The Art Gallery Problem

Stefano Fochesatto

University of Alaska Fairbanks

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Introduction

- Original Art Gallery Problem:
  - Suppose you run an art gallery and want to know what the minimum number of security guards is needed to secure the gallery.
- Mathematical Interpretation:
  - Suppose a simple polygon S, we say that vertex v guards vertex u if the segment $vu \in S$. What is the minimum number of vertices sufficient to guard all of S.

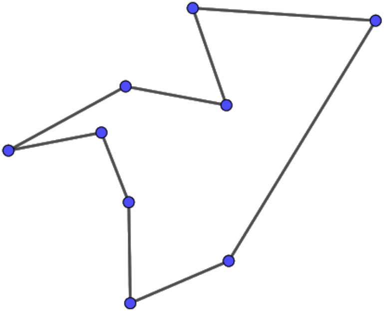
Dr. Victor Klee (1925 - 2007)

Steve Fisk's Solution

- Suppose a simple polygon S.
- Triangulate S such that no new vertices are added.
- The triangulation of S is 3 - colorable via an induction argument.
- Denote colors a, b, c. Let T_a denote the vertices colored by a.
- Without loss of generality we can assume $|T_a| \leq |T_b| \leq |T_c|$.
- Note that every point u lies on some triangle in the triangulation of S and every triangle contains a vertex v from T_a.
- Since triangles are convex we know that $vu \in S$ and therefore T_a guard S.
- Thus $\left\lfloor \frac{n}{3} \right\rfloor$ guards are sufficient to guard an n vertex simple polygon.

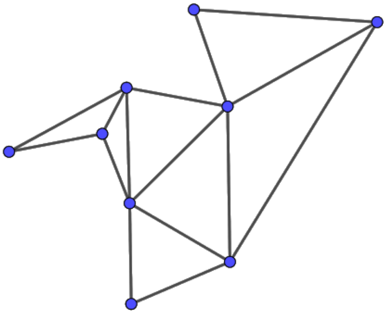
Steve Fisk's Solution Example

Suppose a Simple Polygon S



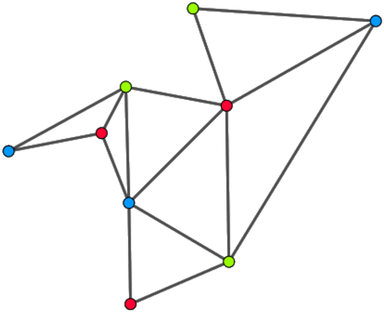
Steve Fisk's Solution Example

Triangulate S



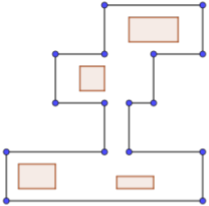
Steve Fisk's Solution Example

Three-color the Triangulation of S



Orthogonal Art Galleries and Thomas Shermer

Most of the time art galleries have walls that are parallel or perpendicular with each other, and the entire floor plan cannot be represented with a simple polygon. This leads us to an open variation of the problem which supposes the art gallery is represented by an orthogonal polygon with holes.



An orthogonal polygon with 4 holes

Orthogonal Art Galleries and Thomas Shermer

- Thomas Shermer's Conjecture:
  - Any orthogonal polygon with n vertices and h holes can always be guarded by $\left\lfloor \frac{n+h}{4} \right\rfloor$ guards.

- Most of all the research I've seen is heavily inspired by Fisk's approach.
  - Quadrilateralization of the polygon.
  - Make some sort of coloring argument on the quadrilateralization or the corresponding dual graph.

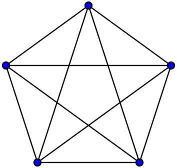
Pawel Zylinski's Cactus Graph Approach

- Published in March 7, 2006 from the University of Gdansk

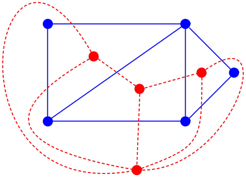
- Pawal's research aims to prove that $\left\lfloor \frac{n+h}{4} \right\rfloor$ vertex guards are always sufficient to see the entire interior of an n-vertex orthogonal polygon with an arbitrary number h of holes if that there exists a quadrilateralization whose dual graph is a cactus.

Review

- Dual Graph
  - Consider a planer graph G, the dual graph of G is a graph that has a vertex for each face of G, two vertices are adjacent if the corresponding faces share an edge.

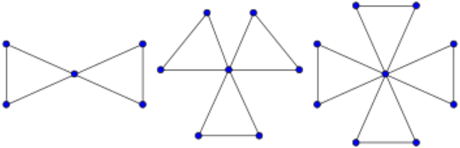


Non-planer Graph



Planer Graph and Dual Graph

- Cactus Graph
  - A graph with the property that any two of its cycles share at most one vertex.

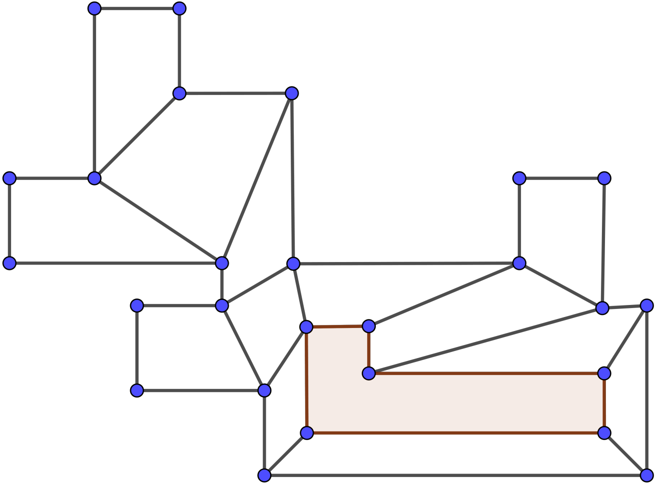


Orthogonal galleries with one hole

- let Q be the quadrilateralization of an n-vertex orthogonal polygon P, with one hole.
- Adding the diagonals to each quadrilateral gives us the quadrilateralization graph G_Q
- Following Fisk, we want to for color G_Q.

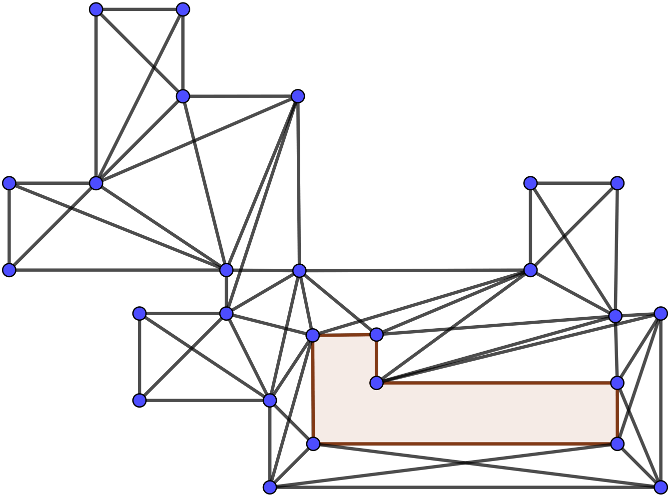
Orthogonal galleries with one hole

Quadrilateralization Q



Orthogonal galleries with one hole

Quadrilateralization Graph G_Q

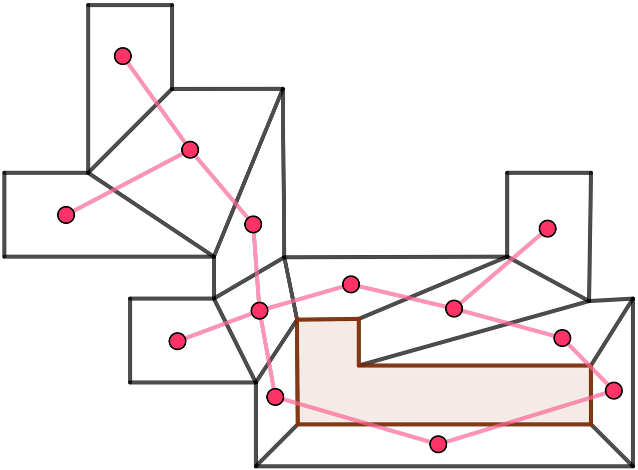


Orthogonal galleries with one hole Cont.

- Consider the dual graph G_D of quadrilateralization Q.
- Note that G_D is always composed of a single cycle and trees.
- Consider a quadrilateral, P in G_Q which corresponds to a leaf in G_D.
- Note that quadrilateral P will always have two vertices u, v that have degree three, and thus if $G_Q^1 = G_Q - u, v$ is 4-colorable, if and only if G_Q is also 4-colorable.
- Note that $G_Q - u, v$ removes a leaf in G_D.
- Repeat until G_D is a cycle, let the corresponding graph be G_Q^k
- G_Q is 4-colorable if and only if G_Q^k is 4-colorable

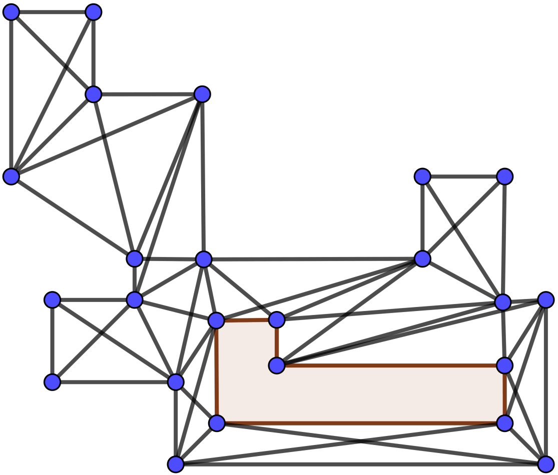
Orthogonal galleries with one hole Cont.

Dual Graph of Q, G_D



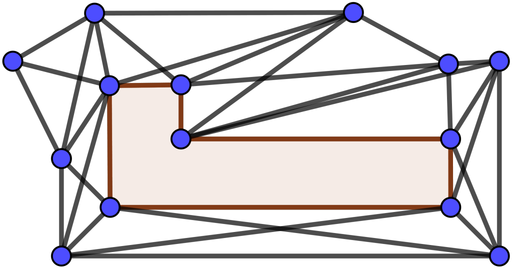
Orthogonal galleries with one hole Cont.

Graph G_Q^1



Orthogonal galleries with one hole Cont.

Resultant G_Q^k Graph



Orthogonal galleries with one hole Cont.

- Remove vertices of degree three from G_Q^k. Same argument as before.
- The resultant graph is colorable or can be made colorable by adding a vertex.
  - Hence
$$\left\lfloor \frac{n+1}{4} \right\rfloor$$
.

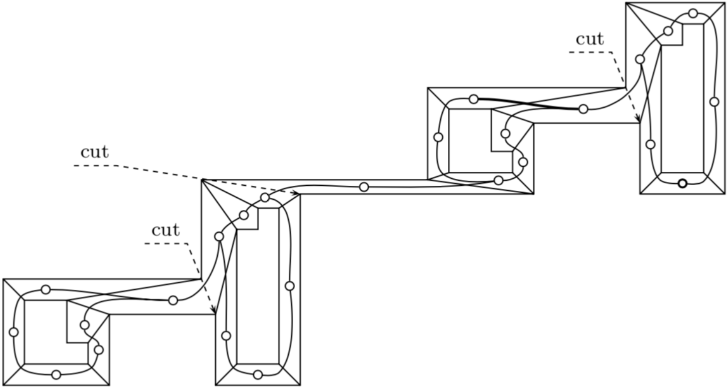
Orthogonal galleries with Cactus Dual Graphs

- Proving that this extends to an arbitrary number of holes.
  - Dual graph of the quadrilateralization contains cycles connected by single edges.
  - Dual graph of quadrilateralization contains cycles that share a vertex.
- Both cases involve splitting the graph up by the number of cycles, and applying the one hole result.
- Each sub-cycle (hole) will require adding at most one vertex we get,

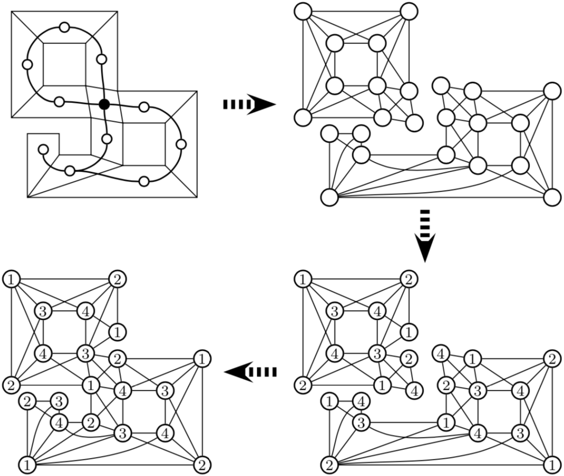
$$\left\lfloor \frac{n+h}{4} \right\rfloor$$

- Note that the family of graphs with the properties are cactus graphs.

Cycles are connected by an Edge.



Cycles are connected by a Vertex.

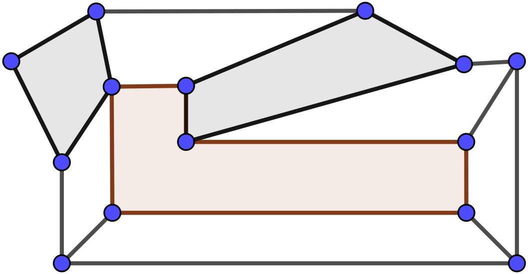


Orthogonal galleries with one hole Cont.

- Note that each quadrilateral in G_Q^k has all four vertices on either the interior or exterior boundary of the polygon.
- We say that a quadrilateral is *balanced* if it has 2 vertices on either boundary, otherwise we call it *skewed*
- Observe that every *skewed* quadrilateral has a vertex v of degree three.
- Let $G_Q^* = G_Q^k - v$. Note that G_Q is 4-colorable if and only if G_Q^* is 4-colorable.

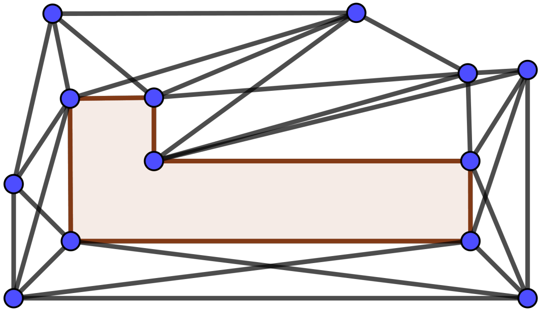
Orthogonal galleries with one hole Cont.

Balanced and Skewed Quadrilaterals



Orthogonal galleries with one hole Cont.

Resultant G_Q^* Graph



Proving G_Q^* is 4-colorable

- Note that skewed quadrilaterals result in two types of triangles.
  - *e*-triangles have two vertices on the exterior boundary.
  - *i*-triangles have two vertices in the interior boundary

- Lemma: The cycle in the dual graph of any quadrilateralization of an orthogonal polygon with one hole has an even number (at least four) of balanced quadrilaterals.

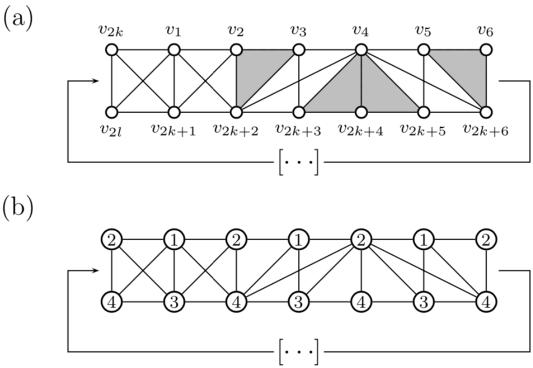
Proving G_Q^* is 4-colorable Cont.

Case 1: G_Q^* contains and even number of *e*-triangles and *i*-triangles.

- Note that G_Q^* contains $m = 2l$ vertices where l is the number of *balanced* quadrilaterals.
- We can color the graph by alternating 2 colors for the outside boundary, and the 2 other colors for the inside boundary.

Proving G_Q^* is 4-colorable Cont.

Resultant G_Q^* Graph



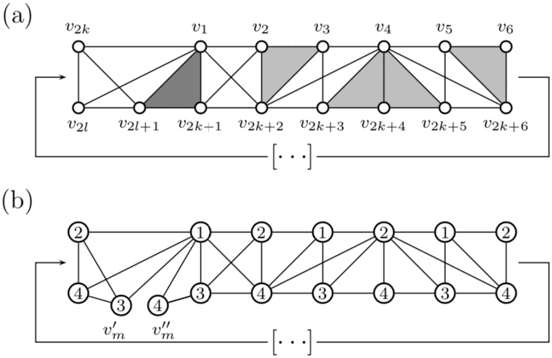
Proving G_Q^* is 4-colorable Cont.

Case 2/3: G_Q^* contains and even number of e-triangles and odd number of i-triangles.

- Note that G_Q^* contains $m = 2l + 1$ vertices where l is the number of *balanced* quadrilaterals.
- Consider the i-triangle with interior vertices A and B and exterior vertex C.
- Split A into A' and A'' where $N(A') = N(A) - B$ and $N(A'') = \{B, C\}$.
- We can color the graph by alternating 2 colors for the outside boundary, and the 2 other colors for the inside boundary.

Proving G_Q^* is 4-colorable Cont.

Resultant G_Q^* Graph



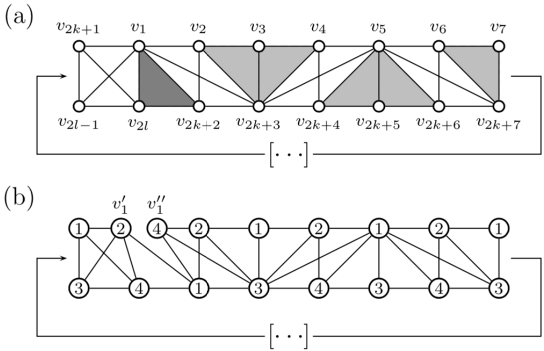
Proving G_Q^* is 4-colorable Cont.

Case 4: G_Q^* contains and odd number of e-triangles and odd number of i-triangles.

- Note there exists either an i-triangle or e-triangle that shares an edge with a K_4 subgraph.
- WLOG assume it to be t_i.
- Label the vertices on the external cycle $v_1, v_2, ..., v_{2k+1}$ in a clockwise manner.
- Label the vertices on the internal cycle $v_{2k+2}, v_{2k+3}, ..., v_m$ in a clockwise manner.
- We can labeled the graph such that t_i is labeled $(v_m, v_{2k+2} v_1)$
- Note that $V(G_Q^*)$ is even.
- Split v_1 into v_1' and v_1'' such that $N(v_1') = N(v_1)/\{v_2, v_{2k+3}\}$ and $N(v_1'') = \{v_2, v_{2k+2}, v_{2k+3}\}$.

Proving G_Q^* is 4-colorable Cont.

Resultant G_Q^* Graph



Proving G_Q^* is 4-colorable Cont

- Thus we have shown that for any quadrilateralization Q of an orthogonal polygon with one hole, the graph G_Q^* is 4-colorable, with the upper-bound on the smallest color class being $\left\lfloor \frac{n+1}{4} \right\rfloor$.