Directions: For all numerical problems, a *complete* solutions involves a calculation that ends in a numerical value and a rationale for that calculation.

(1) Problem 2.1.4 Consider the possible function $f : [7] \to [9]$.

(a) How many have $f(3) = 8$? How many have $f(3) \neq 8$?
Answer: The total number of functions from $[7] \to [9]$ is counted by 9^7. To count the number of function from $[7] \to [9]$ such that $f(3) = 8$ we know that 3 in the domain is already mapped so all we have to do is count the rest of the functions, $9^6 = 531441$. To count the number of functions that don't have 3 in the domain mapped to 8 we can just calculate the complement, $9^7 - 9^6 = 4251528$

(b) How many have $f(1) \neq 5$ and are one-to-one?
Answer: To count the number of one-to-one functions that don't map 1 to 5 all we have to do is assign and count the number of boxes for the number one ball and then assign the rest, so $8 * 8_6 = 161280$.

(c) How many have $f(i)$ even for all i?
Answer: Since there are 4 even numbers in the set $[9]$ we know that each element in the domain $[7]$ is have 4 possible elements to map to, thus $4^7 = 16384$.

(d) How many have $rng(f) = \{5, 6\}$?
Answer: There are $2^7 - 2 = 126$ functions that have a $rng(f) = \{5, 6\}$. 2^7 because each element in the codomain, ie. 5 and 6 can have up to 7 elements mapped. The -2 is ti remove the two functions where the $rng(f) = \{5\}$ and $rng(f) = \{6\}$.

(e) How many in which f^{-1} is not a function?
Answer: Since there exists no bijective functions from $[7] \to [9]$, every function from $[7] \to [9]$ has an inverse which is not a function, $9^7 = 4782969$.

(2) Find the number of onto functions $[k] \to [4]$.
Answer: The number of onto function from $[k] \to [4]$ can be calculated by $S(k, 4) * 4!$ such that $k \geq 4$.

(3) Problem 2.1.11 Give a combinatorial proof: For $n \geq 1$ and $k \geq 1$, $2^{kn} > \max\{n^k, k^n\}$. (Hint: Compare relations to functions.)
Proof: Consider that 2^{kn} counts the number of relations from sets k and n, also note that the number of functions from one set to another is counted by n^k or k^n. We know by how functions and relations are defined that the set of all functions is a subset of the set of all relations. Therefore it must be true that $2^{kn} > \max\{n^k, k^n\}$.

(4) Problem 2.2.4 Give combinatorial or bijective proofs of the following. Part of your job is to determine all values of n, k, and/or m for which the identities are valid.

(a) $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}$

Answer: Consider the total of n - digit ternary numbers, we know that for each digit there is 3 options, so 3^n. We can partition the set of n - digit ternary numbers by the number of k digits that are 2s. Consider $n = 3$, then,

$$\begin{aligned}
3^3 &= \binom{3}{0} * 2^{3-0} + \binom{3}{1} * 2^{3-1} + \binom{3}{2} * 2^{3-2} + \binom{3}{3} * 2^{3-3} \\
27 &= 1 * 2^3 + 3 * 2^2 + 3 * 2^1 + 1 * 2^0 \\
&= 8 + 12 + 6 + 1 \\
&= 27
\end{aligned}$$

We can see that the choose statement serves to identify the k 2s in the ternary number, then the $2^{(n-k)}$ term counts the number of ways to fill the rest of the $n - k$ digits. Note that the last term in the sum will always correspond to the ternary number that contains all 2. For our proof $n \geq 1$ even though the the equation is still satisfied when $n = 0$ it doesn't make sense to have a zero digit ternary number.

(b) $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$

Answer: First consider a k size committee of n people, and then of the k people in that committee choose j as department heads. The right side of the equation simply chooses the n department heads first and then fills the rest of the k size committee.

(d) $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{k+1}{n-1}\!\right)$

Answer: Simplifying the expression above into a choose statement,

$$\begin{aligned}
\left(\!\binom{n}{k}\!\right) &= \left(\!\binom{k+1}{n-1}\!\right) \\
\binom{n+k-1}{k} &= \binom{(k+1)+(n-1)-1}{n-1} \\
\binom{k+n-1}{k} &= \binom{k+n-1}{n-1}
\end{aligned}$$

Now it is clear that, the expression above simply serves to invert the multi-set formula, for example instead of picking k donuts out of n flavors we can pick $n - 1$ donuts out of $k + 1$ flavors. Things become even clearer when we reduce to the combination formula and think of a $k + n - 1$ length binary digit because we can see that when we choose k of those digits (right side of equation) it is the same as choosing $n - 1$ of those digits. $n \geq k \geq j$ such that $n \geq 1$.

(5) Problem 2.2.5 What does $\binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \cdots + \binom{k-1}{k-1}$ equal? Make a conjecture and then give a combinatorial proof.

Proof: Consider Pascal's rule

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Now suppose we iterate Pascal's rule over the last term, thus,

$$\begin{aligned}\binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k} \\ &= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \binom{n-3}{k}\end{aligned}$$

The end sum will be,

(1) $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \binom{n-3}{k} + \ \dots \ + \binom{k-1}{k-1} + \binom{k-1}{k}$$

We can see that following the $\binom{k-1}{k-1}$ term the rest of the iterations will be zero. For this in terms of a combinatorial proof, let's first consider that $\binom{n}{k}$ counts the number of subsets with k elements from a set with n elements. We can partition these subsets on whether or not they contain the nth element, $\binom{n-1}{k-1}$ being the term that contains the nth element and $\binom{n-1}{k}$ begin the element that doesn't contain the nth element. This process is repeated on the second term until there are no more elements left.

(6) Problem 2.2.7 Determine the number of solutions to each of the following equations. Assume all z_i are nonnegative integers unless stated otherwise.

(a) $z_1 + z_2 + z_3 + z_4 = 1$
Answer: There exists only 4, integer solutions such that z_i are nonnegative integers.

(b) $z_1 + z_2 + 10z_3 = 8$
Answer: There exists only $\left(\!\binom{2}{8}\!\right)$, integer solutions simply because z_3 must always be zero, z_i are nonnegative integers.

$$\left(\!\!\binom{2}{8}\!\!\right) = \binom{9}{8} = 9$$

(c) $z_1 + z_2 + \cdots + z_{20} = 401$ where each $z_i \geq 1$
Answer: We can count these integer solutions, given the new inequality by, changing the sum. We can reduce the problem to this,

$$z_1 + z_2 + \cdots + z_{20} = 381 \text{ where each } z_i \geq 0.$$

Then we simply calculate the integer solutions through the multiset formula,

$$\left(\!\!\binom{20}{381}\!\!\right) = \binom{400}{381} = 1.464 * 10^{32}$$

(7) Problem 2.3.2 For any integer $n \geq 2$, how many onto function $[n] \to [n-1]$ are possible? Give an answer that does not involve Stirling numbers.
Answer: For a function to be onto, for every element in the codomain there must be an element in the domain that maps onto it. First lets partition the $[n]$ elements into $[n-1]$ blocks. By the PHP there will always be a block size two and the rest of the elements are in their own blocks, we can count this by $\binom{n}{2}$. Now that we have $n-1$ block all we have to do is map them to the $n-1$ elements in the codomain. Thus the number of onto function $[n] \to [n-1]$ is $\binom{n}{2} * (n-1)!$.
Thus the total number of onto functions from $[n] \to [n-1]$ is $\binom{n}{2} * (n-1)!$

(8) Problem 2.3.7 Find the number of equivalence relations on an n-set.
Answer: There exists a bijection between equivalence relations on the n-set and the number of partitions on that set, when we simply define each partition as an equivalence class. Thus if we count up the possible partitions on an n-set then we have counted all the possible equivalence relations. So the number of equivalence relations on an n-set is simply $B(n)$.