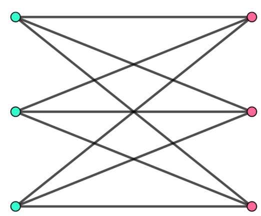
(1) Prove or disprove: For any simple graph G, $\chi(G) \geq \delta(G)$, where $\delta(G)$ denotes the minimum degree of G.

Answer: Let G be a $K_{3,3}$,

Figure 1. Complete bipartite graph on 3 vertices.

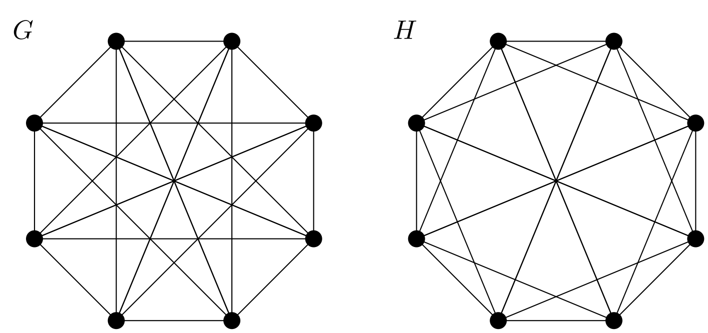


Note that the blue and pink two coloring indicates that $\chi(G) \leq 2$ and since showing that $\chi(G) \neq 1$ is trivial we know that $\chi(G) = 2$. By the definition of complete bipartite graph we know that $\delta(K_{m,n}) = min\{m, n\}$, so therefore $\delta(G) = 3$.

(2) (1.1.5) Prove or disprove: If every vertex of a simple graph G has degree 2, then G must contain a cycle.

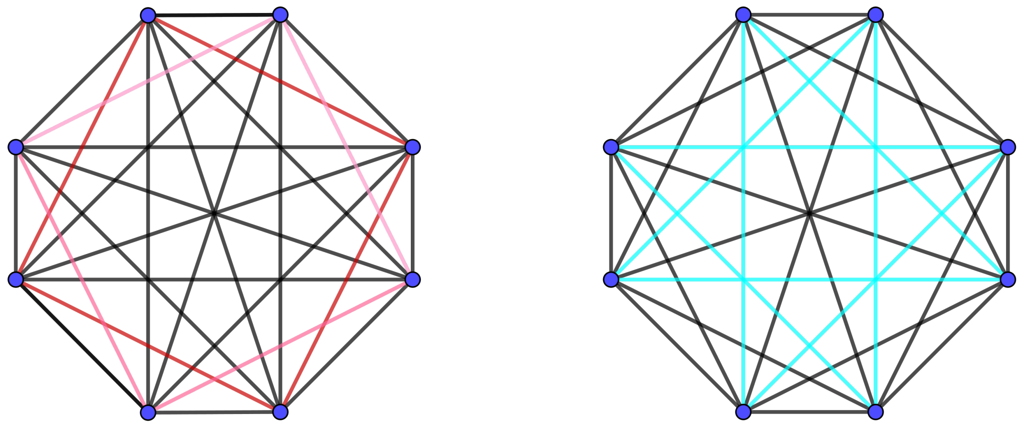
Proof(Contradiction:) Suppose a simple graph G has degree 2, and that G is acyclic. Now consider the longest path l in G, with end vertices a and z. Since each vertex in graph G has degree 2 and G is acyclic, vertices a and z must both be adjacent to vertices not contained inside the path l. Therefore l is both the longest path and not the longest path.

(3) (1.1.16) Determine whether the graphs below are isomorphic. (Full credit will be given only for succinct solutions.)



Answer: Consider the compliment of each graph,

Figure 2. Compliment of graphs G and H.



We can see that the compliment of G is composed of 2 disconnected four cycles and the compliment of H is an 8 cycle. Thus the compliments of graphs G and H are not isomorphic and therefore there cannot exists an edge preserving, vertex bijection between G and H, i.e G and H cannot be isomorphic.

(4) (1.1.26) Let G be a graph of girth 4 in which every vertex has degree n. Prove that G has at least $2n$ vertices. Determine all such graphs with exactly $2n$ vertices.

Answer: Suppose a graph G with girth 4, where every vertex has degree n. Now consider vertex i in G. Note that vertex i must be adjacent to n other vertices, which themselves, cannot be adjacent to each other, in order to avoid a 3 cycle. We can fill in the rest of graph G by making each vertex that is adjacent to i also adjacent to $n-1$ vertices that are not adjacent to i. We do this to avoid a 3 cycle and to give each vertex degree n. Note, that what we have described is a complete bipartite graph $K_{n,n}$ which has girth 4 when $n \geq 2$ and each vertex has degree n. Since graph G must be a complete bipartite graph, if each vertex has degree n there must be at least $2n$ vertices.