3.2 #3,7,9; 3.3 #2,3,4,7

(1) (Problem 3.2.3) Prove: for $n \geq 2$, $\displaystyle\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n}$.

Proof: (Induction) Consider the following statement S_n,

$$S_n = \prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n} \text{ for all } n,\ n \geq 2.$$

Base Case: Let $n = 2$,

$$\begin{aligned}\prod_{j=2}^{2}\left(1-\frac{1}{j^2}\right) &= 1-\frac{1}{2^2}\\ &= \frac{2^2}{2^2}-\frac{1}{2^2}\\ &= \frac{4-1}{2*2}\\ &= \frac{3}{2*2}\\ &= \frac{2+1}{2*2}.\end{aligned}$$

From here it is clear that since S_2 can be reduced to a statement of the form $\frac{n+1}{2n}$ we can conclude that S_2 is true.

Induction: Assume induction hypothesis S_k,

$$S_k = \prod_{j=2}^{k}\left(1-\frac{1}{j^2}\right)=\frac{k+1}{2k} \text{ for all } k,\ k \geq 2.$$

We want to show that S_{k+1} is true,

$$S_{k+1} = \prod_{j=2}^{k+1}\left(1-\frac{1}{j^2}\right)=\frac{(k+1)+1}{2(k+1)}.$$

Through some algebra on the LHS of S_{k+1},

$$\begin{aligned}\prod_{j=2}^{k+1}\left(1-\frac{1}{j^2}\right) &= \prod_{j=2}^{k}\left(1-\frac{1}{j^2}\right)*(1-\frac{1}{(k+1)^2}) \text{ pulling out last product}\\ &= \frac{k+1}{2k}*(1-\frac{1}{(k+1)^2}) \text{ substituting induction hypothesis}\\ &= \frac{k+1}{2k}*\frac{k^2+2k}{(k+1)^2} \text{ final steps are just algebra}\\ &= \frac{1}{2k}*\frac{k^2+2k}{(k+1)}\\ &= \frac{k^2+2k}{2k(k+1)}\\ &= \frac{k(k+2)}{2k(k+1)}\\ &= \frac{k+2}{2(k+1)}\\ &= \frac{(k+1)+1}{2(k+1)}.\end{aligned}$$

Therefore S_{k+1} is true. Thus by induction we have proven that S_n is true for all $n \geq 2$.

(2) (Problem 3.2.7) Give a combinatorial proof: for $n \geq 1$, $\sum_{j=1}^{n} j! < (n+1)!$.

Proof: Consider that the RHS of the inequality is the number of permutations on the set $[n+1]$. We can see by expanding the sum on the LHS,

$$\sum_{j=1}^{n} j! = 1! + 2! + 3! ... n!$$

What we want so prove is that LHS partitions some subset S, of the number of permutations on the set $[n+1]$ such that $|S| < n+1$ for $n \geq 1$.

(3) (Problem 3.2.9) For the recursive relation shown in (3.6) on page 99 we proved $L_n < 2^n$ for $n \geq 1$.

(a) Prove the tighter inequality $L_n \leq 1.7^n$. At what value should you start the induction?

Proof: (Strong Induction) Consider the following statement S_n,

$$S_n = L_n \leq 1.7^n$$

Where $L_n = L_{n-1} + L_{n-2}$ and $L_1 = 1$ and $L_0 = 2$.

Base Case: Let $n = 1$, then by definition $L_1 = 1$ and $1.7^1 = 1.7$ this it follow that $L_1 < 1.7^1$ so S_1 is true. Let $n = 2$, then by definition $L_2 = L_1 + L_0 = 3$ and $1.7^2 = 2.98$ thus it follows that $L_2 \approx 1.7^2$ so S_2 is true.

Induction: Since S_2 is an approximation it is best to start induction when $n = 3$ because it is incorrect to assume the induction hypothesis for $n = 2$. Assume Induction hypothesis S_k,

$$S_k = L_k \leq 1.7^k \text{ when } k \geq 3 \tag{1}$$

We want to show that S_{k+1} is true,

$$S_{k+1} = L_{k+1} \leq 1.7^{k+1}$$

So looking at the LHS of the inequality,

$$\begin{aligned}
L_{k+1} &= L_k + L_{k-1} \text{ by definition of } L_k \\
&\leq 1.7^k + 1.7^{k-1} \text{ substitution of induction hypothesis} \\
&\leq 1.7^{k-1}(1.7 + 1) \\
&\leq 1.7^{k-1} 2.7 \\
&\leq 1.7^{k-1} 2.89 \text{ because } 2.7 < 2.89 \\
&\leq 1.7^{k-1} 1.7^2 \text{ because } 1.7^2 = 2.89 \\
&\leq 1.7^{k+1}
\end{aligned}$$

Therefore S_{k+1} is true. Thus by induction we have proven that S_n is true for all $n \geq 3$.

(b) What is so special about the number 1.7? Adjust your work in part (a) to create the tightest bound that you can.

Proof: Consider the following inequality from our induction step,

$$x + 1 < x^2$$

This inequality illustrates exactly what is so special about the number 1.7, you can see that when we let $x = 1.7$ we get,

$$2.7 < 2.89$$

So in order to create a tighter bound we just need to solve the inequality. When we solve the inequality we get that $x < 1.618$. So the inequality $L_n \leq x^n$ is true for all $x < 1.618$.

(4) (Problem 3.3.2) In each case, find a concise OGF for answer the question and also identify what coefficient you need.

(a) How many ways are there to distribute 14 forks to 10 people so that each person receives one or two forks?
Answer: Each person has either 1 or 2 forks so our generating function is $(x + x^2)^{10}$ and we want the coefficient of x^{14}.

(b) You can buy soda either by the can, or in 6-, 12-, 24-, or 30-packs. How many ways are there to buy exactly k cans of soda?
Answer: Consider the following solutions to k, $x_1 + x_2 + x_3 + x_4 = k$ such that $x_1 \in \{0, 6, 12, 18, ...\}$ $x_2 \in \{0, 12, 24, ...\}$ $x_3 \in \{0, 24, 48, ...\}$ $x_4 \in \{0, 30, 60, ...\}$. Therefore we are looking for the coefficient of x^k of the OGF $(1 + x^6 + x^{12} + x^{18} + ...)(1 + x^{12} + x^{24} + ...)(1 + x^{24} + x^{48} + ...)(1 + x^{30} + x^{60} + ...)$ The concise form of the OGF is $\frac{1}{(1-x^6)(1-x^{12})(1-x^{24})(1-x^{30})}$

(c) How many ways are there to put a total postage of 75 cents on an envelope using 3-, 5-, 10-, and 12-cent stamps?
Answer: Here we do the same thing as we did for the other problem, so the OGF is $\frac{1}{(1-x^3)(1-x^5)(1-x^{10})(1-x^{12})}$ and we want to find the coefficient of x^{75}.

(d) At the movies you select 24 pieces of candy from among five different types. How many ways can you do this if you want at least two pieces of each type?
Answer: There are five different types of candy, and if there are two of each type of candy we can consider the solutions to the following equation, $x_1 + x_2 + x_3 + x_4 + x_5 = 24$ such that $x_i \in \{2, 3, 4, ...\}$ thus, the concise OGF is $\frac{x^10}{(1-x)^5}$ and we want the coefficient of the x^24 term.

(e) How many solutions to $z_1 + z_2 + z_3 = 15$ are there, where thez_i are integers satisfying $0 \leq z_i \leq 8$?
Answer: Consider the OGF $(1 + x + x^2 + x^3 + .. + x^8)^3$, where we want to find the coefficient of the x^{15} term.

(f) How many ways are there to make change for a dollar using only pennies, nickels, dimes and quarters?
Answer: Consider the solutions to the equation $x_1 + x_2 + x_3 + x_4 = 100$ such that $x_1 \in \{0, 1, 2, 3, ...\}$ $x_2 \in \{0, 5, 10, ...\}$ $x_3 \in \{0, 10, 20, ...\}$ $x_4 \in \{0, 25, 50, ...\}$. Then we get the concise generating function of the form $\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$ where we want the coefficient on the x^{100} term.

(5) (Problem 3.3.3) Find the coefficient of ...

(a) x^{60} in $\frac{1}{(1-x)^{23}}$
Answer: $\left(\!\!\binom{23}{60}\!\!\right)$

(b) x^k in $\frac{1+x+x^4}{(1-x)^5}$
Answer: $\left(\!\!\binom{5}{k}\!\!\right)+\left(\!\!\binom{5}{k-1}\!\!\right)+\left(\!\!\binom{5}{k-4}\!\!\right)$

(c) x^3 in $\frac{x}{(1-x)^8}$
Answer: $\left(\!\!\binom{8}{2}\!\!\right)$

(d) x^{50} in $(x^9 + x^{10} + x^{11} + \cdots)^3$
Answer: Simplifying we get x^{50} in $\frac{x^{27}}{(1-x)^3}$. Thus $\left(\!\!\binom{3}{23}\!\!\right)$

(e) x^{k-1} in $\frac{1+x}{(1-2x)^5}$
Answer: $2^{k-1}\left(\!\!\binom{5}{k-1}\!\!\right)+2^{k-2}\left(\!\!\binom{5}{k-2}\!\!\right)$

(6) (Problem 3.3.4) A professor grades an exam that has 20 questions worth five points each. The professor awards zero, two , four, or five points on each problem. Find a concise OGF that can be used to determine the number of ways to obtain an exam score of k points.

Answer: The OGF that enumerates the scores is $(x + x^2 + x^4 + x^5)^{20}$ where the number of ways to get k points is the coefficient on the x^k term.

(7) (Problem 3.3.7) Find the number of solutions to $z_1 + z_2 + z_3 + z_4 = 10$ where the z_i are nonnegative integers such that $z_1 \leq 4$, z_2 is odd, z_3 is prime, and $z_4 \in \{1, 2, 3, 6, 8\}$.

Answer: Consider the OGF $(1 + x + x^2 + x^3 + x^4)(x + x^3 + x^5 + x^7 + x^9)(x^2 + x^3 + x^5 + x^7)(x + x^2 + x^3 + x^6 + x^8)$ then all we need to do is find the coefficient on the x^{10} term. Thus there are 24 solutions.