Exercise P10: Use by-hand calculations to determine SVDs of the following matrices. Note that in the decomposition $A = U\Sigma V^*$, the factor Σ is unique but the factors U, V are not, and thus there will be more than one correct answer.

a.

$$\begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$$

Solution:
Since the given matrix is diagonal, and square we can simply factor into $U\Sigma V^*$, like the following

$$\begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

We could throw the $[0 \; -1]'$ column into either unitary matrix, since Σ must contain all non-negative values and be in decreasing diagonal order.

b.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \end{bmatrix}$$

Solution:
Note that the given matrix would be in the form of Σ simply by swapping the first and third rows. Doing so we get the following decomposition,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c.

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Solution:
Recall from Theorem 5.4 that we can compute the singular values of a matrix by solving for the square root of the eigenvalues of A^*A and AA^*. In our case, let's consider AA^*,

$$AA^* = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that since AA^* is diagonal matrix the eigenvalues are the values on the diagonal. Thus our eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. Solving for the corresponding eigenvectors, we get the following,

$$I\lambda_1 - \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $v_1 = [1; 0]$ and solving for the second eigenvector,

$$I\lambda_2 - \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we get that $v_2 = [0; 1]$. By Theorem 5.4 we know that the singular values of A are $\sigma_1 = \sqrt{2}$ and $\sigma_2 = 0$. Furthermore the eigenvalues to our corresponding sigma form the columns of the matrix U. Putting everything together we get that,

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix},$$

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

To solve for V^* lets first recall the column/row interpretation of the SVD,

$$A = u_i \sigma_i v_i^*.$$

Solving for the row v_i^*,

$$\frac{u_i^* A}{\sigma_i} = v_i^*.$$

Solving for the first row v_1^* we get the following,

$$v_1^* = \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}' \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}'$$

Since we only have one non-trivial singular value, to get the second row v_2^* all we need is an orthonormal vector to v_1^*, so we can consider the following,

$$v_2^* = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}'$$

Thus our V^* matrix is the following,

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

The full SVD for A comes out to,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

d.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Solution:
Again recalling Theorem 5.4 and considering the eigenvalue decomposition of A^*A we get the following,

$$A^*A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

Solving for the eigenvalues,

$$\begin{aligned} det\left(\begin{bmatrix} \lambda - 5 & 4 \\ 4 & \lambda - 5 \end{bmatrix}\right) &= 0 \\ (\lambda - 5)^2 - 4^2 &= 0 \\ \lambda^2 - 10\lambda + 9 &= 0 \\ (\lambda - 9)(\lambda - 1) &= 0. \end{aligned}$$

Therefore we get that the eigenvalues $\lambda_1 = 9$ and $\lambda_2 = 1$. Solving for the eigenvectors we get,

$$\begin{aligned} I\lambda_1 - \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} v_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} v_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Which gives a $v_1 = [1; 1]$, normalizing we get $v_1 = [\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}]$. Solving for the second eigenvector,

$$\begin{aligned} I\lambda_2 - \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} v_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix} v_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Which gives us $v_2 = [-1; 1]$, and normalizing we get $v_2 = [-\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}]$. By Theorem 5.4 we know have values for,

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

$$V^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Considering the column/row interpretation of the SVD we can solve for the columns of U,

$$A = u_i \sigma_i v_i^*.$$

Solving for the column u_i by left multiplying both sides by v_i and dividing by σ_i,

$$\frac{Av_i}{\sigma_i} = u_i.$$

Substituting to get u_1,

$$\frac{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}{3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Substituting to solve for u_2,

$$\frac{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}{1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore the full SVD of A comes out to,

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Exercise P11: Theorem 5.7 interprets the SVD as saying that any matrix A is the sum of rank-one matrices, a sum of outer products from columns of U and V, and weightes by the singular values σ_j. Furthermore Theorem 5.8 shows that the partial sums oare the nearest lower-rank approximations to A in the induced 2-norm. The goal of this exercise is to see these idea.

a. Consider the 4x3 matrix,

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & 6 & 5 \\ 3 & 5 & 8 \end{bmatrix}$$

Using Matlab/Octave to compute the full SVD of A. What is the rank of A?

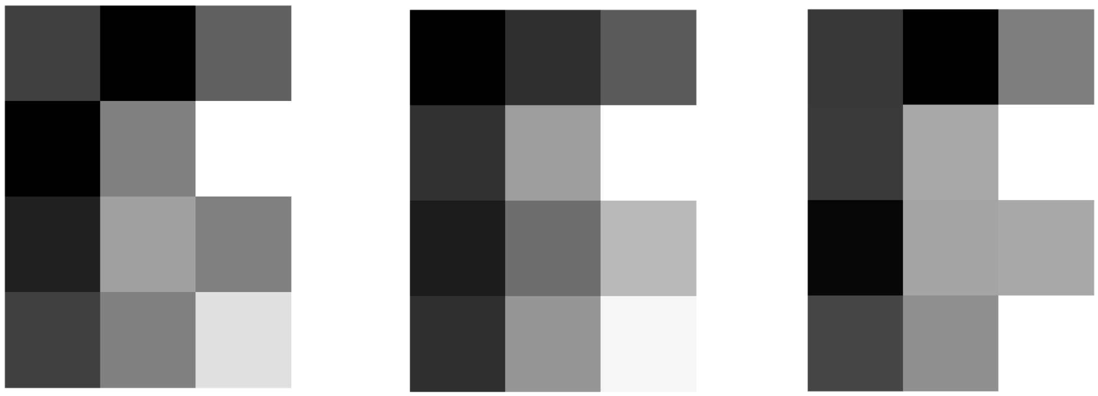
Solution:
Code:

```
>> A = [3 1 4; 1 5 9;2 6 5 ;3 5 8]

>> [U, S, V] = svd(A)

U =

   -0.268    0.770    0.348   -0.463
   -0.607   -0.068   -0.729   -0.309
   -0.462   -0.605    0.571   -0.309
   -0.588    0.193    0.146    0.772


S =

   16.787        0        0
        0    2.925        0
        0        0    2.376
        0        0        0


V =

   -0.244    0.551    0.798
   -0.537   -0.762    0.362
   -0.807    0.340   -0.482
```

b. Part (a) suggests that printing the entries of generic unitary matrixes is not very informative. On the other hand we can view matrixes as images. Generate side-by-side block images of the matrix A, and its rank-two and rank-one best approximations.

Solution:
Using the following function, along with the given blockimage.m script were used to generate the blockimages of the rank one and rank two approximations of A,
Code:

```
function [Ahat] = RankApprox(A, n)
%The following code takes a matrix A and a
%rank n and returns A's best rank n
%approximation.

[U, S, V] = svd(A);
Ahat = U(:,1:n)*S(1:n,1:n)*V(:,1:n)';
```



(a) BlockImage of A (b) Rank 1 Approx. (c) Rank 2 Approx.

Figure 1: Matrix A Approximations

c. Describe in words what the image of any rank-one matrix looks like.

Solution:
A rank-one matrix has the property that all column vectors are linearly dependent, and all row vectors are linearly dependent. Therefore when we look at the block image of a rank one matrix each row will be some multiple of every other row and each column will be some multiple of every other column. More broadly for larger rank-one matrices we might begin to notice a faint plaided pattern, where white lines begin to form in the rows and columns that are generated by a multiple close to zero.

Exercise 3.4: Example 3.6 shows that if E is an outer product $E = uv^*$, then $||E||_2 = ||u||_2||v||_2$. Is the same true for the Frobenius norm, i.e., $||E||_F = ||u||_F||v||_F$? Prove it or give a counter example.

Solution:
Suppose that E is an outer product $E = uv^*$. Now consider the Forbenius Norm of the matrix E and by substitution we get,

$$||E||_F = ||uv^*||_F.$$

Since each term of the outer product can be written in the form $u_i v_j$ we can substitute into the definition of the Forbenius norm to get,

$$\|E\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |u_i v_j|^2}.$$

Through some algebra we get,

$$\begin{aligned}
\|E\|_F &= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |u_i v_j|^2}, \\
&= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |u_i|^2 |v_j|^2}, \\
&= \sqrt{\sum_{i=1}^{m} |u_i|^2 \sum_{j=1}^{n} |v_j|^2}, \\
&= \sqrt{\sum_{i=1}^{m} |u_i|^2}\sqrt{\sum_{j=1}^{n} |v_j|^2}.
\end{aligned}$$

Invoking the definition of the Forbenius norm we finally get,

$$\|E\|_F = \sqrt{\sum_{i=1}^{m} |u_i|^2}\sqrt{\sum_{j=1}^{n} |v_j|^2} = \|u\|_F \|v\|_F.$$

Exercise 4.4: Two matrices $A, B \in \mathbb{C}^{mxm}$ are unitarily equivalent if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{mxm}$. Is it true of false that A and B are unitarily equivalent if and only if they have the same singular values?

Proof. Contradiction ($\rightarrow$): Suppose that $A, B \in \mathbb{C}^{mxm}$ are unitarily equivalent with SVDs $A = U_1\Sigma_1 V_1^*$ and $B = U_2\Sigma_2 V_2^*$ such that, $\Sigma_1 \neq \Sigma_2$. By the definition of unitarily equivalent we know that for some unitary $Q \in \mathbb{C}^{mxm}$,

$$A = QBQ^*.$$

By substitution we get that,

$$A = U_1\Sigma_1 V_1^* = QU_2\Sigma_2 V_2^* Q^*.$$

Note that since Q, U, and V^* are unitary QU_2 and $V_2^*Q^*$ are also unitary. Thus A has two distinct SVDs contradicting Theorem 4.1 (Uniqueness of the SVD). $\square$

Proof. Counter Example ($\leftarrow$): Consider the following matrices,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

One can quickly see that both matrices have the same singular values. Plugging into the the unitary equivalents formula we get,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = Q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Q^* = QQ^* = I.$$

A clear contradiction. $\square$

c

Exercise 5.1: In Example 3.1 we considered the matrix (3.7) and asserted, among other things, that its 2-norm is approximately 2.9208. Using the SVD, work out the exact values for the minimum and maximum singular values.

Solution:
Recall the matrix from example 3.1,

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}.$$

From Theorem 5.4 we know that to solve for the singular values of A need to solve for the square root of tbe eigenvalues of AA^* (or A^*A), setting up the characteristic equation,

$$det\left(\begin{bmatrix} \lambda - 5 & 4 \\ 4 & \lambda - 4 \end{bmatrix}\right) = 0,$$

$$(\lambda - 5)(\lambda - 4) - 4^2 = 0,$$

$$\lambda^2 - 9\lambda + 4 = 0.$$

Solving with the quadratic formula we get that $\lambda_1 = \frac{9+\sqrt{65}}{2}$ and $\lambda_2 = \frac{9-\sqrt{65}}{2}$. Taking the square roots we get the singular values,

$$\sigma_1 = \sqrt{\frac{9 + \sqrt{65}}{2}},$$

$$\sigma_2 = \sqrt{\frac{9 - \sqrt{65}}{2}}.$$

Exercise 5.2: Using the SVD, prove that any matrix in $\mathbb{C}^{mxn}$ is the limit of a sequence of matrices of full rank. In other words, prove that the set of full-rank matrixes is a dense subset of $\mathbb{C}^{mxn}$. Use the 2-norm for your proof.

Proof. Let $A \in \mathbb{C}^{mxn}$ with an SVD such that $A = U\Sigma V^*$. Now suppose some $A_n \in \mathbb{C}^{mxn}$ with an SVD such that $A_n = U(\Sigma - \frac{1}{n}I)V^*$ (full-rank is implicit in this definition). Let $\epsilon > 0$ and note that by the Archimedean Property there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ we know that $N > \frac{1}{\epsilon}$. Consider the following,

$$\|A_n - A\|_2 = \|U\left(\Sigma + \frac{1}{n}I\right)V^* - U\Sigma V^*\|_2 = \|U\left(\frac{1}{n}I\right)V^*\|_2 = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Thus any matrix in $\mathbb{C}^{mxn}$ is the limit of a sequence of matrices of full rank.

□