Automorphism Groups:

Theorems 2.10 and 2.11

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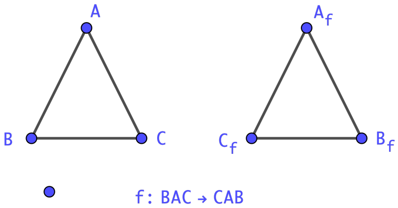
Review

- Automorphism: An isomorphism from a graph G to itself.
  - A permutation of V(G) that preserves adjacency and non-adjacency

f:(BC)



- Automorphism Group: The set of all automorphisms on G, form a group (denoted $Aut(G)$) under function composition.

Theorem 2.10

Theorem 2.10 : For every graph G, $Aut(G) \cong Aut(\overline{G})$

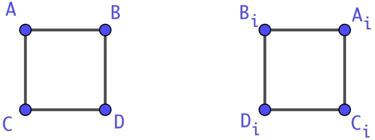
- How do we prove this?

Theorem 2.10

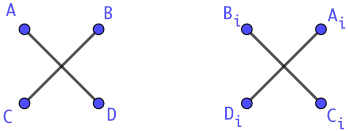
- **Direct** : WTS: For every $i \in Aut(G)$, that $i \in Aut(\overline{G})$.
  - Consider an automorphism $i \in Aut(G)$.
  - By definition $i : V(G) \to V(G)$ that preserves adjacency and non-adjacency.
  - We can apply i to the set $V(\overline{G})$ since $V(G) = V(\overline{G})$.
  - Note a function that preserves adjacency in G will preserve non-adjacency in $\overline{G}$.
  - Similarly a function that preserves non-adjacency in G will preserve adjacency in $\overline{G}$.
  - Therefore by definition i is an automorphism for $\overline{G}$.
  - Thus $i \in Aut(\overline{G})$.

Theorem 2.10 Example:

$i : (AB)(CD)$



$i : (ABCD) \to (BADC)$



Theorem 2.11

Theorem 2.11 : The order of the automorphism group of a graph G with order n is a divisor of $n!$ and equals $n!$ if and only if $G = K_n$ or $G = \overline{K}_n$

- What does that even mean?
  - if $|V(G)| = n$ then $|Aut(G)| \Big| n!$
  - If $G = K_n$ or $G = \overline{K}_n$ then $|Aut(G)| = n!$

Theorem 2.11

Recall

- **Symmetric Group** : The symmetric group S_n is the group of all permutations on n elements. Thus $|S_n| = n!$

$$S_3 = \begin{pmatrix} (1)(2)(3) & (1)(23) \\ (123) & (2)(13) \\ (132) & (3)(12) \end{pmatrix}$$

- **Lagrange′s Theorem** : If H is a subgroup of G, then $|G| = n|H|$ for some $n \in \mathbb{Z}$.
    - This implies $H \Big| G$.

Theorem 2.11

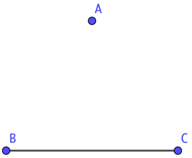
- **Direct** : WTS: if $|V(G)| = n$ then $|Aut(G)|\big|n!$
  - Suppose a graph G such that $|V(G)| = n$.
  - By definition the of an Automorphism Group (permutation) we know that $Aut(G)$ is a group of permutations on n elements that preserves (non)-adjacency,
  - Note that S_n is the group of **all** permutation on a set of n elements.
  - Thus $Aut(G) \triangleright S_n$.
  - By Lagrange's Theorem we know that $|Aut(G)|\big||S_n|$, and by substitution we get $|Aut(G)|\big|n!$.

Theorem 2.11

- **Direct** : WTS: If $G = K_n$ or $G = \overline{K}_n$ then $|Aut(G)| = n!$
  - Suppose K_n,
  - By definition the of an Automorphism Group (permutation) we know that $Aut(K_n)$ is a group of permutations on n elements that preserves (non)-adjacency.
  - Since every vertex in K_n is adjacent with the rest of the vertices, **every** permutation of $V(K_n)$ preserves (non)-adjacency.
  - Therefore $Aut(K_n) \cong S_n$
  - By Theorem 2.10 $Aut(K_n) \cong Aut(\overline{K}_n)$
  - Thus $|Aut(G)| = |Aut(\overline{K}_n| = n!$.

Theorem 2.11 Example:

Graph G:



- $$Aut(G) = \{(a)(b)(c), (a)(bc)\}$$

- $$2 \mid 3!$$

Theorem 2.11 Example:

Graph G: