Exercise P12: a. Give an example of a projector which is not an orthogonal projector.

Solution:
Consider the outer-product of any two, non orthogonal vectors. This will create a projector whose complimentary subspace is not orthogonal to it's own subspace. Consider the following outer-product,

$$P = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Clearly this is not an orthogonal projector since,

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = P^*$$

We could also show that they are not orthogonal by computing the complimentary projector, $I - P$ passing in some vector v to both P and $I - P$ and show that the resulting vectors are not orthogonal via inner product.

b. Show that if P is a projector and λ is an eigenvalue of P then $\lambda = 0$ or $\lambda = 1$.

Solution:
Suppose P is a projector and consider the eigenvector equation for some λ and x,

$$Px = \lambda x$$

Recall that by definition a projector has the following property, $P = P^2$ by substitution we get,

$$\begin{aligned} P^2 x &= \lambda x, \\ PPx &= \lambda x, \\ P(Px) &= \lambda x. \end{aligned}$$

Substituting the eigenvector identity we get,

$$\begin{aligned} P(\lambda x) &= \lambda x, \\ \lambda P x &= \lambda x, \\ \lambda \lambda x &= \lambda x, \\ \lambda^2 x - \lambda x &= 0, \\ (\lambda^2 - \lambda)x &= 0, \\ ((\lambda)(\lambda - 1))x &= 0. \end{aligned}$$

Solving the equation, it must be the case that $\lambda = 0$ or $\lambda = 1$.

c. Show that if a projector is invertible then it is the identity.

Solution:
Suppose a projector P has an inverse such that $PP^{-1} = I$. By definition since P is a projector the following is true,

$$P^2x = Px.$$

Applying P^{-1} to both sides we get,

$$\begin{aligned} P^{-1}PPx &= P^{-1}Px, \\ Px &= Ix, \\ Px &= x. \end{aligned}$$

Thus P must be the matrix multiplicative identity, so $P = I$.

Exercise P13: Just for fun write a Matlab/Octave function
function P = randproj(m,k) which generates a random orthogonal projection $P \in \mathbb{R}^{mxm}$ with $rank(P) = k$. Note that k is an integer in the rank $0 \leq k \leq n$. Verify the function by checking the output is an orthogonal projector and that it has the desired rank.

Solution:
Recall from chapter six, that for an orthogonal projector P there exists some orthonormal basis Q on $\mathbb{R}^{mxm}$, such that,

$$P = \hat{Q}\hat{Q}^*$$

Code:

```
using LinearAlgebra
using Random

function randproj(m,k)
#This function takes a dimension m, and a rank k
#and produces an orthogonal projector P, on a k
#dimension subspace of m
    A = randn(m,m);
    (Q, R)= qr(A)
    P = Q[:,1:k]*Q[:,1:k]'
    return P
end
```

Console:

```
P = randproj(4,3)
ProjectorCheck = norm(P^2 - P)
(U,S,V) = svd(P);
println(ProjectorCheck)
print(S)

>    7.506413274784675e-16
>    [1.0000000000000004, 1.0, 0.9999999999999998, 9.843583432324734e-18]
```

As expected we have a projector since when we computed the norm of $P^2 - P$ we essential got zero minding rounding error. Similarly when we check the SVD of our projector we get 3 non-zero singular values therefore we have a rank-3 projector as expected.

Exercise 5.4: Suppose $A \in \mathbb{C}^{mxm}$ has an SVD $A = U\Sigma V^*$. Find the eigenvalue decomposition of the $2mx2m$ hermitian matrix,

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

Solution:
Since $A = U\Sigma V^*$ we know that,

$$AV = U\Sigma.$$

Concequently we know that $A^* = V\Sigma U^*$, and therefore,

$$A^*U = V\Sigma.$$

Written as a system of block matrices with respect to B we get,

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix}\begin{bmatrix} U\Sigma \\ V\Sigma \end{bmatrix}$$

Similarly the following is true,

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\begin{bmatrix} -U \\ V \end{bmatrix} = \begin{bmatrix} U\Sigma \\ -V\Sigma \end{bmatrix}.$$

Through block matrix multiplication and factoring it follows that,

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\begin{bmatrix} -U & U \\ V & V \end{bmatrix} = \begin{bmatrix} U\Sigma & U\Sigma \\ -V\Sigma & V\Sigma \end{bmatrix} = \begin{bmatrix} U & U \\ -V & V \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}$$

We want an expression of the form, $AX = X\Lambda$ so we can solve for A by right multiplying the expression by X^{-1} and attain the eigenvalue decomposition. Note that by block matrix multiplication the following is also equivalent,

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\begin{bmatrix} -U & U \\ V & V \end{bmatrix} = \begin{bmatrix} -U & U \\ V & V \end{bmatrix}\begin{bmatrix} -\Sigma & 0 \\ 0 & \Sigma \end{bmatrix}$$

Therefore we can left multiply by X^{-1} to get our eigenvalue decomposition,

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = \begin{bmatrix} -U & U \\ V & V \end{bmatrix}\begin{bmatrix} -\Sigma & 0 \\ 0 & \Sigma \end{bmatrix}\begin{bmatrix} -U & U \\ V & V \end{bmatrix}^{-1}.$$

Exercise 6.4: Consider the matrices,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Answer the following questions by hand calculation.

a. What is the orthogonal projector P onto *range*(A), and what is the image under P of the vector $[123]'$?

Solution:
Given the simplicity of A we can quickly compute an orthonormal basis on the range of A. Note that the columns are already orthogonal, all we have to do is normalize the first column. Doing so we get,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Applying $P = QQ^*$ we get,

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Computing $P[123]'$ to find the image of $[123]'$ under P,

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

b. To get an an orthogonal basis on the range of B by hand, we can use Gram-Schmidt, or alternatively use the formula,

$$P = B(B^*B)^{-1}B^*$$

Since there are only two vectors, I will proceed with Gram-Schmidt. Let b_i be the columns if B, and q_i form the orthonormal basis on the range of B,

$$q_1 = \frac{b_1}{||b_1||} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

For the second vector we do the following,

$$\begin{aligned} v_2 &= b_2 - q_1^* b_2 q_1, \\ &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \\ &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \\ &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

$$q_2 = \frac{v_2}{||v_2||} = \frac{v_2}{\sqrt{3}} \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix}.$$

Putting our column vectors together we get our orthonormal basis Q,

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

Computing P with $P = QQ^*$ we get,

$$P = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}.$$

Computing $P[123]'$ to find the image of $[123]'$ under P,

$$\begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Exercise 7.1: Consider again the matrixes A and B of exercise 6.4 (the previous exercise)

a. Using any method you like, determine a reduced and full QR factorization for the matrix A.

Solution:
Recall from the previous problem our orthonormal basis on the range of A,

$$\hat{Q} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Note that this matrix was produced by normalizing the first column of A, undoing that operation we get the following R,

$$\hat{R} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

In order to get a full QR factorization we need an orthonormal basis for the full $\mathbb{C}^{3x3}$ space. Geometrically we can see that the following is a suitable basis,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 * \frac{1}{\sqrt{2}} & \end{bmatrix}.$$

The full R matrix simply adds another row of zeroes, so we get

$$\hat{R} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

b. Recall that in the previous problem we used Gram-Schmidt to find the following, orthonormal basis on the range of B,

$$\hat{Q} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \end{bmatrix}.$$

Recalling the Gram-Schmidt process from the previous problem we can also form our matrix $\hat{[}R]$ by solving for the b_i in each equation,

$$\begin{aligned} q_1 &= \frac{b_1}{||b_1||}, \\ b_1 &= ||b_1||q_1, \\ b_1 &= \sqrt{2}q_1. \end{aligned}$$

$$q_2 = \frac{b_2 - q_1^* b_2 q_1}{||v_2||},$$
$$b_2 = (q_1^* b_2)q_1 + ||v_2|| q_2,$$
$$b_2 = (\sqrt{2})q_1 + (\sqrt{3})q_2.$$

Forming the matrix $\hat{R}$ wih our coefficients, we get

$$\hat{R} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

In order to get a full QR factorization we need an orthonormal basis for the full $\mathbb{C}^{3x3}$ space. To do so lets consider a third step in the Gram-Schmidt process on the vector $b_3 = [100]'$,

$$v_3 = b_3 - (q_1^* b_3)q_1 - (q_2^* b_3)q_2,$$
$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\sqrt{2}}{2} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} - \frac{\sqrt{3}}{3} \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix},$$
$$= \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}.$$

$$q_3 = \frac{v_3}{||v_3||} = \frac{v_3}{1/\sqrt{6}} = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \end{bmatrix}.$$

Therefore our full QR factorization comes out to,

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \end{bmatrix}.$$
$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}.$$

Exercise 7.3: Let A be an mxm matrix, and let a_j be its jth column. Give an algebraic proof of Hadamard's inequality:

$$|det(A)| \leq \prod_{j=1}^{m} ||a_j||_2.$$

Also give a geometric interpretation of this result, making use of the fact tha t the determinant equals the volume of a parallelepiped.

Solution:
Suppose A is an mxm invertible matrix (otherwise the result is trivial). Consider the $A = QR$ factorization, column-wise we know that,

$$a_j = \sum_{k=1}^{m} r_{kj} q_k,$$

where r_{kj} are the entries of R and q_k are the columns of Q. Taking the norm of both sides,

$$||a_j||_2 = ||\sum_{k=1}^{m} r_{kj} q_k||_2 = \sum_{k=1}^{m} |r_{kj}| ||q_k||_2.$$

Note that the $||q_k||_2 = 1$, therefore we get the following,

$$||a_j||_2 = \sum_{k=1}^{m} |r_{kj}|.$$

Considering only the diagonal entries(subtracting non-diagonal terms) in the sum it must follow that,

$$||a_j||_2 \geq |r_{jj}|.$$

Now consider the following,

$$det(A) = det(QR) = |det(Q)||det(R)| = 1|det(R)| = \prod_{j=1}^{m} |r_{jj}| \leq \prod_{j=1}^{m} ||a_j||_2.$$

The fact that $det(Q) = 1$ follows from,

$$\begin{aligned} QQ^* &= I, \\ det(QQ^*) &= 1, \\ det(Q)det(Q^*) &= 1, \\ det(Q)det(Q)^* &= 1, \\ |det(Q)| &= 1. \end{aligned}$$

The substitution of $|det(R)| = \prod_{j=1}^{m} |r_{jj}|$ follows from performing co-factor expansion on a triangular matrix. The final inequality comes from substitution.
Geometrically, the absolute value of the determinant of A is the volume of the parallelepiped where each length is a column vector of A. The product in the Hadamard inequality is the volume of the box where each length is a column vector of A. Consider the equation for the volume of the A is $3x3$ parallelepiped(scalar triple product),

$$Volume = |det(A)| = |a_1||a_2||a_3|cos(\phi)$$

where ϕ is the angle between a_3 and vector perpendicular both a_1 and a_2. Under the constraints of having the lengths be column vectors of A, a box will have the maximum volume since it minimizes ϕ.

Exercise 8.1: Let A be an mxn matrix. Determine the exact numbers of floating point additions, subtractions, multiplcations, and divisions involved in computing the factorization of $A = \hat{Q}\hat{R}$ in MGS.

Solution:
Let's start by first considering the MGS algorithm,
MGS Algorithm:

```
for i = 1:n
    v_i = a_i

for i = 1:n
    r_ii = ||v_1||
    q_i = v_i/r_ii

    for j = i+1:n
        r_ij = q_i^*v_i
        v_j = v_j - r_ijq_i
```

Let's start by counting the total number of additions in the algorithm. Consider the line $r_{ii} = \|v_i\|$ computing the norm of a length m vector requires $m-1$ additions. In the second loop consider the line $r_{ij} = q_i^* v_j$, computing an inner-product on length m vectors requires $m-1$ additions. Taking into account the nesting of the loops we get the following

$$Additions = \sum_{i=1}^{n} m - 1 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} m - 1$$

To count the total number of multiplications, let's start by consider the line $r_{ii} = \|v_i\|$, computing a norm of a length m vector requires m multiplications. In the second loop consider the line, $r_{ij} = q_i^* v_j$, computing an inner-product on length m vectors requires m multiplications. Th second loop also has the line $v_j = v_j - r_{ij} q_i$, applying a scalar to a length m vector also requires m multiplications. Taking into account the nesting of the loops we get the following

$$Multiplications = \sum_{i=1}^{n} m + \sum_{i=1}^{n} \sum_{j=i+1}^{n} 2m$$

Counting the number of subtractions, we can look at line $v_j = v_j - r_{ij} q_i$. Subtracting two length m vectors, requires m subtractions. Given that this line is in a nested for loop we get the following,

$$Subtractions = \sum_{i=1}^{n} \sum_{j=i+1}^{n} m$$

Counting divisions, there is only one line $q_i = v_i / r_i i$. Normalizing a length m vector, requires m divisions. Since the line is in a for loop we get the following,

$$Divisions = \sum_{i=1}^{n} m$$

Exercise 8.2: Write a MATALB funciton $[Q, R] = MGS(A)$ that computes a reduced QR factorizaiton $A = \hat{Q}\hat{R}$ of an mxn matrix A with $m \geq n$ using modified Gram-Schmidt orthogonalization.

Solution:

Code:

```
function mgs(A)
    (m,n) = size(A);
    Q = zeros(m,n);
    R = zeros(n,n);
    V = zeros(m,n);
    for i = 1:n
        V[:,i] = A[:,i];
    end
    for i = 1:n
        R[i,i] = norm(V[:,i]);
        Q[:,i] = V[:,i]/R[i,i];
        for j = i+1:n
            R[i,j] = Q[:,i]'V[:,j];
            V[:,j] = V[:,j] - R[i,j]*Q[:,i];
        end
    end
    return (Q, R)
end
```