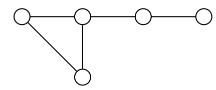
(1) (Problem A) On page 22 of the text you will find a graph from Example 1.2.9. (On the page, the graph is between Example 1.2.9 and Remark 1.2.10.

Is the graph, H, below a component of the graph from Example 1.2.9? Justify your answer. You answer must be formal and, therefore, it must use the definition of *component*.

H

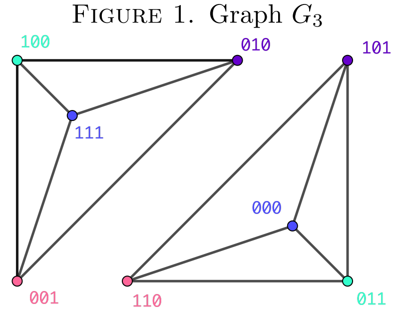


Answer: By definition the components of a graph G are its maximal connected subgraphs. When we look at the graph H above and compare it to the graph on page 22 we can see that it is missing an edge and therefore it is not a maximal subgraph, because it could be made larger.

(2) (Problem B) Let G_n be the graph with vertex set consisting of binary n-tuples. (So $V(G) = \{(a_1, a_2, \cdots, a_n) \mid a_i \in \{0, 1\}\}$.) Two vertices of G_n are adjacent if and only if the n-tuples differ in exactly two coordinates.

(a) Draw G_3.
Answer: Consider the following graph, Color coded by n-tuples that differ by exactly three coordinates.

Figure 1. Graph G_3



(b) For $v \in V(G_n)$, determine $\deg(v)$. (Justify your answer.)
Answer: Consider vertex v. We can determine the number of vertices adjacent to v by counting up the number of ways there is to change an arbitrary n-tuple by exactly two coordinates. Consider $\binom{n}{2}$, since there are n digits in the tuple and we are only choosing two of them to change.

(c) Determine the number of components of G_n. (Justify your answer.)
Answer: In a G_n there is always two components. Consider we partition the vertex set into two parts by if the number of ones that appear in the tuple are even or odd, ie $V_E(G_n)$ contains all even n-tuples and $V_O(G_n)$ contains all odd n-tuples. Consider vertices $u, v \in V_E(G_n)$ we know by the definition of the partition $V_E(G_n)$ that the n-tuples that represent u and v have a difference of $2k$ ones, where $k \in \mathbb{W}$, and therefore there must exist a path between them in G. The same argument applies for vertices $u, v \in V_O(G_n)$. Now consider that vertex $u \in V_E(G_n)$ and $w \in V_O(G_n)$, by the definition of our parts we know that each n-tuple u and w differs by $2k + 1$ ones, where $k \in \mathbb{W}$. Therefore there cannot exist a path between them in G

(3) (Problem 1.2.10) Prove or disprove:

(a) Every Eulerian bipartite graph has an even number of edges.

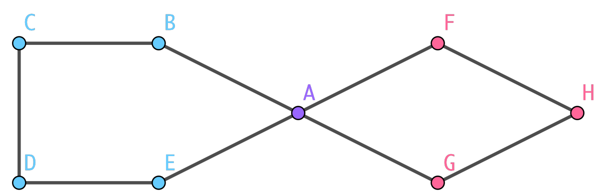
Answer:

Suppose Eulerian bipartite graph G. From Theorem 1.2.26 we know that if a graph is Eulerian the degree of each vertex must be even. Since G must be an even graph we know from Proposition 1.2.27 that G decomposes into cycles. Since G is a bipartite graph we know that it cannot contain an odd cycle from Theorem 1.2.18. Therefore G is composed of even cycles and thus has an even number of edges.

(b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

Answer: By counterexample. Consider the following graph.

Figure 2. Graph H

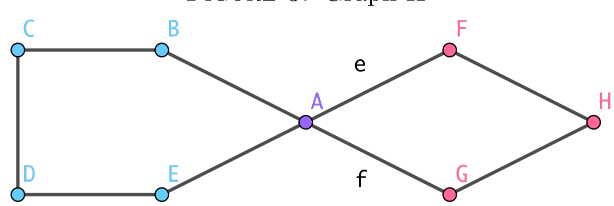


We can see that the graph has an odd number of edges and each vertex has an even degree. consider the Eulerian circuit $[A, D, C, D, A, E, F, G, A]$

(4) (Problem 1.2.11) Prove or disprove: If G is an Eulerian graph with edges e, f that share a vertex, then G has an Eulerian circuit in which e, f appear consecutively.

Answer: Consider Graph H where edges e and f are labeled.

Figure 3. Graph H



We showed that H is Eulerian, and as we can see e and f share vertex A. Note that, constructing a closed trail where edges e and f are consecutive will always result in a maximal trail that does not contain every edge in the graph H. Therefore there cannot exist an Eulerian circuit where e and f are consecutive.

(5) (Problem 1.2.25) Use ordinary induction on the number of edges or vertices to prove that the absence of odd cycles is a sufficient condition for a graph to be bipartite.

Proof: We will proceed by induction on the number of verticesℓ.

Base Case: Suppose Graph G on n vertices and has no odd cycles. Let $\ell = 0$. Note that every 2-partition of $V(G)$ is always disjoint, therefore G is trivially bipartite.

Induction Hypotheses: Suppose a graph G with no odd cycles, and ℓ edges is bipartite.

Induction Step: Suppose a graph G with no odd cycles and $\ell + 1$ edges. Consider an arbitrary edge e in G that is incident to vertices u and v. Now consider $G - e$, we now have a graph with no odd cycles, ℓ edges. By the induction hypothesis $G - e$ is bipartite. Since $G - e$ is bipartite we can partition $V(G - e)$ into two disjoint sets such that no two vertices within the same set are adjacent. Note there will always exists a partition $V(G - e)$ where u and v are in different parts. We can add edge e back into the graph $G - e$ by selecting two vertices, one from each part. Thus we have shown that graph G is bipartite.

(6) (Problem C) Let H be a connected graph with at least three vertices (so u, x and y are assumed to be distinct) and let u be the endpoint of a maximal path in H. Call this path P. Let $x, y \in V(H) - u$. Show that there exists an xy-path in $H - u$.

Case 1: x and y are inside of the path P. Since P is a maximal path The neighbors of endpoint vertex u must also lie on the path. Therefore when we delete u we cannot disconnect the graph because all the neighbors of u lie in P and would therefore still be connected. Thus there must still exist an xy-path in $H - u$.

Case 2: x and y are both outside of the path P. We know that x and y both are in a connected graph H so there must exist an xy-path Q. Suppose Q uses vertices from path P, we know that Q cannot contain the endpoints of P because then P wouldn't be a maximal path. Therefore deleting an endpoint u of path P would not disrupt path Q. Now suppose Q doesn't use vertices in P, then the deletion of vertex u still wouldn't disrupt path Q. Thus Q must still exist an xy-path in $H - u$.

Case 3: WLOG say that x is contained in path P and y is not. Since H is connected there must exist xy-path Q. Note, in Q, If x is directly incident to u then path P is no longer maximal. Therefore the first vertex in Q, from P must be an inner vertex. and since that is the case there must always exist an xy-path that avoids using the endpoints of P. Thus there must still exist an xy-path in $H - u$.

p.s. In the interest of transparency you should know that I worked with Brons for a majority of this homework, and I also helped Brett and Rohan approach the n - tuple and Eulerian problems.