ON APPROXIMATING π

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ABSTRACT. This paper will explore the many ways π had been approximated throughout history. In doing so we will be deriving, or at least describing, the various proofs for each method as well as examining some of the historical context in which the methods were originally derived.

INTRODUCTION EARLY: HISTORY BEFORE N-GONS.

One of the earliest known examples of π being used in calculation comes from a stone tablet called YBC 7302 [9]. This tablet is said to have been used by a trainee Babylonian scribe. On the tablet we see the numbers 3, 45, and 9 etched in Babylonian cuneiform, with 45 at the center of a circle and the rest of the numbers outside the circle.

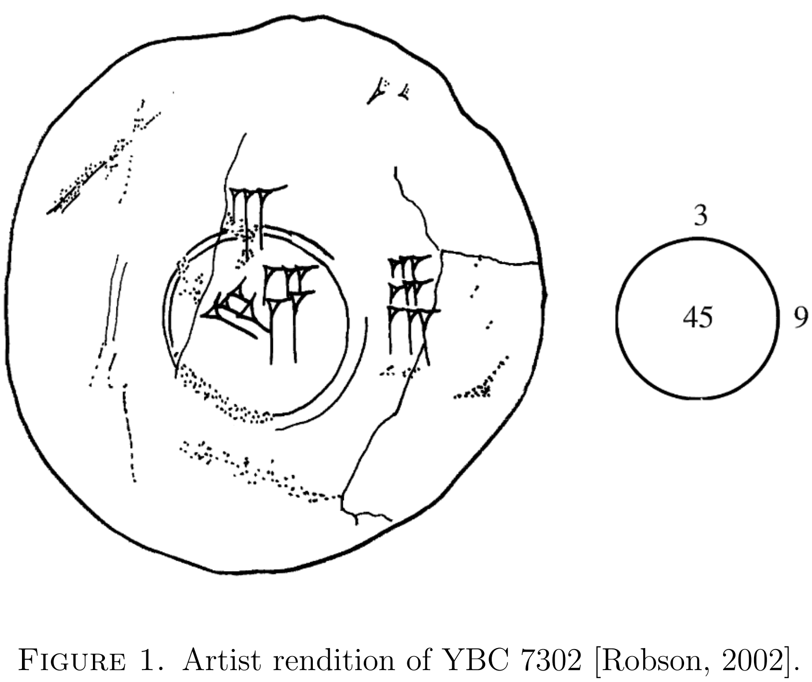

FIGURE 1. Artist rendition of YBC 7302 [Robson, 2002].

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It is largely hypothesized that the 3 represents the circumference of the circle and the 45 is actually meant to represent the area as $;45$ or $45/60$. Solving for the radius we get,

$$3 = 2\pi r,$$
$$r = \frac{3}{2\pi}.$$

Substituting for area we get,

$$\frac{45}{60} = \pi(\frac{3}{2\pi})^2,$$
$$\frac{45}{60} = \frac{9}{4\pi},$$
$$\pi = 3.$$

This calculation is why it is believed that the Babylonians accepted 3 as a suitable approximation for π.

Inscribed and Circumscribed N-gons

The prevailing geometric method for calculating π before the discovery of calculus involved approximating the area or circumference of a circle using inscribed and circumscribed n-gons. Archimedes was the first to explicitly use this sort of geometric method to approximate π around 300 BCE [4]

Archimedes. The approach relies on the following idea: The circumference of any given circle acts as the upper bound for any inscribed regular polygon. Similarly the circumference of the circle also acts as a lower bound for any circumscribed regular polygon.

Theorem 1. *Let C be the circumference of a circle O. If p_n is the perimeter of an n-gon that inscribes O and P_n is the perimeter of an n-regular polygon that circumscribes O then it follows that,*

$$p_3 < p_4 < p_5 < ... < \sup p_n = C = \inf P_N < ... < P_5 < P_4 < P_3$$

From the geometric construction Archimedes discovered that P_{2n} was the harmonic mean of p_n and P_n and that p_{2n} was the geometric mean of p_n and P_n[4].

$$P_{2n} = \frac{2p_n P_n}{p_n + P_n},$$

$$p_{2n} = \sqrt{p_n P_{2n}}.$$

With these two relationships Archimedes was able to compute the values of P_{96} and p_{96} with respect to a given diameter d,

$$(3 + \frac{10}{71})d < \pi < (3 + \frac{1}{7})d$$
$$3.14084507 < \pi < 3.14285714.$$

Archimedes' approximation was only accurate to two decimal places but as we'll see later on this method, and slight variations were used for more then a millennium to calculate π to a high enough accuracy for any conceivable practical purpose.

Liu Hui. A variation on the same approach came about in the third century by Chinese mathematician Liu Hui. Liu Hui estimated the area of a circle by calculated the areas of inscribed $n - gons$ using an iterative process [10]. His method proceeds as follows,

Proof. Consider a circle of radius r and let c_n be the length of the side of an inscribed n-gon. Let a_n be the length of the perpendicular bisector of the chord created by the n-gon. Finally let S_n be the area of the n-gon (See Figure 2).

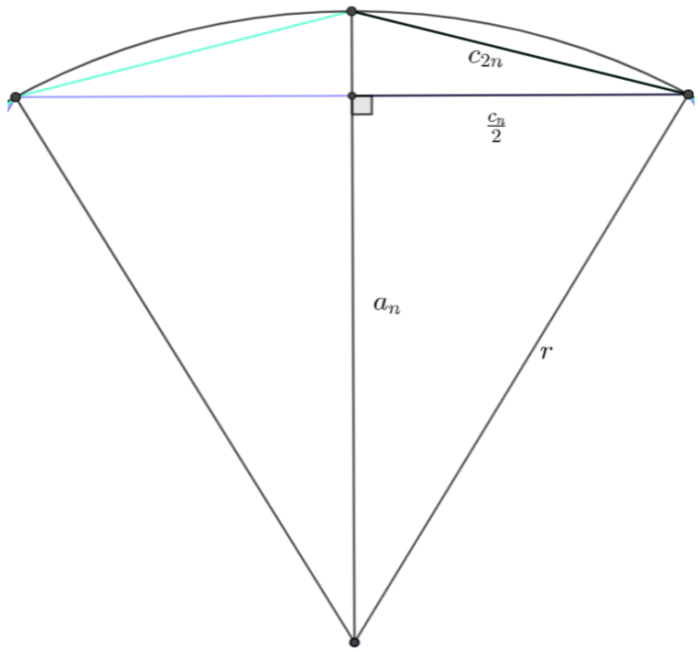

FIGURE 2. Liu Hui's setup for calculating π [10].

Begin by noting that an inscribed hexagon is composed of six equilateral triangles with a side length of r, and thus we know that,

$$c_6 = r$$

By the pythagorean theorem we get that,

$$a_n = \sqrt{r^2 - (\frac{c_n}{2})^2},$$

$$c_2n = \sqrt{(\frac{c_n}{2})^2 + (r - a_n)^2},$$

Using these to formulas we can iteratively calculate the side length of an inscribed n-gon, with each step doubling the value of n. Finally we can calculate the area of the $2n$-gon by noting that each triangle has a base of r and a height of $c_n/2$. Therefore we get the following,

$$S_{2n} = (2n)\frac{1}{2}\frac{c_n}{2}r = \frac{nc_nr}{2}$$

$\square$

Using this method Liu Hui was able to approximate π to the same precision as Archimedes. Although his method relies on solving for the area of an inscribed polygon, both methods use the same geometric concept.

N-gons to the Limit. By the 14th century it was already the case that Hindu mathematicians at the Kerala school had discovered the Leibniz/Mahadva series for approximating π [5]. However since the Leibniz/Madhava series converges so slowly, it was Jamshid Al-Kashi at the Samarkand observatory who held the title for most accurate approximation of π. In about 1436 Al-Kashi calculated π to 17 digits, doing so by calculating the perimeters of a circumscribed and inscribed $3 \cdot 2^{28}$-gon[7].

This approximation would go on uncontested until around 1600 when the Dutch mathematician, Ludoph van-Ceulen would calculate π accurately to 35 digits by way of computing the circumference of a 2^{26}-gon. Van-Ceulen spent 25 years working on his approximation, which can be found engraved on his tombstone. Shortly thereafter, the discovery of calculus would bring about exponentially improved methods for calculating π [2].

Classical Formulas from Analysis

In the following section we will discuss a few of the methods that approximate π that were discovered in 17th century and onwards. We will consider the calculus derivation for the Leibniz/Madhava series, Newton's approximation of π, the Wallis Product, and finally John Manchin's formula will serve as an introduction to modern computing.

Leibniz/Madhava Series. Although it was discovered centuries before, the calculus derivation for the Leibniz/Madhava series serves as an excellent introduction to the many methods that were derived from calculus.

Proof. Recall that,

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2}.$$

Now consider the following geometric sequence, and sum

$$1 - t^2 + t^4 + ... = \sum_{i=0}^{\infty} (-(t)^2)^i = \frac{1}{1+(t)^2}.$$

By substitution we get,

$$\begin{aligned} \tan^{-1} x &= \int_0^x \sum_{i=0}^{\infty} (-(t)^2)^i dt, \\ &= \int_0^x (1 - t^2 + t^4 + ...) dt, \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots. \end{aligned}$$

Letting $x = 1$ we get,

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots.$$

$\square$

Newton's Approximation. Newton discovered a series similar to Leibniz and Madhava but instead it relies on integrating the equation of a circle (Note: Another interpretation involves the $\sin^{-1} x$ Taylor series). Newton's approximation relies on the following geometric diagram [6].

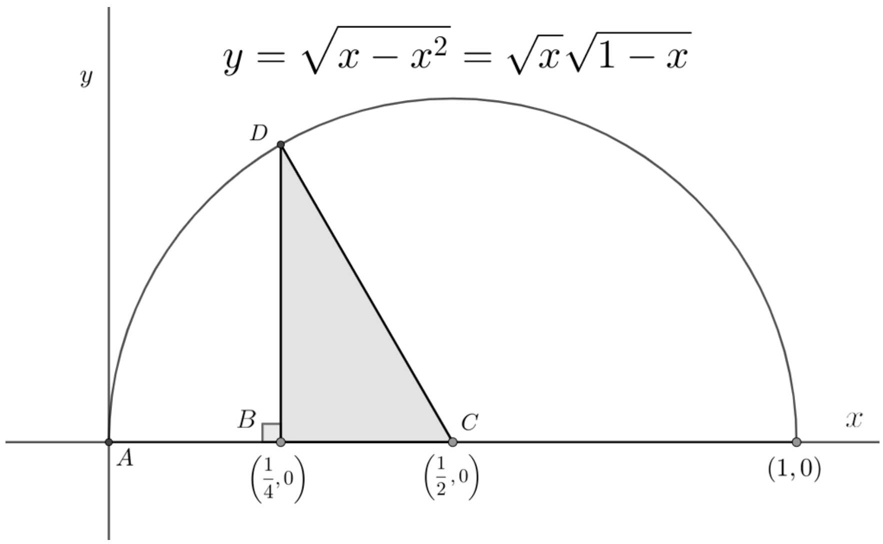

FIGURE 3. Newton's setup for calculating π. [6]

The general concept is that we use calculus to derive a series for the area enclosed by ADB, then use geometry to solve for the area $\triangle DBC$. Summing both areas we get a series that approximates the area of the sector, S_{ACD}. By construction the area of $S_{ACD} = \frac{\pi}{6}\frac{1}{2}^2$ and we can solve for π.

Proof. Consider circle with radius $\frac{1}{2}$ and whose center point is $(\frac{1}{2}, 0)$. Thus the equation of the circle is,

$$(x - \frac{1}{2})^2 + (y-0)^2 = \frac{1}{2}^2$$
$$y = \sqrt{x - x^2} = \sqrt{x}\sqrt{1-x}.$$

We can expand the right $\sqrt{1-x}$ into an infinite series with the Binomial Theorem, and we get that,

$$\begin{aligned} y &= \sqrt{x}(1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \dots) \\ &= x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}} - \dots \end{aligned}$$

Integrating we get that,

$$A_{ADB} = \int_0^{\frac{1}{4}} ydx = \frac{2}{3 \cdot 2^3} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \dots$$

Solving for the area of $\triangle DBC$ we get,

$$Area_{\triangle DBC} = \frac{1}{2}\frac{1}{4}\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{32}$$

Summed together we get the area of S_{ACD} and since S_{ACD} has a central angle of $\pi/3$ we can conclude that,

$$\frac{\pi}{6}\frac{1}{2}^2 = \frac{\sqrt{3}}{32} + \frac{2}{3 \cdot 2^3} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \dots$$

Solving for π we arrive at Newtons approximation of π,

$$\pi = \frac{3\sqrt{3}}{4} + 24(\frac{2}{3 \cdot 2^3} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \dots).$$

$\square$

We can arrive at the general form of the formula by using the general form of the Binomial Theorem to expand the function we are integrating (Note: We can also consider the general form for Taylor Series of $sin^{-1}x$)[3],

$$\pi = \frac{3\sqrt{3}}{4} + 24\sum_{n=1}^{\infty} \frac{-(2n-2)!}{2^{4n-2}(n-1)!^2(2n-3)(2n+1)}$$

Wallis Product (John Wallis). In 1656 John Wallis derived an infinite product that approximates π by considering the following, let $n \in \mathbb{N}$

$$I(n) = \int_0^\pi \sin^n(x)dx.$$

This is the general form of the Wallis' integrals. Wallis showed that the limit of the ratio between even and odd Wallis' integrals converged to 1 and was equivalent to the following infinite product,

$$\lim_{n\to\infty} \frac{I(2n)}{I(2n+1)} = \frac{\pi}{2}\prod_{n=1}^{\infty}\frac{4n^2-1}{4n^2} = 1$$

Solving for $\frac{\pi}{2}$ we get the famous result,

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}$$

This result can be shown through basic (yet lengthy) calculus. A proof and thoughtful convergence analysis can be found in Nouri Al-Othman's paper on π [3].

John Machin and Machin-like formulas. In 1706 John Machin discovered the formula [8], whose basic principles would guide how we computed π until the early 2000s [6]. John Machin's formula goes as follows,

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - tan^{-1}\left(\frac{1}{239}\right).$$

Proof. Proving Machin's formula is relatively straight forward as well, since it relies on a simple trig identity,

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}.$$

First let, $\tan(\alpha) = \frac{1}{5}$ and using the identity consider $\tan(2\alpha)$

$$\tan(2\alpha) = \frac{\tan(\alpha)+\tan(\alpha)}{1-\tan(\alpha)\tan(\alpha)} = \frac{\frac{2}{5}}{\frac{24}{25}} = \frac{5}{12}.$$

Using this result we can solve for $\tan(4\alpha)$ by plugging into the identity again,

$$\tan(4\alpha) = \frac{\tan(2\alpha) + \tan(2\alpha)}{1 - \tan(2\alpha)\tan(2\alpha)} = \frac{\frac{10}{12}}{\frac{119}{144}} = \frac{120}{119}.$$

Now consider passing $\tan(4\alpha - \frac{\pi}{4})$ through the identity,

$$\tan(4\alpha - \frac{\pi}{4}) = \frac{\tan(4\alpha) - \tan(\frac{\pi}{4})}{1 - \tan(4\alpha)\tan(\frac{\pi}{4})} = \frac{\frac{120}{119} - 1}{\frac{120}{119} + 1} = \frac{1}{239}.$$

Finally with some algebra we get,

$$\begin{aligned}
\tan(4\alpha - \frac{\pi}{4}) &= \frac{1}{239}, \\
\tan^{-1}\left(\tan\left(4\alpha - \frac{\pi}{4}\right)\right) &= \tan^{-1}\left(\frac{1}{239}\right), \\
4\alpha - \frac{\pi}{4} &= \tan^{-1}\left(\frac{1}{239}\right), \\
\frac{\pi}{4} &= 4\alpha - \tan^{-1}\left(\frac{1}{239}\right).
\end{aligned}$$

Recall that since $\tan(\alpha) = \frac{1}{5}$ we know that $\alpha = \tan^{-1}\left(\frac{1}{5}\right)$ so by substitution,

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

□

The Leibniz/Madhava Series, John's formula, and all other Machin-like formulas are simply variations on the $\tan^{-1}(x)$ Taylor Series.

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Machin-like formulas take advantage of the idea that as we input values of x that are closer and closer to zero we achieve faster convergence in the Taylor Series.

Convergence Analysis. To get a better sense of how all these formulas converge to π let's compare them using numerical methods. The following plot shows the absolute error for each of the discussed formulas up to the 50^{th} iteration. The absolute error was computed with a 16-digit approximation of π and the data was plotted on a log-log plot to signify digit wise accuracy.

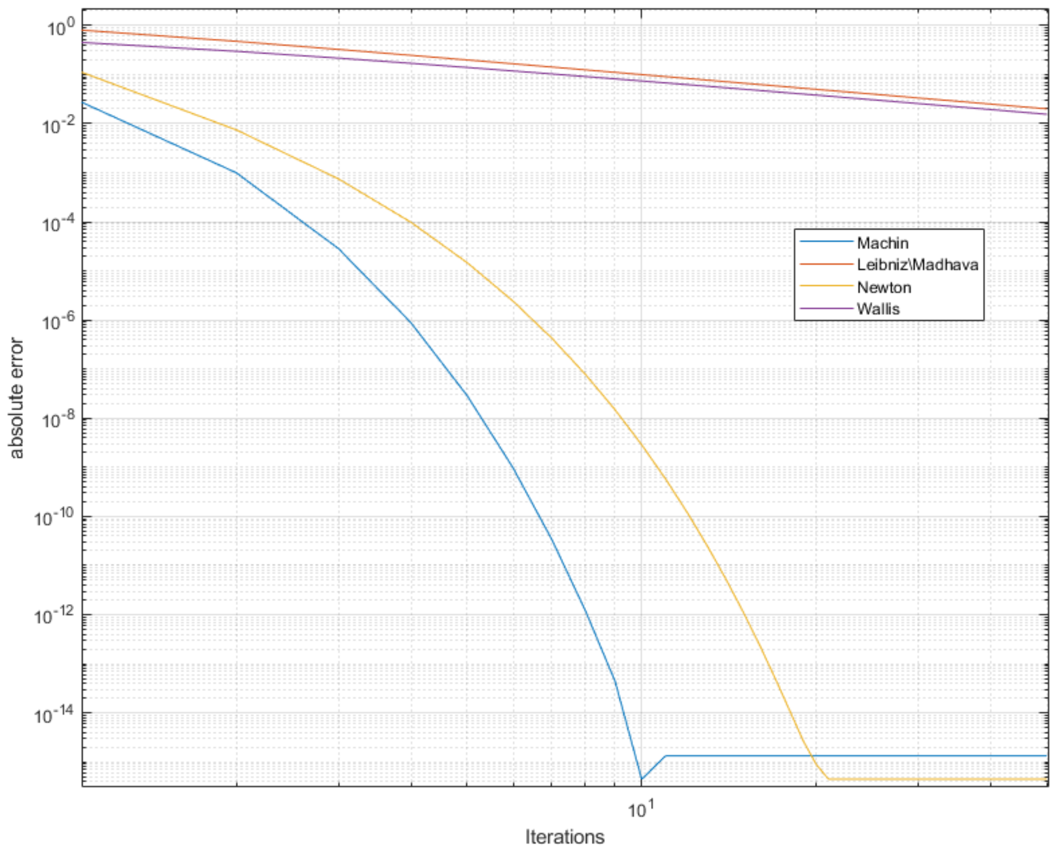

FIGURE 4. Examining Classic Formulas with Numerical Methods.

From the plot we can see that Newton's approximation and John Machin's formula both achieved 16-digits of accuracy before 25^{th} iterations. An impressive result considering that it took Ludoph van-Ceulen 25 years to compute 35 digits. We can also see that at the 10^{th} iteration, John Machin's formula achieved nearly double the accuracy of Newton's approximation with the gap between them continuing to widen.

Modern Computing

Following John Machin's discovery, Machin-like formulas would become the basis for all modern computations of π until the early 2000s [6]. In 1949 the ENIAC(Electronic Numerical Integrator and Computer) a computer built to compute ballistic tables during WWII computed π to more than 2000 digits. This was the first computation of π that took advantage of electronic computers [1]. As electronic computers improved, so did our approximations of π and by 1973 we would have our first million-digit approximation of π. 29 years later, in 2002 we would see a team of computer scientists from the University of Tokyo compute π to a staggering 1.2411 trillion digits.

Kanada's 1.2411 trillion decimal approximation. In December of 2002 Yasamusa Kanada and his team were able to compute π 1.2411 trillion digits with the use of the Hitachi SR8000 Supercomputer. The computation was performed twice in hexadecimal, taking over 600 hours to complete, and used the following Machin-like formulas [6]

$$\pi = 48\arctan\frac{1}{49} + 128\arctan\frac{1}{57} - 20\arctan\frac{1}{239} + 48\arctan\frac{1}{110443},$$

$$\pi = 176\arctan\frac{1}{57} + 28\arctan\frac{1}{239} - 48\arctan\frac{1}{682} + 96\arctan\frac{1}{12943}.$$

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