Exercise Abbott 4.5.2: Provide an example of each of the following, or explain why the request is impossible,

1. A continuous function defined on an open interval with range equal to a closed interval

   Solution:
   Suppose a function $sin(x)$ defined on the open interval $(0, 2\pi)$. Let $\epsilon > 0$. Now consider some $c \in (0, 2\pi)$ and let $\delta = \epsilon$. then for all $x \in (0, 2\pi)$, $|x - c| < \delta$ we get that,

$$\begin{aligned}
|sin(x) - sin(c)| &= |2cos(\frac{x+c}{2})sin(\frac{x+c}{2})| \\
&\leq 2(1)|sin(\frac{x+c}{2})| \\
&\leq 2(1)|\frac{x+c}{2}| \\
&\leq |x+c| \\
&< \delta \\
&< \epsilon.
\end{aligned}$$

   Now note that the range of $sin(x)$ on the interval $(0, 2\pi)$ is $[-1, 1]$ a closed interval.

2. A continuous function defined on a closed interval with a range equal to an open interval.

   Solution:
   Such a request is impossible. Note that a closed interval is a compact set, by Theorem 4.4.1 (Preservation of Compact Sets) we know that if the domain of a continuous function is compact then so is the range.

3. A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbb{R}$.

   Solution:
   Consider the function $f(x) = |tan(x)|$ defined by $f : (-\frac{pi}{2}, \frac{pi}{2}) \to \mathbb{R}$. Note that the range of f is $[0, \infty)$ which is an unbounded closed set different from $\mathbb{R}$.

4. A continuous function defined on all of $\mathbb{R}$ with a range equal to $\mathbb{Q}$

Solution:
Such a request is impossible. Suppose for the sake of contradiction that there exists a function f defined on all $\mathbb{R}$ such that the range of f was equal to $\mathbb{Q}$. Consider some set $K \subseteq \mathbb{R}$ such that $K = [a, b]$. By Theorem 4.4.1 $f(K) = [f(a), f(b)]$. By the density of $\mathbb{I}$ in $\mathbb{R}$ there exists some $i \in \mathbb{I}$ such that $i \in f(K)$. Thus the range of f is not equal to $\mathbb{Q}$.

Exercise Abbott 4.5.5 (b): You may assume that you have found a sequence of nested intervals $I_k = [a_k, b_k]$ with $f(a_k) < 0$ and $f(b_k) \geq 0$ and $|I_{k+1}| = |I_k|/2$, where $|\cdot|$ denotes the length of the interval.

For those of you in Numerical Analysis, this proof of the IVT mirrors the bisection method for finding roots!

Proof. Consider the case where $L = 0$ and we suppose that $f(a) < 0 < f(b)$. As described in the text we have constructed a series of nested intervals $I_k = [a_k, b_k]$ with $f(a_k) < 0$ and $f(b_k) \geq 0$ and $|I_{k+1}| = |I_k|/2$. By the Nested Interval Property we know that,

$$\bigcap_{n=0}^{\infty} I_n \neq \emptyset$$

Let $c \in \cap_{n=0}^{\infty} I_n$. Also note that by the NIP that the sequences $a_k \to c$, $b_k \to c$. Since f is continuous by the sequential criteria for continuity we know that $f(a_k) \to f(c)$ and $f(b_k) \to f(c)$. Recall that by the construciton of our intervals I_k the inequality,

$$f(a_k) < 0 \leq f(b_k),$$

holds for all $k \in \mathbb{N}$. Finally by the squeeze theorem we get that

$$f(c) \leq 0 \leq f(c),$$

and so $f(c) = 0$. $\square$

Exercise Abbott 4.4.3: Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$

Proof. Consider some function on $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x^2$ and let $c \in [1, \infty)$. Consider the following,

$$|f(x) - f(c)| = |\frac{1}{x^2} - \frac{1}{c^2}| = |\frac{x^2 - c^2}{x^2c^2}| = \frac{(x + c)|x - c|}{x^2c^2}.$$

Note that on the domain of $[1, \infty)$, $\dfrac{(x+c)}{x^2c^2}$ is bounded above by 2 which gives us that,

$$|f(x) - f(c)| \leq 2|x - c|.$$

Let $\epsilon > 0$. Choose $\delta = \epsilon/2$. Then, $|x - c| < \delta$ implies,

$$|f(x) - f(c)| \leq 2|x - c| < 2\frac{\epsilon}{2} = \epsilon.$$

$\square$

Proof. Consider some function on $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x^2$. Consider the sequences $x_n, y_n \in (0, 1]$ where $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2} - \frac{1}{n}$. Now consider the following sequence,

$$|x_n - y_n| = \frac{1}{n} - (\frac{1}{n^2} - \frac{1}{n}) = \frac{1}{n^2}.$$

Note that $|x_n - y_n| \to 0$. Now consider the function limit

$$\begin{aligned}
|f(x_n) - f(y_n)| &= |\frac{1}{\frac{1}{n^2}} - \frac{1}{(\frac{1}{n^2} - \frac{1}{n})^2}| \\
&= |\frac{1}{\frac{1}{n^2}} - \frac{1}{\frac{1}{n^4} - \frac{2}{n^3} + \frac{1}{n^2}}| \\
&= |n^2 - (n^4 - \frac{n^3}{2} + n^2)| \\
&= | - n^4 + \frac{n^3}{2}|.
\end{aligned}$$

Clearly this limit is divergent, note that $|f(x_n) - f(y_n)| \geq \frac{1}{2}$ for all $n \geq 1$. Thus by the Theorem 4.4.5 (Sequential Criterion for Absence of Uniform Continuity) we know that f is not uniformly continuous on $(0, 1]$ $\square$

Exercise Abbott 4.2.10: Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting x approach a form the right-hand side."

1. Give a proper definition in the style of Definition 4.2.1 for the right-hand and left-hand limit statements,

$$\lim_{x\to a^+} f(x) = L$$

$$\lim_{x\to a^-} f(x) = L$$

Solution:
Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to a^+} f(x) =$

L provided that , for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < c - x < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Similarly we sat that $\lim_{x \to a^-} f(x) = L$ provided that , for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < x - c < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Prove that $\lim_{x \to a} f(x) = L$ if and only if both right and left-hand limits equal L

Proof. Suppose a function $f : A \to \mathbb{R}$ with the property that $\lim_{x \to a} f(x) = L$. By the definition of the Functional Limit we know that for all $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ we get that $|f(x) - L| < \epsilon$. Consider the case where $x < c$ then we get the inequality $0 < x - c < \delta$ and therefore by definition $\lim_{x \to a^-} f(x) = L$. Now consider the case where $x > c$ we get the inequality $0 < c - x < \delta$ and therefore by definition $\lim_{x \to a^+} f(x) = L$.

Suppose a function $f : A \to \mathbb{R}$ with the property that $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$. By definition this gives us that for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < c - x < \delta$ or $0 < x - c < \delta$ it follows that $|f(x) - L| < \epsilon$. Note that $0 < c - x < \delta$ or $0 < x - c < \delta$ implies that $0 < |x - c| < \delta$ thus by definition $\lim_{x \to a} f(x) = L$. $\square$