Approximating π

A Brief Study on Approximating Methods

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Before the N-gon Approach

- One of the earliest known references to π comes from YBC 7302.
- A tablet said to have been used by a babylonian scribe.
- From diagram etched on the tablet it is largely hypothesized that,
  - the Babylonians simply used $\pi = 3$

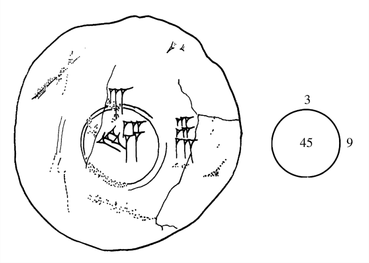


Figure 1: Artist's rendition of YBC 7301

Before the N-gon Approach

- Similarly we know that the Rhind Papyrus has references to π.
- Problem 50 of the Rhind Papyrus asks about the area of a round field.
- The solution given suggests that,
  - the Egyptians used the ratio $\pi = \frac{256}{81} \approx 3.1604..$

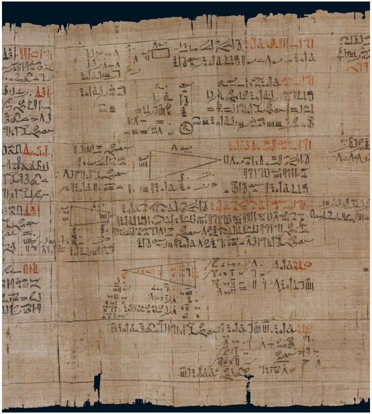
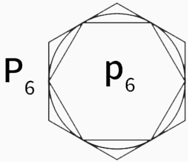


Figure 2: A section of the rhind Papyrus

Inscribed and Circumscibed n-gons

- The method of inscribed and circumscribed n-gons to approximate π became the prevailing method for approximating π
- Archimedes was the first to explicitly use this method in 'On the Evaluation of a Circle.'
- The basic idea goes as follow,
  - Let C be the circumference of a circle. If p_n is the perimeter of an n-regular polygon that inscribes O and P_n is the perimeter of an n-regular polygon that circumscribes O then it follows that,

$$p_3 < p_4 < p_5 < \ldots < \sup p_n = C = \inf P_N < \ldots < P_5 < P_4 < P_3$$



Inscribed and Circumscibed n-gons

- An area centric variation of this method was discovered by Liu Hui.
- Consider the following,
    - r is the radius
    - c_n is the side length of an inscribed n-gon
    - a_n is the length of the bisector formed by an inscribed n-gon
    - S_n is the area of the n-gon

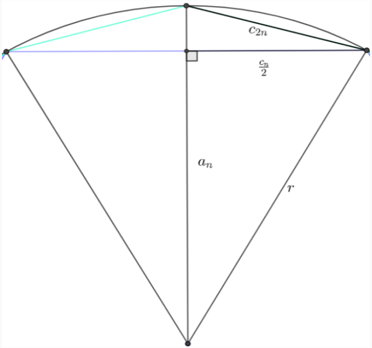


Figure 3: Lui Hui's Setup

$$a_n = \sqrt{r^2 - (\frac{c_n}{2})^2} \rightarrow c_{2n} = \sqrt{(\frac{c_n}{2})^2 + (r - a_n)^2} \rightarrow S_{2n} = \frac{nc_n r}{2}$$

N-gons to the Limit

- By the 15^{th} century Madhava was using series to approximate π.
- In 1436 Al-Kashi calculated π to 17 digits using Archimedean methods.
- To do so he computed the perimeter of the inscribed/circumscribed $3 \cdot 2^{28}$-gon.

Figure 4: Jamshīd al-Kāshī

N-gons to the Limit

- Al-Kashi's approximation would go uncontested until around 1600.
- Ludoph van-Ceulen, spent a majority of his life calculating pi.
- van-Ceulen approximate 35 digits by computing the circumference of a 2^{62}-gon.
- Shortly after the discovery of calculus would spawn exponentially improved methods.

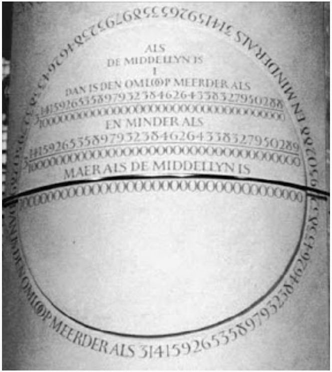


Figure 5: Ludoph van-Ceulen's tombstone

Classical Formulae From Analysis

- Leibniz/Madhava Series (1350 – 1425)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

- Newton's Approximation of π (1666)

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{2}{3 \cdot 2^3} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \dots\right)$$

- Wallis Product (1656)

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \left(\frac{2}{1} \cdot \frac{2}{3}\right) \cdot \left(\frac{4}{3} \cdot \frac{4}{5}\right) \cdot \left(\frac{6}{5} \cdot \frac{6}{7}\right) \cdot \dots$$

- John Machin's Formula (1706)

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

Leibniz/Madhava Series

- Recall that,

$$\arctan x = \int_0^x \frac{dx}{1+t^2}.$$

- Note that by sum of a geometric series,

$$\frac{1}{1+(t^2)} = \sum_{i=0}^{\infty}(-t^2)^i = 1 - t^2 + t^4 - \dots$$

- By substitution,

$$\begin{aligned}\arctan x &= \int_0^x (1 - t^2 + t^4 - \dots)dt,\\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots.\end{aligned}$$

- Letting $x = 1$ we get,

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Newton's Approximation of π

- Consider the following Diagram.

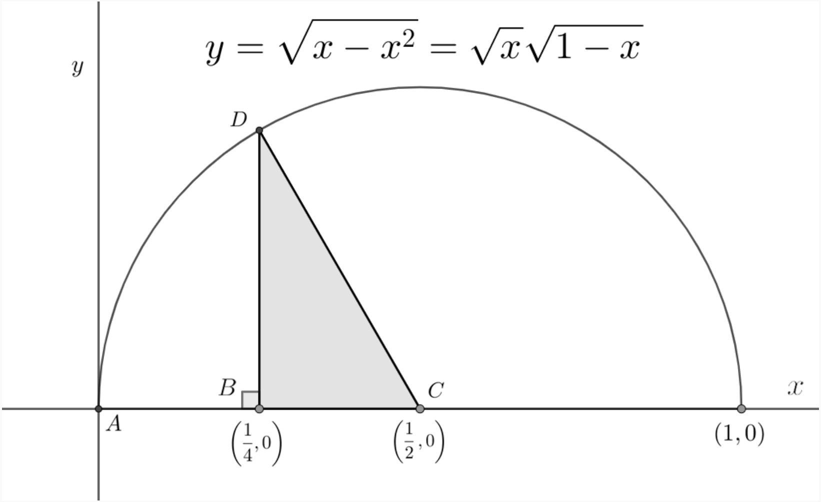


Figure 6: Newton's Setup for Approximating π

Newton's Approximation of π

- The diagram described the area of sector CAD whose central angle is 60° as,

$$\frac{\pi \frac{1}{2}^2}{6} = Area_{DBC} + \int_0^{\frac{1}{4}} \sqrt{x}\sqrt{1-x}dx$$

- Expanding the function with the Binomial Theorem and substituting $Area_{DBC}$

$$\frac{\pi \frac{1}{2}^2}{6} = \frac{\sqrt{3}}{32} + \int_0^{\frac{1}{4}} x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}} - \dots$$

- Integrating and solving for π

$$\frac{\pi}{6}\frac{1}{2}^2 = \frac{\sqrt{3}}{32} + \frac{2}{3 \cdot 2^3} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \dots$$

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{2}{3 \cdot 2^3} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \dots\right).$$

Wallis Product

- We define the Wallis' Integrals as,

$$I(n) = \int_0^{\pi} \sin^n x dx$$

- Through some integration by parts and algebra we get the following,

$$\lim_{n \to \infty} \frac{I(2n)}{I(2n+1)} = \frac{\pi}{2} \prod_{n=1}^{\infty} \frac{4n^2 - 1}{4n^2} = 1$$

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}$$

- This result can be shown through basic (yet lengthy) calculus, Euler's infinite product for sin, and geometrically with complex polynomials.

John Machin's formula

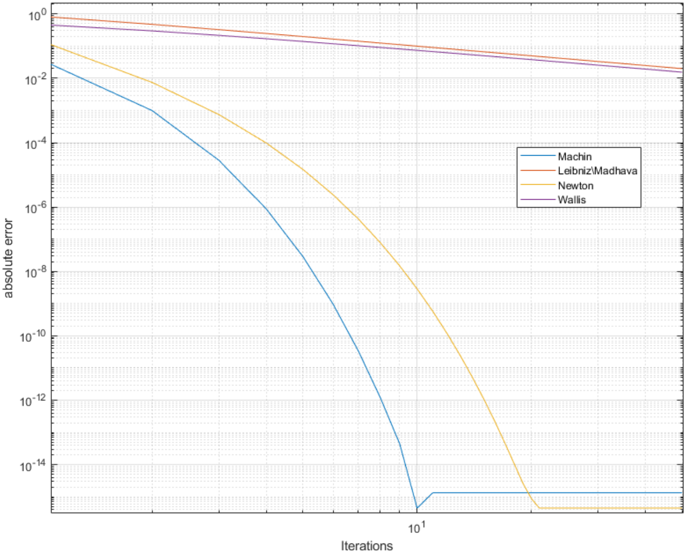
$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

- Machin-like formulas are variations on the Leibniz/Madhava Series.
- Recall that the convergence of the Leibniz/Madhava Series depends on the convergence of the arctan taylor series,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

- So in order to increase the convergence rate we need a smaller input argument.

Convergence Analysis



Modern Iterative Formualas

- By 1949 the first electronic computers were able to compute up to 2000 digits of π.
- By 1973 computer approximations reached a million digits.
- With modern computing implementation is everything (faster convergence does not guarantee better efficiency).
- In 2002 computer scientists at the University of Tokyo computed a record 1.2411 trillion digits.

Figure 8: Yasumasa Kanada with the Hitachi SR8000

Trillion Digit Formulas

$$\pi = 48 \arctan \frac{1}{49} + 128 \arctan \frac{1}{57} - 20 \arctan \frac{1}{239} + 48 \arctan \frac{1}{110443}$$

$$\pi = 176 \arctan \frac{1}{57} + 28 \arctan \frac{1}{239} - 48 \arctan \frac{1}{682} + 96 \arctan \frac{1}{12943}$$

Monte Carlo Approximation Demo

Code and Bibliography

- Convergence Analysis
- MonteCarlo Demonstration
- Bibliography