(1) (Problem 3.4.2) Derive a combinatorial identity via the equation

$$\frac{1}{(1-x)^{m+n}} = \frac{1}{(1-x)^m} \cdot \frac{1}{(1-x)^n}.$$

(You should justify your answer.)

Answer: Consider the equation is an equality of two ordinary generating functions.

$$[\![\frac{1}{(1-x)^{m+n}}]\!] = [\![\frac{1}{(1-x)^m} \cdot \frac{1}{(1-x)^n}]\!]$$

From here along with the convolution formula for ordinary generating function we get,

$$\left(\!\!\binom{n+m}{k}\!\!\right) = \sum_{j=0}^{k} \left(\!\!\binom{m}{j}\!\!\right) \left(\!\!\binom{n}{k-j}\!\!\right)$$

$$\binom{n+m+k-1}{k} = \sum_{j=0}^{k} \binom{m+j-1}{j} \binom{n+k-j-1}{k-j}$$

I suppose that it's some variation of Vandermonde's Identity but with multi-sets.

(2) (Problem 3.4.5) Let a, b, and c be nonzero real numbers. Find the coefficient of x^k in $\dfrac{a}{b+cx}$.

Answer: Through a little bit of algebra we can simplify this problem,

$$\begin{aligned} \frac{a}{b+cx} &= a\frac{1}{b+cx} \\ &= \frac{a}{b}\frac{1}{1+(\frac{c}{b})x} \end{aligned}$$

Because of the linearity of coefficient extraction we can see that the coefficient of x^k will be $\frac{a}{b} * (-\frac{c}{b})^k$

(3) (Problem 3.4.9) Suppose the EGF of $\{c_n\}_{n\geq 0}$ is $(e^x-1)^2$. Find a formula for c_n.

Answer: Again we can just do a little bit of algebra to simplify the EGF,

$$\begin{aligned} (e^x-1)^2 &= (e^x-1)(e^x-1) \\ &= e^{2x} - 2e^x + 1 \end{aligned}$$

Then from here we split up the EGF and solve. We get $c_n = 2^n - 2$.

(4) (Problem 3.4.11) Here is how Euler proved that the binary representation of any nonnegative integer is unique. For $n \geq 0$, let b_n denote the number of ways to write n as a sum of powers of 2. Let $B(x)$ be the OGF of $\{b_n\}_{n\geq 0}$.

(a) Explain why $B(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\cdots$.
Answer: Consider that $n = z_0 + z_1 + z_2 + ..$ and that for each $z_i \in \{0, 2^i\}$ If we wanted to find the number of solutions to n with the given constraint on z_i the corresponding generating function would be, $B(x)$.

(b) Explain why $B(x) = (1+x)B(x^2)$.
Answer: We NTS that the LHS can simplify to

$$B(x) = \prod_{i=0}^{\infty}(1+x^{2^i})$$

We can do this through some algebra,

$$\begin{aligned}(1+x)B(x^2) &= (1+x)(1+x^{2*1})(1+x^{2*4})(1+x^{2*8})... \\ &= (1+x)(1+x^2)(1+x^8)(1+x^{16})... \\ &= \prod_{i=0}^{\infty}(1+x^{2^i})\end{aligned}$$

(c) Use part (b) to prove that $b_n = 1$ for all $n \geq 0$.

Answer: So first of all lets expand the first two terms,

$$\begin{aligned}B(n) &= (1+x)(1+x^2)\prod_{i=2}^{\infty}(1+x^{2^i}) \\ &= (1+x+x^2+x^3)\prod_{i=2}^{\infty}(1+x^{2^i}) \\ &= (1+x+x^2+x^3)(1+x^4)\prod_{i=3}^{\infty}(1+x^{2^i}) \text{ Moving the third term out} \\ &= (1+x+x^2+x^3+x^4+x^5+x^6+x^7)\prod_{i=3}^{\infty}(1+x^{2^i})\end{aligned}$$

As you can see if we continue the infinite product simplifies to a geometric series,

$$\sum_{n=0}^{\infty} x^k$$

and we know the coefficient on each term will always be 1. Thus $b_n = 1$ for all $n \geq 0$.

(5) (Problem 3.5.1) Solve the following recurrence relations using the generating function technique.

(a) $a_0 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 1$

Answer: First suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the OGF that describes the sequence $\{a_n\}_{n\geq 0}$. Now through some algebra on the recurrence we can get a concise form for our OGF,

$$\begin{aligned}
a_n &= 2a_{n-1} + 1 \\
a_n x^n &= 2a_{n-1}x^n + x^n \text{ Multiply through by } x^n \\
\sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} 2a_{n-1}x^n + \sum_{n=1}^{\infty} x^n \text{ Summing over values which the recurrence is defined} \\
\sum_{n=0}^{\infty} a_n x^n - a_0 &= 2x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} + \sum_{n=1}^{\infty} x^n \text{ Simplifying each sum} \\
f(x) &= 2xf(x) + \frac{x}{1-x} \text{ Substituting concise OGF} \\
f(x) &= \frac{x}{(1-x)(1-2x)} \text{ Solving for } f(x) \\
f(x) &= \frac{-x}{(1-x)} + \frac{2x}{1-2x} \text{ Partial Fraction} \\
a_n &= 2^n + 1
\end{aligned}$$

Thus the closed form for the given recurrence relation is $a_n = 2^n + 1$.

(c) $c_0 = 1$ and $c_n = 3c_{n-1} + 3^n$ for $n \geq 1$

Answer: First suppose that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is the OGF that describes the sequence $\{c_n\}_{n\geq 0}$. Now through some algebra on the recurrence we can get a concise form for our OGF,

$$\begin{aligned}
c_n &= 3c_{n-1} + 3^n \\
c_n x^n &= 3c_{n-1}x^n + 3^n x^n \text{ Multiply through by } x^n \\
\sum_{n=1}^{\infty} c_n x^n &= \sum_{n=1}^{\infty} 3c_{n-1}x^n + \sum_{n=1}^{\infty} 3^n x^n \text{ Summing over values which the recurrence is defined} \\
\sum_{n=0}^{\infty} c_n x^n - c_0 &= 3x \sum_{n=1}^{\infty} c_{n-1}x^{n-1} + \sum_{n=1}^{\infty} 3^n x^n \text{ Simplifying each sum} \\
f(x) - 1 &= 3xf(x) + \frac{1}{(1-3x)} - 1 \text{ Substituting concise OGF} \\
f(x) &= \frac{3}{(1-3x)^2} \text{ Solving for } f(x) \\
c_n &= 3^n \left(\!\!\binom{2}{n}\!\!\right) \\
c_n &= (n+1)3^n \text{ Simplifying multiset}
\end{aligned}$$

Thus the closed form for the given recurrence relation is $c_n = (n+1)3^n$.

(e) $e_0 = e_1 = 1$, $e_2 = 2$ and $e_n = 3e_{n-1} - 3e_{n-2} + e_{n-3}$ for $n \geq 3$

Answer: First suppose that $f(x) = \sum_{n=0}^{\infty} e_n x^n$ is the OGF that describes the sequence $\{e_n\}_{n\geq 0}$. Now through some algebra on the recurrence we can get a concise form for our OGF,

$$\begin{aligned}
e_n &= 3e_{n-1} - 3e_{n-2} + e_{n-3} \\
e_n x^n &= 3e_{n-1}x^n - 3e_{n-2}x^n + e_{n-3}x^n \text{ Multiply through by } x^n \\
\sum_{n=3}^{\infty} e_n x^n &= \sum_{n=3}^{\infty} 3e_{n-1}x^n - \sum_{n=3}^{\infty} 3e_{n-2}x^n + \sum_{n=3}^{\infty} e_{n-3}x^n \text{ Summing the recurrence} \\
\sum_{n=0}^{\infty} e_n x^n - (e_0 + e_1 + e_2) &= 3x \sum_{n=0}^{\infty} e_{n-1}x^{n-1} - (e_0 + e_1) - 3x^2 \sum_{n=0}^{\infty} e_{n-2}x^{n-2} - (e_0) + x^3 \sum_{n=3}^{\infty} e_{n-3}x^{n-3} \\
f(x) - (e_0 + e_1 + e_2) &= 3x(f(x) - (e_0 + e_1)) - 3x^2(f(x) - e_0) + x^3 f(x) \text{ Substituting concise OGF} \\
f(x) &= \frac{-3x^2}{(x-1)^3} + \frac{6x}{(x-1)^3} - \frac{4}{(x-1)^3} \text{ Solving for } f(x) \\
e_n &= -3\left(\!\!\binom{3}{n-2}\!\!\right) + 6\left(\!\!\binom{3}{n-1}\!\!\right) - 4\left(\!\!\binom{3}{n}\!\!\right)
\end{aligned}$$

Thus the closed form for the given recurrence relation is $e_n = -3\left(\!\binom{3}{n-2}\!\right) + 6\left(\!\binom{3}{n-1}\!\right) - 4\left(\!\binom{3}{n}\!\right)$.

(6) (Problem 3.5.2) Use an EGF to solve the recurrence relation $a_0 = 2$ and $a_n = na_{n-1} - n!$ for $n \geq 1$.
Answer: First suppose that $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ is the EGF that describes the sequence $\{a_n\}_{n\geq 0}$. Now through some algebra on the recurrence we can get a concise form for our EGF,

$$\begin{aligned}
a_n &= na_{n-1} - n! \\
a_n \frac{x^n}{n!} &= na_{n-1}\frac{x^n}{n!} - n!\frac{x^n}{n!} \text{ Multiply through by } \frac{x^n}{n!} \\
\sum_{n=1}^{\infty} a_n \frac{x^n}{n!} &= \sum_{n=1}^{\infty} na_{n-1}\frac{x^n}{n!} - \sum_{n=1}^{\infty} n!\frac{x^n}{n!} \text{ Summing over recurrence} \\
\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} - a_0 &= x\sum_{n=1}^{\infty} a_{n-1}\frac{x^{n-1}}{(n-1)!} - \sum_{n=1}^{\infty} n!\frac{x^n}{n!} \text{ Simplifying the sum} \\
f(x) - 2 &= xf(x) - (\frac{1}{1-x} - 1) \text{ Substituting concise EGF} \\
f(x) &= \frac{3}{1-x} - \frac{1}{(1-x)^2} \text{ Solving for } f(x) \\
a_n &= 3n! - n!\left(\!\!\binom{2}{n}\!\!\right) \\
a_n &= (2-n)n! \text{ Simplifying multiset}
\end{aligned}$$

Thus the closed form for the given recurrence relation is $a_n = (2-n)n!$

(7) (Problem 3.5.4) Find a formula for the nth term of the sequence defined by the recurrence relation $E_n = nE_{n-1} + (-1)^n$ for $n \geq 1$ and $E_0 = 1$. Also, what is the relationship between E_n and D_n, the

number of derangements of [n]?

Answer: First suppose that $f(x) = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}$ is the EGF that describes the sequence $\{E_n\}_{n\geq 0}$. Now through some algebra on the recurrence we can get a concise form for our EGF,

$$\begin{aligned}
E_n &= nE_{n-1} + (-1)^n \\
E_n \frac{x^n}{n!} &= nE_{n-1}\frac{x^n}{n!} + (-1)^n \frac{x^n}{n!} \text{ Multiply through by } \frac{x^n}{n!} \\
\sum_{n=1}^{\infty} E_n \frac{x^n}{n!} &= \sum_{n=1}^{\infty} nE_{n-1}\frac{x^n}{n!} + \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n!} \text{ Summing over recurrence} \\
\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} - E_0 &= x\sum_{n=1}^{\infty} E_{n-1}\frac{x^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n!} \text{ Simplifying the sum} \\
f(x) - 1 &= xf(x) + (e^-x - 1) \text{ Substituting concise EGF} \\
f(x) &= \frac{1}{(1-x)}e^-x \\
E_n &= \sum_{i=0}^{n} \binom{n}{i}(-1)^i (n-i)! \text{ By convolution of EGF} \\
E_n &= n!\sum_{i=0}^{n} \frac{-1^i}{i!}
\end{aligned}$$

Thus the closed form for the given recurrence relation is $E_n = n!\sum_{i=0}^{n} \frac{-1^i}{i!}$. In homework 6 we showed that the number of derangements of [n] is,

$$D_n = \sum_{k=0}^{n}(-1)^k (n-k)!\binom{n}{k}$$

It should be clear that when we used convolution on $f(x)$ we got the same thing,

$$E_n = \sum_{i=0}^{n}\binom{n}{i}(-1)^i (n-i)!$$

Thus it must also be true that $E_n = D_n$