Chapter 4: 2 (c): For full credit you must write your own version of the secant method. Your function should have the signature

```
function [r,hist] = secant(f,x0,x1,ftol,xtol,Nmax)

end
```

The input values are

- `f`, the function to find a root of.
- `x0`, `x1` the first two iteration values.
- `ftol`, the tolerance for stopping based of the value of `f`
- `xtol`, the tolerance for stopping based on changes in `x`
- `Nmax`, the maximum number of iterations

Your function should exit with an error if more than `Nmax` iterations are used. It should return whenever $|f(x)| < f_{\text{tol}}$ or $|x_n - x_{n-1}| < x_{\text{tol}}$.

The return values should be `r`, the estimate of the root's position, and `hist`, a list of all estimates starting with `x0` and `x1` and ending with the final estimate `r`.

Test that your function works by finding three different ways to call it so that iteration stops for each of the three possible reasons.

To answer the problem in the textbook, you will want to call your function with $x_{\text{tol}} = 0$ to ensure that only the f_{tol} condition is used to stop the iteration.

Code:

```
function [r, hist] = hw4secant(f,x0,x1,ftol ,xtol ,Nmax)
% Takes a function(f), an f(x) tolerance , an x
% tolerance , max number of iterations , two initial root guess and
% returns an approximation , iteration history , and a message describing
% Secant method termination.


format longE
%Initializing variables
hist  = zeros;
count = 1;
```

```
if x0 == x1
    error('initial gueses are the same')
end
x = x0;
%Assigning first value
hist(1) = x;

x = x1;
%Assigning second value
hist = [hist; x];


%Iteration Step
for i = 2:(Nmax - 1)        %initial guess is counted in Nmax
    mk = (f(hist(i)) - f(hist(i - 1)))/(hist(i) - hist(i - 1));
    x = x - f(x)/mk ;
    hist = [hist; x];

    %Checking tolerances

        if abs(f(x)) <= ftol
            disp('Inside of f tolerance')
         break
        end

        if i > 1        % hist must have at least two values to check xtc
            if abs(x - hist(i - 1)) <= xtol
                disp('Inside of x tolerance')
                break
            end
        end

        %termination message for Nmax iterations
        if i == (Nmax - 1)
            disp('Terminated after Nmax iterations')
        end

end

r = x;

end
```

Solution:
Testing all the exit conditions on the function,

$$f(x) = x^2 - 4.$$

Console:

```
[r, hist] = hw4secant(f, -1, 3, 0, 0,5);
Terminated after Nmax iterations
[r, hist] = hw4secant(f, -1, 3, 10e-4, 0,10);
Inside of f tolerance
[r, hist] = hw4secant(f, -1, 3, 0, 10e-4,10);
Inside of x tolerance
diary off
```

Running our secant method with the function,

$$f(x) = (5 - x)e^x - 5,$$

where, $x_0 = 4$,$x_1 = 5$ and our $f(x)$ tolerance set to 10^{-8}.

Console:

```
f = @(x) (5 - x)*exp(x)-5

f =

    @(x)(5-x)*exp(x)-5

[r, hist] = hw4secant(f, 4, 5, 10e-8, 0, 10)
Inside of f tolerance

r =

      4.965114231713327e+00


hist =

      4.000000000000000e+00
      5.000000000000000e+00
      4.908421805556329e+00
      4.963079336311798e+00
      4.965235312126352e+00
      4.965113980657901e+00
      4.965114231713327e+00
```

```
err = f(hist)

err =

   4.959815003314424e+01
  -5.000000000000000e+00
   7.402024407938184e+00
   2.808942198028612e-01
  -1.675049904093306e-02
   3.473149426902467e-05
   4.280998666672531e-09

diary off
```

Using lemma 4.17 from the textbook we can approximate the magnitude of the next few errors,

$$e_{k+1} \approx 1.62(e_k)(e_{k-1})$$

Console:

```
ef = f(hist);
ef(7)*ef(6)*1.62

ans =

   2.408704786647276e-13

ans*ef(7)*1.62

ans =

   1.670489240767246e-21

diary off
```

Thus we can see that after two more iteration through the secant method we will be inside an an $|f(x)| = 10^-16$

Chapter 4: 3: Newton's Method can be used to compute reciprocals, without division. To compute $1/R$, let $f(x) = x^{-1} - R$ so that $f(x) = 0$ when $x = 1/R$. Write down the Newton iteration for this problem, and compute the first few Newton iterates for approximating $1/3$, starting at $x_0 = .5$, and not using any division. What happens if ou start with $x_0 = 1$? For positive R, use theory of fixed point iteration to determine an interval about $1/R$ from which Newton's method will converge to $1/R$.

Solution:
Consider the Newton iteration,

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= x_n + (x_n^{-1} - R)(x^2), \\
x_{n+1} &= 2x_n - Rx_n^2.
\end{aligned}$$

Solving the first few terms for the iteration where $R = 3$ and $x_0 = .5$,

$$\begin{aligned}
x_0 &= .5, \\
x_1 &= .25, \\
x_2 &= .3125, \\
x_3 &= .33203125.
\end{aligned}$$

Solving the first few terms for the iteration where $R = 3$ and $x_0 = 1$,

$$\begin{aligned}
x_0 &= 1, \\
x_1 &= -1, \\
x_2 &= -5, \\
x_3 &= -185.
\end{aligned}$$

It's likely that the sequence will diverge to $-\infty$.

Using Theorem 4.5.1 we can solve for the interval about $1/R$ for which Newton's method will converge. Note that we can write our Newton iteration as a fixed point problem where,

$$x_{n+1} = \phi(x_n).$$

Also note that $\phi(x) \in C^1$. Now consider,

$$|\phi'(x)| < 1.$$

Expanding the inequality and substituting $\phi'(x)$,

$$\begin{aligned}
-1 <& \phi'(x) < 1 \\
-1 <& 2 - 2Rx < 1 \\
-3 <& -2Rx < -1 \\
\frac{1}{2R} <& x < \frac{3}{2R}.
\end{aligned}$$

Thus if x_0 is inside the interval $(\frac{1}{2R}, \frac{3}{2R})$ the fixed point iteration $\phi(x)$ will converge.

Chapter 4: 8: In using the secant method to find a root, $x_0 = 2$, $x_1 = -1$ $x_2 = -2$ with $f(x_1) = 4$ and $f(x_2) = 3$. What is $f(x_0)$?

Solution:
Consider the iteration function for the secant method,

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Through substitution we get,

$$\begin{aligned} -2 &= -1 - \frac{-12}{4 - f(x_0)} \\ 1 &= \frac{-12}{4 - f(x_0)} \\ 4 - f(x_0) &= -12 \\ f(x_0) &= 16. \end{aligned}$$

Chapter 4: 12: Let the function,

$$\phi(x) = \frac{x^2 + 4}{5}$$

a. Find the fixed points of $\phi(x)$.

Solution:
By definition we know that a fixed point is where $\phi(x) = x$, therefore solving for the fixed points,

$$\begin{aligned} x &= \frac{x^2 + 4}{5} \\ 5x &= x^2 + 4 \\ 0 &= x^2 - 5x + 4 \\ 0 &= (x - 1)(x - 4) \end{aligned}$$

Thus the fixed points are $x = 4$ and $x = 1$.

b. Would the fixed point iteration, $x_{k+1} = \phi(x_k)$, converge to a fixed point in the interval $[0, 2]$ for all initial guesses $x_0 \in [0, 2]$?

Solution:
Using Theorem 4.5.1 we can solve for the interval for which $x_{k+1} = \phi(x_k)$ will converge. Note that $\phi(x) \in C^1$. Now consider,

$$|\phi'(x)| < 1.$$

Expanding the inequality and substituting $\phi'(x)$,

$$\begin{aligned}-1 <&\phi'(x) < 1\\ -1 <&\frac{2}{5}x < 1\\ -\frac{5}{2} <&x < \frac{5}{2}\end{aligned}$$

Thus if x_0 is inside the interval $(-\frac{2}{5}, \frac{2}{5})$ the fixed point iteration $\phi(x)$ will converge.

Supplemental 1: Suppose $f(x)$ is a differentiable function on $\mathbb{R}$ and $|f'(x)| \leq 1/2$ for all real numbers. Show that

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|$$

for all $x, y \in \mathbb{R}$.

Hint: Use Taylor's theorem, the zeroth order version, AKA the Mean Value Theorem. Apply it centered at some point x and then see what the theorem says about $f(y)$.

For context, look at equation (4.22) of the text, which defines a **contraction**. You are showing that if $|f'(x)| \leq 1/2$ for all x then f is a contraction. This is interesting because Theorem 4.5.2 says that every contraction has a unique fixed point, and if you perform fixed point iteration on the contraction then the iterates will converge to the fixed point. You can think of Theorem 4.5.2 as a generalization of Theorem 4.5.1 (you can prove Theorem 4.5.1 directly from the more difficult Theorem 4.5.2).

Solution:
Suppose that $f(x)$ is a differentiable function on $\mathbb{R}$ and $|f'(x)| \leq 1/2$ for all real numbers. Now consider the first zeroth order Taylor polynomial,

$$f(x) = f(y) + f'(\xi)(x - y).$$

Where $\xi, y \in \mathbb{R}$. Now through some algebra we get that,

$$\begin{aligned}f(x) &= f(y) + f'(\xi)(x - y),\\ f(x) - f(y) &= f'(\xi)(x - y),\\ |f(x) - f(y)| &= |f'(\xi)(x - y)|,\\ |f(x) - f(y)| &= |f'(\xi)||(x - y)|.\end{aligned}$$

Note that since $|f'(x)| \leq \frac{1}{2}$ then we know that,

$$|f(x) - f(y)| = \frac{1}{2}|(x - y)|,$$

for all values $x, y \in \mathbb{R}$.

Chapter 4: 13: Consider the $a = y - \epsilon sin(y)$, where $0 < \epsilon < 1$ is given and $a \in [0, \pi]$ is given. Write this in the form of a fixed point problem for the unknown solution of y and show that it has a unique solution. [Hint: You will want to use Theorem 4.5.2!]

Consider the equation,

Solution:
Consider the fixed point equation,

$$\phi(y) = a + \epsilon sin(y)$$

Now consider the derivative of $\phi(y)$

$$\phi'(y) = \epsilon cos(y).$$

Note that since the function $cos(y) \in [-1, 1]$ and $0 < \epsilon < 1$ we know that for all ϵ,

$$\phi'(y) < 1.$$

Now consider the zeroth order taylor polynomial for the function ϕ,

$$\begin{aligned}
\phi(x) &= \phi(y) + \phi'(\xi)(x - y), \\
\phi(x) - \phi(y) &= \phi'(\xi)(x - y), \\
|\phi(x) - \phi(y)| &= |\phi'(\xi)(x - y)|, \\
|\phi(x) - \phi(y)| &= |\phi'(\xi)||(x - y)|.
\end{aligned}$$

Recall that since,

$$\phi'(y) < 1$$

is true for all values of ϵ we know, by Theorem 4.5.2 that ϕ is a contraction and therefore it has a unique fixed point x_* and the corresponding iteration function,

$$x_{k+1} = \phi(x_k),$$

converges to x_* for any x_0.

Chapter 4: 14: If you enter a number into handheld calculator and repeatedly press the cosine button, what number will approximately will appear? Provide a proof.

Solution:

We can model the action of repeatedly pressing the cosine button as an iteration function,

$$x_{k+1} = cos(x_k), \tag{1}$$

Where the corresponding fixed point equation,

$$\phi(x) = cos(x).$$

Note, that any fixed point for this equation must be contained where $x \in [-1, 1]$. Considering again the zeroth order taylor polynomial for the function ϕ,

$$cos(x) = cos(y) + -sin(\xi)(x - y),$$
$$cos(x) - cos(y) = -sin(\xi)(x - y),$$
$$|cos(x) - cos(y)| = |-sin(\xi)(x - y)|,$$
$$|cos(x) - cos(y)| = |-sin(\xi)||(x - y)|.$$

Since $|-sin(\xi)| > 1$ for all values $\xi \in [-1, 1]$ then, by Theorem 4.5.2 that ϕ is a contraction and therefore it has a unique fixed point. Approximating that fixed point using matlab,

Console:

```
x = 12

count = 1

hist = []


while count< 10
    x = cos(x)
    hist = [hist; x]
    count = count+1
end



x =

    7.436336063810779e-01


hist =

    8.438539587324921e-01
    6.645880521849051e-01
    7.871709145124681e-01
    7.058521464152847e-01
    7.610590556006372e-01
    7.241059982749000e-01
    7.490919670019339e-01
    7.323075176281578e-01
    7.436336063810779e-01
```