The Misspecified and Semiparametric lower bounds and their application to inference problems with Complex Elliptically Symmetric (CES) distributed data

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Outline – Part I: The Misspecified CRB

1. Motivations and history

2. Formal description of a misspecified problem

3. The Misspecified Cramér-Rao Bound (MCRB)

4. The Mismatched Maximum Likelihood (MML) estimator

5. Simple examples: variance and power estimation

6. Radar applications:

   ❖ Direction of Arrivals (DOAs) estimation

   ❖ Scatter matrix estimation for Complex Elliptically Simmetric (CES) distributed data

Outline – Part II: The Semiparametric CRB

1. Why semiparametric models?

2. CRB for parametric models with finite-dimensional nuisance parameters:

   ❑ Classical approach,

   ❑ "Hilbert-space-based" approach.

3. Extension to semiparametric models.

4. Semiparametric interpretation of Real and Complex ES Distributions.

5. Examples.

Why study lower bounds (LBs)?

☐ **Lower Bounds (LBs)** provide a benchmark of comparison for the performance of any **unbiased** estimator.

☐ If the performance of a certain estimator achieves a relevant LB, then it is established that no other *unbiased* estimators can do better.

☐ A LB is said to be **tight** if it reasonably predicts the performance of the Maximum Likelihood (ML) estimator.

☐ The **Cramér-Rao Bound (CRB)** is an *asymptotically tight* bound for any *unbiased* estimator.

Model misspecification (1/2)

❑ **Classical "matched" assumption**: the true data model and the model assumed to derive the estimation algorithm are the same, i.e. <u>the model is *correctly specified*</u>.

❑ All the results on the ML estimator and the CRB rely on this implicit assumption.

❑ However, much evidence from engineering practice shows that this assumption is often violated.

❑ **Model misspecification**: the assumed data model (i.e. the data pdf) differs from the true model.

Model misspecification (2/2)

❑ There are two main reasons for model misspecification:

1. An **imperfect knowledge** of the true data model that leads to a wrong specification of the data pdf.

2. The true data model is known but it is **too involved** to pursue the optimal "matched" estimator.

❑ One may be forced (scenario 1) or may prefer (scenario 2) to derive an estimator by assuming a **simpler** but **misspecified** data model.

❑ This suboptimal procedure may lead to some degradation in the overall system performance.

Misspecified Lower Bounds (MLBs)

❑ Assessing the impact of **model misspecification** on the estimation performance is crucial to guarantee the reliability of the (mismatched) estimator.

❑ **Misspecified LBs** allow the assumed and true models to differ, yet establishing performance limits on estimation error covariance.

❑ **Misspecified LBs** indicate how the model mis-specification affects estimation performance.

❑ Properties of the **Mismatched ML estimator**: Huber [1] (1967), Akaike [2] (1972) and White [3] (1982).

❑ Generalization to the **Bayesian** framework: Berk [5] (1966), Bunke and Milhaud [6] (1998), Richmond and Basu [7] (2016).

❑ **Cramér-Rao inequality** under **model misspecification**: Vuong [4] (1986), Richmond and Horowitz [8] (2015), S. Fortunati, F. Gini, M. S. Greco [11] (2016).

❑ A tutorial introduction to the **Misspecified CRB** has been proposed in:

S. Fortunati, F. Gini, M. S. Greco and C. D. Richmond, "Performance Bounds for Parameter Estimation under Misspecified Models: Fundamental Findings and Applications," *IEEE Signal Processing Magazine*, vol. 34, no. 6, pp. 142-157, Nov. 2017.

❑ Recent applications of **misspecified LBs**:

1. Direction of Arrivals estimation in sensor arrays [8] and MIMO radars [9].

2. Covariance matrix estimation in non-Gaussian data ([10], [11], [12] and [23]).

3. Radar-communication systems coexistence [7].

4. Waveform parameter estimation in the presence of uncertainty in the propagation model [13].

5. Time of Arrivals (ToA) estimation problem for UWB signals in the presence of interferences [14].

6. Sparse Bayesian estimation [27].

7. Spectral estimation [28].

8. Estimation of hybrid sinusoidal FM-PPS signals [29].

❑ Let $\mathbf{x}_m \in \mathbb{C}^N$ be an N-dimensional measurement vector.

❑ Let $p_X(\mathbf{x}_m) \in \mathcal{P}$ its true probability density function (pdf) belonging to a possibly non-parametric model $\mathcal{P}$.

❑ To characterize the statistical behavior of $\mathbf{x}_m$, we adopt a parametric pdf, say $f_X(\mathbf{x}_m \mid \boldsymbol{\theta})$ with $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^d$.

❑ The assumed pdf $f_X(\mathbf{x}_m \mid \boldsymbol{\theta})$ is implicitly assumed to belong to a *parametric* model:

$$\mathcal{F} = \left\{ f_X \mid f_X(\mathbf{x}_m \mid \boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta \subset \mathbb{R}^d \right\}$$

❑ The classical "matched" estimation theory requires the existence of a parameter vector $\overline{\boldsymbol{\theta}} \in \Theta$ such that:

$$p_X(\mathbf{x}_m) = f_X(\mathbf{x}_m \mid \overline{\boldsymbol{\theta}}) \quad \text{or, equivalently,} \quad p_X(\mathbf{x}_m) \in \mathcal{F}$$

❑ <u>If this assumption is violated, the model is **misspecified**.</u>

<u>Model misspecification</u>:

$f_X(\mathbf{x}_m \mid \boldsymbol{\theta})$ differs from $p_X(\mathbf{x}_m)$ for every $\boldsymbol{\theta} \in \Theta$.

$$p_X(\mathbf{x}_m) \neq f_X(\mathbf{x}_m \mid \boldsymbol{\theta}), \forall \boldsymbol{\theta} \in \Theta \quad \rightarrow \quad \mathcal{P} \not\subset \mathcal{F}$$

❑ Suppose to collect *M independent, identically distributed* (i.i.d.) *N*-dimensional measurement vectors:

$$\mathbf{x} = \left\{ \mathbf{x}_m \right\}_{m=1}^{M}, \quad \mathbf{x}_m \sim p_X\left(\mathbf{x}_m\right), \ m = 1,\ldots,M$$

❑ Due to the independence assumption, the true joint pdf of the dataset **x** is:

$$p_X\left(\mathbf{x}\right) = \prod_{m=1}^{M} p_X\left(\mathbf{x}_m\right)$$

❑ The assumed joint pdf of the dataset **x** is:

$$f_X\left(\mathbf{x}\,|\,\boldsymbol{\theta}\right) = \prod_{m=1}^{M} f_X\left(\mathbf{x}_m\,|\,\boldsymbol{\theta}\right)$$

Two fundamental questions

❑ This <u>misspecified scenario</u> raises two main questions:

1. Is it still possible to derive LBs on the error covariance of any mismatched estimator of the parameter vector $\boldsymbol{\theta}$?

2. How will the classical statistical properties of an estimator, e.g. *unbiasedness*, *consistency* and *efficiency*, change in this misspecified model framework?

❑ The **Misspecified Cramér-Rao Bound (MCRB)** provides answers to these questions.

Regular models

❑ As for the classical CRB, in order to guarantee the existence of the MCRB, some <u>regularity conditions</u> need to be imposed.

❑ Specifically, the assumed parametric model $\mathcal{F}$ has to be **regular** with respect to (wrt) the true one $\mathcal{P}$.

❑ Among the rather technical assumptions that $\mathcal{F}$ has to satisfy to be regular wrt $\mathcal{P}$, the most important is:

> ❖ Existence and uniqueness of the pseudo-true parameter vector $\boldsymbol{\theta}_0$

The pseudo-true parameter vector

❑ If $\mathcal{F}$ is regular wrt $\mathcal{P}$, then there exist a unique interior point $\boldsymbol{\theta}_0$ of Θ, such that:

$$\boldsymbol{\theta}_0 \triangleq \arg\min_{\boldsymbol{\theta} \in \Theta}\left\{-E_p\left\{\ln f_X\left(\mathbf{x}_m \mid \boldsymbol{\theta}\right)\right\}\right\} = \arg\min_{\boldsymbol{\theta} \in \Theta}\left\{D\left(p_X \| f_X\right)\right\}$$

where $E_p\{\cdot\}$ is the expectation operator wrt the true pdf $p_X(\mathbf{x}_m)$ and

$$D\left(p_X \| f_X\right) \triangleq \int \ln\left(\frac{p_X(\mathbf{x}_m)}{f_X(\mathbf{x}_m \mid \boldsymbol{\theta})}\right) p_X(\mathbf{x}_m) d\mathbf{x}_m$$

is the **Kullback-Leibler divergence (KLD)** between the true pdf and the assumed pdf.

Generalization of the FIM

❑ The vector $\boldsymbol{\theta}_0$ is the point the minimizes the KLD between the true pdf $p_X(\mathbf{x}_m)$ and the assumed pdf $f_X(\mathbf{x}_m | \boldsymbol{\theta})$.

❑ Let $\mathbf{A}_{\boldsymbol{\theta}_0}$ be the matrix whose entries are given by:

$$\left[\mathbf{A}_{\boldsymbol{\theta}_0} \right]_{ij} \triangleq E_p \left\{ \frac{\partial^2}{\partial\theta_i\partial\theta_j} \ln f_X \left(\mathbf{x}_m | \boldsymbol{\theta} \right) \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right\}$$

If $\mathcal{F}$ is regular wrt $\mathcal{P}$, then the matrix $\mathbf{A}_{\boldsymbol{\theta}_0}$ is non singular

❑ In the matrix $\mathbf{A}_{\boldsymbol{\theta}_0}$ we can recognize a sort of generalization of the Fisher Information Matrix (FIM).

Another generalization of the FIM

❑ The second generalization of the FIM is given by:

$$\left[\mathbf{B}_{\boldsymbol{\theta}_0}\right]_{ij} \triangleq E_p\left\{\left.\frac{\partial \ln f_X\left(\mathbf{x}_m \mid \boldsymbol{\theta}\right)}{\partial \theta_i}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \cdot \left.\frac{\partial \ln f_X\left(\mathbf{x}_m \mid \boldsymbol{\theta}\right)}{\partial \theta_j}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right\}.$$

❑ If the model is **misspecified**, i.e. it does not exists a parameter vector $\bar{\boldsymbol{\theta}} \in \Theta$ such that $p_X(\mathbf{x}_m) = f_X(\mathbf{x}_m \mid \bar{\boldsymbol{\theta}})$,

then: $\mathbf{B}_{\boldsymbol{\theta}_0} \neq -\mathbf{A}_{\boldsymbol{\theta}_0}$

❑ The Misspecified (MS)-unbiasedness property generalizes the classical notion of unbiased estimators.

❑ In the misspecified model framework, unbiasedness is defined wrt the pseudo-true parameter vector $\mathbf{\theta}_0$.

❑ Let $\mathbf{x} = \left\{ \mathbf{x}_m \right\}_{m=1}^{M}$ be the available dataset and let $\hat{\mathbf{\theta}}(\mathbf{x})$ be an estimator derived by assuming the misspecified model $\mathcal{F}$.

$\hat{\mathbf{\theta}}(\mathbf{x})$ is an **MS-unbiased** estimator iff:

$$E_p\left\{ \hat{\mathbf{\theta}}(\mathbf{x}) \right\} = \int \hat{\mathbf{\theta}}(\mathbf{x}) p_X(\mathbf{x}) d\mathbf{x} = \mathbf{\theta}_0$$

❑ Let $\mathcal{F}$ be a misspecified parametric model that is regular wrt a true model $\mathcal{P}$.

> ***Theorem 1 (Vuong 1986)***: Let $\hat{\boldsymbol{\theta}}(\mathbf{x})$ be an MS-unbiased estimator derived under the misspecified model $\mathcal{F}$ from a dataset $\mathbf{x} = \{\mathbf{x}_m\}_{m=1}^{M}$. Then, for every possible $p_X(\mathbf{x}_m) \in \mathcal{P}$:
>
> $$\mathbf{C}_p\left(\hat{\boldsymbol{\theta}}(\mathbf{x}), \boldsymbol{\theta}_0\right) \geq \frac{1}{M}\mathbf{A}_{\boldsymbol{\theta}_0}^{-1}\mathbf{B}_{\boldsymbol{\theta}_0}\mathbf{A}_{\boldsymbol{\theta}_0}^{-1} \triangleq \mathrm{MCRB}(\boldsymbol{\theta}_0)$$
>
> where
>
> $$\mathbf{C}_p\left(\hat{\boldsymbol{\theta}}(\mathbf{x}), \boldsymbol{\theta}_0\right) \triangleq E_p\left\{\left(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}_0\right)\left(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}_0\right)^T\right\}$$
>
> is the error covariance matrix of the mismatched estimator.

[4] Q. H. Vuong, "Cramér-Rao bounds for misspecified models", *Working paper 652, Division of the Humanities and Social Sciences*, Caltech, October 1986.

The Misspecified Cramér-Rao Bound (2/2)

❑ The MCRB is a local lower bound (LB) on the error variance of any MS-unbiased estimator of the pseudo-true parameter vector $\boldsymbol{\theta}_0$.

❑ MCRB for constrained estimation problem [20]:

S. Fortunati, F. Gini, M. S. Greco, "The Constrained Misspecified Cramér-Rao Bound," *IEEE Signal Process. Letters*, Vol. 23, No. 5, pp. 718-721, May 2016.

❑ Extension to complex parameters [21]:

S. Fortunati, "Misspecified Cramér-Rao Bounds for Complex Unconstrained and Constrained Parameters," *EUSIPCO 2017*, Kos, Greece, 28 Aug. 2017–2 Sept. 2017.

❑ If the model is **correctly specified**, i.e. if there exists a parameter vector $\overline{\boldsymbol{\theta}} \in \Theta$, such that $p_X(\mathbf{x}_m) = f_X(\mathbf{x}_m | \overline{\boldsymbol{\theta}})$, then:

$$\boldsymbol{\theta}_0 = \overline{\boldsymbol{\theta}} \text{ and } \mathbf{B}_{\boldsymbol{\theta}_0} = -\mathbf{A}_{\boldsymbol{\theta}_0} = -\mathbf{A}_{\overline{\boldsymbol{\theta}}} \qquad \Rightarrow \qquad \text{MCRB} = \text{CRB}$$

$$E_p\left\{\left(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \overline{\boldsymbol{\theta}}\right)\left(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \overline{\boldsymbol{\theta}}\right)^T\right\} \geq -\frac{1}{M}\mathbf{A}_{\boldsymbol{\theta}_0}^{-1} = \text{MCRB}\left(\overline{\boldsymbol{\theta}}\right) = \text{CRB}\left(\overline{\boldsymbol{\theta}}\right)$$

> The misspecified framework is consistent with the classical matched theory!

❑ As before, suppose to collect *M independent, identically distributed* (i.i.d.) *N*-dimensional measurement vectors:

$$\mathbf{x} = \{\mathbf{x}_m\}_{m=1}^{M}, \quad \mathbf{x}_m \sim p_X(\mathbf{x}_m), m = 1, \ldots, M$$

❑ The log-likelihood function for the dataset **x** under a misspecified model $\mathcal{F}$ is given by:

$$L_M(\boldsymbol{\theta}) \triangleq \frac{1}{M} \sum_{m=1}^{M} \ln f_X(\mathbf{x}_m \mid \boldsymbol{\theta}), \quad \mathbf{x}_m \sim p_X(\mathbf{x}_m)$$

❑ The **MML estimator** is the point that maximizes $L_M(\boldsymbol{\theta})$:

$$\hat{\boldsymbol{\theta}}_{MML}(\mathbf{x}) \triangleq \arg\max_{\boldsymbol{\theta} \in \Theta} L_M(\boldsymbol{\theta}) = \arg\max_{\boldsymbol{\theta} \in \Theta} \sum_{m=1}^{M} \ln f_X(\mathbf{x}_m \mid \boldsymbol{\theta})$$

Theorem 2 (Huber 1967, White 1982): Under suitable regularity conditions, it can be shown that:

$$\hat{\boldsymbol{\theta}}_{MML}(\mathbf{x}) \xrightarrow[M\to\infty]{a.s.} \boldsymbol{\theta}_0.$$

Moreover:

$$\sqrt{M}\left(\hat{\boldsymbol{\theta}}_{MML}(\mathbf{x}) - \boldsymbol{\theta}_0\right) \underset{M\to\infty}{\overset{d}{\sim}} \mathcal{N}\left(\mathbf{0}, \mathbf{A}_{\boldsymbol{\theta}_0}^{-1}\mathbf{B}_{\boldsymbol{\theta}_0}\mathbf{A}_{\boldsymbol{\theta}_0}^{-1}\right).$$

Huber "sandwich"
covariance matrix = MCRB

- $\xrightarrow[M\to\infty]{a.s.}$ indicates the *almost sure* (*a.s.*) convergence.

- $\underset{M\to\infty}{\overset{d}{\sim}}$ indicates the convergence *in distribution*.

❑ The MML estimator is asymptotically MS-unbiased and its asymptotic error covariance is equal to the MCRB, i.e. it is an *asymptotically efficient* estimator wrt the MCRB.

❑ Consistent with the *classical* "matched" ML estimator:

If the model $\mathcal{F}$ is correctly specified, i.e. if there exists

$$\overline{\boldsymbol{\theta}} \in \Theta \text{ such that: } p_X(\mathbf{x}_m) = f_X(\mathbf{x}_m \mid \overline{\boldsymbol{\theta}})$$

then:

$$\hat{\boldsymbol{\theta}}_{ML}(\mathbf{x}) \xrightarrow[M \to \infty]{a.s.} \boldsymbol{\theta}_0 = \overline{\boldsymbol{\theta}}$$

CRB

$$\sqrt{M}\left(\hat{\boldsymbol{\theta}}_{ML}(\mathbf{x}) - \overline{\boldsymbol{\theta}}\right) \underset{M \to \infty}{\overset{d}{\sim}} \mathcal{N}\left(\mathbf{0}, -\mathbf{A}_{\overline{\boldsymbol{\theta}}}^{-1}\right)$$

❑ Let us define the following data-dependent matrices:

$$\left[\mathbf{A}_M(\boldsymbol{\theta}) \right]_{ij} \triangleq M^{-1} \sum_{m=1}^{M} \frac{\partial^2 \ln f_X(\mathbf{x}_m \,|\, \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}$$

$$\left[\mathbf{B}_M(\boldsymbol{\theta}) \right]_{ij} \triangleq M^{-1} \sum_{m=1}^{M} \frac{\partial \ln f_X(\mathbf{x}_m \,|\, \boldsymbol{\theta})}{\partial \theta_i} \cdot \frac{\partial \ln f_X(\mathbf{x}_m \,|\, \boldsymbol{\theta})}{\partial \theta_j}$$

$$\mathbf{C}_M(\boldsymbol{\theta}) \triangleq [\mathbf{A}_M(\boldsymbol{\theta})]^{-1} \mathbf{B}_M(\boldsymbol{\theta}) [\mathbf{A}_M(\boldsymbol{\theta})]^{-1}$$

❑ It can be shown (see [3, Theo 3.2]) that:

$$\left[\mathbf{C}_M(\hat{\boldsymbol{\theta}}_{MML}) \right]_{i,j} \xrightarrow[M \to \infty]{a.s.} \left[\mathbf{C}_{\boldsymbol{\theta}_0} \right]_{i,j} = \left[\mathrm{MCRB}(\boldsymbol{\theta}_0) \right]_{i,j} \quad \forall i, j = 1, \ldots, \dim(\boldsymbol{\theta}_0)$$

A test for model misspecification

❑ It is possible to infer from the collected dataset whether or not the assumed model $\mathcal{F}$ is correctly specified.

❑ Recall that, if $\mathcal{F}$ is correctly specified, then $\mathbf{B}_\theta = -\mathbf{A}_\theta$.

❑ Since $\mathbf{A}_\theta$ and $\mathbf{B}_\theta$ are unobservable, we can exploit their sample estimate to implement a composite hypothesis testing:

$$\begin{cases} H_0: & \mathbf{A}_M(\hat{\boldsymbol{\theta}}_{MML}) + \mathbf{B}_M(\hat{\boldsymbol{\theta}}_{MML}) = \mathbf{0} \\ H_1: & \mathbf{A}_M(\hat{\boldsymbol{\theta}}_{MML}) + \mathbf{B}_M(\hat{\boldsymbol{\theta}}_{MML}) \neq \mathbf{0} \end{cases}$$

❑ A Wald-type test can be derived to discriminate between the two hypotheses: *correct specification (H_0) vs model misspecification (H_1)* (see [3, Sec. 4]).

❑ **Problem**: we want to estimate the <u>variance</u> of a Gaussian data set in the presence of misspecified mean value, e.g. <u>we erroneously assume that the mean value is zero</u>.

❑ True data set:

$$\mathbf{x} = \left\{ x_m \right\}_{m=1}^{M}, \quad x_m \sim p_X \left(x_m \right) \equiv \mathcal{N} \left(\bar{\mu}, \bar{\sigma}^2 \right), \quad \bar{\mu} \neq 0$$

❑ Assumed data model:

$$\mathcal{F} = \left\{ f_X \middle| f_X \left(x_m \middle| \theta \right) \equiv \mathcal{N} \left(0, \theta \right) \ \forall \theta \in \mathbb{R}^+ \right\}$$

❑ Note that, as long as $\bar{\mu} \neq 0$,

$$p_X \left(x_m \right) \notin \mathcal{F}$$

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❑ Is the misspecified model $\mathcal{F}$ regular wrt $p(x_m)$?

❑ We have to check if:

1. there exists the pseudo-true parameter θ_0;

2. the matrix $\mathbf{A}_{\boldsymbol{\theta}_0}$ is non singular.

❑ The KLD can be expressed as:

$$D\left(p_X \| f_X\right) = \frac{\bar{\mu}^2}{2\theta} + \frac{1}{2}\left(\frac{\bar{\sigma}^2}{\theta} - 1 - \ln\frac{\bar{\sigma}^2}{\theta}\right)$$

❑ Its minimum point, i.e. the pseudo-true parameter, *exists* and is *unique*:

$$\theta_0 = \bar{\sigma}^2 + \bar{\mu}^2, \quad \bar{\mu} \neq 0$$

❑ The scalar A_{θ_0} can be evaluated as:

$$A_{\theta_0} \triangleq E_p \left\{ \left. \frac{\partial^2 \ln f_X(x_m | \theta)}{\partial \theta^2} \right|_{\theta = \theta_0} \right\} = \frac{1}{2\theta_0^2} - \frac{1}{\theta_0^3} E_p \left\{ x_m^2 \right\} = -\frac{1}{2\theta_0^2}$$

that is always different from zero, since $\bar{\mu} \neq 0, \bar{\sigma}^2 \in \mathbb{R}^+$.

Since θ_0 exists and is unique and the scalar $A_{\theta_0} \neq 0$,

then $\mathcal{F}$ is regular wrt $p(x_m)$.

❑ The scalar B_{θ_0} can be evaluated as:

$$B_{\theta_0} \triangleq E_p \left\{ \left(\frac{\partial \ln f_X(x_m | \theta)}{\partial \theta} \right)^2 \bigg|_{\theta = \theta_0} \right\} = \frac{2\bar{\sigma}^4 + 4\bar{\sigma}^2 \bar{\mu}^2}{4\theta_0^4}$$

❑ By collecting the previous results, the MCRB is given by:

$$\text{MCRB}(\theta_0) = \frac{2\bar{\sigma}^4}{M} + \frac{4\bar{\sigma}^2 \bar{\mu}^2}{M} \geq \text{CRB}(\bar{\sigma}^2) = \frac{2\bar{\sigma}^4}{M}$$

❑ <u>The MCRB is always greater than the CRB</u>, as expected.

❑ The MCRB equates the CRB when there is no model misspecification, i.e. when $\bar{\mu} = 0$.

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❑ Regarding the <u>MML estimator</u>, we have that:

$$\hat{\theta}_{MML}(\mathbf{x}) = \frac{1}{M}\sum\nolimits_{m=1}^{M} x_m^2 \xrightarrow[M\to\infty]{a.s.} \theta_0 = \bar{\sigma}^2 + \bar{\mu}^2 \neq \bar{\sigma}^2$$

❑ The MML estimator is <u>not consistent</u>, since it converges to a value different from the true variance.

❑ However, the MML is <u>MS-unbiased</u>, since:

$$E_p\left\{\hat{\theta}_{MML}(\mathbf{x})\right\} = E_p\left\{\frac{1}{M}\sum\nolimits_{m=1}^{M} x_m^2\right\} = \bar{\sigma}^2 + \bar{\mu}^2 = \theta_0$$

❑ Hence, <u>the error variance of this MML estimator is lower bounded by the MCRB</u>.
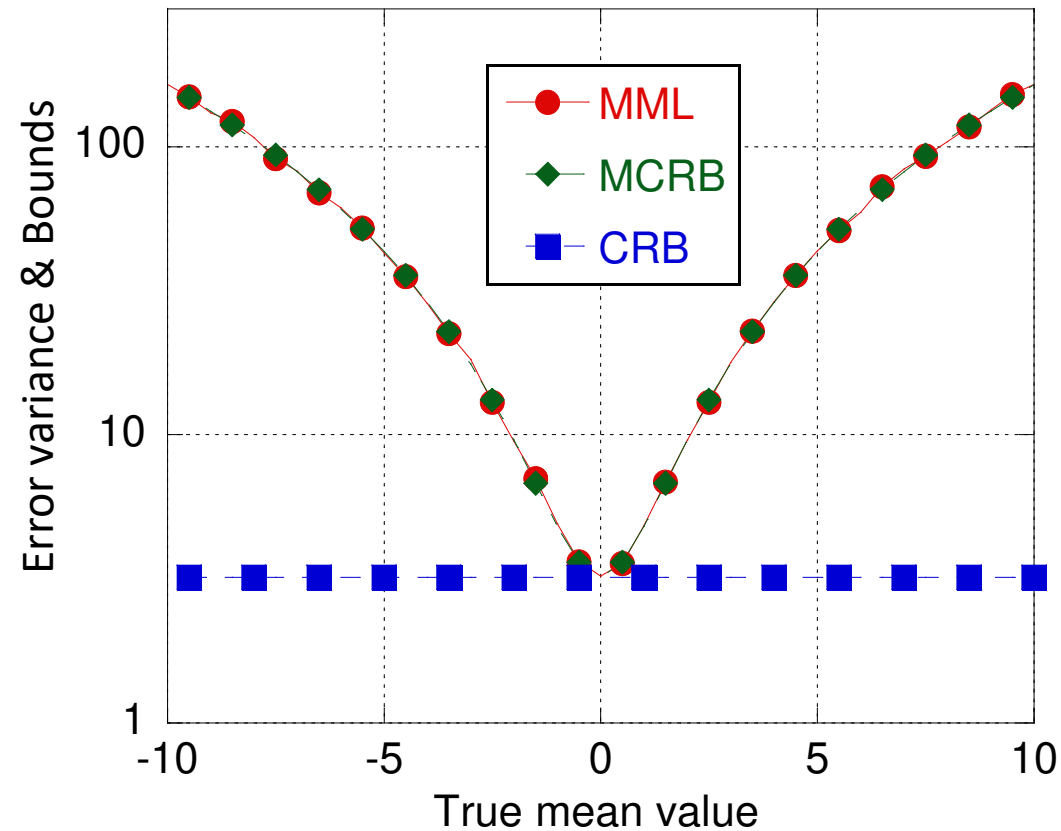
$$\bar{\sigma}^2 = 4$$

$$M = 10$$

❑ MCRB = CRB when $\bar{\mu} = 0$: matched case.

❑ The MML estimator is efficient wrt the MCRB.

❑ **Problem**: we want to estimate the <u>statistical power</u> of a multivariate Gaussian data set under misspecification of the correlation structure.

❑ True dataset:

$$\mathbf{x} = \{\mathbf{x}_m\}_{m=1}^M \qquad p_X(\mathbf{x}_m) \equiv \mathcal{N}\left(\mathbf{0}, \bar{\sigma}^2 \boldsymbol{\Sigma}\right) \in \mathcal{P} \qquad [\boldsymbol{\Sigma}]_{ij} = \rho^{|i-j|}$$

❑ Assumed data model:

$$\mathcal{F} = \left\{ f_X \mid f_X(\mathbf{x}_m \mid \theta) \equiv \mathcal{N}\left(\mathbf{0}, \theta \mathbf{I}_N\right) \ \forall \theta \in \mathbb{R}^+ \right\}$$

❑ Note that, as long as $\rho \neq 0$,

$$p_X(\mathbf{x}_m) \notin \mathcal{F}$$

❑ Is the misspecified model $\mathcal{F}$ regular wrt $p(\mathbf{x}_m)$?

❑ As before, we have to check if:

1. there exists the pseudo-true parameter θ_0;

2. the scalar A_{θ_0} is non singular.

❑ The KLD can be expressed as:

$$D\left(p_X \| f_X\right) = \frac{1}{2}\left[\operatorname{tr}(\theta^{-1}\bar{\sigma}^2 \mathbf{\Sigma}) - N + \ln\theta - \ln\det(\bar{\sigma}^2 \mathbf{\Sigma})\right]$$

❑ Its minimum point, i.e. the pseudo-true parameter, *exists* and is *unique*: $\theta_0 = \bar{\sigma}^2$

❑ The scalar A_{θ_0} can be evaluated as:

$$A_{\theta_0} \triangleq E_p \left\{ \frac{\partial^2 \ln f_X(\mathbf{x}_m \mid \theta)}{\partial \theta^2} \bigg|_{\theta=\theta_0} \right\} = \frac{N}{2\theta_0^2} - \frac{1}{\theta_0^3} E_p \left\{ \mathbf{x}_m^T \mathbf{x}_m \right\} = -\frac{N}{2\bar{\sigma}^4}$$

that is always different from zero, since $\bar{\sigma}^2 \in \mathbb{R}^+$.

> Since θ_0 exists and is unique and the scalar $A_{\theta_0} \neq 0$, then $\mathcal{F}$ is regular wrt $p(\mathbf{x}_m)$.

❑ The scalar B_{θ_0} can be evaluated as:

$$B_{\theta_0} = \frac{N\theta_0^2 + E_p\left\{(\mathbf{x}_m^T\mathbf{x}_m)^2\right\} - 2N\theta_0 E_p\left\{\mathbf{x}_m^T\mathbf{x}_m\right\}}{4\theta_0^4} = \frac{\mathrm{tr}\left(\mathbf{\Sigma}^2\right)}{2\bar{\sigma}^4}$$

❑ By collecting the previous results, the MCRB is given by:

$$\mathrm{MCRB}(\theta_0) = \mathrm{MCRB}(\bar{\sigma}^2) = \frac{2\bar{\sigma}^4}{MN^2}\,\mathrm{tr}\left(\mathbf{\Sigma}^2\right) \geq \mathrm{CRB}(\bar{\sigma}^2) = \frac{2\bar{\sigma}^4}{MN}$$

❑ <u>The MCRB is always greater than the CRB</u>, as expected.

❑ MCRB = CRB when there is no model misspecification, i.e. when $\rho = 0$.

❑ Regarding the <u>MML estimator</u>, we have that:

$$\hat{\theta}_{MML}(\mathbf{x}) = \frac{1}{M} \sum_{m=1}^{M} \frac{\mathbf{x}_m^T \mathbf{x}_m}{N} \xrightarrow[M \to \infty]{a.s.} \theta_0 = \frac{E_p\{\mathbf{x}_m^T \mathbf{x}_m\}}{N} = \frac{\bar{\sigma}^2}{N} \mathrm{tr}(\mathbf{\Sigma}) = \bar{\sigma}^2$$
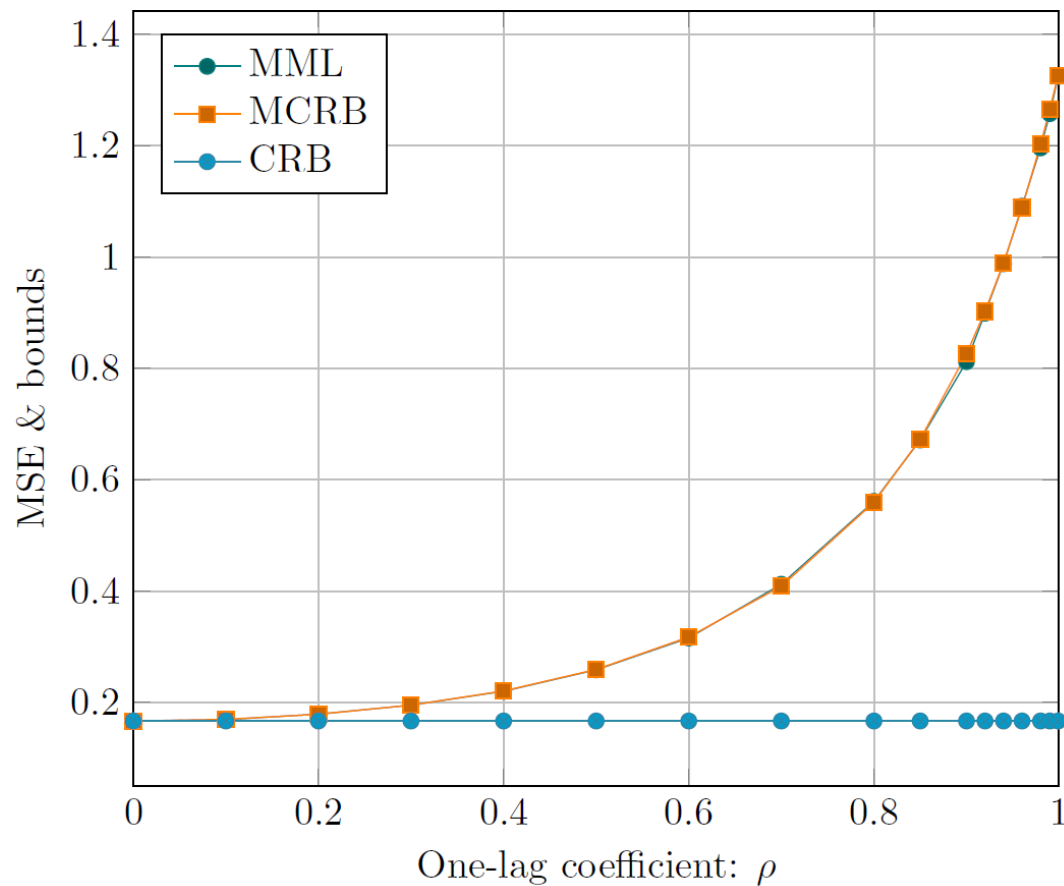
❑ The MML estimator is <u>consistent</u>, since it converges to the true value of the statistical power.

❑ Moreover, the MML is <u>MS-unbiased</u>, since:

$$E_p\{\hat{\theta}_{MML}(\mathbf{x})\} = \frac{1}{M} \sum_{m=1}^{M} \frac{E_p\{\mathbf{x}_m^T \mathbf{x}_m\}}{N} = \bar{\sigma}^2 = \theta_0$$

❑ Hence, <u>the error variance (and the MSE) of this MML estimator is lower bounded by the MCRB</u>.

$$\overline{\sigma}^2 = 4$$

$$M = 3N$$

$$N = 8$$

❑ MCRB = CRB when $\rho = 0$: matched case.

❑ The MML estimator is efficient wrt the MCRB.

❑ **Problem**: we want to estimate the Directions of Arrival (DOAs) of plane-waves signals by means of an array of sensors.

❑ This is a core research within the SP community [15].

❑ The fundamental prerequisite for any DOA estimation algorithm is that the positions of the sensors in the array are known exactly.

❑ Many authors have investigated the impact on the DOA estimation performance of an imperfect knowledge of the sensor positions or of the misscalibration of the array itself (see e.g. [16] and [17]).

❑ Some authors have proposed *hybrid* or *modified* CRB with the aim to predict the lowest MSE of the DOA estimators in the presence of the position uncertainties ([18], [19]).

❑ All these classical approaches, although reasonable, are application-dependent and not general.

❑ The application of the general misspecified estimation framework to DOA estimation problems has been firstly proposed by Richmond and Horowitz in their seminal paper:

[8] Richmond, C.D.; Horowitz, L.L., "Parameter Bounds on Estimation Accuracy Under Model Misspecification," *IEEE Trans. on Signal Process.*, vol.63, no.9, pp.2263-2278, 2015

- ❑ Following [8], consider a ULA of M sensors and a single plane wave signal impinging on the array from a conic angle $\bar{\theta}$.

- ❑ Due to array misscalibration, the true position vector of the mth sensor is known up to an error term modeled as a zero-mean Gaussian random vector $\mathbf{e}_m \sim \mathcal{N}(0, \sigma_e^2 \mathbf{I}_3)$.

- ❑ Define as $\mathbf{u}(\bar{\theta})$ the unit vector pointing at the direction of the impinging plane wave.

- ❑ Define $\mathbf{k}_{\bar{\theta}} \triangleq (2\pi/\lambda)\mathbf{u}(\bar{\theta})$ where λ is the wavelength.

z

$\mathbf{e}_1$

Plane wave

$\mathbf{t}_1$

$\mathbf{u}(\overline{\theta})$

$\mathbf{p}_1$

$\mathbf{p}_2$

$\mathbf{p}_m$

y

x

◼ : Nominal position of the sensors $\mathbf{p}_m$

⬤ : True position of the sensors $\mathbf{t}_m = \mathbf{p}_m + \mathbf{e}_m$
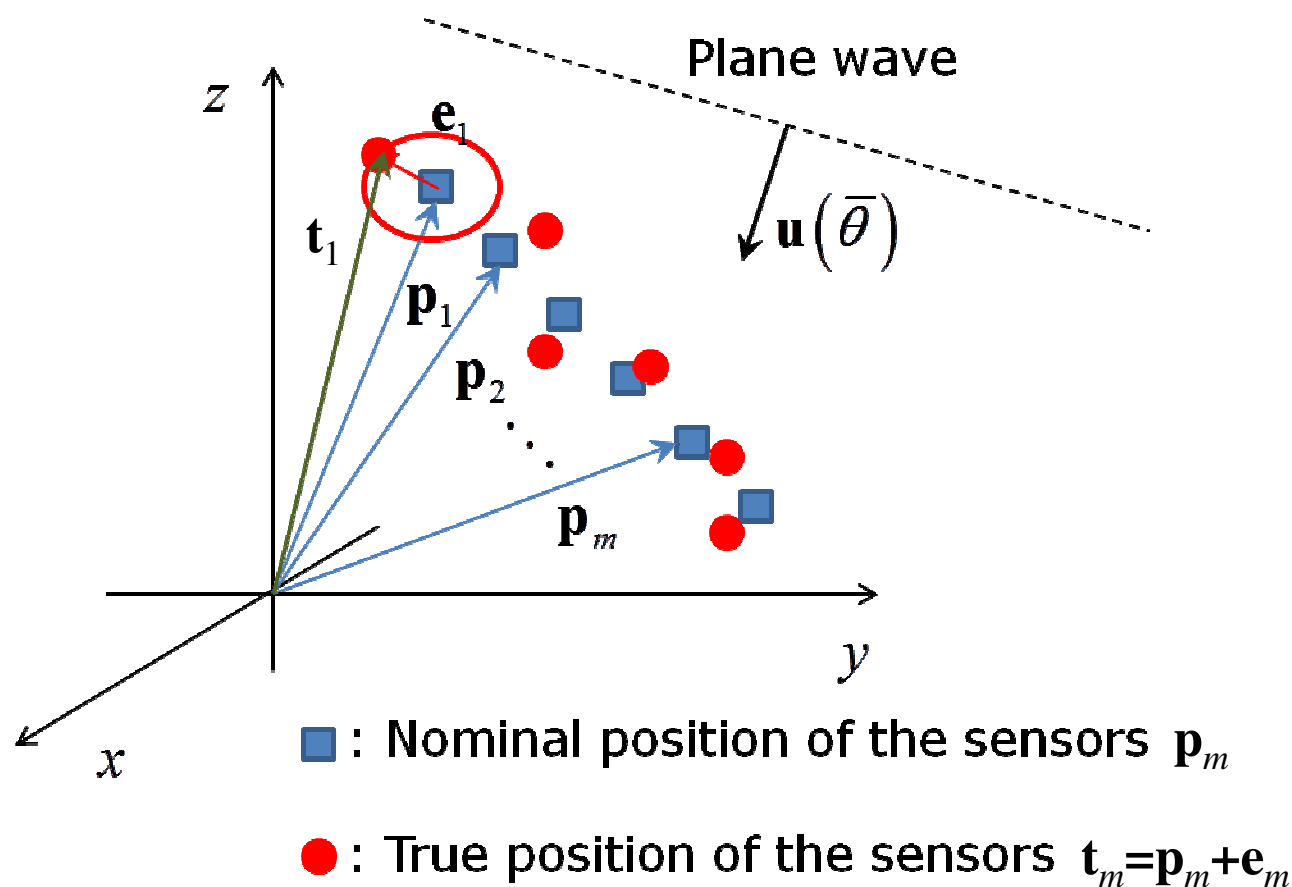
❑ The true (perturbed) steering vector is given by:

$$[\mathbf{d}(\overline{\theta})]_m = \exp\left(j\mathbf{k}_{\overline{\theta}}^T (\mathbf{p}_m + \mathbf{e}_m) \right), \quad m = 1,\ldots,M$$

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❑ The signal received at the *m*th sensor is:

$$x_m = \overline{s} \cdot [\mathbf{d}(\overline{\theta})]_m + [\mathbf{c}]_m, \quad m = 1, \ldots, M$$

❑ $\overline{s}$ is the deterministic unknown complex amplitude.

❑ $\mathbf{c} = \mathbf{n} + \mathbf{j}$ is the disturbance vector composed of a white Gaussian noise $\mathbf{n}$ and possibly also of a jammer $\mathbf{j}$.

❑ Given particular realizations of $\mathbf{e}_m$, the disturbance can be modeled as:

$$\mathbf{c} \sim \mathcal{N}\left(0, \sigma_n^2 \mathbf{I}_M + \sigma_j^2 \mathbf{d}(\theta_j)\mathbf{d}^H(\theta_j)\right)$$

Power and DOA of the jammer

❑ Since the particular realizations of $\mathbf{e}_m$ is generally unknown, one may decide to assume the <u>nominal steering vector</u> in the estimation algorithm:

$$[\mathbf{v}(\theta)]_m = \exp(j\mathbf{k}_\theta^T \mathbf{p}_m), \quad m = 1,\ldots,M$$

❑ The true (but unknown) data model is:

$$p_X(\mathbf{x}) = \mathcal{N}\left(\bar{s}\mathbf{d}(\bar{\theta}), \sigma_n^2 \mathbf{I}_M + \sigma_j^2 \mathbf{d}(\theta_j)\mathbf{d}^H(\theta_j)\right) \in \mathcal{P}$$
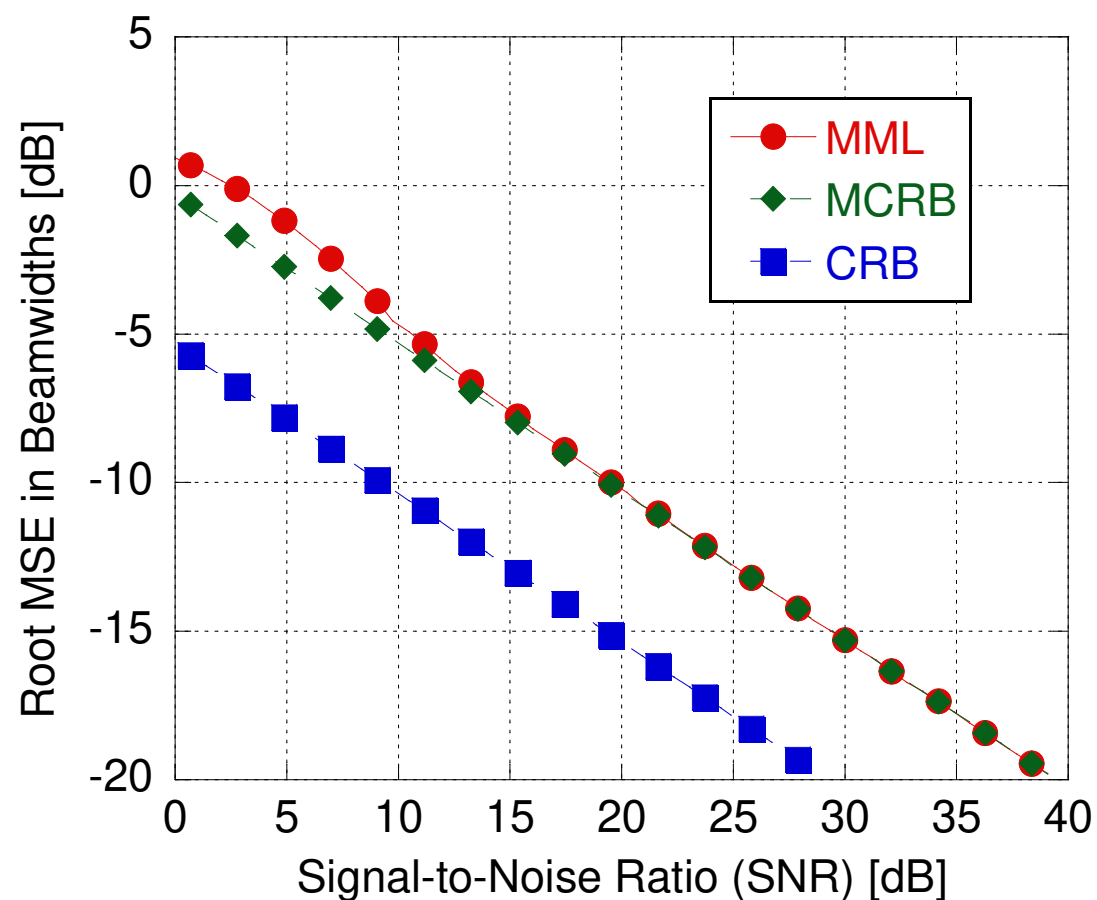
❑ The assumed parametric data model is:

$$\mathcal{F} = \left\{ f_X \mid f_X(\mathbf{x} \mid s, \theta) \equiv \mathcal{N}\left(s\mathbf{v}(\theta), \sigma_n^2 \mathbf{I}_M + \sigma_j^2 \mathbf{v}(\theta_j)\mathbf{v}^H(\theta_j)\right) \right\}$$

where $s \in \mathbb{C}, \theta \in [0, 2\pi)$

❑ The MCRB can predict how large is the performance loss in the estimation of $\bar{\theta}$ due to this model mismatch.
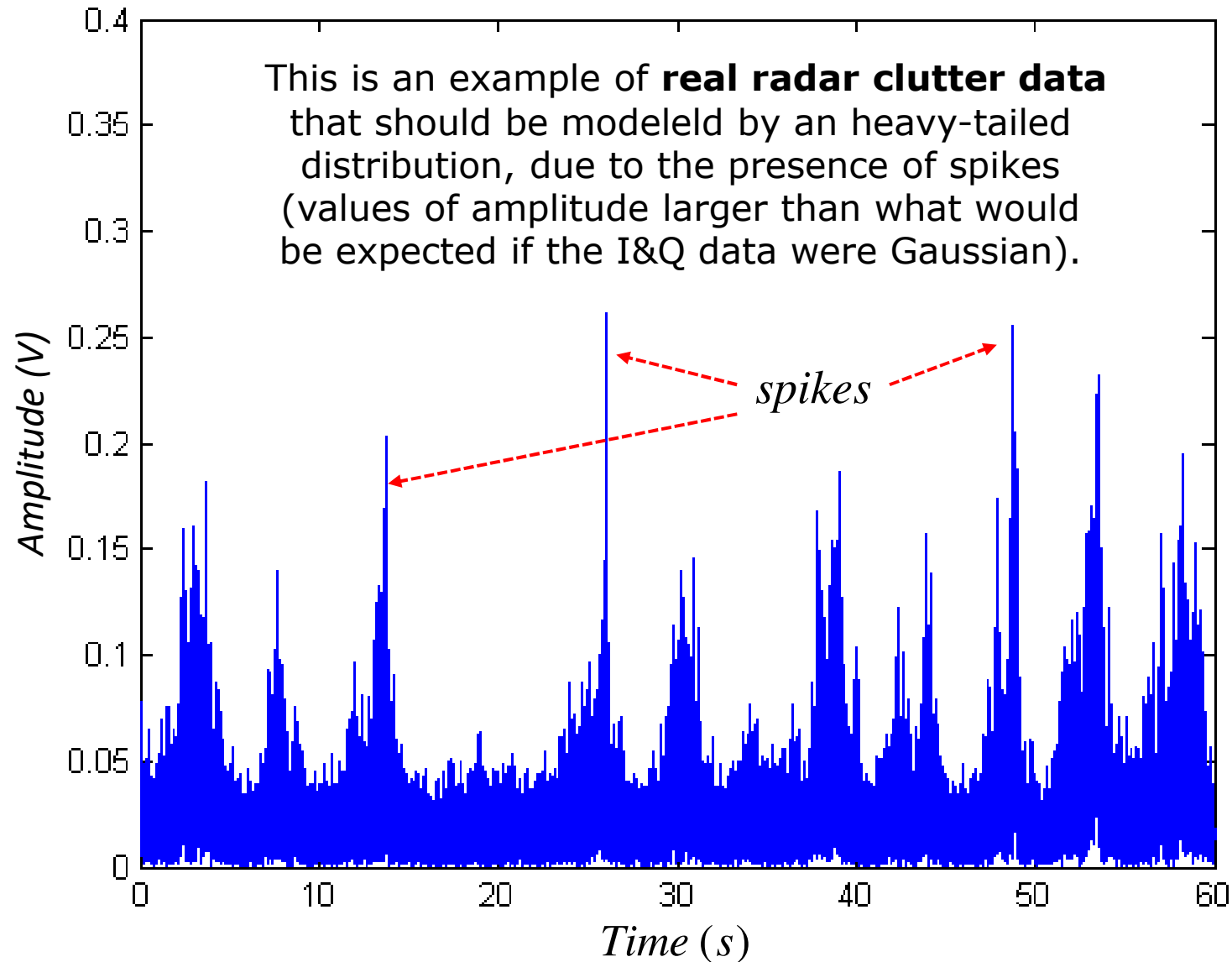
❑ <u>The MCRB accurately predicts performance of the Mismatched ML (MML) estimator</u>.

❑ If the system goal is a 10-to-1 beamsplit ratio, i.e. -10dB RMSE in beamwidths, then this could be accomplished with an *SNR* of ~10dB when the model is perfectly known.

❑ However, not knowing precisely the true sensor positions requires an additional ~10dB of *SNR* to achieve the same goal [8].

[8] Richmond, C.D.; Horowitz, L.L., "Parameter Bounds on Estimation Accuracy Under Model Misspecification," *IEEE Trans. on Signal Process.*, vol.63, no.9, pp.2263-2278, 2015
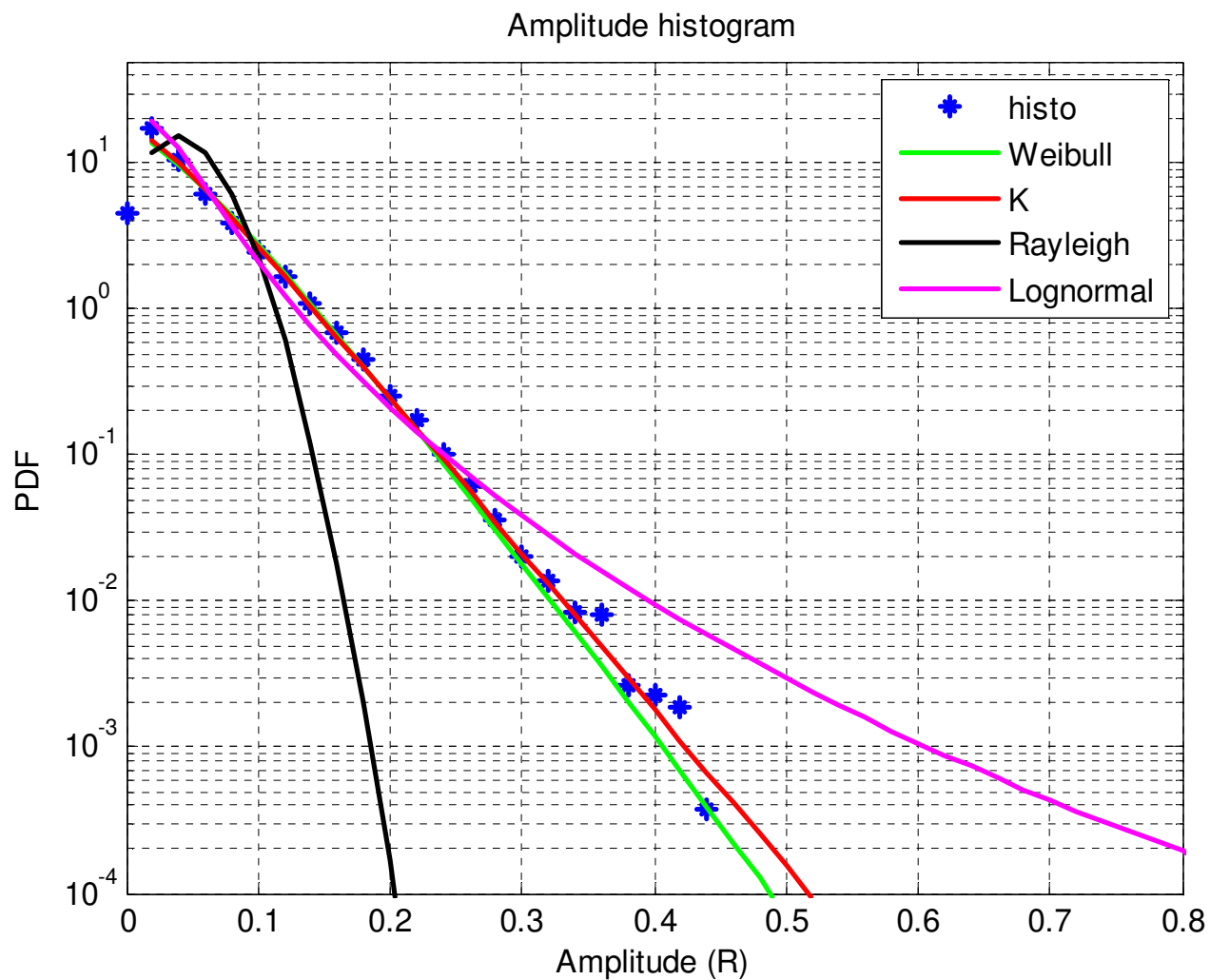
❑ The **estimation of the correlation structure**, i.e. the scatter or covariance matrix, of a dataset is another common problem in many SP applications:

- ❑ Adaptive detection in radar and sonar systems,

- ❑ DOA estimation in array processing,

- ❑ Principal Component Analysis (PCA),

- ❑ Interference cancellation,

- ❑ Portfolio optimization.

❑ Even if the data may come from disparate applications, they often share a <u>non-Gaussian</u> <u>heavy-tailed</u> statistical behavior (in radar and sonar applications, typically it is due to the high-resolution of the receiving sensor).

This is an example of **real radar clutter data** that should be modeleld by an heavy-tailed distribution, due to the presence of spikes (values of amplitude larger than what would be expected if the I&Q data were Gaussian).

spikes

Clutter amplitude : $R = \left| X_I + jX_Q \right|$

Scatter matrix estimation

❑ Estimating the scatter matrix of a set of non-Gaussian data is generally not a trivial task.

❑ The statistical characterization of non-Gaussian data requires additional parameters that generally have to be jointly estimated with the scatter matrix.

❑ The joint ML estimator of all unknown parameters often encounters computational difficulties and convergence (or even existence) issues.

❑ One could be led to use a simpler (mismatched) model.

A family of non-Gaussian distributions

❑ A popular family of non-Gaussian pdf's is the class of **Complex Elliptically Symmetric (CES)** distributions.

❑ Thanks to their flexibility, CES distributions represent a reliable data model in many areas such as radar, sonar, and communications [22].

❑ The complex Normal, Generalized Gaussian, K-distribution, complex t-distribution and all the compound-Gaussian pdf's belong to the CES class.

❑ The statistical behavior of high-resolution radar clutter can be accurately characterized by using the CES model ([24,25]).

❏ A complex N-dimensional random vector $\mathbf{x}_m$ is <u>Complex Elliptically Symmetric (CES)</u> distributed if its pdf is [22]:

$$p_X(\mathbf{x}_m) = \left|\bar{\boldsymbol{\Sigma}}\right|^{-1} g\left((\mathbf{x}_m - \boldsymbol{\gamma})^H \bar{\boldsymbol{\Sigma}}^{-1} (\mathbf{x}_m - \boldsymbol{\gamma})\right) \in CES_N\left(\boldsymbol{\gamma}, \bar{\boldsymbol{\Sigma}}, g\right)$$

❏ g is the density generator and γ the mean vector.

❏ $\bar{\Sigma}$ is the full-rank scatter matrix, that is a scaled version of the covariance matrix $\bar{\mathbf{M}}$.

❏ A typical constraint is $\mathrm{tr}\left(\bar{\boldsymbol{\Sigma}}\right) = N$. As a consequence:

$$\bar{\mathbf{M}} \triangleq E\left\{(\mathbf{x}_m - \boldsymbol{\gamma})(\mathbf{x}_m - \boldsymbol{\gamma})^H\right\} = \sigma_X^2 \bar{\boldsymbol{\Sigma}}, \qquad \sigma_X^2 \triangleq \frac{E\left\{\mathbf{x}_m^H \mathbf{x}_m\right\}}{N} = \frac{\mathrm{tr}\left(\bar{\mathbf{M}}\right)}{N}$$

❏ **Problem**: estimate the **scatter matrix** of a zero-mean CES distributed random vector in the presence of misspecified modeling, as described below.

❏ Set of M independent CES-distributed data: $\mathbf{x} = \{\mathbf{x}_m\}_{m=1}^{M} \in \mathbb{C}^N$

❏ True (unknown) data pdf: $p_X(\mathbf{x}_m) \in CES_N(\mathbf{0}, \bar{\Sigma}, g)$

❏ Assumed data pdf: $f_X(\mathbf{x}_m; \boldsymbol{\theta}) \in CES_N(\mathbf{0}, \Sigma, h_\varsigma)$

$$\boldsymbol{\theta} = \left(\text{vecs}(\Sigma)^T \quad \varsigma^T \right)^T$$

❏ The <u>misspecification is in the choice of the density generator</u> (which is parametrized by ς).

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A misspecified scenario

❑ A possible mismatched scenario in coherent radar [12]:

➢ the **true** data pdf is a **complex *t*-distribution**;

➢ the ML estimator of the scatter matrix (and of the statistical power) is derived under the **Gaussian model assumption**.

❑ The ML estimator of the scatter matrix under the Gaussian model assumption is the well-known **Sample Covariance Matrix (SCM)**.

1. Is the SCM a (misspecified) consistent estimator?

2. Is it efficient wrt the MCRB?

3. How large is its performance loss wrt the matched case?

❑ **True model**: the *heavy-tailed* complex *t*-distribution.

$$p_X\left(\mathbf{x}_m \mid \bar{\boldsymbol{\Sigma}}, \lambda, \eta\right) \triangleq \frac{1}{\pi^N |\boldsymbol{\Sigma}|} \cdot \frac{\Gamma(N+\lambda)}{\Gamma(\lambda)} \left(\frac{\lambda}{\eta}\right)^\lambda \left(\frac{\lambda}{\eta} + \mathbf{x}_m^H \bar{\boldsymbol{\Sigma}}^{-1} \mathbf{x}_m\right)^{-(N+\lambda)}$$

❑ λ and η are the <u>shape and scale parameters</u>; they should be jointly estimated with the scatter matrix.

❑ The *statistical power* of $\mathbf{x}_m$ is: $\bar{\sigma}^2 = \dfrac{\lambda}{\eta(\lambda-1)}$

❑ The true, or "matched", parameter vector:

$$\boldsymbol{\tau} \triangleq [\mathrm{vecs}(\bar{\boldsymbol{\Sigma}})^T \quad \underbrace{\lambda \quad \eta}_{\zeta}]^T, \quad \mathrm{tr}(\bar{\boldsymbol{\Sigma}}) = N$$

Misspecified scatter matrix estimation

❏ **Assumed model**: the complex Gaussian distribution.

$$f_X\left(\mathbf{x}_m \middle| \boldsymbol{\theta}\right) \triangleq f_X\left(\mathbf{x}_m \middle| \boldsymbol{\Sigma}, \sigma^2\right) = \frac{1}{\left(\pi\sigma^2\right)^N |\boldsymbol{\Sigma}|} \exp\left(-\frac{\mathbf{x}_m^H \boldsymbol{\Sigma}^{-1} \mathbf{x}_m}{\sigma^2}\right)$$

❏ The "assumed" parameter vector to be estimated is:

$$\boldsymbol{\theta} = [\mathrm{vecs}(\boldsymbol{\Sigma})^T \quad \sigma^2]^T$$

❏ under the constraint:

$$\mathbf{f}(\boldsymbol{\theta}) = \mathbf{0} \quad \Rightarrow \quad \mathrm{tr}(\boldsymbol{\Sigma}) - N = 0$$

❏ Without such (or similar) constraint, $\boldsymbol{\Sigma}$ and σ^2 cannot be jointly estimated → there is an ambiguity.

❑ The **constrained MML (CMML) estimator** is derived as:

$$\hat{\boldsymbol{\theta}}_{CMML}(\mathbf{x}) \triangleq \arg\max_{\boldsymbol{\theta} \in \Theta} \ln f_X(\mathbf{x};\boldsymbol{\theta})$$

assumed complex
Gaussian distribution

$$= \arg\max_{\boldsymbol{\theta} \in \Theta} \sum_{m=1}^{M} \ln f_X(\mathbf{x}_m;\boldsymbol{\theta}), \quad \mathbf{x}_m \sim p_X(\mathbf{x}_m)$$

true (unknown)
complex t-distribution

❑ The closed form expression is:

$$\begin{cases} \hat{\boldsymbol{\Sigma}}_{CMML} = \dfrac{N}{Tr\{\mathbf{SCM}\}}\mathbf{SCM} = \dfrac{N}{Tr\left\{\dfrac{1}{M}\sum_{m=1}^{M}\mathbf{x}_m\mathbf{x}_m^H\right\}} \cdot \dfrac{1}{M}\sum_{m=1}^{M}\mathbf{x}_m\mathbf{x}_m^H \\[3em] \hat{\sigma}^2_{CMML} = \dfrac{1}{M}\sum_{m=1}^{M}\dfrac{\mathbf{x}_m^H\hat{\boldsymbol{\Sigma}}_{CMML}^{-1}\mathbf{x}_m}{N} \end{cases}$$

❑ The **CMML estimator** under the Gaussian assumption:

$$
\begin{cases}
\hat{\boldsymbol{\Sigma}}_{CMML} = \dfrac{N}{\sum_{m=1}^{M} \mathbf{x}_m^H \mathbf{x}_m} \sum_{m=1}^{M} \mathbf{x}_m \mathbf{x}_m^H \\[3em]
\hat{\sigma}^2_{CMML} = \dfrac{1}{NM} \sum_{m=1}^{M} \mathbf{x}_m^H \hat{\boldsymbol{\Sigma}}_{CMML}^{-1} \mathbf{x}_m
\end{cases}
$$

Which is the convergence point of the CMML estimator?

$$
\hat{\boldsymbol{\theta}}_{CMML}(\mathbf{x}) \xrightarrow[M \to \infty]{a.s.} \boldsymbol{\theta}_0 = ?
$$

$$
\hat{\boldsymbol{\theta}}_{CMML} = \left[\operatorname{vecs}\left(\hat{\boldsymbol{\Sigma}}_{CMML} \right)^T \quad \hat{\sigma}^2_{CMML} \right]^T
$$

❏ The pseudo-true parameter vector is:

$$\boldsymbol{\theta}_0 = \arg\min_{\boldsymbol{\theta} \in \tilde{\Theta}} \left\{ -E_p \left\{ \ln f_X(\mathbf{x}_m; \boldsymbol{\theta}) \right\} \right\}$$

❏ For the case study at hand, it can be shown that:

$$\boldsymbol{\theta}_0 = [\operatorname{vecs}(\overline{\boldsymbol{\Sigma}})^T \quad \overline{\sigma}^2]^T \qquad \text{and} \qquad \det\left(\mathbf{A}_{\boldsymbol{\theta}_0}\right) \neq 0$$

The **CMMLE** converges **a.s.** to:

$$\hat{\sigma}^2_{CMML}(\mathbf{x}) \overset{a.s.}{\underset{M \to \infty}{\to}} \overline{\sigma}^2 = \frac{\lambda}{\eta(\lambda - 1)}, \quad \hat{\boldsymbol{\Sigma}}_{CMML}(\mathbf{x}) \overset{a.s.}{\underset{M \to \infty}{\to}} \overline{\boldsymbol{\Sigma}}$$

Hence, it provides **consistent** estimates for both the statistical power and the scatter matrix.

❑ Bias of the estimator $\hat{\sigma}^2_{CMML}$: $\quad b_{\hat{\sigma}^2_{CMML}} \triangleq E\left\{\hat{\sigma}^2_{CMML} - \bar{\sigma}^2\right\}$

Simulation parameters:

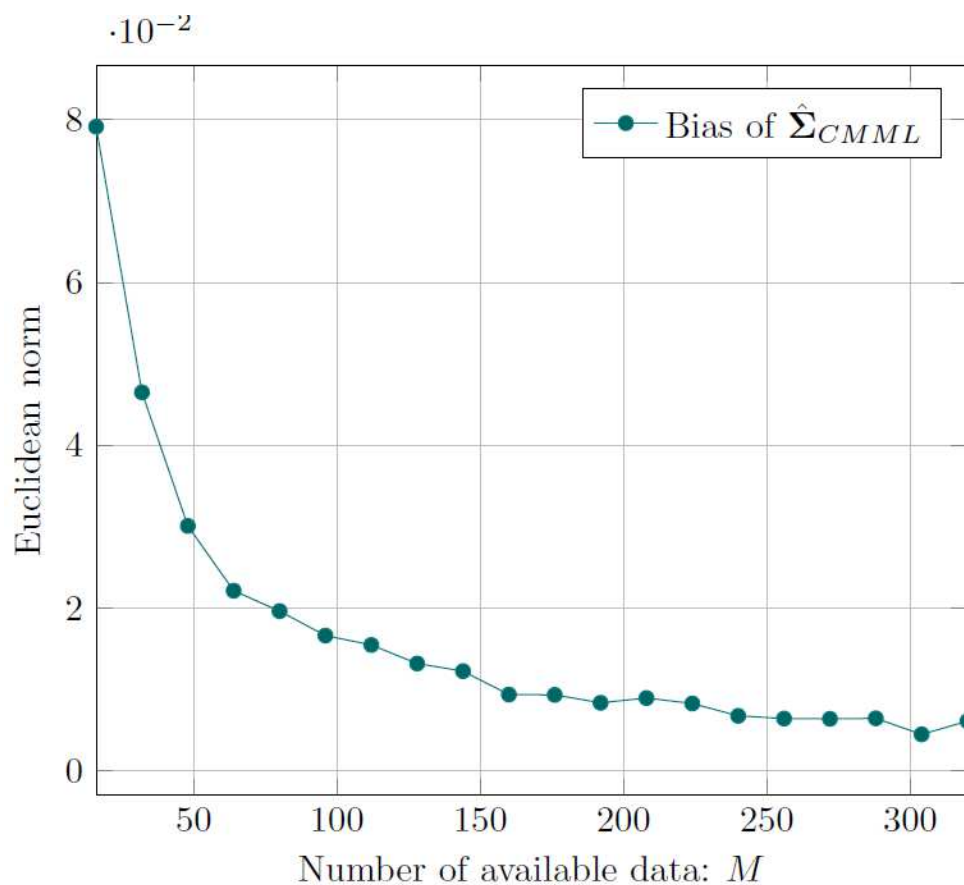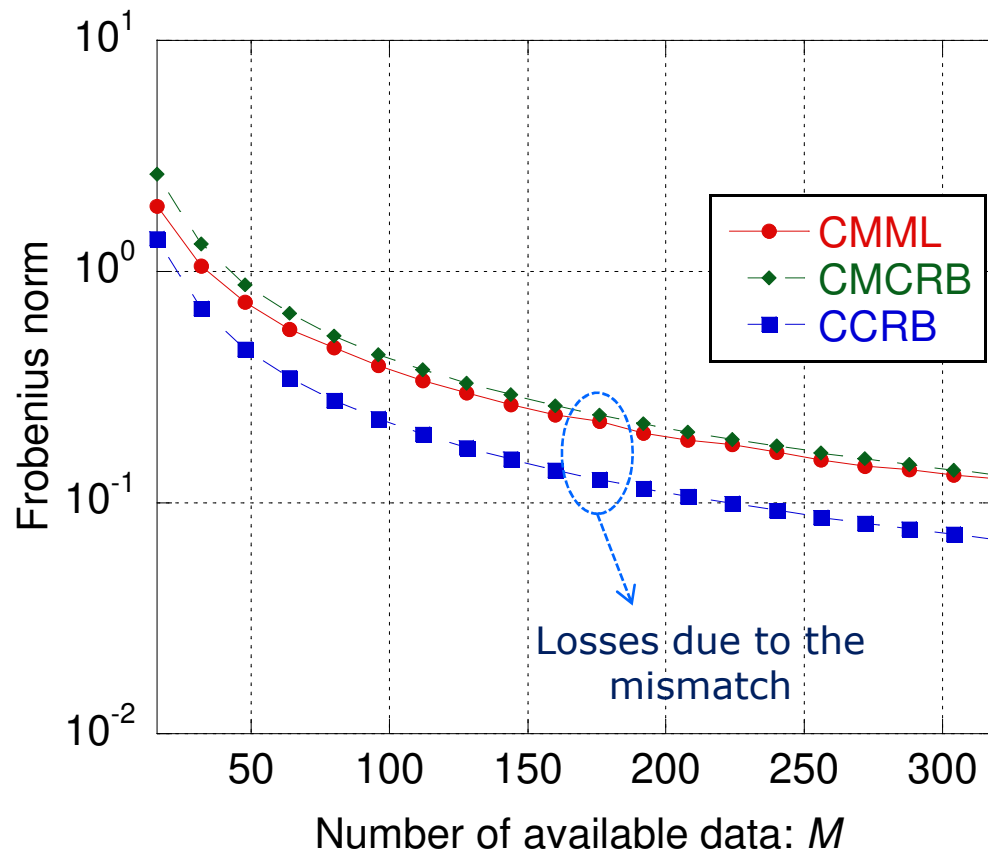$$[\bar{\mathbf{\Sigma}}]_{i,j} = \rho^{|i-j|}$$

$$\rho = 0.8$$

$$N = 16$$

$$\lambda = 3$$

$$\eta = 1$$

$$\bar{\sigma}^2 = \frac{\lambda}{\eta(\lambda-1)} = \frac{3}{2}$$

❑ Bias of the estimator $\hat{\boldsymbol{\Sigma}}_{CMML}$: $b_{\hat{\boldsymbol{\Sigma}}_{CMML}} \triangleq \left\| E\left\{ \mathrm{vecs}(\hat{\boldsymbol{\Sigma}}_{CMML} - \bar{\boldsymbol{\Sigma}}) \right\} \right\|_2$

Simulation parameters:

$$[\bar{\boldsymbol{\Sigma}}]_{i,j} = \rho^{|i-j|}$$

$$\rho = 0.8$$

$$N = 16$$

$$\lambda = 3$$

$$\eta = 1$$

$$\bar{\sigma}^2 = \frac{\lambda}{\eta(\lambda - 1)} = \frac{3}{2}$$

$$\varepsilon_{\hat{\bar{\Sigma}}_{CMML}} \triangleq \left\| E\left\{ \text{vecs}(\hat{\Sigma}_{CMML} - \bar{\Sigma})\text{vecs}(\hat{\Sigma}_{CMML} - \bar{\Sigma})^T \right\} \right\|_F$$
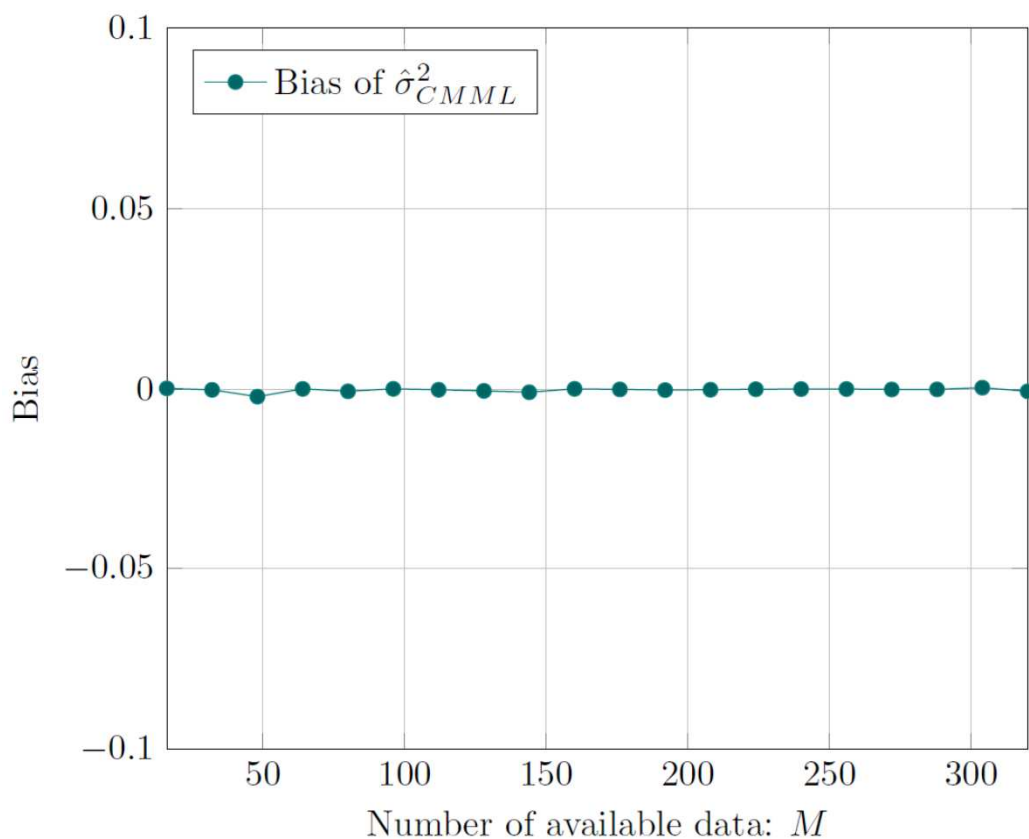
Simulation parameters:
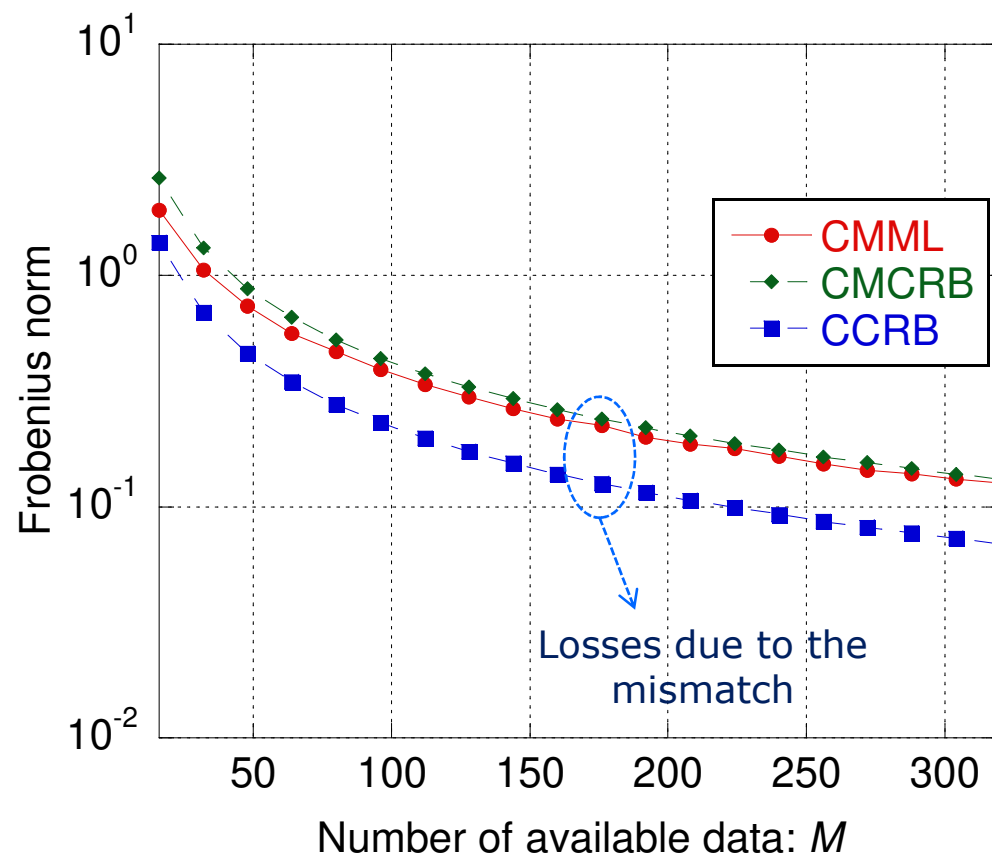
$$[\bar{\Sigma}]_{i,j} = \rho^{|i-j|}$$

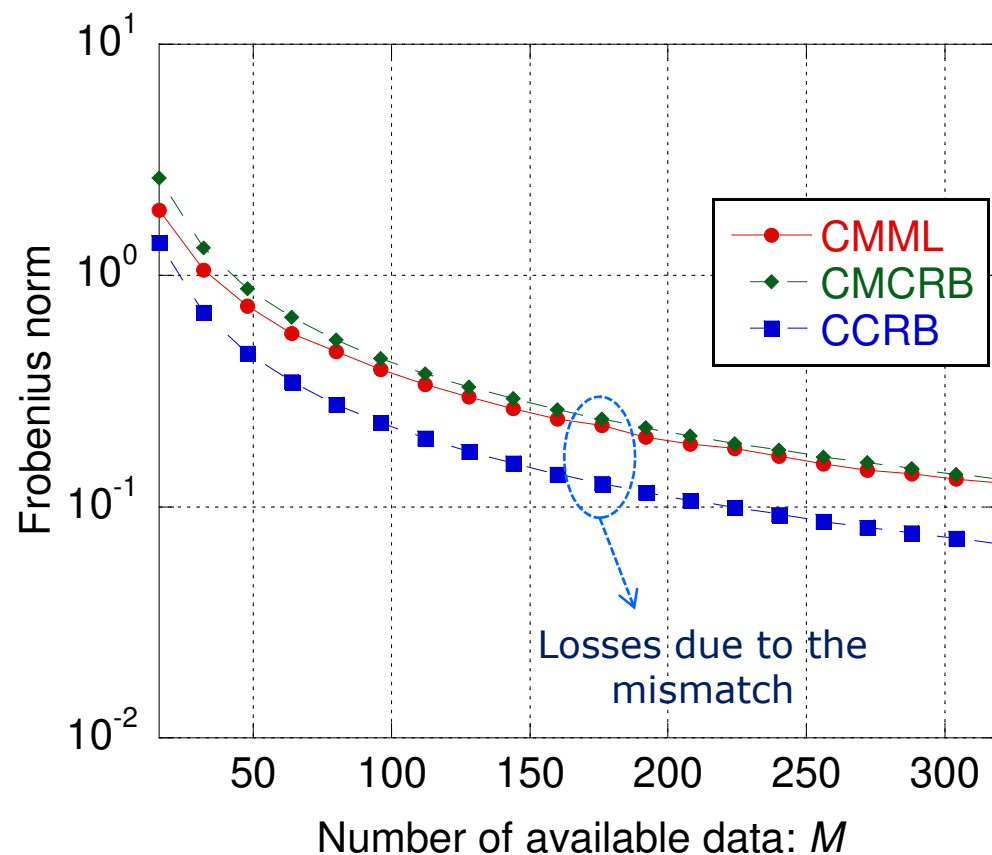$$\rho = 0.8$$

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$$\bar{\sigma}^2 = \frac{\lambda}{\eta(\lambda-1)} = \frac{3}{2}$$
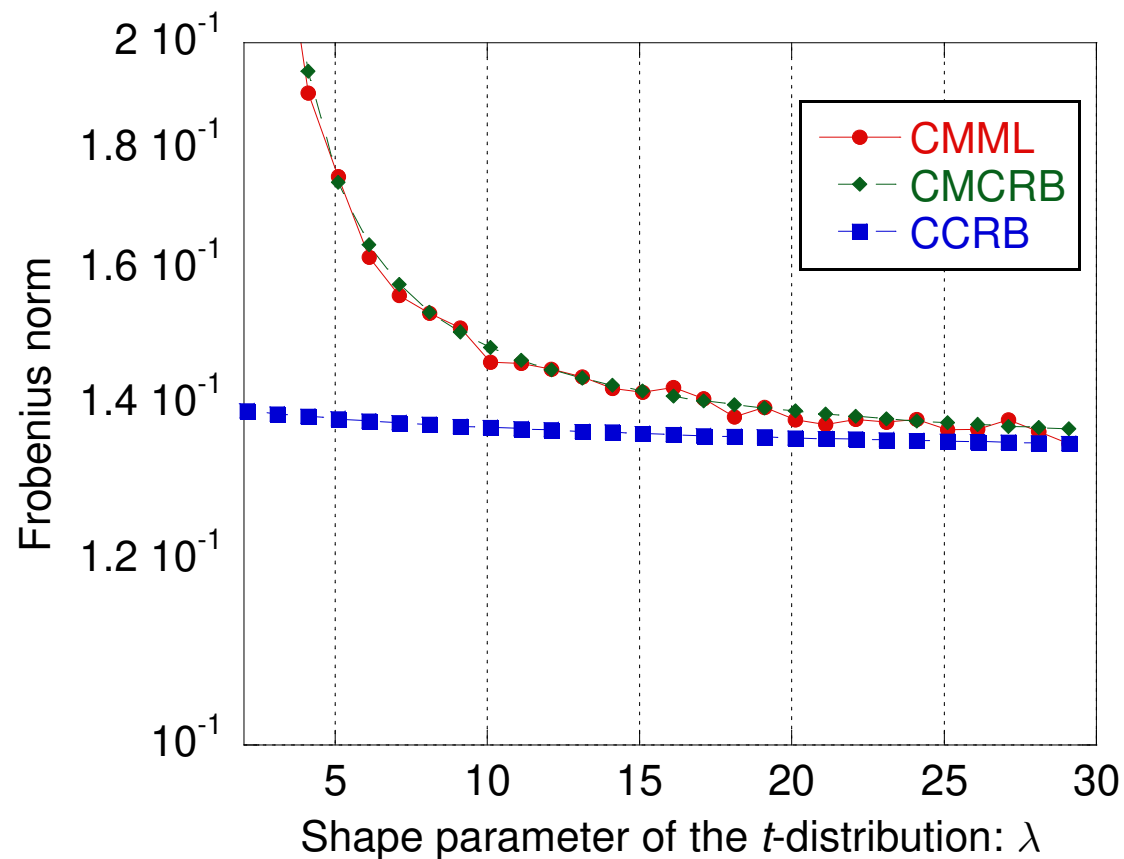
Misspecified scatter matrix estimation

□ **CCRB** is the **constrained** "matched" **CRB** [30,31,32] on the joint estimation of $\bar{\Sigma}$, λ, and η (for t-distributed data).
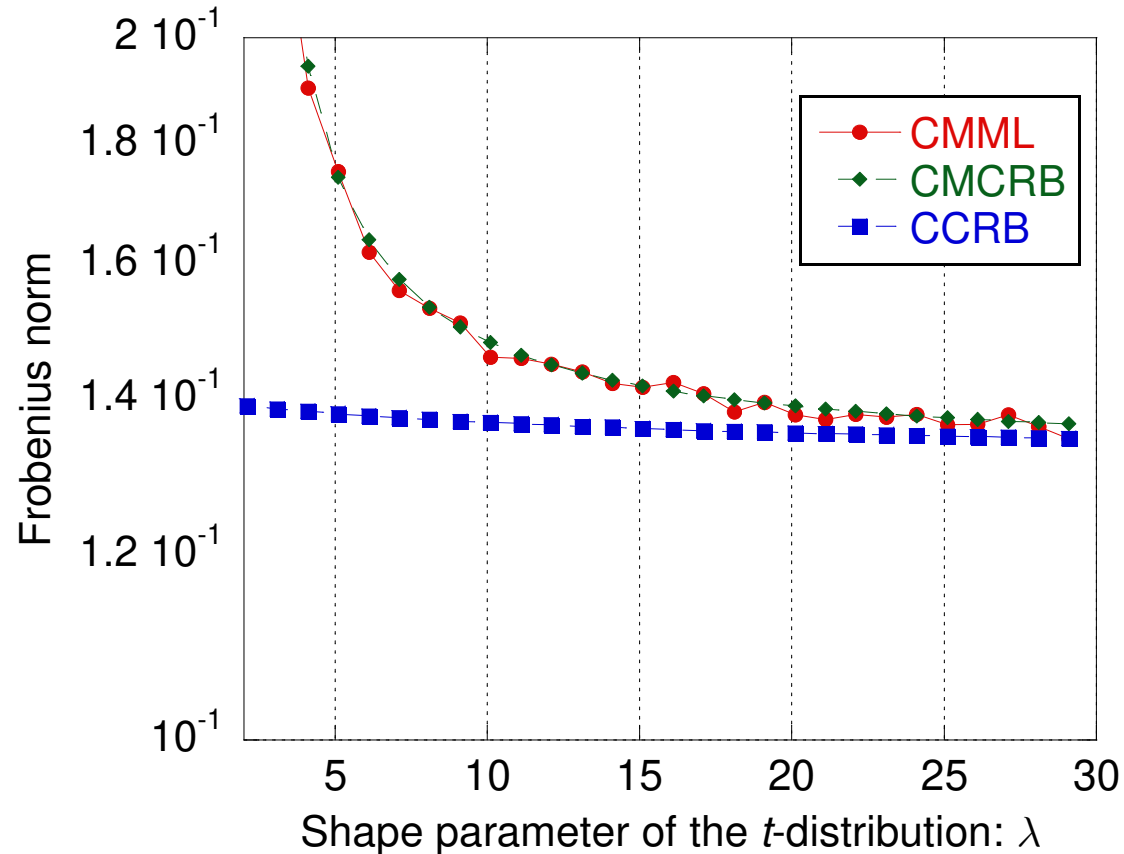
Misspecified scatter matrix estimation

□ **CMCRB** is the **constrained MCRB** [20] on the joint estimation of $\bar{\Sigma}$ and $\bar{\sigma}^2$ (under model misspecification).

Misspecified scatter matrix estimation

☐ When $\lambda \to 0$ (extremely spiky data), the estimation losses due to model mismatching rapidly increase.

Misspecified scatter matrix estimation

When $\lambda \to \infty$ the data tend to be Gaussian distributed and the MSE of the **CMMLE**, the **CMCRB**, and the **CCRB** tend to coincide.

Concluding remarks for Part I

❑ We summarized the fundamental concepts about lower bounds and efficient estimators in the presence of <u>model misspecification</u>.

❑ The MML estimator is asymptotically MS-unbiased and its error covariance matrix asymptotically equates the MCRB.

❑ We showed how to apply these theoretical findings to two well-known problems:

1. Direction of Arrivals (DOAs) estimation with an array of antennas;

2. Estimation of the the disturbance scatter matrix in complex t-distributed data.

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