconsistent. **Solution:** First let's describe the dimensions of all the spaces of the associated matrix. Let A be the associated matrix, since there are two free variable we know that the $dimnullA = 2$. Then we can use the rank-nullity theorem to find the rank of the matrix, $rankA = 8 - 2 = 6$ Thus we know that the $dimColA = 6$ which means there are pivot positions. Since there is a pivot in every row ,ie nor zero rows we cannot make the system inconsistent by changing the RHS.

Exercise 4.6.28: Justify the following equalities(let A be m x n).

$$dimRowA + dimNulA = n \tag{4}$$

$$dimColA + dimNulA^T = M \tag{5}$$

Solution: We can explain the first equality by simply noting that the $dimRowA = dimColA$ this comes from the property that the dimension for both spaces is determined by the number of pivot position (a pivot is a pivot for a row and a column just the same.) The the equality become the same as the rank - nullity theorem.

The second is just a variation once note that $dimColA = dimRowA^T$.

Section 4.7:

Exercise 4.7.7: Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. find the change of coordinates matrix from $B \to C$ and from $C \to B$

$$b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

Solution: In order to solve for the change of coordinates matrix we take the same approach we used for finding the inverse of a matrix, for example consider, the change of coordinates matrix from $B \to C$

$$\begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix} \approx \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & -8 \end{bmatrix} \approx \begin{bmatrix} 7 & -3 \\ 40 & -16 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix} \tag{9}$$

Thus the change of coordinates matrix from $B \to C$ is,

$$\begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

. The same technique is done to get the change of coordinates matrix from $C \to B$,

$$\begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix} \approx \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix} \tag{11}$$

Thus the change of coordinates matrix from $C \to B$,

$$\begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

Exercise 4.7.9: Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. find the change of coordinates matrix from $B \to C$ and from $C \to B$

$$b_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, c_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, c_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix},$$

Solution: Here we can take the same approach as we did for the last problem. Consider the change of coordinates matrix from $B \to C$,

$$\begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix} \approx \begin{bmatrix} -6 & 2 \\ -1 & 0 \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \approx \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} -3 & 1 \\ -4 & 1 \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} \tag{15}$$

Thus the change of coordinates matrix from $B \to C$ is,

$$\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$$

Consider the change of coordinates matrix from $C \to B$,

$$\begin{bmatrix} -6 & 2 \\ -1 & 0 \end{bmatrix} \approx \begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \tag{17}$$

Thus the change of coordinates matrix from $B \to C$,

$$\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$$

Exercise 4.7.14: In P_2 find the change of coordinates matrix from the basis $B = \{1-3t^2, 2+t-5t^2, 1+2t\}$ to the standard basis $C = \{1, t, t^2\}$ then find the B-coordinate vector for $-1+2t$. **Solution:** Since C is the standard basis, the coordinates of a polynomial from the standard basis are simply the coefficients. Therefore,

$$[b_1]1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$$[b_2]_c = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$$

$$[b_3]_c = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Therefore the change of coordinates matrix from $B \to C$ is,

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

In order to calculate B-coordinate vector for $-1+2t$ we need the change of coordinates matrix from $C \to B$. A quick way to do so in this case is to just take the inverse of the change of coordinates matrix from $B \to C$.Therefore the change of coordinates matrix from $C \to B$,

$$\begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix}$$

Then all we need to do to calculate the B-coordinate vector for $-1+2t$ is just multiply the coordinates to $-1+2t$ by the change of coordinate matrix from $C \to B$,

$$\begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -20 \\ 12 \\ -5 \end{bmatrix}$$