Section 1.4:

Exercise 1.4.12: Given A and b in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation $Ax = b$. Then solve the system and write the solution as a vector.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{1}$$

Solution: The corresponding augmented matrix looks like this,

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \tag{2}$$

Then we want to get the matrix in RREF to find the solution,

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix}, 3r_1 + r_2 \tag{3}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{5} \\ 0 & 5 & 3 & -1 \end{bmatrix}, \frac{1}{5}r_2 \tag{4}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{5} \\ 0 & 0 & -2 & -2 \end{bmatrix}, -5r_2 + r_3 \tag{5}$$

$$\begin{bmatrix} 1 & 0 & -1 & -\frac{2}{5} \\ 0 & 1 & 1 & \frac{1}{5} \\ 0 & 0 & -2 & -2 \end{bmatrix}, -2r_2 + r_1 \tag{6}$$

$$\begin{bmatrix} 1 & 0 & -1 & -\frac{2}{5} \\ 0 & 1 & 1 & \frac{1}{5} \\ 0 & 0 & 1 & 1 \end{bmatrix}, -\frac{1}{2}r_3 \tag{7}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{5} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & 1 \end{bmatrix}, r_3 + r_1, -r_3 + r_2 \tag{8}$$

Now that we have achieved RREF it is clear to see that,

$$x = \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 1 \end{bmatrix} \tag{9}$$

Exercise 1.4.24: In Exercises 23 and 24, mark each statement True or False. Justify each answer.

(a) Every matrix equation $Ax = b$ corresponds to a vector equation with the same solution set.

Answer: True. The matrix equation, vector equation and augmented matrix forms are all equivalent forms to the same problem.

(b) Any linear combination of vectors can always be written in the form Ax for a suitable matrix A and vector x.

Answer: True. From our definition of the matrix equation from p.35 that Ax is the linear combination of the columns of A using the corresponding entries in x as weight.

(c) The solution set of a linear system whose augmented matrix is $\begin{bmatrix} a_1 & a_2 & a_3 & b \end{bmatrix}$ is the same as the solution set of $Ax = b$ if $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$.

Answer: True. See Theorem 3 on p.36.

(d) If the equation $Ax = b$ is inconsistent, then b is not in the set spanned by the columns of A.

Answer: True. If the equation $Ax = b$ is inconsistent, then the set spanned by the columns of A is empty.

(e) If the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ has a pivot position in every row, then the equation $Ax = b$ is inconsistent.

Answer: False. If the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ has a pivot position in every row, we cannot say whether the system is inconsistent or consistent. If the statement were to be altered stating that the coefficient matrix $\begin{bmatrix} A \end{bmatrix}$ has a pivot position in every row then we could conclude that the system is consistent.

Exercise 1.4.30: Construct a 3x3 matrix, not in echelon form, whose columns do not span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

Solution: To construct a matrix with whose columns do not span $\mathbb{R}^3$ all we have to do is make sure the last row reduces to all zeroes, that way the vector $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is excluded from the span. Let A be,

$$A = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\1 & 1 & 1\end{bmatrix}, \tag{10}$$

Then we get A in RREF,

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, r_1 - r_3, r_2 - r_3, r_1 \to r_2 \tag{11}$$

From here it is clear that $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is not in the span of the three columns of the matrix A.

Section 1.5:

Exercise 1.5.24: In Exercises 23 and 24, mark each statement True or False. Justify each answer.

(a) If x is a nontrivial solution of $Ax = 0$ then every entry in x is nonzero
Answer: False. Just one entry needs to not be zero in order for the solution to be considered nontrivial.

(b) The equation $x = x_2 u + x_3 v$, with x_2 and x_3 free (and neither u nor v a multiple of each other), describe a plane through the origin.
Answer: True. The solution set is defined as Span{u,v} , and we can see this in section 1.3 pg. 30.

(c) The equation $Ax = b$ is homogenous if the zero vector is a solution
Answer: True. By the definition of homogenous p.43. If zero vector is in solution we can say that $A0 = b$ which then implies that

(d) The effect of adding p to a vector is to move the vector in a direction parallel to p.
Answer: False. The effect of adding p, or a particular solution to the general solution gives a solution set that is parallel to the general solution set, and not the particular solution.

Exercise 1.5.26: Suppose $Ax = b$ has a solution. Explain why the solution is unique precisely when $Ax = 0$ has only the trivial solution.

Solution: The theorem in the book states that "The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable."(p.44) If this is true we can also state that the contrapositive is true. The equation $Ax = b$ has no free variables if and only if the homogeneous equation $Ax = 0$ has no non trivial solutions. Since we have an if and only if statement then the reverse is true as well. If the homogeneous equation $Ax = 0$ has no non trivial solutions then the equation $Ax = b$ has no free variables. Since $Ax = b$ has no free variables its solution must be unique.

Exercise 1.5.31: Suppose A is a 3X2 matrix with two pivot positions.

(a) Does the equation $Ax = 0$ have a nontrivial solution.
Answer: It does not have a nontrivial solution, because there is a pivot in each column which means there are no free variables.

(b) Does the equation $Ax = b$ have at least one possible solution for every possible b.
Answer: The equation $Ax = b$ will only have a solution when

$$b = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} \tag{12}$$

If $b_3 \neq 0$ then $Ax = b$ has no solution.