Exercise 1.3.9: (a) If $\sup A < \sup B$ then show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not necessarily the case if we we only assume $\sup A \leq \sup B$.

Proof (a). Suppose $\sup A < \sup B$, and let $x = supA$ and $y = supB$. First consider σ such that $0 < \sigma < y - x$, and therefore through algebra, $x < y - \sigma$. By Lemma 1.3.8, we know that $y = \sup B$ then there exists some $b \in B$ such that $y - \sigma < b$. Thus,

$$x < y - \sigma < b < y.$$

Since x is an upper bound for A and $x \leq b$ then we know that b is also an upper bound for A. $\square$

Example for (b): When we assume $supA \leq supB$, consider $A = [0, 2]$ and $B = [0, 2)$. In this case $supA = supB$ and since any $x \in \mathbb{R}$ such that $0 \leq x < 2$ is also in A.

Exercise 1.3.11: Decide if the following statements are true. Give a short proof for the true statements and a counterexample for the false statements.

a. If A and B are nonempty, bounded, and satisfy $A \subseteq B$ then $\sup A \leq \sup B$.

Solution: (True) Suppose A and B are nonempty, bounded, and satisfy $A \subseteq B$. Consider $a \in A$, and note that since $A \subseteq B$ we know that $a \in B$. Note that by definition of upper bound we know that $a \leq \sup B$ and therefore $\sup B$ is an upper bound for the set A. Since $\sup A$ is the least upper bound for the set A we get $\sup A \leq \sup B$. $\square$

b. If $\sup A < \inf B$ for sets A and B, then there exists $c \in \mathbb{R}$ such that $a < c < b$ for all $a \in A$ and $b \in B$.

Solution: (True) Suppose $\sup A < \inf B$ for sets A and B. Now consider $c \in \mathbb{R}$ such that,

$$c = \frac{\sup A + \inf B}{2}$$

Note that this produces the following inequality,

$$\sup A < c < \inf B.$$

Therefore by definition we get that for all $a \in A$ and $b \in B$,

$$a < c < b.$$

□

c. If there exists $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$ then $\sup A < \inf B$.

Solution: (False) Let $A = x \in \mathbb{R}; x < 1$ and $B = x \in \mathbb{R}; x > 1$. Note that $c = 1$ satisfies the inequality $a < c < b$, for all $a \in A$ and $b \in B$, while clearly $\sup A = \inf B = 1$.

Exercise First Edition 1.4.1: Recall that $\mathbb{I}$ stands for the set of irrational numbers.

a. Show that if $a, b \in \mathbb{Q}$ then ab and $a + b \in \mathbb{Q}$ as well.

Proof: Suppose that $a, b \in \mathbb{Q}$. By definition we know that $a = \frac{i}{j}$ and $b = \frac{n}{m}$ where $i, n \in \mathbb{Z}$ and $j, m \in \mathbb{Z}^*$. Consider the following,

$$ab = \frac{i}{j}\frac{n}{m} = \frac{in}{jm}$$

Note that, $in \in \mathbb{Z}$ and $jm \in \mathbb{Z}^*$. Therefore, by definition, $ab \in \mathbb{Q}$. Now consider,

$$a + b = \frac{i}{j} + \frac{n}{m} = \frac{mi + jn}{jm}$$

Note that, $mi + jn \in \mathbb{Z}$ and $jm \in \mathbb{Z}^*$. Therefore, by definition, $a + b \in \mathbb{Q}$. □

b. Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$ and if $a \neq 0$ then $at \in \mathbb{I}$ as well.

Proof: Let $a \in \mathbb{Q}$ such that $a = \frac{i}{j}$ where $i, j \in \mathbb{Z}$ and $j \neq 0$. Also let $t \in \mathbb{I}$. Now suppose for the contrary that both $a + t \in \mathbb{Q}$. Note,

$$a + t = \frac{i}{j} + t = \frac{n}{m},$$

for some $n, m \in \mathbb{Z}$ where $m \neq 0$. Through algebra we get that,

$$\begin{aligned} \frac{i}{j} + t &= \frac{n}{m}, \\ t &= \frac{n}{m} - \frac{i}{j}, \\ &= \frac{jn - mi}{jm}. \end{aligned}$$

Note that, $jn - mi \in \mathbb{Z}$ and $jm \neq 0$. Therefore $t \in \mathbb{I}$ and $t \notin \mathbb{I}$.
Now suppose for the contrary that $t \in \mathbb{Q}$. Note,

$$at = \frac{i}{j}t = \frac{n}{m},$$

for some $n, m \in \mathbb{Z}$ where $m \neq 0$.Through algebra we get that,

$$\begin{aligned} \frac{i}{j}t &= \frac{n}{m} \\ t &= \frac{n}{m}\frac{j}{i} \\ &= \frac{nj}{mi} \end{aligned}$$

Note that, $nj, mi \in \mathbb{Z}$ and $mi \neq 0$. Therefore $t \in \mathbb{I}$ and $t \notin \mathbb{I}$. □

c. Part (a) says that the rational numbers are closed under multiplication and addition. What can be said about st and $s + t$ when $s, t \in \mathbb{I}$?

Proof: We can show that the irrational numbers are not closed with respect to addition and multiplication. For a counterexample let $s = \sqrt{2}$ and $t = -\sqrt{2}$. Note that $s + t = 0$ which is rational, and $st = -2$ which is again a rational number.

Exercise First Edition 1.4.2: Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + (1/n)$ is an upper bound for A but $s - (1/n)$ is not an upper bound for A. Show that $s = \sup A$.

Proof: Let $A \subseteq \mathbb{R}$ be nonempty and bounded above also let $s \in \mathbb{R}$ have the property such that for all $n \in \mathbb{N}$, $s + (1/n)$ is an upper bound for A but $s - (1/n)$ is not an upper bound for A. Now suppose to the contrary that there exists $x \in A$ such that $x > s$. From the Archimedean Property we get that there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < x - s$. Note,

$$\begin{aligned} \frac{1}{n} &< x - a \\ s + \frac{1}{n} &< x \end{aligned}$$

We have shown that $s + \frac{1}{n}$ is both an upper bound and not an upper bound, and thus by contradiction we have proven that for all $x \in A$, $x \leq s$.

Suppose to the contrary that there exists some upper bound of A, t such that $t < s$. From the Archimedean Property we get that there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < s - t$. Note,

$$\begin{aligned} \frac{1}{n} &< s - t \\ t &< s - \frac{1}{n} \end{aligned}$$

We have shown that $s - \frac{1}{n}$ is both an upper bound and not an upper bound, and thus by contradiction we have proven that for all upper bounds t of A, $s \leq t$ □

Exercise First Edition 1.4.3: Show that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

Proof: Suppose $A_n = (0, \frac{1}{n})$ and $A = \cap_{n=1}^{\infty} A_n$. Suppose there exists some $x \in A$. Note that by the definition of $A_1 = (0, 1)$, it is the case that $x > 0$. Recall that by the Archimedean Property we know that if $x > 0$, there exists an $m \in \mathbb{N}$ which satisfies,

$$\frac{1}{m} < x.$$

Therefore we know that for all $n \geq m$, $x \notin A_n$ and therefore $x \notin A$. □

Exercise First Edition 1.4.4: Let $a < b$ be real numbers and let $T = [a, b] \cap \mathbb{Q}$. Show that $\sup T = b$.

Proof: Suppose $x \in T$. Note by the definition of T, $x \leq b$ and thus b is an upper bound.

Suppose to the contrary that there exists an upper bound, u for the set T such that $u < b$. Recall that the Density of $\mathbb{Q}$ in $\mathbb{R}$ shows us that between the two real numbers u and b there exists a rational number r such that $u < r < b$. By definition $r \in T$ and therefore u is not an upper bound. Thus we have shown that all upper bounds u, of T satisfy $b \leq u$ and tha $\sup A = b$. □

Exercise First Edition 1.4.5: Use Exercise 1.4.1 to provide a proof of Corollary 1.4.4 (Density of Rational Numbers) by considering real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Proof: Suppose that $a, b \in \mathbb{R}$ and that $a < b$. Consider the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$. Note that by the Density of Rational Numbers Theorem we know that there exists $x \in \mathbb{Q}$ such that,

$$a - \sqrt{2} < x < b - \sqrt{2}.$$

Through algebra we get,

$$a < x + \sqrt{2} < b.$$

Note that by Exercise 1.4.1 we know that $t = x + \sqrt{2} \in \mathbb{I}$. Thus there exist an irrational number t satisfying $a < t < b$. □