(1) (Problem A) Rigorously prove that the relation R (defined both below and on page 40) is an equivalence relation.

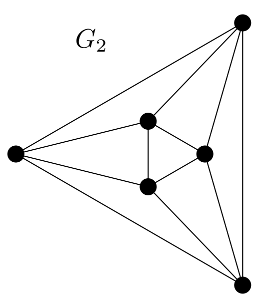
Let G be a simple graph with vertex set V. Let R be the relation on V defined as uRv if there exists $\phi \in Aut(G)$ such that $\phi u = v$.

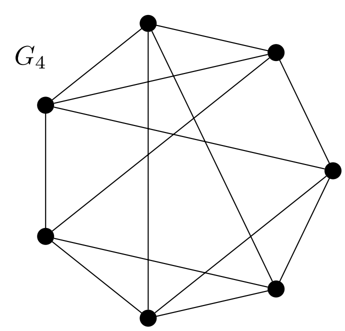
Proof: Reflexive: (Direct) WTS that for uRu. Every automorphism group must contain the identity, which maps every vertex to itself. Therefore uRu.

Symmetric: (Direct): WTS that if uRv then uRv. Suppose $u, v \in V(G)$ such that uRv. By our the definition of R we know that there exists an automorphism $\phi \in Aut(G)$ such that $\phi(u) = v$. Since groups are closed with respect to inverses we know that $\phi^{-1} \in Aut(G)$. Therefore it follows that $\phi^{-1}(v) = u$ and thus vRu.

Transitive: (Direct) WTS if uRv and vRw then uRw. Suppose $u, v, w \in V(G)$, uRv and vRw. By our definition of R we know that there exists an automorphism $\phi \in Aut(G)$ such that $\phi(u) = v$ and $\lambda \in Aut(G)$ such that $\lambda(v) = w$. Since $Aut(G)$ is a group closed with respect to function composition we know that $\lambda(\phi) \in Aut(G)$. Note that $\lambda(\phi(u)) = w$, thus uRw.

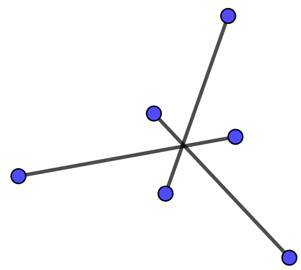
(2) (Problem B:) A graph that contains a single orbit is called *vertex transitive*. **Prove** that the regular graph G_2 is vertex transitive and G_4 is not vertex transitive.





Proof: Consider the compliments for each graph.

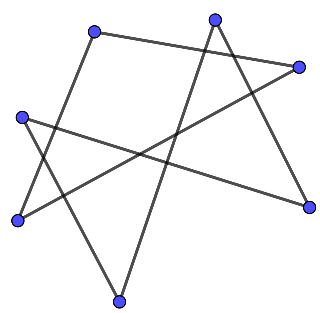
Figure 1. Graph $\overline{G_2}$



Consider a vertex $v \in V(\overline{G_2})$.
Case 1: WTS: There exists $\phi(v) = u$ such that $u \in N(v)$. Since $\overline{G_2}$ is composed of 3, K_2 components we know that $\phi : (uv)$ must preserve adjacency and non adjacency. Thus $\phi \in Aut(\overline{G_2})$ and from Theorem 2.10 we know that $Aut(\overline{G_2}) \cong Aut(G_2)$ thus $\phi \in \overline{G_2}$.

Case 2: WTS: There exists $\phi(v) = u$ such that $u \in \overline{N(v)}$. Note $\overline{G_2}$ is composed of 3, K_2 components. Let $w \in N(v)$ and $x \in N(u)$. Consider the mapping $\phi : (vu)(wx)$. Note that $N(\phi(v)) = w$, $N(\phi(v)) = x$ therefore adjacency and non adjacency is preserved. Thus $\phi \in Aut(\overline{G_2})$ and from Theorem 2.10 we know that $Aut(\overline{G_2}) \cong Aut(G_2)$ thus $\phi \in \overline{G_2}$.

Figure 2. Graph $\overline{G_4}$



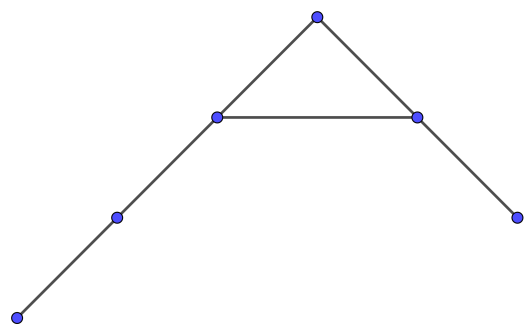
Note that $\overline{G_4}$ has two components a C_4 and C_3. Let $u \in C_4$ and $v \in C_3$. Note that any mapping with the permutation (uv) would fail to preserve *totaldistance* for either vertex. Thus $\overline{G_4}$ is not vertex transitive. From Theorem 2.10 we know that $Aut(\overline{G_4}) \cong Aut(G_4)$ and therefore G_4 cannot be vertex transitive.

(3) (Problem C) For which pairs k and n of positive integers with $k \leq n$ does there exist a graph G of order n having k orbits?

Proof:

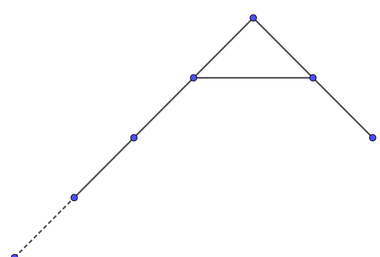
Algorithm 1: Consider the tuple (k, n) where $k = n \geq 6$. Now I will describe an algorithm which produces a graph G for which the tuple applies.
Input: A tuple (k, n) where $k = n \geq 6$.
Initialization: Consider the graph G, for which the tuple $(6, 6)$ applies.

FIGURE 3



Iteration: If (k, n) doesn't apply append a vertex to the graph, incident to the leftmost vertex.

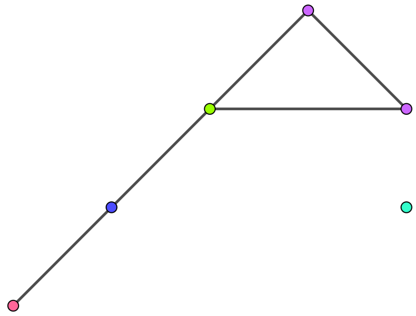
FIGURE 4



Note that the graph now corresponds to the tuple $(7, 7)$. Thus the algorithm will terminate after $n - 6$ iterations and produce a graph that corresponds to the (k, n) tuple. Thus all same number tuples greater than 5 apply.

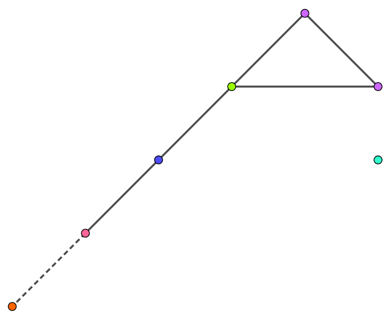
Algorithm 2: Now consider the tuple, (k, n) where $n > k$ and $k \geq 6$. Now I will describe an algorithm which produces a graph G for which the tuple applies.
Input: A tuple (k,n) where $n > k$ and $k \geq 6$.
Initialization: Consider the graph G with m orbits and i vertices. Note that $(6, 7)$ applies to G.

FIGURE 5



Iteration: if $m \neq k$ append a vertex to the graph, incident to the leftmost vertex.

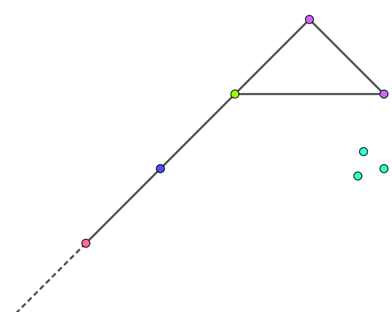
FIGURE 6



Note that G now has $m+1$ orbits and, $i+1$ vertices. This part of the algorithm, terminates after $k-6$ iterations and produces a graph with k orbits and $m = i+k-6$ vertices.

Iteration 2: Add trivial components to G until $m = n$. Note that adding these vertices does not add to the number of orbits yet increases the number of vertices. Finally the algorithm terminates when $m = n$ and produces a graph which corresponds to the tuple (k, n).
For example when we input the tuple $(6, 9)$ the algorithm produces the following graph,

FIGURE 7



So we have shown that we can produce a graph for every tuple $6 \leq k, n \leq \infty$. There are more and I think they require special cases like $(1, n)$ which corresponds to K_n or $\overline{K_n}$.

P.S: I did exactly what you suggested in your email, and I saw a few patterns but not any that would produce an algorithm regardless of the choices for k and n. Describing how to produce a graph for when $2 \leq k \leq 5$ was difficult even though through exploring those graphs I know that tuples like $(2, 2), (3, 3), (4, 4)$, and $(5, 5)$ don't exist. Also when I discussed this question with a group of other students the idea of allowing multi-graphs came up but I felt that it would make the problem too trivial. A pattern I found that was really interesting is that we can prove the tuple $(k, k + 1)$ exists by induction on k starting on $k = 2$ by picking the graph that uses a trivial component, taking the compliment and then adding a trivial component.

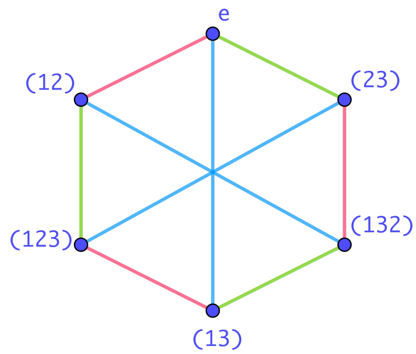
(4) (Problem D) For which pairs k and n of positive integers with $k \leq n$ does there exist a graph G of order n and a vertex v of G such that there are exactly k vertices similar to v? (Recall that u and v are *similar* if v is in the same orbit as u.

Proof: Consider a tuple (k, n) of positive integers with $k \leq n$. Take k vertices and construct a K_k, let the rest of the $n - k$ vertices be trivial components. This construction produces a graph for every tuple except $(1, n)$ since any K_1 would be included in the orbit of trivial components. To construct a graph for the tuples $(1, n)$ consider a K_{n-1} and a single trivial component. This construction produces a graph for every tuple $(1, n)$ except for $(1, 2)$ since $K_{2-1} = K_1$ and thus any graph on 2 vertices must have an orbit with 2 elements. Therefore, with the exception of $(1, 2)$ there exists a graph for every tuple (k, n) where $k \leq n$.

(5) (Problem E) On page 44, the text graphs the Cayley color graph of the group S_3 with generators $\alpha = (123)$ and $\beta = (12)$. Determine (i.e. draw the graph of...) the Cayley color graph of the group S_3 with generators $\beta = (12)$, $\gamma = (23)$, and $\delta = (13)$. Prove that the automorphism group of this graph is isomorphic to S_3.

Proof: First note that we can get the permutation (132) by the following compositions: (12)(23), (13)(12), and (23)(13). We can get the permutation (123) by the following compositions: (23)(12), (12)(13), and (13)(23). Note that every permutation in $\Delta = \{(12), (23), (13)\}$ is order two and thus every edge in $D_\Delta(S_3)$ is bi-directional. Thus the following graph $D_\Delta(S_3)$ is,

Figure 8



By Theorem 2.14 we know that the color preserving automorphism group of $D_\Delta(S_3)$ is isomorphic to S_3.