3.6 #1,2ad,4,8; 6.1 #3,6,7,10

(1) (Problem 3.6.1) Find a_{30} for the recurrence relation $a_0 = -1$, and $a_n = 1 - 3a_{n-1}$.

Answer: First we need to get this recurrence relation in a closed form. Note that $a_n = 1 - 3a_{n-1}$ is a first order linear recurrence relation, therefore we can use the sweet formula from p.(135) of the textbook,

$$a_n = a_0\alpha^n + \beta\frac{1-\alpha^n}{1-\alpha}.$$

Thus the solution to the given recurrence relation is,

$$\begin{aligned} a_n &= (-1)(-3)^n + \frac{1-(-3)^n}{1-(-3)} \\ &= -(-3)^n + \frac{1-(-3)^n}{4} \\ &= -(-3)^n - \frac{(-3)^n}{4} + \frac{1}{4} \\ &= (-3)^n(-1 - \frac{1}{4}) + \frac{1}{4} \\ &= (-\frac{5}{4})(-3)^n + \frac{1}{4}. \end{aligned}$$

And plugging in $n = 30$ we get,

$$a_{30} = (-\frac{5}{4})(-3)^{30} + \frac{1}{4} = -257363915118311$$

(2) (Problem 3.6.2.a) Solve the recurrence $a_0 = 3$, $a_1 = 7$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$.

Answer: Here we can take the same approach as we did for the last problem, because we can see that this recurrence relation is a second-order linear homogenous recurrence relation. Note that we have a case of distinct roots, because the characteristic equation that models this recurrence relation, is

$$x^2 - 3x + 2 = (x-2)(x-1)$$

Where $x = 2, 1$. Therefore plugging into the formula all we need to do is solve the following system of equations for A and B,

$$\begin{aligned} 3 &= A2^0 + B1^0 \\ 7 &= A2^1 + B1^1. \end{aligned}$$

Doing so we get that the closed form of the given recurrence relation is,

$$a_n = 4(2)^n - 1.$$

(3) (Problem 3.6.2.d) Solve the recurrence $d_0 = 10$, and $d_n = 11d_{n-1} - 10$ for $n \geq 1$.

Answer: Here we can take the exact same approach as the first problem, because we have a first order linear recurrence relation. Recall,

$$d_n = d_0\alpha^n + \beta\frac{1-\alpha^n}{1-\alpha}.$$

By substitution and some algebra,

$$\begin{aligned} d_n &= 10(11)^n + (-10)\frac{1-(11)^n}{1-(11)} \\ &= 10(11)^n - 10\frac{1-(11)^n}{-10} \\ &= 10(11)^n + 1 - (11)^n \\ &= 9(11)^n + 1. \end{aligned}$$

Thus the closed form of the given recurrence relation is,

$$d_n = 9(11)^n + 1.$$

(4) (Problem 3.6.4) Find a formula for the nth term of the sequence defined by the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$, where $a_0 = 1$ and $a_1 = 2$.
Answer: Again we can take the same approach we took for the second problem because we can see that the given recurrence is a second order linear homogenous recurrence relation. We can see what type of case we have by factoring the characteristic equation for the recurrence relation,

$$\begin{aligned} 0 &= x^2 - 2x - 3 \\ &= (x+1)(x-3) \end{aligned}$$

Since $x = -1, 3$ we have a case of distinct roots. So know all we have to do is solve the following system of equations and then 1 have a closed form of the recurrence relation,

$$\begin{aligned} 1 &= A(-1)^0 + B(3)^0 \\ 2 &= A(-1)^1 + B(3)^1. \end{aligned}$$

Doing so we get that, the closed form of the recurrence relation is,

$$a_n = \frac{1}{4}(-1)^n + \frac{3}{4}(3)^n.$$

(5) (Problem 3.6.8) Find $\lim\limits_{\alpha\to 1}\dfrac{1-\alpha^2}{1-\alpha}$. Then, explain why this clarifies the relationship between the formulas for a_n shown in Theorem 3.6.1.

Answer: We can see that to evaluate the given limit it is necessary to recall L'Hospital's Rule. So we can see that by simply plugging in $\alpha = 1$ we get,

$$\frac{1-(1)^2}{1-(1)} = \frac{0}{0}$$

Since the limit evaluates to an indeterminate form we can use L'Hospital's Rule,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

So,

$$\begin{aligned} \lim_{\alpha \to 1} \frac{1-\alpha^2}{1-\alpha} &= \lim_{\alpha \to 1} \frac{-2\alpha}{-1} \\ &= 2 \text{ by plugging in } \alpha = 1 \end{aligned}$$

Now consider Theorem 3.6.1 which provides a formula that gives a closed form to first order linear recurrence relations,

$$a_n = \begin{cases} a_0\alpha^n + \beta(\frac{1-\alpha^n}{1-\alpha}) & \alpha \neq 1 \\ a_0\alpha^n + \beta n & \alpha = 1 \end{cases}$$

Now we can see the connection between the limit calculated before and Theorem 3.6.1, consider the following,

$$\lim_{\alpha \to 1} \frac{1-\alpha^n}{1-\alpha} = \lim_{\alpha \to 1} \frac{-n\alpha}{-1} = n$$

So that's where the βn term comes from.

(Problem 6.1.3) Prove: If G is a connected graph with n vertices and $n-1$ edges where $n \geq 2$, then G has at least two vertices of degree 1.

Answer: Proof by contradiction: Suppose that If G is a connected graph with n vertices and $n-1$ edges where $n \geq 2$, then G has at most one vertices of degree 1.

Case 1: G has 0 vertices degree one. By the handshaking lemma,

$$\begin{aligned} 2e(G) &= \sum_{v \in VG} d(v) \\ 2n-2 &= \sum_{v:d(v)even} d(v) + \sum_{v:d(v)odd} d(v) \end{aligned}$$

(Problem 6.1.6) Let $G = (V, E)$ be a graph. The **complement** of G is that graph $\overline{G} = (V, E^C)$ where E^C is the complement of E relative to the edge set of $K_{n(G)}$. In other words, for all $i, j \in V(G)$ we have $\{i, j\} \in E^C$ if and only if $\{i, j\} \notin E$.

Prove that if G is isomorphic to $\overline{G}$ then either $n(G) \equiv 0(\text{mod } 4)$ or $n(G) \equiv 1(\text{mod } 4)$.

Answer:

(Problem 6.1.7) Prove that if $\delta(G) \geq k$, then G contains a path of length at least k.

Answer:

(Problem 6.1.10) Use graphs to give combinatorial proofs of the following results.

(a) $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$

Answer:

(b) Suppose $n_1, n_2, \cdots, n_k$ are positive integers. If $\sum_{i=1}^{n} n_i = n$, then

$$\sum_{i=1}^{n} \binom{n_i}{2} \leq \binom{n}{2}.$$

When does equality hold?

Answer: