(1) (problem 1.2.5) Let v be the vertex of a connected simple graph G. Prove that v has a neighbor in every component of $G - v$. Conclude that no graph has a cut-vertex of degree 1.

Proof: Suppose a simple connected graph G with vertex v. Now consider $G - v$. Suppose $G - v$ is still one component, since we know that v cannot be a trivial component we know that v must have a neighbor in the one component. Now suppose $G - v$ has more components then G ie more than one, then by definition we know that v is a cut vertex. Since v is a cut vertex it must have neighbors in each of the components of $G - v$.

(2) (problem 1.2.8) Determine the values of m and n such that $K_{m,n}$ is Eulerian.

Proof: From Theorem 1.2.26 we know that for a graph to be Eulerian it must have at most one non-trivial component and all the vertices must have an even degree. Since every complete bipartite graph has at most one non-trivial component, all we have to worry about is the degree of each vertex. In a complete bipartite graph $K_{m,n}$ with vertex partitions M, N then we know every vertex in the partition M will have degree n and and vertices in N have degree m. Let m, n be even natural numbers.

(3) (problem 1.2.20) Let v be a cut vertex of a simple graph G. Prove that $\overline{G} - v$ is connected.

Proof: We have to show that the graph $\overline{G} - v$ is connected. Suppose $u, w \in V(G - v)$, since v is a cut vertex we know that $G - v$ must have more that one non-trivial component. Consider the case where u and w are in different components in $G - v$, then we know for certain that they are not neighbors and there for the edge uw exists in $\overline{G} - v$. Now consider the case where lie in the same connected component in $G - v$. Since v is a cut vertex we know that a vertex x that lies in a different component from u, w cannot be neighbor to either u or w. Thus then we look at $\overline{G} - v$ we know that there has to exist a path $[u - x - w]$. Thus we have show that $\overline{G} - v$ is connected

(4) (problem 1.3.1) Prove or Disprove: If u and v are the only vertices of odd degree in a graph G, then G contains a u, v-path.

Proof: (Contradiction:) Suppose u and v are the only vertices of odd degree in a graph G, and G does not contain a u, v-path. If G does not contain a u, v-path we know that u and v must lie in separate components U and V. Now consider subgraph U, we can use the degree sum formula to get,

$$\sum_{v \in V(U)} dv = 2e(U)$$

Since u is the only vertex of odd degree in graph U we have a contradiction because the sum of the degrees cannot be an even number.

(5) (problem 1.3.3) Let u and v be adjacent vertices in a simple graph G. Prove that edge uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G.

Proof: To count the number of triangle is equivalent to counting the number of neighbors that are shared between vertices u and v. Consider a graph G where every vertex is a neighbor to either u or v. Continuing by inclusion-exclusion we can count the neighbors of u by $d(u)$ and similarly with $d(v)$, however we have counted the neighbors that are shared by both vertices twice (including themselves), and the unshared neighbors once so we subtract away the total number of vertices $n(G)$ to get the total number of shared vertices. In this case we have counted exactly the number of shared neighbors between u and v,

$$neighbors(u, v) = d(u) + d(v) - n(G)$$

There is the case, as with most graphs that there are vertices that are neither neighbor to u or v and therefore do not get counted at all during the inclusion step, but they do get counted during the exclusion step, therefore the result of our count will always be a lower bound,

$$neighbors(u, v) \geq d(u) + d(v) - n(G)$$