Exercise 1.4.7: Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ contradicts the assumption that $\alpha = \sup A$.

Proof. Consider the set,

$$A = \{a \in \mathbb{R} : a^2 < 2\}.$$

Let $\alpha = \sup A$. Suppose to the contrary that $\alpha^2 > 2$. Consider an element of A that is smaller than α, like $(\alpha - \frac{1}{n})$, where $n > \frac{2\alpha}{\alpha^2 - 2}$.

$$\begin{aligned}(\alpha - \frac{1}{n})^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2},\\ &> \alpha^2 - \frac{2\alpha}{n},\\ &> \alpha^2 - (\alpha^2 - 2),\\ &= 2.\end{aligned}$$

Thus we have shown that $(\alpha - \frac{1}{n})$ is greater than a for all $a \in A$ and therefore $(\alpha - \frac{1}{n})$ is an upper bound. Since $(\alpha - \frac{1}{n}) < \alpha$ we have contradicted $\alpha = \sup A$.

□

Exercise Supplemental 1: Give a from-scratch proof of the following facts:

(a) If $f : A \to B$ has an inverse function g, then f is injective.

(b) If $f : A \to B$ has an inverse function g, then f is surjective.

Proof (a). Suppose $f : A \to B$, whose inverse is $g : B \to A$, now consider $a_i, a_j \in A$ such that $f(a_i) = f(a_j)$. Using g as an intermediary we get the equality,

$$\begin{aligned}f(a_i) &= f(a_j),\\ g(f(a_i)) &= g(f(a_j)),\\ a_i &= a_j.\end{aligned}$$

Thus we have shown f is an injective function. □

Proof (b). Suppose $f : A \to B$, whose inverse is $g : B \to A$. Consider some $b \in B$, by definition of g we know that there exists some $a \in A$ such that, $g(b) = a$. Taking the inverse of both sides we get,

$$\begin{aligned}f(g(b)) &= f(a)\\ b &= f(a).\end{aligned}$$

Since $a \in A$ we have shown that for every $b \in B$ there exists some $a \in A$ such that $f(a) = b$ thus f is surjective. □

Exercise Supplemental 2: Show that the sets $[0, 1)$ and $(0, 1)$ have the same cardinality.

Proof. Suppose the function $f : [0,1) \to (0,1)$ defined by,

$$f(x) = \begin{cases} 1 - \frac{1}{n+1} & 1 - \frac{1}{n}, n \in \mathbb{N} \\ x & \text{Otherwise} \end{cases}$$

Suppose $a, b \in [0,1)$ such that $f(a) = f(b)$. For the case where $f(x) = x$ the function is trivially injective. let a, b be of the form $1 - \frac{1}{n}$ such that, $a = 1 - \frac{1}{n}$ and $b = 1 - \frac{1}{m}$, where $n, m \in \mathbb{N}$. Now consider $f(a) = f(b)$, by definition,

$$\begin{aligned} f(a) &= f(b), \\ 1 - \frac{1}{n+1} &= 1 - \frac{1}{m+1}, \\ \frac{1}{n+1} &= \frac{1}{m+1}, \\ n+1 &= m+1, \\ n &= m. \end{aligned}$$

Thus we have shown that $n = m$ and therefore by substitution $a = b$. Thus f is injective.

Note that for the case where $f(x) = x$ the function is trivially surjective. Suppose some $b \in (0,1)$. Let b be of the form $b = 1 - \frac{1}{n+1}$, where $n \in \mathbb{N}$. Consider $a = 1 - \frac{1}{n}$ and note that, $f(a) = b$. Observe that $a \in [0,1)$, thus f is surjective. □

Exercise 1.5.10 (a) (c): (Wait until after Wednesday to start this one)

(a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(c) Determine, with proof, if the same statement remains true replacing uncountable with infinite.

Proof (a). Suppose $C \subseteq [0,1]$ is uncountable. Suppose for the sake of contradiction that for all $a \in (0,1)$, $C \cap [a,1]$ is countable. Let $a = \frac{1}{n}$. Note that $C \cap [\frac{1}{n}, 1]$ is countable. By Theorem 1.5.8 we know that since $C \cap [\frac{1}{n}, 1]$ is countable the infinite union is also countable. Through set theory

$$\begin{aligned} \bigcup_{n-1}^{\infty} C \cap [\frac{1}{n}, 1] &= C \cap (\bigcup_{n-1}^{\infty} [\frac{1}{n}, 1]) \\ &= C \cap (0,1] \end{aligned}$$

Therefore, $C \cap (0,1] \cup \{0\} = C$ is also countable. □

Proof (c). Suppose the countably infinite set $C = \{\frac{1}{n}, n \in \mathbb{N}\}$. By Archimedean Principle we know that for all $a \in (0,1)$ we can find some $\frac{1}{n} < a$, and therefore we force the set $C \cap [a,1]$ to be finite.

□

Exercise Supplemental 3: (Wait until after Wednesday to start this one) Suppose for each $k \in \mathbb{N}$ that A_k is at most countable. Use the fact that $\mathbb{N} \times \mathbb{N}$ is countably infinite to show that $\cup_{k=1}^{\infty} A_k$ is at most countable. Hint: take advantage of surjection.

Proof. Suppose for each $k \in \mathbb{N}$ that A_k is at most countable. Recall that since all A_k are at most countable there must exists a surjection $g_k : \mathbb{N} \to A_k$. Consider the function $f : \mathbb{N} \times \mathbb{N} \to \cup_{k=1}^{\infty} A_k$ defined such that, $f(n, m) = g_n(m)$. Let $a \in \cup_{k=1}^{\infty} A_k$ and by definition we know that a must exist in some set A_i, where $i \in \mathbb{N}$. Furthermore, since g_i, is a surjection we know that there exists some $j \in \mathbb{N}$ where $g_i(j) = a$. Therefore we know that $f(i, j) = g_i(j)$ where $i, j \in \mathbb{N} \times \mathbb{N}$. Thus f is a surjection, and it follows that since $\mathbb{N} \times \mathbb{N}$ is countably infinite then $\cup_{k=1}^{\infty} A_k$ is at most countable. $\square$