Section 1.9:

Exercise 1.9.20: In Exercises 17–20, show that T is a linear transformation by finding a matrix that implements the mapping. Note that $x_1, x_2, ...x_n$ are not vectors but are entries in vectors. $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_3 - 4x_4$ such that $T : \mathbb{R}^4 \to \mathbb{R}$.

Solution: First we must note that matrix transformation that facilitates the $T : \mathbb{R}^4 \to \mathbb{R}$ must be a $1x4$ matrix. In order to find the elements in the matrix transformation we can simply look to the coefficients on each x term. Consider,

$$\begin{bmatrix} 2 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 0 \\ 3x_3 \\ -4x_4 \end{bmatrix} \tag{1}$$

Exercise 1.9.24: In Exercises 23 and 24, mark each statement True or False. Justify each answer.

(1) Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.
Answer: False. For every linear transformation we can build a unique matrix A where $T(x) = Ax$ such that $A = [T(e_1)...T(e_n)]$.

(2) The columns of a standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the n x n identity matrix.
Answer: True. As previously stated for every linear transformation we can build a unique matrix A where $T(x) = Ax$ such that $A = [T(e_1)...T(e_n)]$.

(3) The standard matrix of linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects points through the horizontal axis, the vertical axis, or the origin has the form,

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \tag{2}$$

where a and d are +-1.

Answer: True. Seen in Table 1 p.74. Reflection through the $x - axis$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3}$$

Reflection through the $y - axis$,

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4}$$

Reflection about the origin

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

(4) A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one if each vector $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$
Answer: False. For $T : \mathbb{R}^n \to \mathbb{R}^m$ to be one-to-one each vector in $\mathbb{R}^n$ has to map onto a distinct vector in $\mathbb{R}^m$. The statement above "if each vector $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$" just describes a function not a one-to-one function.

(5) If A is a 3 x 2 matrix, then the transformation $x \to Ax$ cannot map $\mathbb{R}^2$ to $\mathbb{R}^3$
Answer: True. For the transformation $x \to Ax$ to map $\mathbb{R}^2$ to $\mathbb{R}^3$ then A must be a 2 x 3 matrix.

Exercise 1.9.32: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. with A its standard matrix. Complete the following statement to make it true; "T maps $\mathbb{R}^n$ to $\mathbb{R}^m$ if and only if A has BLANK pivot columns." Find some theorems that explain why the statement is true.

Solution: The statement is only true when A has a pivot in every row. By Theorem 12, "T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$". For A, an m x n matrix to span $\mathbb{R}^m$ there must be a pivot in every row (Theorem 4).

Section 2.1:

Exercise 2.1.10: Let,

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}, C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} \tag{6}$$

Verify that $AB = AC$ and yet $B \neq C$.

Solution: First lets calculate AB,

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} * \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 2(8) - 3(5) & 2(4) - 3(5) \\ -4(8) + 6(5) & -4(4) + 6(5) \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} \tag{9}$$

Then lets calculate AC,

$$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} * \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} 2(5) - 3(3) & 2(-2) - 3(1) \\ -4(5) + 6(3) & -4(-2) + 6(1) \end{bmatrix} \tag{11}$$

$$= \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} \tag{12}$$

Thus $AB = AC$ and $B \neq C$

Exercise 2.1.16: Exercises 15 and 16 concern arbitrary matrices A, B, and C for which the indicated sums and products are defined. Mark each statement True or False. Justify each answer.

(1) If A and B are 3x3 and $B = [b_1, b_2, b_3]$, then $AB = [Ab_1 + Ab_2 + Ab_3]$
Answer: False. When calculating the product of matrixes we don't sum the columns, the appropriate calculation is $AB = [Ab_1, Ab_2, Ab_3]$.

(2) The second row of AB is the second row of A multiplied on the right by B.
Answer: True. The second row of AB is calculated by,

$$AB_2 = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{13}$$

(3) $(AB)C = (AC)B$
Answer: False. Matrix multiplication is not communicative, the statement would be correct if it was $(AB)C = A(BC)$ showcasing the associative law of multiplication.

(4) $(AB)^T = A^T B^T$
Answer: False. Theorem 3 states that $(AB)^T = B^T A^T$.

(5) The transpose of a sum of matrices equals the sum of their transposes.
Answer: True. By Theorem 3 which states that $(A + B)^T = A^T + B^T$

Exercise 2.1.18: Suppose the first two columns, b_1 and b_2, of B are equal. what can you say about the columns of AB (if AB is defined)? Why?
Solution: Let $B = [b_1, b_2, b_3]$ such that $b_1 = b_2$. Consider $AB = A[b_1, b_2, b_3]$, by substitution $AB = A[b_1, b_1, b_3]$. Through distribution $AB = [Ab_1, Ab_1, Ab_3]$, therefore we can say that that if there exists two columns in B that are identical there must be two columns in AB that are identical.

Exercise 2.1.22: Show that if the columns of B are linearly dependent, then so are the columns of AB.
Solution: If the columns in B are linearly dependent, then there exists a non-zero vector x such that $Bx = 0$ by the dependence relation. Thus $A(Bx) = A0$ and $(AB)x = 0$. Therefore the columns of $AB$are linearly dependent.

Section 2.2:

Exercise 2.2.8: Use matrix algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$.
Solution: Suppose $AD = I$ multiplying both sides from A^{-1},

$$A^{-1}(AD) = (I)A^{-1} \tag{14}$$
$$(A^{-1}A)D = A^{-1} \tag{15}$$
$$(I)D = A^{-1} \tag{16}$$
$$D = A^{-1} \tag{17}$$

Thus $D = A^{-1}$.

Exercise 2.2.10:

A product of invertible nxn matrices is invertible, and the inverse of the product is the product of the inverses in the same order.
Answer: False. A product of invertible nxn matrices is invertible but the inverse of the product is the product of the inverses in reverse order. Theorem 6.

If A is invertible, then the inverse of A^{-1} is A itself.
Answer: True. Theorem 6.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad = bc$, then A is not invertible.
Answer: True. Theorem 4.

If Acan be row reduced to the identity matrix, then A must be invertible.
Answer: True. Theorem 7.

If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.
Answer: False. The elementary row operations that reduce A to the identity I_n, reduce I_n to A^{-1}.

Exercise 2.2.12: Let A be an invertible nxn matrix, and let B be an nxp matrix. Explain why $A^{-1}B$ can be computed by row reduction:

$$if[AB] \ ... \ [IX], \text{then} X = A^{-1}B \tag{18}$$

If A is larger than 2 x 2, then row reduction of [A B] is much faster than computing both A^{-1} and $A^{-1}B$.
Solution: By reducing [A B] to [I X] we can say that $AX = B$. We can solve for X by multiplying both sides by A^{-1} we get $X = A^{-1}B$.

Exercise 2.2.14: Suppose $(B - C)D = 0$, where B and C are both m x n matrices and D is invertible. Show that $B = C$.
Solution: Since D is invertible, we can multiply both sides of $(B - C)D = 0$ by D^{-1}.

$$(B - C)D = 0 \tag{19}$$

$$D^{-1}(B - C)D = 0D^{-1} \tag{20}$$

$$(B - C)I = 0 \tag{21}$$

$$(B - C) = 0 \tag{22}$$

$$B = C \tag{23}$$

Thus $B = C$.