Exercise Abbott 4.2.1 a,b: 1. Supply the details for how Corollary 4.2.4ii follows from the Sequential Criterion for Functional Limit Theorem for sequences proved in Chapter 2.

Proof. Suppose f and g are functions defined on the domain $A \subseteq \mathbb{R}$, and lets assume that for some point $c \in A$,

$$\lim_{x \to c} f(x) = L,$$

$$\lim_{x \to c} g(x) = M.$$

By Theorem 4.2.3 (Sequential Criterion for Functional Limits) we know that for all sequences $(x_n) \subseteq A$ which satisfy $x_n \neq c$ and $(x_n) \to c$, it must be the case that,

$$f(x_n) \to L,$$

$$g(x_n) \to M.$$

By the ALT we know that when we sum these sequences, the limit becomes the sum of the limits, therefore for all $(x_n) \subseteq A$, where $x_n \to c$ and $x_n \neq c$,

$$f(x_n) + g(x_n) \to L + M.$$

Using Theorem 4.2.3 we get back that,

$$\lim_{x \to c} f(x) + g(x) = L + M.$$

□

2. Now write another proof of Corollary 4.2.4ii directly from Definition 4.2.1 without using the Sequential Criterion in Theorem 4.2.3

Proof. Suppose f and g are functions defined on the domain $A \subseteq \mathbb{R}$, and lets assume that for some point $c \in A$,

$$\lim_{x \to c} f(x) = L,$$

$$\lim_{x \to c} g(x) = M.$$

By Definition 4.2.1 we know that for all $\epsilon > 0$ there exists a $\delta_f > 0$ such that $0 < |x - c| < \delta_f$ where it follows that,

$$|f(x) - L| < \frac{\epsilon}{2}.$$

Similarly we also know that there exists a $\delta_g > 0$ such that $0 < |x - c| < \delta_g$ where it follows that,

$$|g(x) - M| < \frac{\epsilon}{2}.$$

Let $\epsilon > 0$ and consider a $\delta = \min\{\delta_f, \delta_g\}$ to ensure we can fit inside the tolerance ϵ. Therefore whenever $0 < |x - c| < \delta$ we get,

$$\begin{aligned}
|(f(x) + g(x)) - (L + M)| &= |f(x) + g(x) - L - M|, \\
&= |f(x) - L + g(x) - M|, \\
&\leq |f(x) - L| + |g(x) - M|, \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}, \\
&< \epsilon.
\end{aligned}$$

□

Exercise Abbott 4.2.5 a,b: Use Definition 4.2.1 to supply a proper proof for the following statements.

1. $\lim_{x\to 2}(3x + 4) = 10$

   Proof. let $\epsilon > 0$. Through some algebra we get that,

   $$|3x + 4 - 10| = |3x - 6| = 3|x - 2|$$

   Now consider $\delta = \frac{\epsilon}{3}$, therefore whenever $0 < |x - 2| < \delta$,

   $$\begin{aligned}
   |3x + 4 - 10| &= 3|x - 2|, \\
   &< 3\frac{\epsilon}{3}, \\
   &< \epsilon.
   \end{aligned}$$

   □

2. $\lim_{x\to 0} x^3 = 0$

   Proof. let $\epsilon > 0$. Now consider $\delta = \epsilon^{\frac{1}{3}}$, therefore whenever $0 < |x - 0| < \delta$,

   $$\begin{aligned}
   |x^3| &= |x|^3, \\
   &< \delta^3, \\
   &< \epsilon.
   \end{aligned}$$

   □

Exercise Abbott 4.2.7: Let $g : A \to R$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$. Show that if $\lim_{x\to c} g(x) = 0$ then $\lim_{x\to c} g(x)f(x) = 0$.

Proof. Suppose that $g : A \to R$, where $\lim_{x\to c} g(x) = 0$ and that f is a bounded function on A. By the definition of bounded there exists some $M > 0$ such that $|f(x)| \leq M$. Note that by Definition 4.2.1 we know that for all $\epsilon > 0$ there exists an $\delta_g > 0$ where for all $0 < |x - c| < \delta$,

$$|g(x)| < \frac{\epsilon}{M}.$$

Now let $\epsilon > 0$ and consider $\delta = \delta_g$, therefore, for all $0 < |x - c| < \delta$

$$\begin{aligned} |g(x)f(x)| &= |g(x)||f(x)|, \\ &\leq |g(x)|M, \\ &< \frac{\epsilon}{M}M, \\ &< \epsilon. \end{aligned}$$

□

Proof. Suppose that $g : A \to R$, where $\lim_{x\to c} g(x) = 0$ and that f is a bounded function on A. By the definition of bounded there exists some $M > 0$ such that $|f(x)| \leq M$. Consider a sequence $x_n \subseteq A$ and note the following inequality,

$$|f(x_n)| \leq M$$

therefore $f(x_n)$ must be a bounded sequence. By Theorem 4.2.3 we know that for all $x_n \subseteq A$ where $x_n \neq c$ and $x_n \to c$ that $g(x_n) \to 0$. Recall that in exercise 2.3.9 we showed that for all x_n if $f(x_n)$ and $g(x_n) \to 0$ then $g(x_n)f(x_n) \to 0$ and thus by Theorem 4.2.3 we know that $\lim_{x\to c} g(x)f(x) = 0$. □

Exercise Abbott 4.2.11: Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) \to L$ and $\lim_{x\to c} h(x) \to L$ at some point c of A, show that $\lim_{x\to c} g(x) = L$

Proof. Suppose f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A and that $\lim_{x\to c} f(x) \to L$ and $\lim_{x\to c} h(x) \to L$ at some point c of A. By Theorem 4.2.3 we know that for all $x_n \subseteq A$ where $x_n \neq c$ and $x_n \to c$ that $f(x_n) \to L$ and $h(x_n) \to L$. Note that for all $x_n \subseteq A$,

$$f(x_n) \leq g(x_n) \leq h(x_n),$$

therefore by the Squeeze Theorem we know that $g(x_n) \to L$. Thus it follow by Theorem 4.2.3 that $\lim_{x\to c} g(x) = L$. □

Exercise Abbott 4.3.3: 1. Supply a proof for Theorem 4.3.9 using the $\epsilon - \delta$ characterization of continuity.

Proof. Suppose a function $f : A \to \mathbb{R}$ and $f : B \to \mathbb{R}$ and assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f = g(f(x))$ is defined on A. Let f be continuous at point $c \in A$ and g continuous at $f(c) \in B$. By the continuity of g we know that for all $\epsilon > 0$ there exists a δ_g such that whenever $|f(x) - f(c)| < \delta_g$ we know that $|g(f(x)) - g(f(c))| < \epsilon$. By the continuity of f we know that for all tolerances $\delta_g > 0$ there exists a δ_f such that whenever $|x - c| < \delta_f$ we get that $|f(x) - f(c)| < \delta_g$. Therefore for all ϵ there exists a δ_f where whenever $|x - c| < \delta_f$ we know that $|g(f(x)) - g(f(c))| < \epsilon_g$. □

2. Give another proof of Theorem 4.3.9 using the Sequential Characterization of Continuity.

Proof. Suppose a function $f : A \to \mathbb{R}$ and $f : B \to \mathbb{R}$ and assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f = g(f(x))$ is defined on A. Let a_n be a sequence in A where $a_n \to c$. By the Sequential Characterization of Continuity of f we know that $f(a_n) \to f(c)$. By our definition of the range of f we know that the sequence defined by $f(a_n), f(c) \in B$ therefore by the Sequential Characterization of Continuity of G we have that $g(f(a_n)) \to g(f(a)$. Thus by Theorem 4.3.2 we have shown that the composition $g \circ f$ is continuous at c. □

Exercise Abbott 4.3.5: Show using Definition 4.3.1 that if c id an isolated point of $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

Proof. Suppose that c is an isolated point in A. By the definition of isolated point we know that there must exist some $V_\delta(c)$ where,

$$V_\delta(c) \cap A\{c\} = \emptyset.$$

Let $\epsilon > 0$. Consider the $|x - c| < \delta$ where $V_\delta(c)$ has the above property. Since $x \in V_\delta(c)$ it must be the case that $x = c$ which means

$$|f(x) - f(c)| = 0 < \epsilon.$$

Thus by definition 4.3.1 f is continuous at point c □