Section 5.5:

Exercise 5.5.4: Let the matrix A act on C^2 Find the eigenvalues and a basis for each eigenspace in C^2.

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

Answer: First we need to find the characteristic polynomial,

$$\begin{aligned} 0 &= det(A - \lambda I) \\ &= (5-\lambda)(3-\lambda) + 2 \\ &= \lambda^2 - 8\lambda + 17 \end{aligned}$$

Then using the quadratic formula we get that $\lambda = 4-i, 4+i$. Now we want to find the eigenvector for $4-i$. Let $\lambda = 4-i$

$$\begin{aligned} 0 &= (A - (4-i)I) \\ &= \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (1-i)x_1 - 2x_2 &= 0 \\ x_1 + (-1-i)x_2 &= 0 \end{aligned}$$

From here we can see that $x_1 = (1+i)x_2$ and x_2 is free. Therefore the basis for the eigenspace is,

$$\{\begin{bmatrix} 1+i \\ 1 \end{bmatrix}\}$$

Since we've solved for the eigenspace corresponding to the eigenvalue $\lambda = 1-i$ all we have to do to find the eigenspace for the complex conjugate $\lambda = 1+i$ is take the complex conjugate of the eigenspace we just found. Therefore the basis for the eigenspace of $\lambda = 1+i$ is,

$$\{\begin{bmatrix} 1-i \\ 1 \end{bmatrix}\}$$

Exercise 5.5.8: List the eigenvalues of A(using example 3). Give the angle δ of the rotation, where $-\pi < \delta \leq \pi$, and give a scale factor r.

$$A = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$$

Answer: We can see from applying example 3 that the eigenvalues for A are $\lambda = \sqrt{3}+3i, \sqrt{3}-3i$. We can find the magnitude or radius, from the following equation,

$$r = |\lambda| = \sqrt{\sqrt{3}^2 + (-3)^2} = 2\sqrt{3}$$

We can also find the angle δ using some basic trigonometry,

$$\begin{aligned} tan(\delta) &= \frac{-3}{\sqrt{3}} \\ \delta &= tan^{-1}(\frac{-3}{\sqrt{3}}) \\ \delta &= -\frac{\pi}{3} \end{aligned}$$

Exercise 5.5.23: Let A be any nxn real matrix with the property that $A^T = A$ Let x be any vector in C^n and let $q = \overline{x}^T Ax$. the equality below show that q is a real number by verifying that $\overline{q} = q$, Give a reason for each step,

$$\overline{q} = \overline{\overline{x}^T Ax} = x^T \overline{Ax} = x^T A\overline{x} = (x^T A\overline{x})^T = \overline{x}^T A^T x = q$$

(1) $\overline{\overline{x}^T Ax} = x^T \overline{Ax}$
Answer: The transpose of the compliment of a vector is the same at taking the compliment of the the transposed vector, ie $\overline{x}^T = \overline{x^T}$

(2) $x^T \overline{Ax} = x^T A\overline{x}$
Answer: A is a real matrix, so taking the compliment, results in $\overline{A} = A$

(3) $x^T A\overline{x} = (x^T A\overline{x})^T$
Answer: Since $x^T A\overline{x}$ gives a scalar we can take the transpose without changing anything.

(4) $(x^T A\overline{x})^T = \overline{x}^T A^T x$
Answer: Commutativity of transpose, $(AB)^T = B^T A^T$

(5) $\overline{x}^T A^T x = q$
Answer: By definition of A, $A^T = A$ and definition of q.

Section 6.1:

Exercise 6.1.10: Find the unit vector in the direction of the given vector.

$$v = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}$$

Answer: First we need to find the magnitude of the vector,

$$||v|| = \sqrt{-6^2 + 4^2 + (-3)^2} = \sqrt{61}$$

Then we can get the unit vector by dividing each component by the magnitude,

$$u = \frac{1}{\sqrt{61}} \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{-6}{\sqrt{61}} \\ \frac{4}{\sqrt{61}} \\ \frac{3}{\sqrt{61}} \end{bmatrix}$$

Exercise 6.1.14: Find the distance between vectors u and z.

$$u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, z = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$$

Answer: First we subtract z from u to get vector $u - z$,

$$u - z = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}$$

Then we take the magnitude of the resultant vector,

$$||u - z|| = \sqrt{4^2 + (-4)^2 + (-6)^2} = 2\sqrt{17}$$

Therefore the distance between vectors u and z is $2\sqrt{17}$.

Exercise 6.1.20: All vectors are in R^n, mark each statement true or false.

(a) $u \cdot v - v \cdot u = 0$
Answer: True. Since dot product is commutative, we know that $u \cdot v = v \cdot u$ and through a little algebra we can get the statement above.

(b) For any scalar c, $||cv|| = ||cv||$
Answer: False. $||cv||$ can never be negative, while $c||v||$ can for negative values of c.

(c) If x is orthogonal to every vector in the subspace W, then x is in $W^{\perp}$
Answer: True. By the definition of $W^{\perp}$.

(d) If $||u||^2 + ||v||^2 = ||u + v||^2$ then u and v are orthogonal.
Answer: True. By Theorem 2 in Chapter 6.

(e) For an mxm matrix A, vectors in the null space of A are orthogonal to the vectors in the row space of A
Answer: True. By Theorem 3 in Chapter 6.

Exercise 6.1.24: Verify the parallelogram law for vectors u and v in R^n,

$$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||u||^2$$

Answer: We can see prove this through some algebra,

$$\begin{aligned} ||u + v||^2 + ||u - v||^2 &= (u + v) \cdot (u + v) + (u - v) \cdot (u - v) \text{ by dot product} \\ &= (u \cdot u + v \cdot u + u \cdot v + v \cdot v) + (u \cdot u - v \cdot u - u \cdot v + v \cdot v) \\ &= 2u \cdot u + 2v \cdot v \\ &= 2||u||^2 + 2||u||^2 \end{aligned}$$

Exercise 6.1.27: Suppose a vector y is orthogonal to vectors u snd v. Show that y is orthogonal to the vector $u + v$ **Answer:** Since y is orthogonal to both u and v we know that $y \cdot u = 0$ and $y \cdot v = 0$. Now consider $y \cdot (u + v) = y \cdot u + y \cdot v = 0$. Therefore y is orthogonal to $u + v$.

Section 6.2:

Exercise 6.2.24: All vectors are in R^n. Mark true or false.

(i) Not every orthogonal set in R^n is linearly independent. **Answer:** True. An orthogonal set can contain the zero vector.

(ii) If a set $S = \{u_1, ..., u_p\}$ has the property that $u_j \cdot u_i = 0$ S.T $i \neq j$ then S is orthonormal
Answer: False. S is an orthogonal set, but there is no guarantee that S is orthonormal.

(iii) If the columns of an m x n matrix A are orthonormal, then the linear mapping $x \to Ax$ preserves lengths.
Answer: True. The columns of A are formed by vectors whose magnitude is one. So Ax is a fancy multiplication by one.

(iv) The orthogonal projection of y onto v is the same as the orthogonal projection of y onto cv
Answer: False. Projection is dependent on the direction of the projected on vector not the magnitude, so if we let $c \leq 0$ then the projection will be zero.

(v) An orthogonal matrix is invertible.
Answer: False. It is possible to have an orthogonal matrix that isn't square.

Exercise 6.2.30: Let U be an orthogonal matrix, and V by interchanging some columns of U. Explain why Y is an orthogonal matrix.

Answer: Since U is an orthogonal matrix all pair wise columns are orthogonal to each other. If we interchange the columns this property still stands, so Y must also be orthogonal.