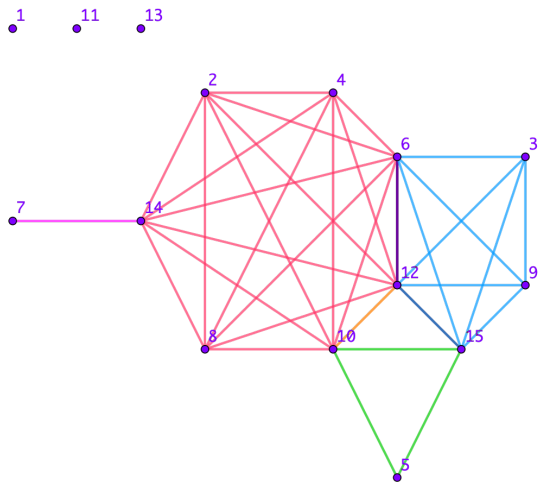
(1) (problem 1.2.3) Let G be the graph with vertex set $\{1, 2, 3, \cdots, 15\}$ in which i and j are adjacent if and only if their greatest common factor exceeds 1. Count the components of G and determine the maximum length of a path in G.

Answer: Consider graph G, (color coded by common primes.)

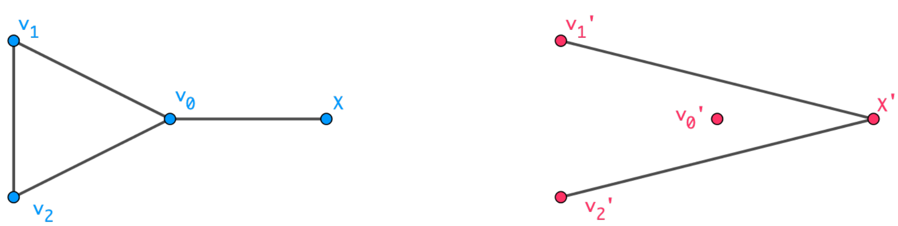
Figure 1. Graph G



We can see that the maximum length path contains every vertex in the non trivial component and therefore their cannot exist a larger path. Consider the path $7, 14, 8, 10, 5, 15, 12, 9, 3, 6, 4, 2$ which has a length of 11 and contains every vertex in the non-trivial component.

(2) (problem 1.2.6) In the graph below (the paw), find all the maximal paths, maximal cliques, and maximal independent sets. Also find all the maximum paths, maximum cliques and maximum independent sets. (Note: I added labels to the vertices since that may make your answers easier to write in some cases.)

Figure 2. Graph H and $\overline{H}$



Answer:

Graph H		
	Maximal	Maximum
Path	x, v_0, v_1, v_2 x, v_0, v_2, v_1 v_2, v_0, v_1	x, v_0, v_1, v_2 x, v_0, v_2, v_1
Clique	v_0, v_2, v_1 x, v_0	v_0, v_2, v_1
Independent Set	v_0 x, v_1 x, v_2	x, v_1 x, v_2

(3) (problem 1.2.13) Alternative proofs of Lemma 1.2.5, that every u, v-walk contains a u, v-path.

(a) (ordinary induction) Given that every walk of length $\ell - 1$ contains a path from its first vertex to its last, prove that every walk of length ℓ also satisfies this.

Proof:
Base Case: Suppose $\ell = 1$. Now consider all walks W length $\ell - 1 = 1 - 1 = 0$. Having no edge $u - v$ walk W must consist of a single vertex such that $u = v$ and therefore W contains a u, v path length 0 that consists of this single vertex.

Induction Hypothesis: Suppose that every u, v-walk W, with length at most $\ell - 1$ contains a u, v-path. Now consider the walk, $W = [v_1, v_2, ...v_\ell]$.

Induction Step: We can partition the walk into two separate walks such that any cycles or repeated vertices are contained in each part, in order to avoid overlapping paths. Consider $W' = [v_1, v_2, ...v_k]$ and $W'' = [v_{k+1}, v_{k+2}, ...v_\ell]$. Note that $|W'|, |W''| \leq \ell - 1$, therefore we can use our induction hypotheses to say that there exists a path from vertices v_1 to v_k and from v_{k+1} to v_ℓ. Also note since v_k and v_{k+1} are adjacent in walk W there must exist a path through $[v_1, v_2, ...v_\ell]$.

(b) (extremality) Given a u, v-walk W, consider a shortest u, v-walk contained in W.

Proof: (Contradiction) Suppose the shortest u, v-walk W, and that W is not a u, v-path. Now consider $W = [v_1, v_2, ...v_k]$, since we know that W is not a path we can say that there exists some $v_i, v_j \in W$ where $v_i = v_j$ and $i \neq j$. Now consider walk W' that is composed by

$$W' = W - [v_{i+1}, ...v_j] = [v_1, v_2, ..., v_{i+1}, ..., v_j, v_{j+1}, ..., v_k]$$

Thus W is the shortest u, v-walk and also not the shortest u, v-walk.