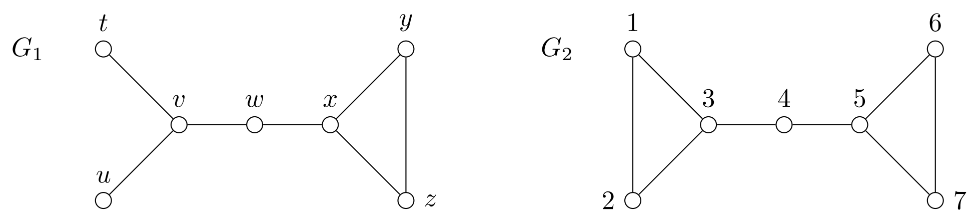
(1) (Problem A:) For the graphs G_1 and G_2 below, describe the automorphisms of G_1 and G_2 in terms of permutation cycles. Explain why your answer is correct. (See Figure 2.3 on page 39 of the handout for an example of the automorphisms of C_4 described in terms of permutation cycles.)



Answer:
$Aut(G_1) = \{\epsilon, (ut), (yz), (ut)(yz)\}$ We can see that the function (ut) preserves (non)-adjacency and similarly with (yz). Since $Aut(G_1)$ is a group it must be closed under function composition $(ut)(yz)$ must also be included.

$Aut(G_2) = \{\epsilon, (12), (67), (12)(67), (16)(27)(35), (17)(62)(35), (1627)(35), (1726)(34)\}$ This whole group can be composed from a set of 3 base permutation $\{(12), (67), (16)(27)(35), \epsilon\}$, which correspond to the symmetries on the graph. I can't seem to find any more reflection symmetries and any rotational symmetry is a composition the reflection symmetries I listed before.

(2) (Problem B) Prove that for $n \geq 3$, $Aut(C_n) \cong D_n$ where D_n is the dihedral group with $2n$ elements.

Answer: The dihedral group is defined as the group of symmetries of a regular polygon. We know that $|D_n| = 2n$ we get n from the rotational symmetries and n from reflection symmetries. Reflection symmetries are different depending on the parity of n. When n is odd the axis of reflection is always between a vertex and the midpoint of the opposite edge. When n is even the axis of reflection is either between two opposite vertices or two opposite edges. From here it clear that for cases $n \leq 2$ the symmetries are trivial. The isomorphism comes from the idea that any C_n can be redrawn as an n-vertex regular polygon and therefore contains the same symmetries. Since every permutation in $Aut(C_n)$ can be described with a symmetry or composition of symmetries we get that the groups are the same.

This is my attempt at an isomorphism proof.
Proof: Consider the isomorphism $f : Aut(C_n) \to D_n$, such that $f(i) = i$.

Injection: Suppose $a, b \in Aut(C_n)$ such that $f(a) = f(b)$. By the definition D_n we know that $f(b)$ and $f(a)$ correspond to the same symmetry of an n-vertex regular polygon. Note that $Aut(C_n)$ contains (non)-adjacency preserving permutations of the set $V(C_n)$ and any C_n can be redrawn as an n-vertex regular polygon. Since we know that there is an injective correspondence between symmetries and permutations it must be the case that $a = b$.

Surjection: Suppose $b \in D_n$. Let a be the permutation in $Aut(C_n)$ which corresponds to the symmetry b. Thus $f(a) = b$.

Homomorphism: Suppose $f(ab)$ where $a, b \in Aut(C_n)$. By definition $f(ab) = ab = f(a)f(b)$.

(3) (Problem C) Prove that for $m > n \geq 1$, $Aut(K_{m,n}) \cong S_n \times S_m$ where S_k is the symmetric group of all permutations of a set of order k.

Proof: Suppose $f : Aut(K_{m,n}) \to S_n \times S_m$ such that $f(i) = (i_n, i_m)$ where $i = i_n i_m$.

Injection: By the definition of a complete bipartite graph we know every vertex in $V(N)$ is non-adjacent to every other vertex in $V(N)$ and adjacent to every vertex in $V(M)$. Thus **every** permutation of $V(N)$ is an automorphism on $K_{m,n}$ and similarly **every** permutation of $V(M)$ is an automorphism on $K_{m,n}$. Note that there is no permutation that maps between parts since that would not preserve adjacency. Thus for every $i \in Aut(K_{m,n})$ there exists an $i_m \in S_m$ and an $i_n \in S_n$ such that $i = i_m i_n$. Note that each permutation decomposes into a unique tuple since $m > n \geq 1$. Since our function maps this decomposition to the corresponding tuple it must be injective.

Surjection: Suppose a tuple $(i_m, i_n) \in S_n \times S_m$. Let $i = i_m i_n$. Note, that $i \in Aut(K_{m,n})$ by what we showed previously. Thus $f(i) = f(i_m i_n) = (i_m, i_n)$

Homomorphism: Suppose $f(ij)$. Note,

$$\begin{aligned}
f(ij) &= f(ij_m ij_n) \text{ Decomposing the } ij \text{ permutation} \\
&= (ij_m, ij_n) \text{ Passing through the function we get the corresponding } ij \text{ tuple} \\
&= (i_m, i_n)(j_m, j_n) \text{ By the operation on the group } S_n \times S_m \\
&= f(i)f(j) \text{ Applying function definition}
\end{aligned}$$

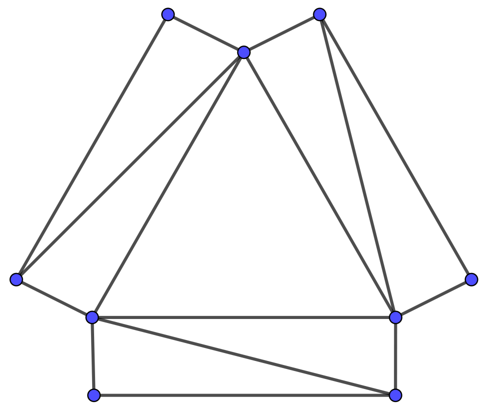
Thus f is closed with respect to group operations.

(4) (Problem D)

(a) Give an example with brief explanation of a graph G such that $|Aut(G)| = 3$.

Answer: Consider the following graph,

FIGURE 1. Graph G

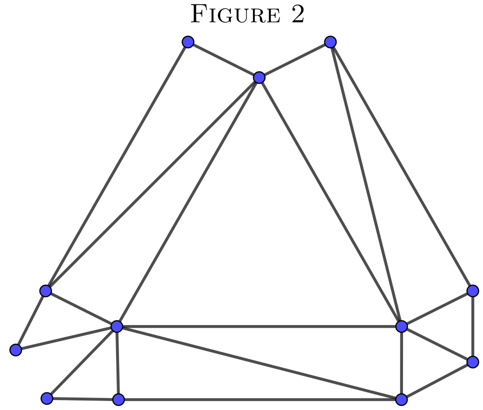


Note that graph G has no reflection symmetries and 3 rotational symmetries. The automorphism group could be described as the identity, a 60° rotation and a 120° rotation.

(b) Give an example of a graph G on 12 vertices with minimum degree 2 such that $|Aut(G)| = 1$.

Answer: Consider the following graph,

FIGURE 2



Taking the graph from before and appending three more vertices such that we remove the two rotation symmetries and also create no reflection symmetries.

(5) (Problem E) Recall that the *distance between two vertices u and v*, denoted $d(u, v)$, is the length of the shortest uv-path. Prove that automorphisms preserve distance. That is prove the statement below.
Let G is a graph. For all $u, v \in V(G)$ and for all $\phi \in Aut(G)$, $d(u, v) = d(\phi(u), \phi(v))$.

Proof:(Direct) Let $d(u, v) = d$ and $d(\phi(u), \phi(v)) = e$.

We want to show that $e \leq d$.
Consider the shortest uv-path $p = u, u_1, u_2, ..., u_{d-1}, v$. Note that since ϕ is an automorphism the $\phi(u)\phi(v)$-path $\phi(p) = \phi(u), \phi(u_1), \phi(u_2), ..., \phi(u_{d-1}), \phi(v)$ must exist. Since we have proven the existence of a $\phi(u)\phi(v)$-path size d it must be the case that $e \leq |\phi(p)| = d$.

We want to show that $d \leq e$.
Now consider the shortest $\phi(u)\phi(v)$-path $\phi(p) = \phi(u), \phi(u_1), \phi(u_2), ..., \phi(u_{d-1}), \phi(v)$. Note that since $\phi^{-1}$is an automorphism the uv-path $p = u, u_1, u_2, ..., u_{d-1}, v$ must exist. Since we have proven the existence of a uv-path size e it must be the case that $d \leq |p| = e$. Therefore $d = e$.