Assignment #10

Due Monday 29 November, 2021 at the start of class

Please read Lectures 22, 23, 24, and 25 in the textbook *Numerical Linear Algebra* by Trefethen and Bau. Then do the following exercises.

P23. A *circulant matrix* is one where constant diagonals "wrap around":

$$C = \begin{bmatrix} c_1 & c_m & \dots & c_3 & c_2 \\ c_2 & c_1 & c_m & & c_3 \\ \vdots & c_2 & c_1 & \ddots & \vdots \\ c_{m-1} & & \ddots & \ddots & c_m \\ c_m & c_{m-2} & \dots & c_2 & c_1 \end{bmatrix} \tag{1}$$

Each entry of $C \in \mathbb{C}^{m\times m}$ is thus a function of the row/column index difference:

$$C_{jk} = \begin{cases} c_{j-k+1}, & j \ge k, \\ c_{m+j-k+1}, & j < k. \end{cases}$$

Here $c_1, \dots, c_m$ are the entries of a column vector, namely the first column of C. Specifying the first column of a circulant matrix describes it completely.

Here is an extraordinary fact about circulant matrices: Every circulant matrix has a complete set of eigenvectors *that are known in advance*, without knowing the eigenvalues. Specifically, define $f_k \in \mathbb{C}^m$ by

$$(f_k)_j = \exp\left(-i(j-1)(k-1)\frac{2\pi}{m}\right) = e^{-i2\pi(k-1)(j-1)/m}, \tag{2}$$

where, as usual, $i = \sqrt{-1}$. These vectors are *waves*, i.e. combinations of familiar sines and cosines.

After some warm-up exercises you will show in part **(e)** that $Cf_k = \lambda_k f_k$.

(a) Define the *periodic convolution* $u * w \in \mathbb{C}^m$ of vectors $u, w \in \mathbb{C}^m$ by

$$(u * w)_j = \sum_{k=1}^{m} u_{\mu(j,k)} w_k \qquad \text{where} \qquad \mu(j,k) = \begin{cases} j-k+1, & j \ge k, \\ m+j-k+1, & j < k. \end{cases}$$

Show that $u * w = w * u$.

(b) Show that $Cu = v * u$ if C is a circulant matrix and v is the first column of C.

(c) Show that the vectors $f_1, \dots, f_m$ defined in (2) are orthogonal.

(d) For $m = 20$, use Matlab to plot the real parts of the vectors $f_1, \dots, f_5$, together in a single figure. (*They should look like discretized waves.*)

(e) For the general circulant matrix C in (1) above, give a formula for the eigenvalues λ_k, in terms of the entries $c_1, \dots, c_m$. That is, show via by-hand calculation that

$$Cf_k = \lambda_k f_k.$$

(f) Download this MATLAB function, which builds a circulant matrix with a given first column; notice how it uses the `mod()` function:

`http://bueler.github.io/M614F21/matlab/circu.m`

Generate the circulant matrix C with first column consisting of 20 random numbers of your choice. Use the result of **(d)** to compute the eigenvalues λ_k, and compare these against the result of `eig()`. (*They should be the same to high accuracy!*) Also, generate f_5 from (2) and verify that $Cf_5 = \lambda_5 f_5$ to high accuracy; use a vector norm.

Exercise 22.1.

Exercise 22.2.

Exercise 23.2.

Exercise 23.3. *Hint. Flowchart at* `www.mathworks.com/help/matlab/ref/mldivide.html`. *By the way,* your *timings will both be much faster and much harder to understand, both aspects having to do with complicated memory hierarchies. 1991 computers were simpler.*

Exercise 24.1.