Adaptive Mesh Refinement for Obstacle Problems

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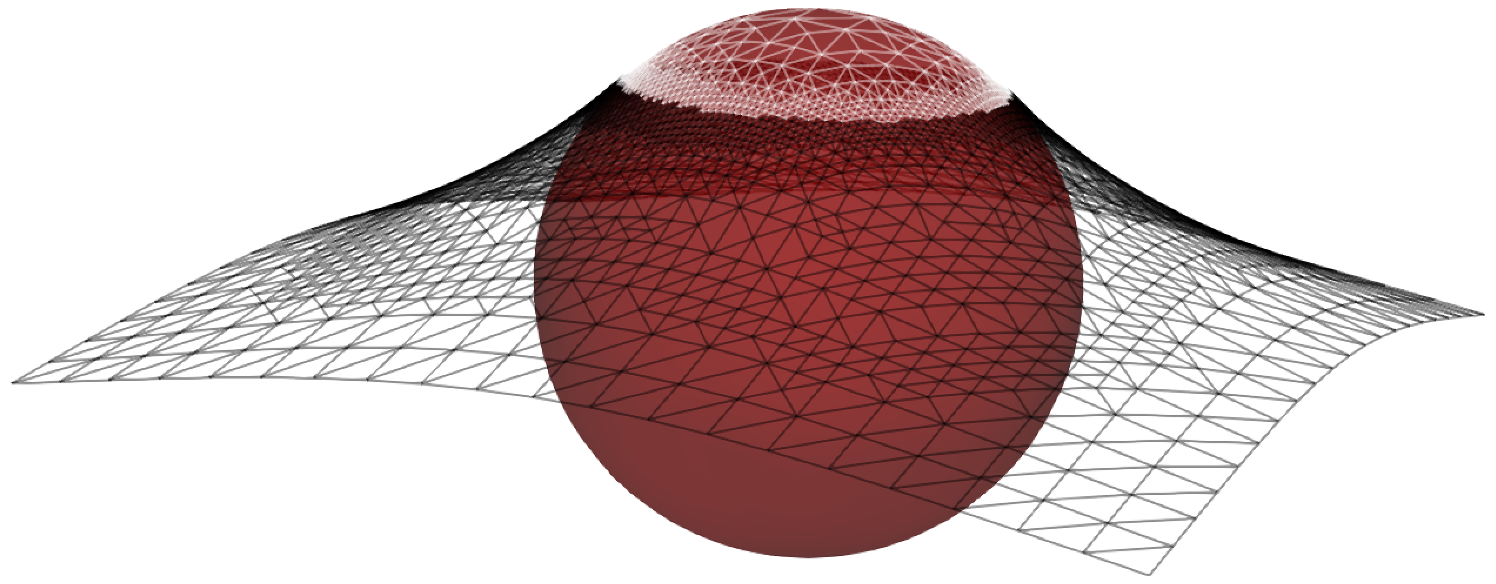
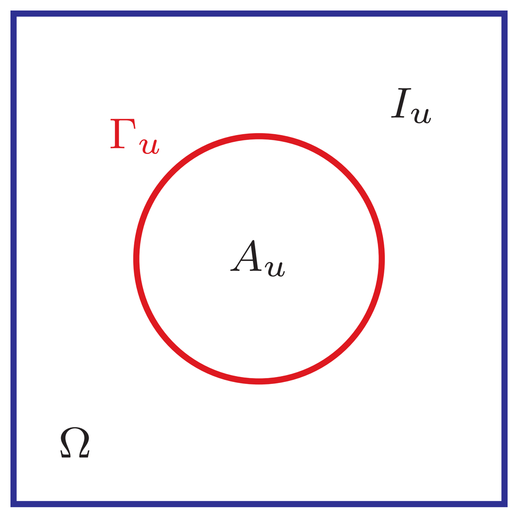
Introduction: Classical Obstacle Problem



Solve for the displacement of an elastic membrane $u(x,y)$ over a region Ω which minimizes elastic potential energy, subject to a distributed load $f(x,y)$, $u|_{\partial\Omega} = g$ and $u \geq \psi$ [4].

- **Energy Minimization:**
  Let $K_\psi = \{v \in H_g^1(\Omega) | v \geq \psi\}$,
  $$\underset{u \in K_\psi}{\text{minimize:}}\ I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - fu$$

- **Complementarity Problem (CP):**
  For $u \in C(\overline{\Omega}) \cap C^2(\Omega)$, the following hold over Ω a.e.:
  $$-\nabla^2 u - f \geq 0$$
  $$u - \psi \geq 0$$
  $$(-\nabla^2 u - f)(u - \psi) = 0$$



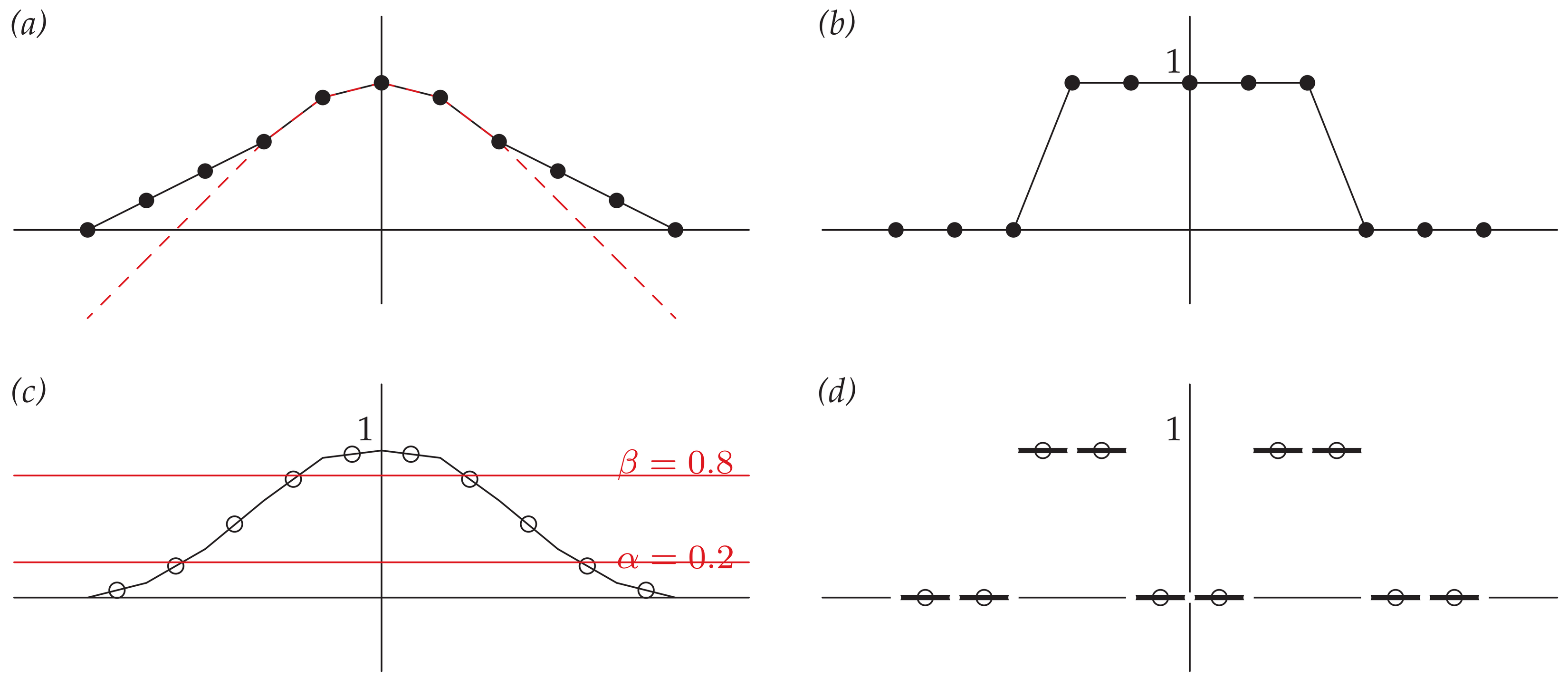
- The solution u defines:
  - Active Set $A_u = \{u = \psi\}$ (Data)
  - Inactive Set $I_u = \{u > \psi\}$ (PDE region)
  - Free Boundary $\Gamma_u = \partial I_u \cap \Omega$
- On the free boundary Γ_u:
  - $u = \psi$
  - $u' = \psi'$

Motivations and Approach: Free Boundary Adaptive Mesh Refinement (AMR)

- AMR for PDEs
  - Refine mesh in regions of high solution error, coarsen in regions of low error, to efficiently capture solution features.
- AMR for Variational Inequalities (VIs)
  - Often the simulation goal is localization of the free boundary Γ_u.
  - Resolution in a stabilized A_u is unnecessary since $u = \psi$.
  - Standard PDE error estimators cannot be applied to all of Ω in a VI problem.
  - Poorly localized Γ_u can produce dominating error in I_{u_h}.
- Our Approach: Free Boundary Aware AMR
  - Use Grid Sequencing and refine near Γ_{u_h}, the discrete free boundary.
  - Use PDE error indicators to refine in I_{u_h}, the discrete inactive set.

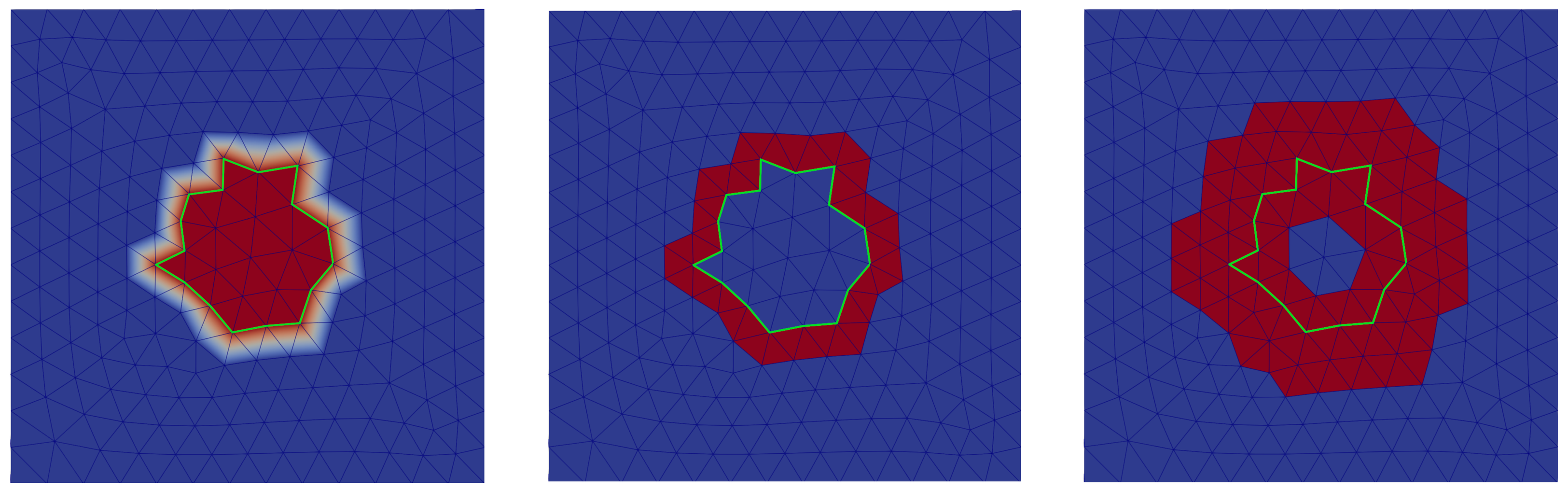
Methods: VCD and UDO Strategies

- **Variable Coefficient Diffusion (VCD)**



(a) ψ in red, u_h in black. (b) Nodal A_{u_h} indicator. (c) Smoothed A_{u_h} indicator with thresholds α and β. (d) Final element indicator for refinement.

- **Unstructured Dilation Operator (UDO)**



(a) (b) (c)

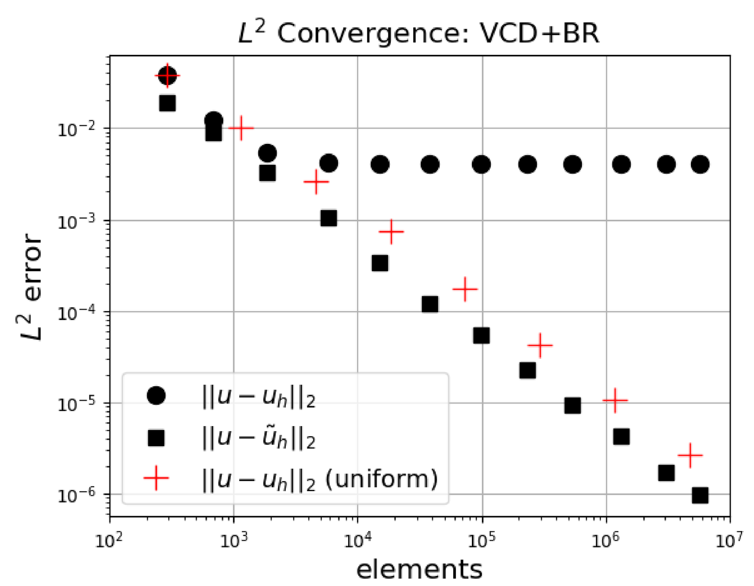
(a) u_h with Γ_{u_h} in green. (b) Computed border elements indicator. (c) Final one-neighbor refinement indicator.

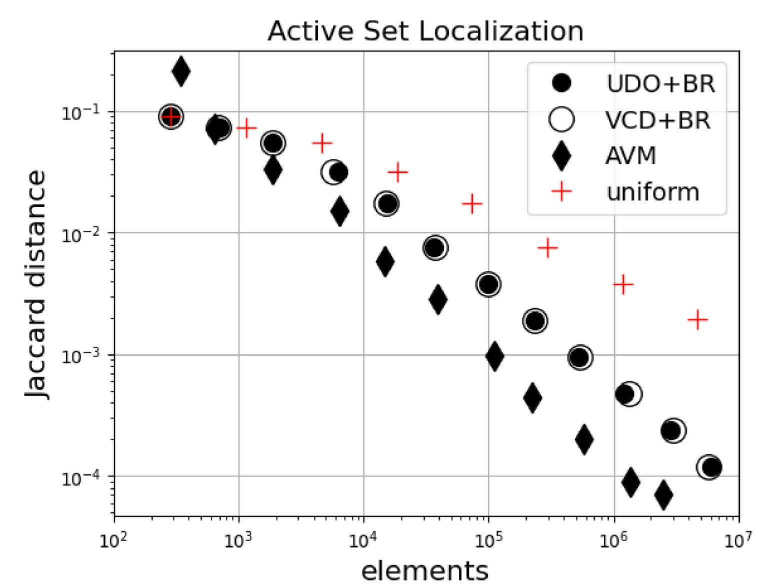
Results: Metric for Γ_{u_h} Convergence

- **Preferred Approximation:**
  $$\tilde{u}_h(x) = \begin{cases} \psi(x) & x \in A_u^h \\ u_h(x) & \text{otherwise} \end{cases}$$

- **Jaccard Distance [5]:**
  Let $S, T \subset \Omega$ be measurable sets, then
  $$d(S,T) = 1 - \frac{|S \cap T|}{|S \cup T|}$$
  where $|\cdot|$ is Lebesgue measure.

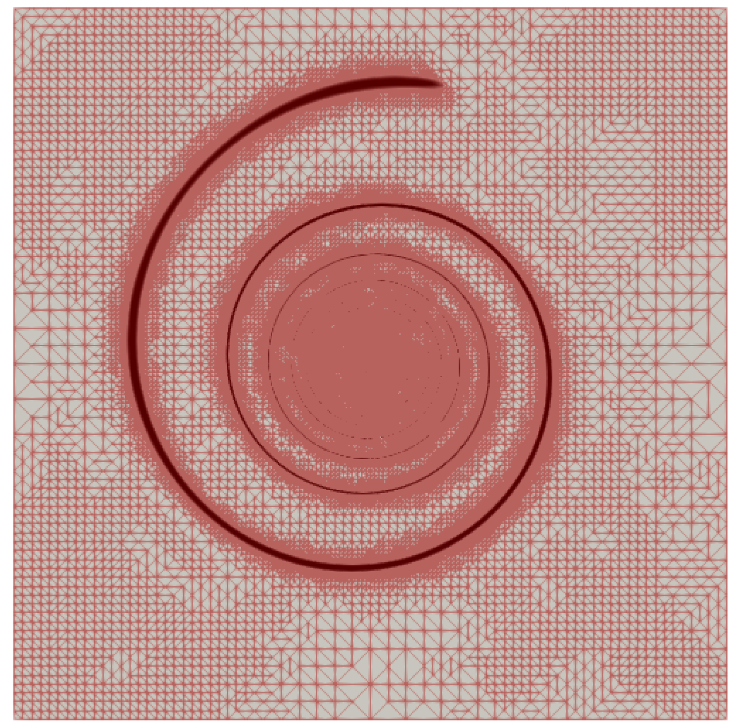
- **Reference Sphere Obstacle Problem (Chapter 13, [2]).**



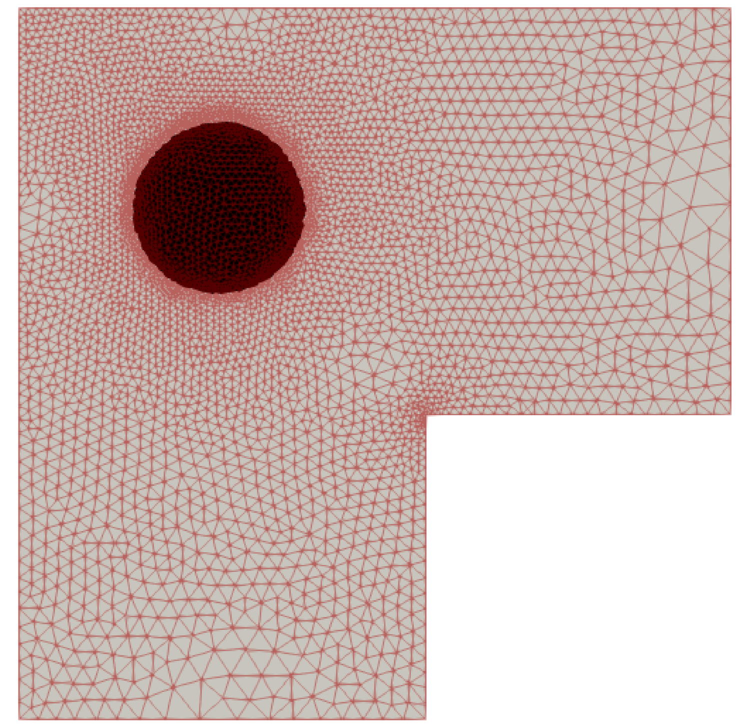


Gallery: Benchmark Problems

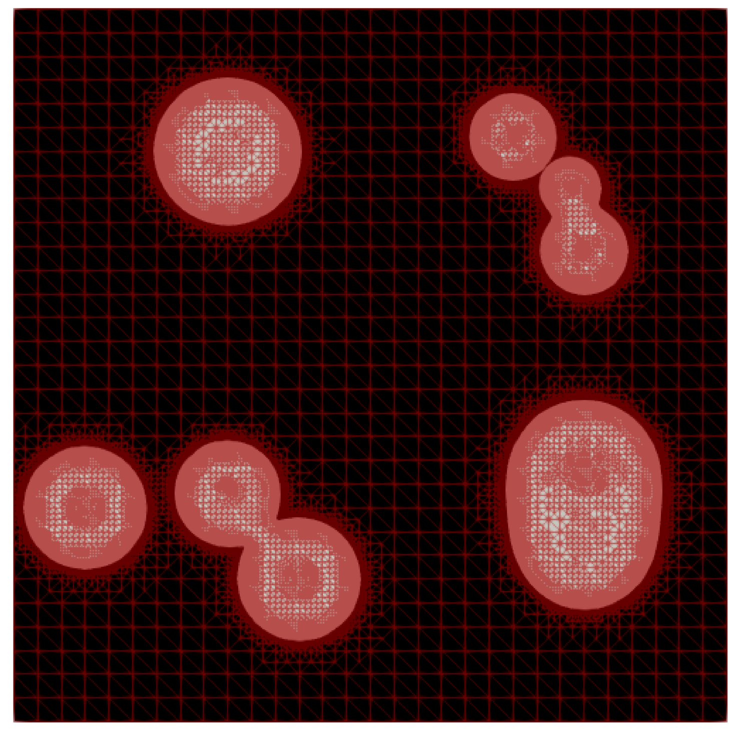
Meshes generated by our methods for several benchmark problems. Mesh in red, A_{u_h} in black, I_{u_h} in grey.



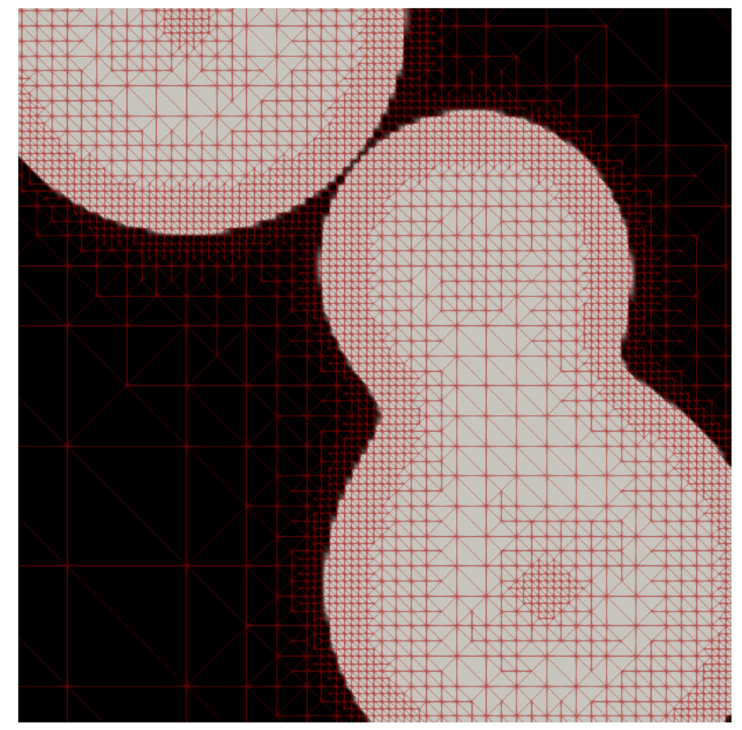
Spiral [3]



L-shaped



Blisters

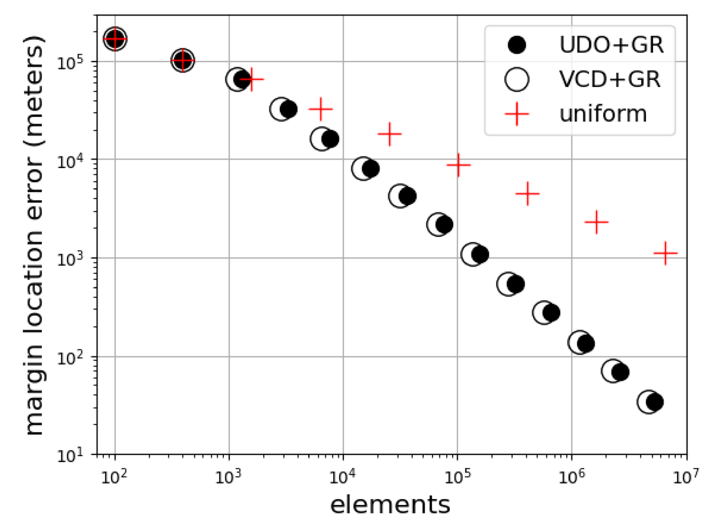


Blisters (zoomed)

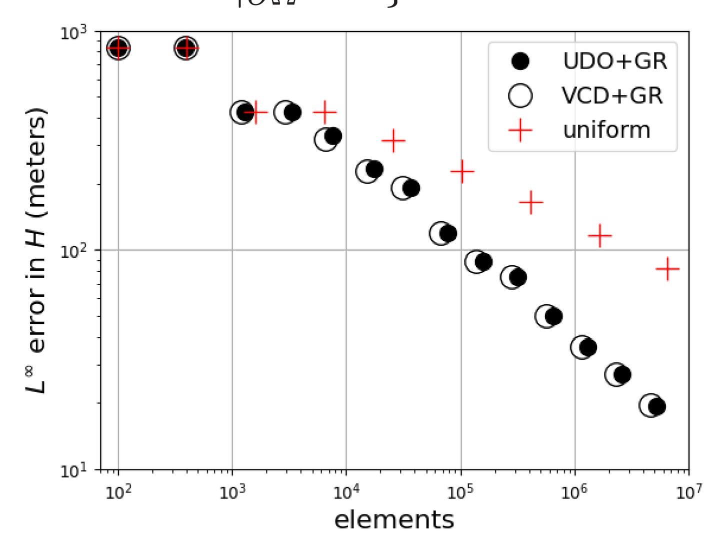
Application: Predicting Glaciated Land Areas

- **Model [1]:** Shallow Ice Approximation (SIA)
  $$\int_\Omega \Gamma|\nabla u + \beta(u)|^2(\nabla u + \beta(u)) \cdot \nabla(v - u) - \tilde{a}(u)(v - u)\, dx \geq 0$$

- **Physical ice thickness**: $H = u^{3/8}$
- **Ice surface elevation**: $s = u^{3/8} + b$
- **Bedrock elevation**: $b(x,y)$
- **Surface mass-balance**: $a(x,y,s)$
- **Ice softness**: $\Gamma > 0$
- **Tilt term**: $\beta(u) = \frac{8}{3}u^{5/8}\nabla b$
- $\tilde{a}(u) = a(x,y,s)$
- $u \in \mathcal{K} = \{u \in W^{1,4}(\Omega) : u \geq 0 \text{ and } u|_{\partial\Omega} = 0\}$



Maximum radial error of the glacier margin.



Max error in the ice thickness H.

References

[1] E. Bueler. Stable finite volume element schemes for the shallow ice approximation. *J. Glaciol.*, 62(232):230–242, 2016.

[2] Ed Bueler. *PETSc for Partial Differential Equations: Numerical Solutions in C and Python*. Number 31 in Software, Environments, and Tools. SIAM Press, Philadelphia, 2021.

[3] Carsten Gräser and Ralf Kornhuber. Multigrid methods for obstacle problems. *Journal of Computational Mathematics*, 27(1):1–44, 2009.

[4] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and their Applications*. Academic Press, New York, 1980.

[5] M. Levandowsky and D. Winter. Distance between sets. *Nature*, 234:34–35, 1971.