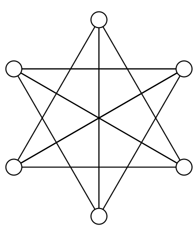
(1) (Problem 6.3.3) Prove that $\chi(G) \leq \Delta(G) + 1$ where $\Delta(G)$ is the maximum degree of G.

Proof: (Induction):
Base Case: Suppose G_n is a graph with one vertex. Then $\chi(G) = 1$ and $\Delta(G) + 1 = 1$, since $1 \leq 1$ the base case is true. We will proceed by induction on the number of vertices.

Induction Step: Suppose $\chi(G_n) \leq \Delta(G_n) + 1$ is true for any graph G_n on n vertices, we want to show that $\chi(G_{n+1}) \leq \Delta(G_{n+1}) + 1$. Now suppose graph G_{n+1}. Note that if we remove vertex from graph G_{n+1}, we get G_n. Say we remove some vertex i from G_{n+1}. Now we have vertex i, whose $deg(i) \leq \Delta(G_{n+1})$ and graph G_n, which by the induction hypothesis has a chromatic number $\chi(G_n) \leq \Delta(G_n) + 1$ which means we can add back in another vertex j to G_n whose color had previously gone unused (because our color set is greater than or equal to the maximum degree of the vertex removed). Thus we have a proper coloring for $G_n + 1$ on at most $\Delta G_{n+1} + 1$ colors, thus $\chi(G_{n+1}) \leq \Delta(G_{n+1}) + 1$

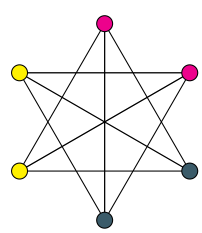
(2) (Problem 6.3.6) Determine, with proof, the chromatic numbers of the graphs below:

G_1



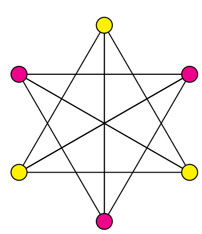
Proof: Consider graph H, which illustrates a proper coloring on 3 colors.

H

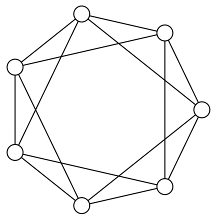


If we try to make a proper coloring of the graph G_1 on 2 colors we have a problem because of the odd cycles in A below,

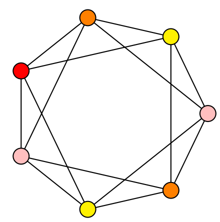
A



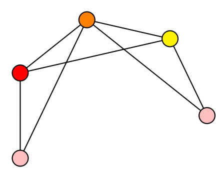
Thus $\chi(G_1) = 3$.

G_2 

Proof: Consider graph J, that illustrates a proper coloring of G_2 on 4 colors.

J 

Now consider the subgraph I,

I 

There exists no proper coloring of subgraph I with 3 colors, because of the three C_3 contained within. Thus $\chi(G_2) = 4$.

(3) (Problem 6.3.11) Find $p(K_n - e, k)$ where e is any edge of K_n.

Proof: The chromatic polynomial for any K_n is denoted by,

$$p(K_n, k) = x(x-1)(x-2)...(x-n-1)$$

This from the fact that since a completed graph has every vertex adjacent to the others we have to let $\chi(K_n) = n$, and the way to do that is to make sure that $p(K_n, 0 \leq k \leq n-1) = 0$. From here all we have to do is use Theorem 6.3.3,

$$p(G, k) + P(G \cdot e, k) = p(G - e, k)$$

And we get,

$$p(G - e, k) = x(x-1)(x-2)...(x-n-2) + x(x-1)(x-2)...(x-n-1).$$

(4) (Problem 6.3.15) In the chromatic polynomial of a graph G, prove that if k^m is the smallest power of k that has a nonzero coefficient, then G has m components.

Proof: (Induction):
Induction HypothesidInduction Hypothesis: If k^m is the smallest power of k that has a nonzero coefficient of the chromatic polynomial of graph G, then G has m components.

Base Case: Suppose $e(G) = 0$ then a graph with zero edges on n vertices has the chromatic polynomial, $P(G, k) = k^n$. Since k^n is the smallest (only) power of k with a non zero coefficient(coefficient is 1), and we know that G has n components, because it has n vertices and no edges, Induction Hypothesis is true, when $e(G) = 0$. We will proceed by induction on the number of edges.

Induction Step: Now let $e(G) = i$ such that $i \geq 1$ and that Induction Hypothesis holds for all G where $e(G) = i - 1$, such that $i \geq 1$. By Theorem 6.3.3,

$$p(G, k) = p(G - e, k) - P(G \cdot e, k)$$

Since both graphs $G - e$ and $G \cdot e$ have $i - 1$ edges we can apply proposition A. Here there are 2 cases, Case 1: where $G - e$ results in a disconnected graph. Case 2: Where $G - e$ results in a connected graph.

Case 1: $G - e$ is disconnected. If $G - e$ is disconnected we can assume that the chromatic polynomials are of the form,

$$p(G - e, k) = k^n - (e(g) - 1)k^{n-1} + \sum_{i=2}^{n-2} (-1)^{n-1} a_i k^i$$

$$p(G \cdot e, k) = k^{n-1} - (e(g) - 1)k^{n-2} + \sum_{j=1}^{n-3} (-1)^{n-1-j} b_j k^j$$

Expanding, the sums so we can see the last few terms,

$$p(G - e, k) = k^n - (e(g) - 1)k^{n-1} + ... - a_3 k^3 + a_2 k^2$$

$$p(G \cdot e, k) = k^{n-1} - (e(g) - 1)k^{n-2} + ... - b_2 k^2 + b_1 k$$

We can see that the chromatic polynomial for $G - e$ has k^2 as the smallest power of k which makes since because Induction Hypothesis holds for graphs G such that $e(G) = i - 1$ and since G is a connected graph and $G - e$ is disconnected, $G - e$ must have 2 components. Applying Theorem 6.3.3 we get,

$$\begin{aligned} p(G, k) &= (k^n - (e(g) - 1)k^{n-1} + ... - a_3 k^3 + a_2 k^2) - (k^{n-1} - (e(g) - 1)k^{n-2} + ... - b_2 k^2 + b_1 k) \\ &= k^n - (e(g) - 1 + 1)k^{n-1} + (a_{n-2} + (e(g) - 1)k^{n-2} - ... - b_1 k \end{aligned}$$

Since b_1 is non zero we have shown that G is one component, which is true given our claim that $G - e$ is disconnected. Thus Induction Hypothesis holds in the case that $G - e$ is disconnected.

Case 2: $G - e$ is a connected graph. If $G - e$ is connected we can assume that the chromatic polynomials are of the form,

$$p(G - e, k) = k^n - (e(g) - 1)k^{n-1} + \sum_{i=1}^{n-2} (-1)^{n-1} a_i k^i$$

$$p(G \cdot e, k) = k^{n-1} - (e(g) - 1)k^{n-2} + \sum_{j=1}^{n-3} (-1)^{n-1-j} b_j k^j$$

Expanding, the sums so we can see the last few terms,

$$p(G-e,k) = k^n - (e(g)-1)k^{n-1} + ... - a_3k^3 + a_2k^2 - a_1k$$
$$p(G\cdot e,k) = k^{n-1} - (e(g)-1)k^{n-2} + ... - b_2k^2 + b_1k$$

Applying Theorem 6.3.3 we get,

$$\begin{aligned} p(G,k) &= (k^n - (e(g)-1)k^{n-1} + ... - a_3k^3 + a_2k^2 - a_1k) - (k^{n-1} - (e(g)-1)k^{n-2} + ... - b_2k^2 + b_1k) \\ &= k^n - (e(g)-1+1)k^{n-1} + (a_{n-2} + (e(g)-1)k^{n-2} - ... - (a_1+b_1)k \end{aligned}$$

Since $a_1 + b_1$ is non zero we have shown that G is one component, which is true given our claim that $G-e$ is connected. Thus Induction Hypothesis holds in the case that $G-e$ is connected.