(1) (problem A) Prove that if the n-vertex graph G has k components each of which is acyclic, then $e(G) = n - k$. (Note: a graph is *acyclic* if it has no cycles.)

Proof (Direct:) Suppose a graph G, with k acyclic components. Consider the connected, acyclic graph H that is created by adding edges that connect the components of G. Note that $e(H) = e(G) + (k-1)$ and $n(H) = n(G)$. Since H is acyclic and connected it is a tree, by Theorem 2.1.4, H has $e(H) = n(H) - 1$. Through algebra,

$$\begin{aligned} e(H) &= e(G) + (k-1), \\ n(H) - 1 &= e(G) + (k-1), \\ n(G) - 1 &= e(G) + (k-1), \\ n(G) - k &= e(G). \end{aligned}$$

□

(2) (problem B) Prove that if L is a minimum edge cover of graph G, then every component of L is a star. (Note: a *star* is isomorphic to $K_{1,r}$ for some r.)

Proof (Contradiction:) Suppose L is a minimum edge cover of graph G, and there exists some component K, of L that is not a star. Since K is not a star there must be at least two vertices in K with at least degree 2. Consider vertex v where $d(v) \geq 2$, since k is a connected component and not a star there must be another vertex u that is adjacent to a neighbor of v in order to fulfill the degree requirements, and therefore we know that K contains a p_3 subgraph or a K_3 is $u \in N(v)$. Thus L could be made smaller. □

(3) (problem 3.1.4) For each of α, α', β, and β' characterize the simple graphs for which the value of the parameter is 1.

Proof (Direct:)
$\alpha = 1$ Consider any K_n. Every independent set can only contain one vertex otherwise the graph wouldn't be complete.

$\alpha' = 1$ Consider any $K_{1,n}$. Since every matching can would saturate the the partition with just one vertex, and the matching would contain one edge.

$\beta = 1$ Consider any star graph. Note that for a star graph a central vertex is incident to all edges, therefore the minimum vertex cover in any star graph is size one.

$\beta' = 1$ Consider a K_2, for a simple graph every edge is incident to exactly two vertices, thus the only way to have an edge cover size one is if the graph is a K_2.

$\square$

(4) (problem 3.1.5) Prove that $\alpha \geq \dfrac{n}{\Delta + 1}$ where $\alpha = \alpha(G)$, $n = n(G)$ and $\Delta = \Delta(G)$.

Proof (Direct:) Let the maximum independent set be X and minimum vertex cover be Y. From Lemma 3.1.21 we know that every vertex $v \in X$ is adjacent to at least one vertex $u \in Y$. Summing over the degrees of each vertex in X we get,

$$\beta \leq \sum_{v \in X} d(v)$$

Since each vertex in X has at most degree Δ we get,

$$\beta \leq \alpha\Delta.$$

Finally substituting Lemma 3.1.21 which states $\beta = n - \alpha$,

$$\begin{aligned}
\beta &\leq \alpha\Delta, \\
n - \alpha &\leq \alpha\Delta, \\
n &\leq \alpha\Delta + \alpha, \\
n &\leq \alpha(\Delta + 1), \\
\frac{n}{(\Delta + 1)} &\leq \alpha.
\end{aligned}$$

$\square$

(5) (problem 3.1.9) Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

Proof (Direct:) Suppose some maximal matching M. Let S be the set of vertices in G saturated by M.

Proving $\overline{S}$ is an independent set: Suppose that the $\overline{S}$ is not an independent set. Since vertices in $\overline{S}$ are unsaturated, it must be the case that there exists an edge incident to two unsaturated vertices that could be added to the maximal matching M. Thus $\overline{S}$ is an independent set.

Note that $n = |S| + |\overline{S}|$ and through algebra we get,

$$\begin{aligned} n &= |S| + |\overline{S}|, \\ |\overline{S}| &= n - |S|. \end{aligned}$$

Since $|S| = 2|M|$,

$$\begin{aligned} |\overline{S}| &= n - 2|M|, \\ \alpha &\geq n - 2|M|, \\ 2|M| &\geq n - \alpha. \end{aligned}$$

By Lemma 3.2.21,

$$2|M| \geq \beta.$$

Since $\beta \geq \alpha'$,

$$\begin{aligned} 2|M| &\geq \alpha', \\ |M| &\geq \frac{\alpha'}{2}. \end{aligned}$$

$\square$