Reminder: Any time you are asked to answer a question, you need to provide a rigorous justification of your answer. (ie Don't just give a bald answer!)

(1) (Problem 3.1.7) Let D_n be the number of derangement for a general n.

(a) Calculate $\lim_{n\to\infty} \frac{D_n}{n!}$. Interpret your result.

Answer: Consider that D_n counts the number of permutations of $[n]$ such that there are no fixed elements. Therefore using inclusion-exclusion principle, we can find a function for D_n,

$$D_n = \sum_{k=0}^{n} (-1)^k (n-k)! \binom{n}{k}$$

From here we can substitute, into the given limit,

$$\lim_{n\to\infty} \frac{1}{n!} * \sum_{k=0}^{n} (-1)^k (n-k)! \binom{n}{k}$$

From here we can simplify the sum for D_n,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n!} * \sum_{k=0}^{n} (-1)^k (n-k)! \binom{n}{k} &= \lim_{n\to\infty} \frac{1}{n!} * \sum_{k=0}^{n} (-1)^k (n-k)! \frac{n!}{k! * (n-k)!} \\
&= \lim_{n\to\infty} \frac{1}{n!} * \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \\
&= \lim_{n\to\infty} \frac{1}{n!} * (n!) \sum_{k=0}^{n} \frac{(-1)^k}{k!} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}
\end{aligned}$$

Here we need to recognize that the sum that we have is in the form of the power series expansion for e^x.

$$e^x = \sum_{k=0}^{\infty} \frac{(x)^k}{k!}$$

We can see that since $x = -1$ we get,

$$\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}$$

What we have shown is that $\frac{D_n}{n!}$ is equivalent to the n^{th} degree Taylor Polynomial of $\frac{1}{e}$. Combinatorially we have show that the ratio of derangements to total permutations converges to $\frac{1}{e}$.

(b) Prove that for any n, D_n equal the closest integer to $n!/e$.

Answer: Let's recall the Alternating Series Remainder Theorem, which states that if an alternating series converges to S, then the n^{th} partial sum S_n and the corresponding remainder R_n can be defined as follows,

$$S_n + R_n = S$$

Such that,

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$$
$$R_n = \sum_{k=n+1} (-1)^{k+1} a_k$$

Then by substitution we know that,

$$R_n = S - \sum_{k=1}^{n} (-1)^{k+1} a_k$$

What we are being asked to prove is,

$$|\frac{n!}{e} - D_n| < \frac{1}{2}$$

For all n. Since we have proven that D_n converges to $\frac{n!}{e}$, it must be true that as n increases the remainder gets smaller and smaller, as one would expect. So let $n = 1$ and, through some algebra we get,

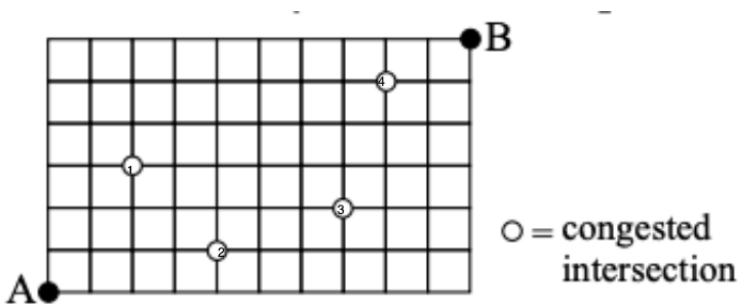
$$\frac{1!}{e} - n! \sum_{k=0}^{1} \frac{(-1)^k}{k!} = \frac{1!}{e} - (1!) * (1 - 1)$$
$$= \frac{1}{e}$$

Thus since $\frac{1}{e} < \frac{1}{2}$ the statement holds for all $n \geq 1$.

(2) (Problem 3.1.9) Generalize the previous problem: In how many ways can you distribute k identical objects to n distinct recipients so that each recipient receives at most r objects?
Answer: Let P_i be the property that i^{th} recipient receives $r+1$ or more objects. Therefore we want a formula for $N_=(\theta)$. Using Inclusion-Exclusion we get

$$N_=(\theta) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(\binom{n}{k - i(r+1)}\right)$$

(3) (Problem 3.1.17) A taxi drives from the intersection labeled A to the intersection labeled B in the grid of streets shown below. The driver only drives north (up) ie east (right).



Traffic reports indicate that there is heavy congestion at the intersections identified. How many routes from A to B can the driver take that ...

(a) avoid all congested intersections?

Answer: Let P_i be the property that a route has the i^{th} congested intersection. Therefore we want a formula for $N_=(\theta)$.

$$\begin{aligned} N_=(\theta) &= p_0 - ((p_1 + p_2 + p_3 + p_4) - (p_{14} + p_{23} + p_{24} + p_{34}) - 2(p_{123})) \\ &= 8008 - 9081 + 3022 + 480 \\ &= 2429 \end{aligned}$$

$5 \left(3\binom{11}{3} - \binom{5}{1}\binom{11}{5} - \binom{9}{2}\binom{7}{4} - \binom{13}{5}\binom{3}{1} \right)$

(b) pass through at most one congested intersection?

Answer: Consider inclusion exclusion such that we want $N_\leq(1)$. Therefore

$$\begin{aligned} N_\leq(1) &= p_0 - ((p_{14} + p_{23} + p_{24} + p_{34}) - (2p_{123})) \\ &= 8008 - 3022 + 480 \\ &= 5466 \end{aligned}$$