

**Exercise 1:** Consider the 3x3 real matrix,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 0 & -2 \\ 2 & 2 & 3 \end{bmatrix}$$

1. Compute the eigenvalues of A,

**Solution:**

First we consider the characteristic equation of A,

$$|A - I\lambda| = 0$$

$$\left| \begin{bmatrix} 2-\lambda & 1 & 1 \\ 4 & -\lambda & -2 \\ 2 & 2 & 3-\lambda \end{bmatrix} \right| = 0$$

Solving for  $\lambda$  when the determinant is zero using co-factor expansion we get the following,

$$\begin{aligned} (2-\lambda)[- \lambda(3-\lambda) + 2(2)] - [4(3-\lambda) + 2(2)] + [4(2) + 2\lambda] &= 0 \\ -\lambda^3 + 5\lambda^2 - 10\lambda + 8 + 4\lambda - 16 + 2\lambda + 8 &= 0 \\ -\lambda^3 + 5\lambda^2 - 4\lambda &= 0 \\ \lambda(-\lambda^2 + 5\lambda - 4) &= 0 \\ \lambda(\lambda - 1)(\lambda - 4) &= 0 \end{aligned}$$

Finally we get that the eigenvalues for the matrix A are  $\lambda = 0, 1, 4$

2. Compute the rank of A,

**Solution:**

To compute the rank of A let's first get the matrix in row echelon form,

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 0 & -2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & -4 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are only 2 non-zero pivot values in the matrix we that the rank of the matrix is 2.

3. Compute the determinant of A,

**Solution:**

We can compute the determinant of A by hand using co-factor expansion,

$$\det(A) = 2(0 + 4) - 1(12 + 4) + 1(8 + 0) = 0$$

We get that the determinant of the matrix A is 0. This is expected as the matrix is not full rank and therefore not invertible.

4. Compute the inverse of  $A$  (if possible),

**Solution:**

As we have demonstrated,  $A$  is not full rank and it has the property that  $\det(A) = 0$ . Therefore  $A$  is not invertible.

5. Compute the inverse of  $B = A(2 : 3, 1 : 2)$

**Solution:**

Consider the matrix  $B$ ,

$$B = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

We can quickly see that the  $\det(B) = 8$ , and thus  $B$  is invertible. We can compute the inverse fairly quickly with the  $2 \times 2$  matrix inverse formula,

$$B^{-1} = \frac{1}{8} \begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

6. Solve the linear system of  $Ax = b$  where  $b = [-1, 8, -6]^*$ ,

**Solution:**

Consider the following augmented matrix,

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & -1 \\ 4 & 0 & -2 & 8 \\ 2 & 2 & 3 & -6 \end{array} \right]$$

Reducing to REF we get that,

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & -1 \\ 4 & 0 & -2 & 8 \\ 2 & 2 & 3 & -6 \end{array} \right] = \left[ \begin{array}{ccc|c} 2 & 1 & 1 & -1 \\ 0 & -2 & -4 & 10 \\ 2 & 2 & 3 & -6 \end{array} \right] = \left[ \begin{array}{ccc|c} 2 & 1 & 1 & -1 \\ 0 & -2 & -4 & 10 \\ 0 & 1 & 2 & -5 \end{array} \right] = \left[ \begin{array}{ccc|c} 2 & 1 & 1 & -1 \\ 0 & -2 & -4 & 10 \\ 0 & 0 & 0 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Note that  $x_3$  is a free variable, so in our solution we get the following,

$$Ax = x_3 \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$$

7. Now check your solutions with MATLAB.

**Terminal:**

```
>> A = [2 1 1; 4 0 -2; 2 2 3]
A =
     2     1     1
     4     0    -2
     2     2     3

>> format long
>> eig(A)
ans =
 4.000000000000000
-0.000000000000002
 1.000000000000001

>> rank(A)
ans =
     2

>> det(A)
ans =
     0

>> inv(A)
Warning: Matrix is singular to working precision.
ans =
   Inf   Inf   Inf
   Inf   Inf   Inf
   Inf   Inf   Inf

>> b = [-1 8 -6]
b =
    -1     8    -6

>> A\b'
Warning: Matrix is singular to working precision.
ans =
   NaN
   NaN
   NaN
```

**Exercise 2:** Write a Matlab script which generates 10 random matrixes of size  $m \times m$  for each of these powers of two:  $m : 2, 4, 8, \dots, 256$ . Every matrix will have entries which are random real numbers uniformly distributed on  $[-10, 10]$ . For each of these matrices compute the rank, the 2-norm, and the absolute value of the determinant. Communicate these data using plots in reasonable ways; a significant part of your script will be devoted to generating plots.

**Solution:**

**Code:**

```
StoreMatrix = {}; %Initilize Storage Cell

%Generating Data
for i = 1:8

    %Initilizing individual data storage cells
    MatrixVec = {};
    NormVec = {};
    RankVec = {};
    DetVec = {};

    %Generating Matrices and Computing data
    for k = 1:10
        j = 20.*rand(2^i)-10; %Generate 2^ix2^i matrix

        % Data Computation and Storage to individual cells
        MatrixVec{1,k} = j;
        RankVec{1,k} = rank(j);
        NormVec{1,k} = norm(j);
        DetVec{1,k} = abs(det(j));
    end

    %Saving generated matrices
    StoreMatrix{i,1} = MatrixVec;
    %Saving the ranks
    StoreMatrix{i, 2} =RankVec;
    %Saving the 2-norms
    StoreMatrix{i, 3} = NormVec;
    %Saving the determinants
    StoreMatrix{i, 4} = DetVec;

end

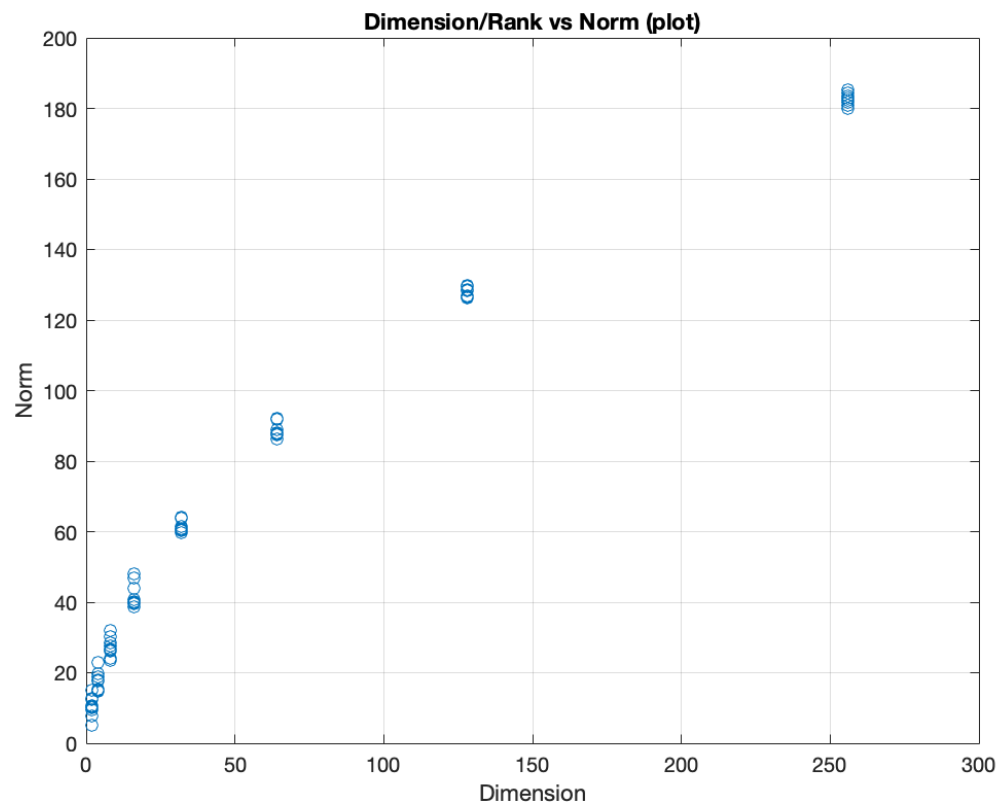
%Plotting Data
```

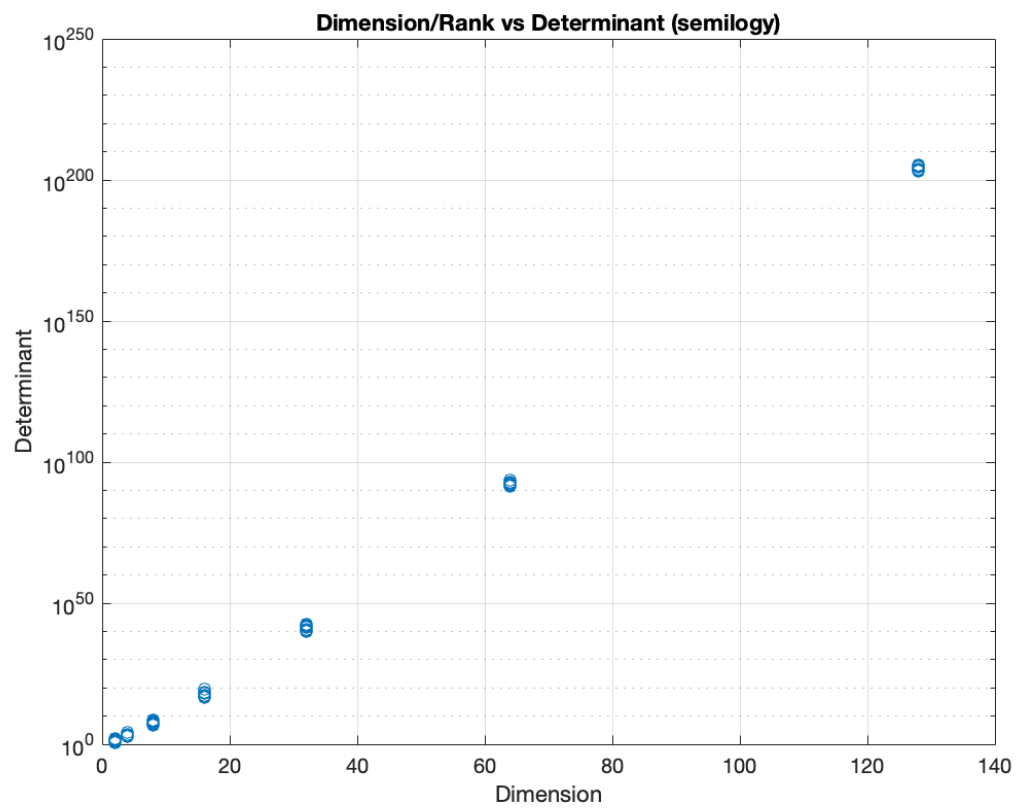
```
%Converting storage cells to plotable vectors
DegreeData = cell2mat([ StoreMatrix{: ,2}]);
NormData = cell2mat([ StoreMatrix{: ,3}]);
DetData = cell2mat([ StoreMatrix{: ,4}]);

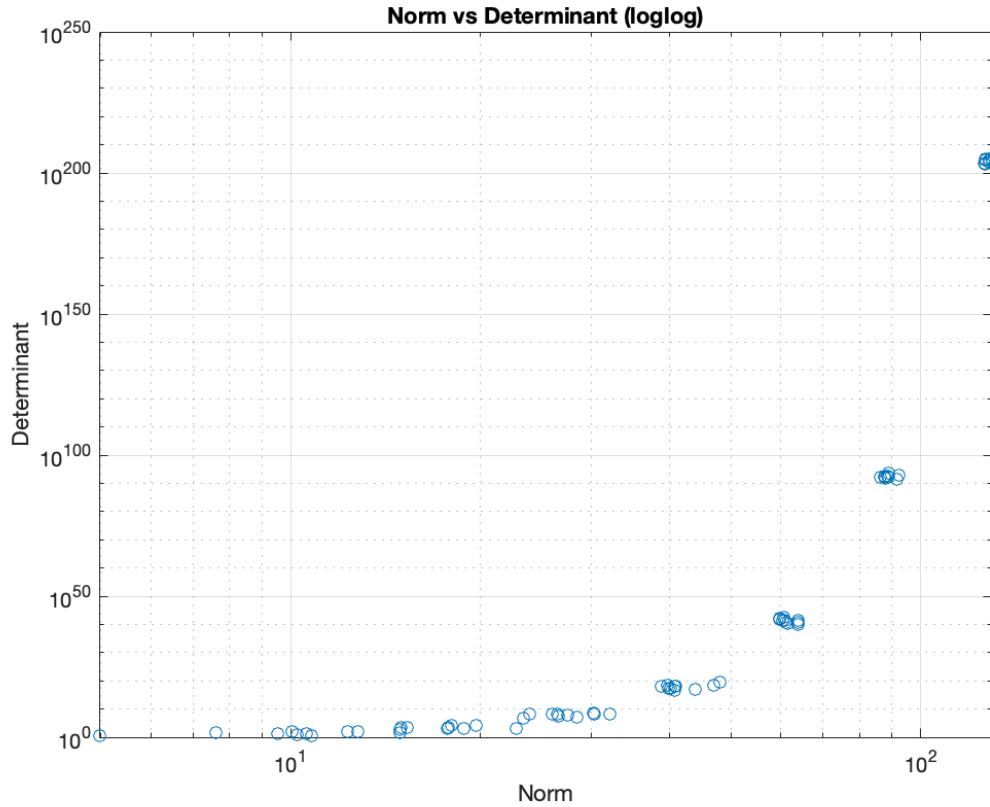
tiledlayout(1,3)
ax1 = nexttile;
semilogy(ax1,DegreeData, NormData, 'o')
title(ax1,'Dimension/Rank vs Norm (plot)')
xlabel(ax1,'Dimension')
ylabel(ax1,'Norm')
grid on

ax2 = nexttile;
semilogy(ax2,DegreeData, DetData, 'o')
title(ax2,'Dimension/Rank vs Determinant (semilogy)')
xlabel(ax2,'Dimension')
ylabel(ax2,'Determinant')
grid on

ax3 = nexttile;
loglog(ax3,NormData, DetData, 'o')
title(ax3,'Norm vs Determinant (loglog)')
xlabel(ax3,'Norm')
ylabel(ax3,'Determinant')
grid on
```

**Plots:**





**Exercise 3:** 1. Consider the algorithm which computes the product of a rectangular matrix  $A \in \mathbb{C}^{m \times n}$  and a column vector  $v \in \mathbb{C}^{n+1}$ . Count the number of floating point operations exactly, i.e as an expression in terms of  $m$  and  $n$ .

**Solution:**

Recall the definition of matrix vector multiplication, denoted as 1.2 in the reading

$$b = Av = \sum_{i=1}^n v_i a_i$$

Note that  $A$  is an  $m \times n$  matrix and  $a_i$  denotes the  $i^{\text{th}}$  column of the matrix  $A$ . Since each  $a_i$  has  $m$  terms, each  $v_i a_i$  term contains  $m$  multiplications. Over all  $n$  columns there are  $mn$  multiplications. Similarly, since there are only  $n - 1$  additions described in the sum, and each  $a_i$  column has  $m$  terms there are a total of  $m(n - 1)$  additions. In summation with  $mn$  multiplications and  $m(n - 1)$  additions we get the following,

$$Total_{FLOPS} = mn + m(n - 1) = m(2n - 1)$$



2. Implement the algorithm in a program *matvec.m* to multiply a matrix  $A$  and vector  $v$ , include error checking. Test your implementation against Matlab.

**Code:**

```
function [Count,x] = MatVec(A,v)
% This function takes a matrix A and a vector v
% and returns the product Av and the # of FLOPs.

if size(A,2) ~= size(v)
    error('Dimension Mismatch')
end

Count = 0; %Init Count
x = zeros(size(A,1),1);

    for m = 1:size(A,1) %Traverses Rows of A
        sum = 0;
        Count = Count-1; % Adjustment for initial addition

            for n = 1:size(v,1)% Traverses vector v
                sum = sum + A(m,n)*v(n);
                Count = Count + 2; % 1 multiplication and 1 addition
            end

        x(m) = sum;
    end
end
```

**Terminal:**

```
>> A = rand(4,3)
A =
    0.3967    0.6648    0.3827
    0.9691    0.9111    0.2267
    0.5269    0.2030    0.1816
    0.0176    0.5845    0.3452

>> v = rand(3,1)
v =
    0.4955
    0.9074
    0.6154
```

```
>> [FLOPs, x] = MatVec(A, v)
FLOPs =
    20
x =
    1.0353
    1.4464
    0.5570
    0.7515
```

```
>> x - (A*v)
ans =
     0
     0
     0
     0
```

3. Write a function *matmat.m* for the product  $C = AB$  of matrices  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times k}$ . Count the number of operations. Check your code against the Matlab results for some example  $m = 3$ ,  $n = 4$ , and  $k = 3$ .

**Solution:**

If we consider the columns of  $B$  as individual vectors we get the following idea,

$$AB = A[b_1|b_2|\dots|b_k] = [Ab_1|Ab_2|\dots|Ab_k].$$

Note that  $A$  is an  $m \times n$  matrix and each column vector  $b_i$  is  $1 \times n$ . Recall from the previous problem that the total number of flops for a matrix-vector product is,

$$m(2n - 1).$$

Since it takes  $k$  matrix-vector products to produce  $AB$  we know that,

$$Total_{FLOPS} = km(2n - 1).$$

**Code:**

```
function [Count, x] = MatMat(A, v)
% This function takes a matrix A and another matrix v
% and returns the product Av and the # of FLOPs.

if size(A,2) ~= size(v,1)
    error('Dimension Mismatch')
end
```

```

Count = 0; %Init Count
x = zeros(size(A,1),size(v,2));

for k = 1:size(v,2) %Traverses Columns of V

    for m = 1:size(A,1) %Traverses Rows of A
        sum = 0;
        Count = Count-1; % Adjustment for initial addition

        for n = 1:size(v,1)% Traverses vector v
            sum = sum + A(m,n)*v(n,k);
            Count = Count + 2; % 1 multiplication and 1 addition
        end
        x(m,k) = sum;
    end
end
end
end

```

**Terminal:**

```

>> A = rand(4,3)
A =
    0.9412    0.5140    0.1830
    0.1128    0.6439    0.9694
    0.3806    0.8492    0.3358
    0.4997    0.9873    0.9615

>> v = rand(3,5)
v =
    0.8081    0.4580    0.8211    0.4860    0.6447
    0.9542    0.1672    0.9689    0.3047    0.8514
    0.7766    0.4877    0.4285    0.3143    0.5161

>> [FLOPs, x] = MatMat(A,v)
FLOPs =
    100
x =
    1.3931    0.6062    1.3492    0.6715    1.1388
    1.4583    0.6321    1.1318    0.5557    1.1211
    1.3787    0.4801    1.2792    0.5493    1.1417
    2.0925    0.8629    1.7788    0.8459    1.6589

>> x - (A*v)
ans =

```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Exercise 4:** Let  $B$  be any  $4 \times 4$  matrix to which we apply the following operations in turn:

1. Interchange rows 1 and 3;

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B$$

2. Interchange Columns 2 and 4;

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

3. Double column 3;

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Add row 3 to row 1;

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. Subtract row 2 from each of the other rows

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. Replace column 3 with column 4;

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

7. Delete row 1;

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

8. Simplify the product so it becomes a product of 3 matrices  $ABC$  where  $B$  is the same.

**Solution:**

Using the commutativity of matrix multiplication we can work from the inside out to simplify the product.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Similarly we can compute  $C$ ,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Finally we get that,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

**Exercise 1.3:** Generalizing Example 1.3, we say that a square or rectangular matrix  $R$  with entries  $r_{i,j}$  is upper-triangular if  $r_{i,j} = 0$  for  $i > j$ . By considering what space is spanned by the first  $n$  columns of  $R$  and using (1.8), show that if  $R$  is a nonsingular  $m \times m$  upper-triangular matrix, then  $R^{-1}$  is also upper triangular.

**Solution:**

Suppose that if  $R$  is a nonsingular  $m \times m$  upper-triangular matrix. Note that by definition  $R$  is a square matrix in row echelon form with no pivot values and is therefore full-rank and fully invertible. Consider that by (1.8),

$$\begin{aligned} RR^{-1} &= I, \\ R[r_1^{-1} r_2^{-1} \dots r_m^{-1}] &= I. \end{aligned}$$

So for some  $k \in (1, m)$  we know that,

$$Rr_k^{-1} = e_k.$$

Note that since  $e_k$  has all zeros below the  $k^{th}$  row we know that the same must be true for  $r_k^{-1}$ , since  $R$  is lower triangular, a non zero value below the  $k^{th}$  row of  $r_k^{-1}$  would show up in  $e_k$ . This property extends to all columns in  $R^{-1}$  and therefore  $R^{-1}$  must be upper triangular.