Annals of Operations Research (2022) 315:1455–1476
https://doi.org/10.1007/s10479-019-03499-2

S.I.: BUSINESS ANALYTICS AND OPERATIONS RESEARCH

Optimal scale sizes in input–output allocative data envelopment analysis models

Hajar Haghighatpisheh[1] · **Sohrab Kordrostami**[1] · **Alireza Amirteimoori**[2] · **Farhad Hosseinzadeh Lotfi**[3]

Published online: 11 December 2019

Abstract
In production theory, industrial units do business in such a way that they use minimum amount of resources to produce maximum amount of products. So, inefficient units decrease their inputs level and increase their outputs level to meet the efficient frontier. By changing inputs and outputs, achieving an optimal scale size (OSS) in industrial units is one of the most important attempts and has attracted considerable attention among researchers. In this paper, an optimal scale size in input–output allocative DEA model is defined to each production firm in which the costs of inputs and the revenues of outputs are considered. We first rearrange the average-revenue efficiency measure that combines scale and output allocative efficiencies. Next, we simultaneously consider both of inputs and outputs in a new average-cost/revenue efficiency measure (ACRE). It has been shown that the proposed ACRE measure is the ratio of the profitability efficiency to ray average productivity. A numerical heuristic procedure is proposed to calculate a relatively good approximation of the new OSS in a convex and continuous technology set. To illustrate the real applicability of the proposed approach, we use a real case on 39 electricity distribution companies.

Keywords Data envelopment analysis · Scale size · Input/output · Efficiency

1 Introduction

An interesting and important issue in all of the industrial units is performance measurement using parametric or nonparametric benchmarking. A well-known and well-established tool in performance analysis in benchmarking literature is data envelopment analysis (DEA) that started from the seminal work of Charnes et al. (1978) and extended by Banker et al. (1984). DEA is a linear programming based technique that has been widely accepted as a

✉ Alireza Amirteimoori
ateimoori@iaurasht.ac.ir; aamirteimoori@gmail.com

[1] Department of Applied Mathematics, Islamic Azad University, Lahijan Branch, Lahijan, Iran

[2] Department of Applied Mathematics, Islamic Azad University, Rasht Branch, Rasht, Iran

[3] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

competing methodology to evaluate DMU's performance. In the last 3 decades, DEA is widely used to evaluate the relative efficiency of production firms, the nature of returns to scale and the productivity changes (see Cooper et al. 2004; Amirteimoori and Emrouznejad 2011; An et al. 2017; Zhu et al. 2018; Emrouznejad and Yang 2018).

The notion of scale size and returns to scale are well-organized and well-established in the framework of DEA. Discussion on these two subjects is attractive and a lot of DEA-based studies are given in the literature. In what follows, we just review some of these works.

Erbetta and Rappuoli (2008) have used the DEA technique to determine optimal scale in the Italian gas distribution industry. To estimate the nature of returns to scale, the envelopment and multiplier forms of the BCC model of Banker et al. (1984) are useful and since the sign of the dual variable corresponding to the convexity constraint in classic DEA models is used to estimate the nature of returns to scale, Ouellette et al. (2012) showed that how the presence of regulatory constraints affect on the estimation of returns to scale in such models. Sahoo et al. (2016) have studied the problem of returns to scale and most productive scale size in DEA context when the inputs and outputs data are negative. Lee (2016) has mixed DEA and multi-objective mathematical programming to set an efficient target to the DMUs that shows a trade-off between MPSS points and demand fulfilment. Podinovski (2017) has studied the notion of one-sided scale elasticities in convex technologies and he established the equivalence of local and global scale characteristics. Assani et al. (2018) extended the concept of MPSS to a multi-stage system in such a way that the MPSS of the whole system is decomposed into the individual stages.

In all of these studies, what is important is that how we can determine a good and appropriate scale size to each operational unit. To achieve an optimal scale size by reducing inputs and increasing outputs of production firms is one of the important attempts to decision makers and has attracted considerable attention among researchers. As the foregoing studies show, two important issues related to the size of the DMUs that are frequently studied in DEA, are returns to scale in qualitative form and scale elasticity in quantitative form. Returns to scale or scale elasticity permits us to scale up or scale down a DMU in order to achieve an optimal size. In the field of economics, to achieve potential cost and advantage, it is recommended to determine the optimal scale that is equivalent to determining the most productive scale size (MPSS). MPSS points present points maximizing the ratio of total outputs to total inputs and there is no guarantees that all productive scale sizes are optimal.

As Banker (1984) stated, MPSS points are those that are located on the efficient benchmark of constant returns to scale frontier. The seminal definition of MPSS provided by Banker (1984) has been given without any knowledge on the input and output prices. When we have the cost of inputs and/or the revenue of outputs, we can provide a strong definition of scale size. Cesaroni and Giovannola (2015) have introduced the average-cost efficiency measure based on the cost of inputs, under fairly general assumptions. Their measure of efficiency combines allocative and scale efficiencies and it is consistent with the expected properties of efficiency measure. By using this new efficiency measure, an optimal scale size to each DMU is introduced. The scale size provided by Cesaroni and Giovannola (2015) was cost-based and the revenue of outputs does not affect on the form of scale size. Clearly, if instead of inputs cost, we consider the revenue of outputs, an alternative one-sided scale size may be defined to each operational unit and normally, it is different from that of obtained in cost-based approach. Although, we do not claim that this is a drawback or shortcoming of the Cesaroni and Giovannola (2015) approach, however, providing an alternative OSS with minimum cost and maximum revenue may be useful. In this sense, we will show that when our knowledge on inputs and outputs are complete (both of inputs cost and outputs price are known), we can define a more powerful scale size to each decision making unit. In this paper, we first extend the average-cost efficiency measure to the case of revenue maximizing and an alternative OSS is defined to each operational unit. In the following, we take both, cost of inputs and revenue of outputs, into consideration and a new average-cost/revenue efficiency (ACRE) measure is defined that combines scale and input/output allocative efficiencies. We will show that the new ACRE measure is the ratio of the profitability efficiency to ray average productivity. As we will show, the complexity of all of the proposed models depends on the structure of the underlying technology set. In the following, a numerical heuristic procedure is proposed to calculate an estimation of the new OSS in a convex and continuous technology set such as variable returns to scale technology. To illustrate the real applicability of the proposed approach, we use a real case on electricity distribution companies.

The rest of paper is organized as follows: Sect. 2 presents definitions of the average cost efficiency (ACE) of Cesaroni and Giovannola (2015) and we will rearrange it to average revenue efficiency (ARE). In Sect. 3, an alternative definition of OSS is given in cost and revenue orientations. A real application on 39 electricity distribution companies are given in Sect. 4. Conclusions appear in Sect. 5.

2 Average-cost and average-revenue efficiencies

In this section we review the average-cost efficiency of Cesaroni and Giovannola (2015) and we will rearrange it to revenue orientation when we have the prices of outputs instead of cost of inputs. Suppose we have n DMUs and each DMU consumes m inputs to produce s outputs. In particular, $DMU_k: \quad k = 1, \dots, n$ consumes inputs $x_k = (x_{1k}, \dots, x_{mk}) \geq 0$ to produce outputs $y_k = (y_{1k}, \dots, y_{sk}) \geq 0$.

The set of all feasible input/output vectors is represented by $T = \{$*(x, y): x can produce y*$\}$. To represent the algebraic formulation of T, we shall assume axioms such as free disposability of inputs and outputs, convexity, minimal set, variable returns to scale and inclusion. By assuming these axioms, the mathematical formulation of T_v in variable returns to scale case is as follows:

$$T_v = \left\{ (x, y) : \ x \geq \sum_{k=1}^{n} \lambda_k x_k, \ y \leq \sum_{k=1}^{n} \lambda_k y_k, \ \sum_{k=1}^{n} \lambda_k = 1 \ , \ \ \lambda_k \geq 0 \ , \ \ k = 1, \dots, n \right\} \quad (1)$$

in which $\lambda_k \geq 0 \ , \ \ k = 1, \ \dots, \ n$ are intensity weights for constructing convex combination of the DMUs. As Baumol (1977) stated, the ray average cost (RAC) of DMU_j is defined as follows:

$$RAC(y_j) = \frac{C(ty_j)}{t} \qquad t > 0, \quad t \neq 1 \quad (2)$$

in which C(y) is the total cost required to produce output y. The average cost efficiency of DMU_j is now defined as:

$$R_j^{(C)} = \frac{p\,\bar{x}}{p\,x_j} \gamma_j \quad (3)$$

in which $\gamma_j = Max_{1\le r\le s}\left\{\frac{y_{rj}}{\bar{y}_r}\right\} \in (0,\infty]$, $(\bar{x},\ \bar{y})$ is an arbitrary reference point and $p=(p_1,\dots,p_m)\ge 0$ is the inputs price.

Cesaroni and Giovannola (2015) have given their definition of optimal scale size (OSS) as follow:

Definition 1 For any given DMU_j, an OSS is a production possibility $(\bar{x},\bar{y})\in T$ that minimizes $R_j^{(C)}$.

In practice, an OSS is a solution to the following mathematical programming problem:

$$\begin{aligned} &Min\ R_j^{(C)} \\ &s.t.\ \ (\bar{x},\bar{y})\in T \end{aligned} \tag{4}$$

It should be noted that depending on the structure of the technology under consideration, $(\bar{x},\ \bar{y})$ is determined. For example, in a FDH technology, $(\bar{x},\ \bar{y})$ is one of the observed production possibilities and in a convex and continuous technology, $(\bar{x},\ \bar{y})$ is a frontier point that is not necessarily an observed production unit. So, the complexity of model (4) depends on the structure of the underlying technology T. In a FDH technology set, it is not hard to solve model (4). However, in a convex and continuous technology such as T_v, model (4) becomes a nonlinear programming problem. In what follows, we propose a procedure to solve model (4) when the underlying technology is T_v. Consider model (4) in the following form:

$$\begin{aligned} &Min\ R_j^{(C)} = \frac{p\,\bar{x}}{p\,x_j}\ \underset{1\le r\le s}{Max}\left\{\frac{y_{rj}}{\bar{y}_r}\right\} \\ &s.t.\ \ (\bar{x},\bar{y})\in T_v \end{aligned} \tag{5}$$

Note that $R_j^{(C)}$ consists of two terms: $\frac{p\,\bar{x}}{p\,x_j}$ and $Max_{1\le r\le s}\left\{\frac{y_{rj}}{\bar{y}_r}\right\}$. In what follows, a leader–follower procedure is given to solve this model. We assume that the cost minimization is leader and the second term in the objective function of model (5) is follower. So, at the first step, the first term $\frac{p\,\bar{x}}{p\,x_j}$ is minimized. In this case, we solve the following linear model:

$$\begin{aligned} &CE_j^{(*)} = Min\frac{\sum_{i=1}^m p_i\bar{x}_i}{\sum_{i=1}^m p_i x_{ij}} \\ &s.t. \\ &\quad \sum_{k=1}^n \lambda_k x_{ik} \le \bar{x}_i, \qquad i=1,\dots m, \\ &\quad \sum_{k=1}^n \lambda_k y_{rk} \ge y_{rj}, \qquad r=1,\dots s, \\ &\quad \sum_{k=1}^n \lambda_k = 1, \\ &\quad \lambda_k,\ \bar{x}_i \ge 0,\ \ for\ all\ i\ and\ k. \end{aligned} \tag{6}$$

Suppose $CE_j^{(*)}$ is the optimal objective value to model (6). Now, we proceed to the second step by considering the following program:

$$
\begin{aligned}
&R_j^{(C)} = Min\left\{ Max\left\{ \frac{y_{rj}}{\bar{y}_r} \ : \ r = 1, \ldots, s \right\} \right\} \\
&s.t. \\
&\quad \sum_{k=1}^{n} \lambda_k x_{ik} = \bar{x}_i, \qquad i = 1, \ldots, m, \\
&\quad \sum_{k=1}^{n} \lambda_k y_{rk} = \bar{y}_r, \qquad r = 1, \ldots, s, \\
&\quad CE_j^{(*)} = \frac{\sum_{i=1}^{m} p_i \bar{x}_i}{\sum_{i=1}^{m} p_i x_{ij}}, \\
&\quad \sum_{k=1}^{n} \lambda_k = 1 \\
&\quad \lambda_k, \ \bar{x}_i, \ \bar{y}_r \geq 0, \ \ for \ all \ i, \ r \ and \ k.
\end{aligned}
\tag{7}
$$

Note that the third constraint in model (7) is given to preserve the optimality of step 1. Model (7) is not a linear programming problem. It is easy to see that $Max\left\{ \frac{y_{rj}}{\bar{y}_r} \ : \ r = 1, \ldots, s \right\} \neq 0$.

Suppose $Max\left\{ \frac{y_{rj}}{\bar{y}_r} \ : \ r = 1, \ldots, s \right\} = \frac{1}{Min\left\{ \frac{\bar{y}_r}{y_{rj}} \ : \ r=1,\ldots,s \right\}}$ and set $Min\left\{ \frac{\bar{y}_r}{y_{rj}} \ : \ r = 1, \ldots, s \right\} = \phi$. In this case, we must have $\phi \leq \frac{\bar{y}_r}{y_{rj}} \ ; \ \ r = 1, \ldots, s$ or equivalently $\phi y_{rj} \leq \bar{y}_r \ ; \ \ r = 1, \ldots, s$. Hence, model (7) can be re-stated in the following linear form:

$$
\begin{aligned}
&\phi^* = Max \ \phi \\
&s.t. \\
&\quad \phi y_{rj} \leq \bar{y}_r, \qquad r = 1, \ldots, s \\
&\quad \sum_{k=1}^{n} \lambda_k x_{ik} = \bar{x}_i, \qquad i = 1, \ldots, m \\
&\quad \sum_{k=1}^{n} \lambda_k y_{rk} = \bar{y}_r, \qquad r = 1, \ldots, s \\
&\quad CE_j^{*} = \frac{\sum_{i=1}^{m} p_i \bar{x}_i}{\sum_{i=1}^{m} p_i x_{ij}}, \\
&\quad \sum_{k=1}^{n} \lambda_k = 1 \\
&\quad \lambda_k, \ \bar{x}_i, \ \bar{y}_r \geq 0, \ for \ all \ i, r \ and \ k.
\end{aligned}
\tag{8}
$$

It is easy to show that at optimality of model (8), one of the first s constraints is binding and in this sense, $\phi^* = Min\left\{ \frac{\bar{y}_r^*}{y_{rj}} \ : \ r = 1, \ldots, s \right\}$. The OSS is now the production possibility $(\bar{x}^*, \bar{y}^*)$ as the optimal solution of model (8) and $R_j^{(C)} = \phi^* * CE_j^{(*)}$.

The average cost efficiency (ACE) measure of Cesaroni and Giovannola (2015) was cost-oriented and in what follows, we redefine an OSS considering the revenue of the outputs. Suppose *R(x)* is the total revenue of consuming input *x*. The ray average revenue of DMU_j is defined by

$$RAR\ (x_j) = \frac{R(tx_j)}{t}, \quad t > 0,\ t \neq 1 \tag{9}$$

Let $\rho_j = Min_{1 \le i \le m}\left\{ \frac{x_{ij}}{\bar{x}_i} \right\} \in [0,\ \infty)$ and suppose $q = (q_1, \ldots, q_s) \ge 0$ is the outputs price. The average revenue efficiency of DMU_j (ARE_j) is defined as

$$R_j^{(R)} = \frac{q\bar{y}}{qy_j} \cdot \rho_j \tag{10}$$

Note that ρ_j is the radial scaling factor obtained by ratio between the input of DMU_j and that of reference unit. Now, we redefine the optimal scale size (OSS) in revenue orientation as follows:

Definition 2 For any given DMU_j, an OSS is a production possibility $(\bar{x}, \bar{y}) \in T$ that maximizes $R_j^{(R)}$.

In other words, an OSS is a solution to the following mathematical programming problem:

$$\begin{aligned} & Max\, R_j^{(R)} \\ & s.t.\ (\bar{x}, \bar{y}) \in T \end{aligned} \tag{11}$$

It should be noted that model (11) is feasible and bounded. Again, to determine an OSS in T_v, the following two-step procedure is proposed when revenue maximization is leader.

Step 1 Solve the following model:

$$\begin{aligned} & RE_j^{(*)} = Max \frac{\sum_{r=1}^{s} q_r \bar{y}_r}{\sum_{r=1}^{s} q_r y_{rj}} \\ & s.t. \\ & \quad \sum_{k=1}^{n} \lambda_k x_{ik} \le x_{ij}, \qquad i = 1, \ldots, m, \\ & \quad \sum_{k=1}^{n} \lambda_k y_{rk} \ge \bar{y}_r, \qquad r = 1, \ldots, s, \\ & \quad \sum_{k=1}^{n} \lambda_k = 1, \\ & \quad \lambda_k,\ \bar{y}_r \ge 0, \ \ for\ all\ r\ and\ k. \end{aligned} \tag{12}$$

Step 2 Solve the following model:

$$\begin{aligned} & Max\ \rho_j \\ & s.t. \\ & \quad 0 \le (\bar{x},\ \bar{y}) \in T_v \\ & \quad RE_j^{(*)} = \frac{\sum_{r=1}^{s} q_r \bar{y}_r}{\sum_{r=1}^{s} q_r y_{rj}}, \end{aligned} \tag{13}$$

Model (13) can be restated in the following linear form:

$$\begin{aligned} & Min\ \pi \\ & s.t. \\ & \quad \pi x_{ij} \ge \bar{x}_i, \qquad i = 1, \dots, m \\ & \quad \sum_{k=1}^{n} \lambda_k x_{ik} \le \bar{x}_i, \qquad i = 1, \dots, m \\ & \quad \sum_{k=1}^{n} \lambda_k y_{rk} \ge \bar{y}_r, \qquad r = 1, \dots, s \\ & \quad RE_j^{(*)} = \frac{\sum_{r=1}^{s} q_r \bar{y}_r}{\sum_{r=1}^{s} q_r y_{rj}}, \\ & \quad \sum_{k=1}^{n} \lambda_k = 1, \\ & \quad \lambda_k,\ \bar{x}_i, \bar{y}_r \ge 0,\ for\ all\ i,\ r\ and\ k. \end{aligned} \tag{14}$$

It is easy to show that at optimality of model (14), one of the first m constraints is binding and in this sense, $\pi^* = Max\left\{ \frac{\bar{x}_i^*}{x_{ij}} \ : \quad i = 1, \dots, m \right\}$. The OSS is now the production possibility $(\bar{x}^*, \bar{y}^*)$ as the optimal solution of model (14) and $RAR_j = \pi^* * RE_j^{(*)}$.

Clearly, the above two different viewpoints lead to two different OSS. If we focus on cost minimization or revenue maximization, two different OSS is determined to each DMU_j. These different results in determining OSS show that there is a need to provide an alternative definition of OSS by taking both, cost minimization and revenue maximization into consideration, simultaneously. It should be pointed out that although we can mix the cost and revenue efficiencies in profit efficiency, but, our mix average-cost/revenue efficiency (ACRE) is different from one that can be extended by taking profit efficiency into account.

3 Average cost/revenue efficiency (ACRE)

So far, we have focused on efficiency analysis with less information, either we know inputs costs or we know the outputs prices, and in this sense, two different definitions of OSS are given, one was cost-based and another one was revenue-based and clearly we should not expect these two orientations lead to same OSS. We believe that this is not a drawback or shortcoming of the Cesaroni and Giovannola (2015) approach, however, providing an alternative OSS in cost and revenue orientations may be useful. In some real situations, we know a prior the prices that can be assigned to both of the inputs and outputs. This permits us to make a more powerful evaluation. In this case, another scale size is defined by taking both, inputs costs and outputs prices, simultaneously. To define an expression of the average cost/revenue efficiency (ACRE) measure, we use both γ_j and ρ_j, defined in (3) and (10) and a new expression to ACRE measure is defined as follow:

$$R_j^{(CR)} = \frac{R_j^{(R)}}{R_j^{(C)}} = \frac{\left[\frac{q\bar{y}}{qy_j} \times Min_{1\leq i\leq m}\left\{\frac{x_{ij}}{\bar{x}_i}\right\}\right]}{\left[\frac{p\bar{x}}{px_j} \times Max_{1\leq r\leq s}\left\{\frac{y_{rj}}{\bar{y}_r}\right\}\right]} \tag{15}$$

$R_j^{(CR)}$ can be interpreted as the product of two gains of DMU_j when its scale size is changed to $(\bar{x},\ \bar{y})$: a gain in average cost and another one in average revenue. So, in defining ACRE ratio, we took, both, cost of inputs and revenue of outputs, into consideration. Proceeding the manner of Cesaroni and Giovannola (2015), we can see that

$$R_j^{(CR)} = \frac{\left[\frac{1}{\gamma_j} + \frac{q\,s_j^+}{q\,y_j}\right]\rho_j}{\left[\frac{1}{\rho_j} - \frac{p\,s_j^-}{p\,x_j}\right]\gamma_j} \tag{16}$$

The numerator in (16) is the sum of two terms: the inverse of ray average productivity of DMU_j and the ratio of the revenue of output slacks multiplied by ρ_j. Moreover, the denominator is the difference of ray average productivity of DMU_j and the cost of input slacks multiplied by γ_j. So, $R_j^{(CR)}$ is the ratio of these two terms. This means that if DMU_j is projected onto its OSS candidate, reallocations of inputs and outputs are needed: the input bundle and output bundle of DMU_j must be projected onto that of OSS candidate. Now, we look at the definition of $R_j^{(CR)}$ more precisely. We can see that the profitability of DMU_j is appeared in a part of this definition as $PR_j = \frac{\frac{q\bar{y}}{p\bar{x}}}{\frac{qy_j}{px_j}}$. Moreover, by letting $\pi_j = \frac{Max_{1\leq r\leq s}\left\{\frac{y_{rj}}{\bar{y}_r}\right\}}{Min_{1\leq i\leq m}\left\{\frac{x_{ij}}{\bar{x}_i}\right\}} > 0$, it is easy to see that π_j is the ray average productivity of DMU_j as defined by Cesaroni and Giovannola (2015). So, the average cost/revenue efficiency measure $R_j^{(CR)}$ is the ratio of profitability efficiency to ray average productivity of DMU_j.

Now, we define our new definition of OSS to a given DMU_j as follow:

Definition 3 Consider a given DMU_j. An OSS is a production possibility $(\bar{x},\ \bar{y})$ that maximizes $R_j^{(CR)}$.

Theorem 1 *An OSS is a MPSS.*

Proof Suppose $D\bar{M}U_j : (\bar{x}^*,\ \bar{y}^*)$ is an OSS to DMU_j. It suffices to show that $D\bar{M}U_j : (\bar{x}^*,\ \bar{y}^*)$ is non-dominated in T_c (the technology with constant returns to scale). Suppose in contrary that $(\bar{x}^*,\ \bar{y}^*)$ is dominated by some $(\widehat{x},\ \widehat{y}) \in T_c$. So, we must have $(-\widehat{x},\ \widehat{y}) \geq (-\bar{x}^*,\ \bar{y}^*)$ and $(-\widehat{x},\ \widehat{y}) \neq (-\bar{x}^*,\ \bar{y}^*)$. This is a contradiction because $(\bar{x}^*,\ \bar{y}^*)$ is optimal. This completes the proof. □

Based on the Definition 3, an OSS of a specific DMU_j is the production possibility $(\bar{x},\ \bar{y})$ as a solution to the following program:

$$\begin{aligned} &R_j^{(*)(CR)} = Max R_j^{(CR)} \\ &s.t. \quad (\bar{x}, \bar{y}) \in T \end{aligned} \tag{17}$$

As we have shown in Theorem 1, our proposed OSS is a MPSS and in this sense, it maximizes the average productivity. Moreover, the new OSS maximizes the profitability of the DMU under consideration (it should be pointed out that the profitability efficiency measure differs from profit efficiency measure that the latter is the difference between revenue and cost). This definition to OSS maximizes the average cost/revenue efficiency measure and this optimizes the cost of inputs and revenue of outputs, simultaneously. In other words, if a given DMU be in OSS, it produces the most profitable products by using the lowest cost resources.

Again, the complexity of model (17) depends on the structure of the underlying technology T. In a convex and continuous technology set T_c or T_v, model (17) becomes a nonlinear programming problem. If the profitability is leader, an approach similar to the one that proposed in the previous section can be put forward. However, we assume that there is no preference and in this case, to derive the optimal solution to model (17) on T_v, we have to solve this model. However, instead of determining the exact solution to model (17), we may be satisfied with an estimated solution that is sufficiently close to the exact solution. In what follows, we propose a numerical heuristic procedure to obtain an estimation to the solution of model (17). Consider model (17) in the following form:

$$
\begin{aligned}
& R_j^{(*)(CR)} = Max \ \frac{\left[\frac{q\bar{y}}{qy_j} \times Min_{1\le i\le m}\left\{\frac{x_{ij}}{\bar{x}_i}\right\}\right]}{\left[\frac{p\bar{x}}{px_j} \times Max_{1\le r\le s}\left\{\frac{y_{rj}}{\bar{y}_r}\right\}\right]} \\
& s.t. \ \ (\bar{x},\ \bar{y}) \in T_v
\end{aligned} \tag{18}
$$

Note that $R_j^{(CR)}$ is a fractional term with $R_j^{(R)}$ in numerator and $R_j^{(C)}$ in denominator. In another look, $R_j^{(CR)}$ consists of the profitability of DMU_j and two radial scaling factors, one for inputs and another one for outputs. To avoid the nonlinearity of model (18), at the first step, the profitability of DMU_j is maximized as follows:

$$
\begin{aligned}
& PR_j^{Max} = Max \ \frac{\left(\frac{\sum_{r=1}^{s} q_r\tilde{y}_r}{\sum_{i=1}^{m} p_i\tilde{x}_i}\right)}{\left(\frac{\sum_{r=1}^{s} q_r y_{rj}}{\sum_{i=1}^{m} p_i x_{ij}}\right)} \\
& s.t. \quad \sum_{k=1}^{n} \lambda_k x_{ik} \le \tilde{x}_i \ , \quad i = 1, \dots, m, \\
& \qquad \sum_{k=1}^{n} \lambda_k y_{rk} \ge \tilde{y}_r \ , \quad r = 1, \dots, s, \\
& \qquad \sum_{k=1}^{n} \lambda_k = 1, \\
& \qquad \lambda_k,\ \tilde{x}_i,\ \tilde{y}_r \ge 0,\ \ for\ all\ k,\ i,\ r.
\end{aligned} \tag{19}
$$

In the objective function of model (19), the profitability of DMU_j is maximized. Clearly, this model is a linear fractional programming problem. It can easily be transformed into a linear form by using the method of Charnes and Cooper (1962) as follows:

$$
\begin{aligned}
PR_j^{Max} = Max\ & \frac{\sum_{r=1}^{s} q_r \bar{y}_r}{\left(\frac{\sum_{r=1}^{s} q_r y_{rj}}{\sum_{i=1}^{m} p_i x_{ij}}\right)} \\
s.t.\quad & \sum_{i=1}^{m} p_i \bar{x}_i = 1 \\
& \sum_{k=1}^{n} \bar{\lambda}_k x_{ik} \le \bar{x}_i\ ,\ i = 1, \ldots m, \\
& \sum_{k=1}^{n} \bar{\lambda}_k y_{rk} \ge \bar{y}_r\ ,\ r = 1, \ldots, s, \\
& \sum_{k=1}^{n} \bar{\lambda}_k = T, \\
& \bar{\lambda}_k,\ \bar{x}_i,\ \bar{y}_r \ge 0,\ \ for\ all\ k,\ i,\ r.
\end{aligned}
\tag{20}
$$

in which $\sum_{i=1}^{m} c_i \tilde{x}_i = \frac{1}{T}$, $T\tilde{x}_i = \bar{x}_i$, $T\tilde{y}_r = \bar{y}_r$, $T\lambda_k = \bar{\lambda}_k$. Suppose PR_j^{Max} is the optimal objective value of model (20). Clearly, the optimal value of $PR_j^{(*)}$ to model (18) must satisfy $PR_j^{(*)} \in (0,\ PR_j^{Max}]$. A sequence of linear models are solved to achieve a good approximation to $PR_j^{(*)}$ and at the same time, a good approximation to $R_j^{(*)(CR)}$ is derived. To do this, we proceed to the second step by solving a sequence of linear programming problem as the following form:

$$
\begin{aligned}
R_j^{(*)(CR)} \approx Max\ & (PR_j^{Max} - k\sigma) \frac{Min_{1 \le i \le m} \left\{ \frac{x_{ij}}{\bar{x}_i} \right\}}{Max_{1 \le r \le s} \left\{ \frac{y_{rj}}{\bar{y}_r} \right\}} \\
s.t. \qquad & \frac{\sum_{r=1}^{s} q_r \bar{y}_r}{\sum_{i=1}^{m} p_i \bar{x}_i} = PR_j^{Max} - k\sigma, \\
& \sum_{k=1}^{n} \lambda_k x_{ik} = \bar{x}_i, \quad i = 1, \ldots m \\
& \sum_{k=1}^{n} \lambda_k y_{rk} = \bar{y}_r, \quad r = 1, \ldots s \\
& \sum_{k=1}^{n} \lambda_k = 1 \\
& \lambda_k,\ \bar{x}_i,\ \bar{y}_r \ge 0, \quad for\ \ all\ \ i,\ k,\ r.
\end{aligned}
\tag{21}
$$

σ is a user-defined value to show the step size. To solve the above mathematical fractional problem, suppose

$$0 \neq Min\left\{\frac{x_{ij}}{\bar{x}_i} \ : \ i = 1, \ldots, m\right\} = \frac{1}{Max\left\{\frac{\bar{x}_i}{x_{ij}} \ : \ i = 1, \ldots, m\right\}}$$
$$\Rightarrow \quad \alpha = Max\left\{\frac{\bar{x}_i}{x_{ij}} \ : \ i = 1, \ldots, m\right\} \Rightarrow \alpha \geq \frac{\bar{x}_i}{x_{ij}} \ : \ i = 1, \ldots, m,$$
$$0 \neq Max\left\{\frac{y_{rj}}{\bar{y}_r} \ : \ r = 1, \ldots, s\right\} = \frac{1}{Min\left\{\frac{\bar{y}_r}{y_{rj}} \ : \ r = 1, \ldots, s\right\}}$$
$$\Rightarrow \quad \mu = Min\left\{\frac{\bar{y}_r}{y_{rj}} \ : \ r = 1, \ldots, s\right\} \Rightarrow \mu \leq \frac{\bar{y}_r}{y_{rj}} \ : \ r = 1, \ldots, s$$

In this case, model (21) can be re-stated in the following form:

$$\begin{aligned}
& R_j^{(*)(CR)} \approx Max\ (PR_j^{Max} - k\sigma)\frac{\mu}{\alpha} \\
& s.t. \quad \alpha \geq \frac{\bar{x}_i}{x_{ij}} \ : \ i = 1, \ldots, m, \\
& \qquad \mu \leq \frac{\bar{y}_r}{y_{rj}} \ : \ r = 1, \ldots, s, \\
& \qquad \frac{\sum_{r=1}^{s} q_r \bar{y}_r}{\sum_{i=1}^{m} p_i \bar{x}_i} = PR_j^{Max} - k\sigma, \\
& \qquad \sum_{k=1}^{n} \lambda_k x_{ik} = \bar{x}_i \qquad i = 1, \ldots m, \\
& \qquad \sum_{k=1}^{n} \lambda_k y_{rk} = \bar{y}_r \qquad r = 1, \ldots s, \\
& \qquad \sum_{k=1}^{n} \lambda_k = 1, \\
& \qquad \lambda_k,\ \bar{y}_r,\ \bar{x}_i \geq 0,\ for\ all\ i,\ k,\ r.
\end{aligned} \tag{22}$$

Using the method of Charnes and Cooper (1962), model (22) can be transformed into the following linear form:

$$
\begin{aligned}
R_j^{(*)(CR)} \approx & \; (PR_j^{Max} - k\sigma) Max \; \bar{\mu} \\
s.t. \quad & 1 \geq \frac{\bar{x}_i}{x_{ij}} \; : \; i = 1, \ldots, m, \\
& \bar{\mu} \leq \frac{\bar{y}_r}{y_{rj}} \; : \; r = 1, \ldots, s, \\
& \frac{\sum_{r=1}^{s} q_r \bar{y}_r}{\sum_{i=1}^{m} p_i \bar{x}_i} = PR_j^{Max} - k\sigma, \\
& \sum_{k=1}^{n} \lambda_k x_{ik} = \bar{x}_i \qquad i = 1, \ldots, m \\
& \sum_{k=1}^{n} \lambda_k y_{rk} = \bar{y}_r \qquad r = 1, \ldots, s \\
& \sum_{k=1}^{n} \lambda_k = T \\
& T, \; \lambda_k, \; \bar{y}_r, \; \bar{x}_i \geq 0, \; for \; all \; i, \; k, \; r.
\end{aligned}
\tag{23}
$$

Model (23) is solved for each $k = 0, 1, 2, \ldots$. Clearly, the global optimal solution of model (17) is $R_j^{(*)(CR)} \approx Max_{1 \leq k \leq n} \left\{ R_j^{(k)(CR)} \right\}$.

4 A real application on Iranian electricity distribution company

In this section, we demonstrate the real applicability of the approach proposed in this paper to a real case on Iranian electricity distribution companies. These companies do business as branches of the TAVANIR Company (Iran Power Generation, Transmission and Distribution Management Company). TAVANIR as a central decision maker has 39 branches in different parts of Iran and in this paper each company is considered as an independent DMU. Data on these 39 companies is derived from operations during 2015. A lot of input and output variables are selected and studied in the first draft but after calculating the Pearson correlation coefficients of the variables, we concluded that six variables are the most important variables and we cannot ignore these six variables. So, to evaluate the relative performances of these companies, we have used six variables from data set as inputs and outputs as follows:

Input variables Transformer capacity (MVA) (x_1), low and medium voltage network (km) (x_2) and Number of employees (x_3).

Output variables Energy delivery (million KWh) (y_1), Number of customers (*1000) (y_2) and Number of lights of street lighting (*1000) (y_3). The input costs $p_1, \; p_2$ and p_3, and the output prices $q_1, \; q_2$ and q_3 are assumed to be equal across all companies. These factors are calculated as $(p_1, \; p_2, \; p_3) = (1.5, \; 1, \; 0.8)$ and $(q_1, \; q_2, \; q_3) = (2.5, \; 2, \; 3)$. The data set are listed in Table 1. We first calculated the BCC, cost, revenue and profitability efficiencies of these 39 companies. The results of different efficiency values for these 39 companies are listed in Table 2. As the results show, seventeen companies are technically efficient in variable returns to scale environment, five companies are cost efficient, thirteen companies are revenue efficient and finally just one company (Ahwaz) is profitability efficient. It

Table 1 Data set for 39 power plants

Company	x1	x2	x3	y1	y2	y3
Tabriz	1686	8283	572	3816	3520	873
Azarbayejansharghi	1731	103,608	700	3296	2882	1019
Azarbayejangharbi	2330	25,928	725	5059	4288	1102
Ardebil	853	12,983	299	1640	1403	480
Ostan Esfahan	5069	35,958	512	9564	8780	1279
Isfahan	2498	13,004	295	5409	5068	1040
Chaharmahal-o-B.	971	10,851	158	1651	1425	315
Markazi	2215	19,183	315	4787	4221	614
Hamedan	2118	17,482	369	3521	2996	663
Lorestan	1797	15,834	221	2926	2533	561
Alborz	2659	12,047	349	6286	5188	1159
Tehran	10,756	30,768	1742	20,512	19,046	4267
Ostan Tehran	7592	31,292	720	12,654	10,546	1966
Ghom	1406	6859	250	3188	2753	486
Mashhad	2596	14,456	396	6477	5947	1378
Khorasanrazavi	2998	39,082	572	7613	6913	1121
Khorasanjonobi	908	17,045	189	1539	1403	329
Khorasanshomali	699	9922	196	1205	1106	311
Ahwaz	4228	9263	419	8957	7145	502
Khozestan	7322	28,928	507	16,195	13,450	914
Kohkiloyeh-o-B.	1083	8006	174	1583	1140	216
Zanjan	1363	13,294	232	3099	2795	393
Ghazvin	1750	11,648	236	4412	4104	530
Semnan	1241	11,054	183	2530	2353	339
Sistan-o-B.	2460	34,049	671	5165	4322	687
Kermanshah	1922	17,708	296	3142	2501	678
Kurdestan	1273	15,131	232	2620	1889	563
Illam	857	6860	112	1320	1010	194
Shiraz	3680	22,827	530	5807	4948	928
Fars	3621	33,672	356	6977	6333	842
Boshehr	2970	12,874	227	5486	4628	389
Shomal-e-kerman	2019	18,611	380	4090	3595	546
Jonob-e-kerman	2708	30,580	368	5500	4683	502
Gilan	2890	27,176	621	5080	4407	1255
Mazandaran	3331	24,582	624	5941	4897	1172
Garb-e-mazandaran	1691	10,033	232	2131	1810	502
Golestan	2021	14,066	348	3258	2847	636
Hormozgan	3750	22,194	447	819	6818	587
Yazd	1765	17,066	349	4491	4197	578

is to be noted that five companies (Tabriz, Tehran, Qom, Ahwaz and Ilam) are efficient in all of the three aspects cost, revenue and technical, and only one company Ahwaz is efficient in whole sense. Figure 1 is used to show all efficiencies in one diagram.

Table 2 The efficiency scores in different models

Company	Technical efficiency	Cost efficiency	Revenue efficiency	Profitability efficiency
Tabriz	1	1	1	0.7111
Azarbayejansharghi	1	0.1205	0.8175	0.0667
Azarbayejangharbi	0.8934	0.4637	0.8534	0.3413
Ardebil	1	0.6309	0.9992	0.2402
Ostan Esfahan	0.8991	0.4838	0.8658	0.4301
Isfahan	1	0.8463	0.9714	0.6580
Chaharmahal-o-B.	0.9826	0.6934	0.9178	0.2660
Markazi	0.8578	0.5152	0.8304	0.4081
Hamedan	0.7102	0.4859	0.6462	0.3343
Lorestan	0.868	0.5119	0.7126	0.3138
Alborz	1	0.9636	1	0.7565
Tehran	1	1	1	0.8831
Ostan Tehran	1	0.6363	0.8019	0.5657
Ghom	1	1	1	0.6799
Mashhad	1	0.9107	1	0.7205
Khorasanrazavi	1	0.4014	1	0.3433
Khorasanjonobi	0.9565	0.467	0.8817	0.1718
Khorasanshomali	1	0.7737	1	0.2310
Ahwaz	1	1	1	1
Khozestan	1	0.8981	1	0.7261
Kohkiloyeh-o-B.	0.8613	0.8564	0.6819	0.2942
Zanjan	0.9679	0.5947	0.9392	0.3903
Ghazvin	1	0.7845	1	0.6011
Semnan	0.9783	0.6855	0.9410	0.3850
Sistan-o-B.	0.8257	0.3182	0.7758	0.2576
Kermanshah	0.7943	0.4902	0.6411	0.2984
Kurdestan	0.9774	0.5565	0.8457	0.2912
Illam	1	1	1	0.2992
Shiraz	0.6338	0.4874	0.6574	0.3946
Fars	0.8832	0.4004	0.8513	0.3458
Boshehr	1	0.6887	1	0.5754
Shomal-e-kerman	0.8117	0.4832	0.7721	0.3624
Jonob-e-kerman	0.8189	0.3482	0.7668	0.2942
Gilan	0.8106	0.4823	0.7218	0.3297
Mazandaran	0.7128	0.5103	0.7273	0.3908
Garb-e-mazandaran	0.8371	0.7256	0.5595	0.3421
Golestan	0.7187	0.5745	0.6409	0.3783
Hormozgan	0.8554	0.557	0.4195	0.2584
Yazd	1	0.5803	1	0.4459

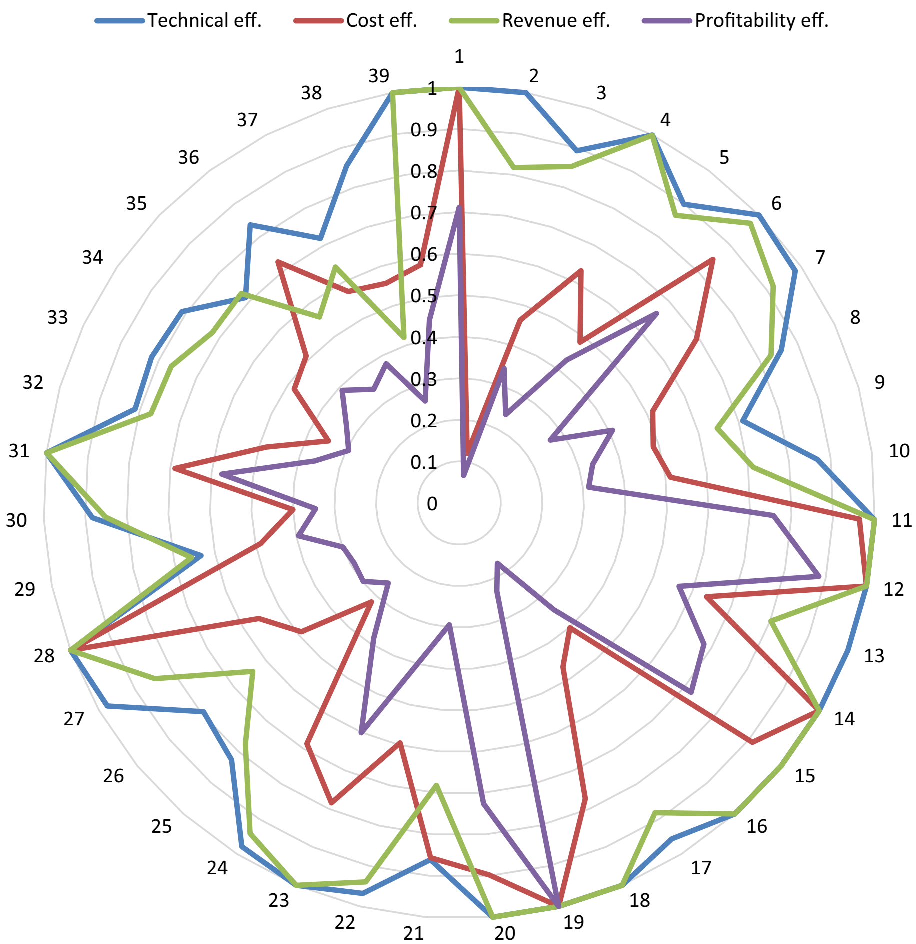

Fig. 1 The efficiency scores in different models

We have also computed the MPSS points to all companies and the results are listed in Table 3. The set of MPSS points are

$$MPSS\ \ set = \left\{ \begin{array}{c} \text{Tabriz, Azarbaijansharghi, Ardebil, Esfahan, Alborz, Tehran,} \\ \text{Mashhad, Khorasanrazavi, Ahwaz, Khozestan, Ghazvin, Yazd} \end{array} \right\}$$

The ACE_j and ARE_j of each DMU_j are calculated by models (8) and (14), respectively. The ACE and ARE measures along with the corresponding OSS points are given in Tables 4 and 5, respectively. The second column of Table 4 shows the ACE_j measure and the second column of Table 5 shows the ARE_j measure of each DMU_j and the last six columns shows the new input–output values of corresponding OSS points.

Finally, the $ACRE$ measure of each DMU_j is calculated by our proposed model (23). The results are given in Table 6. As the table shows, only one company (Ahwaz) performs in optimal size.

Table 3 The MPSS points corresponding to each companies

Company	x1	x2	x3	y1	y2	y3
Tabriz	1686	8283	572	3816	3520	873
Azarbayejansharghi	1731	103,608	700	3296	2882	1019
Azarbayejangharbi	2330	21,561.94	395.26	5719.86	5238.63	1245.96
Ardebil	853	12,983	299	1640	1403	480
Ostan Esfahan	5069	23,471.76	512	11,538.23	10,111.53	1472.97
Isfahan	2498	13,004	295	5409	5068	1040
Chaharmahal-o-B.	971	7489.91	158	2435.68	2230.36	464.71
Markazi	2215	16,224.59	315	5585.77	5177.97	716.45
Hamedan	2118	17,482	352.89	5318.88	4865.61	1001.54
Lorestan	1797	9397.89	221	4009.45	3724.59	768.73
Alborz	2659	12,047	349	6286	5188	1159
Tehran	10,756	30,768	1742	20,512	19,046	4267
Ostan Tehran	6629.29	31,292	720	14,748.72	12,997.89	2291.45
Ghom	1406	6859	199.12	3399.79	3048.27	627.94
Mashhad	2596	14,456	396	6477	5947	1378
Khorasanrazavi	2998	39,082	572	7613	6913	1121
Khorasanjonobi	908	5738.92	141.68	2258.02	2072.2	482.71
Khorasanshomali	699	9922	182.04	1533.66	1366.18	384.16
Ahwaz	4228	9263	419	8957	7145	502
Khozestan	7322	28,928	507	16,195	13,450	914
Kohkiloyeh-o-B.	1083	8006	169.26	2733.69	2541.55	373.01
Zanjan	1363	11,858.49	232	3453.87	3212.72	438
Ghazvin	1750	11,648	236	4412	4104	530
Semnan	1241	8686.09	183	3129.07	2907.85	418.94
Sistan-o-B.	2460	25,713.28	482.45	6256.48	5808.41	832.18
Kermanshah	1922	11,309.25	296	4788.77	4395.99	1020.88
Kurdestan	1273	9078.21	203.42	3154.47	2893.29	677.85
Illam	857	4952.69	112	2107.35	1918.49	309.72
Shiraz	3680	22,827	525.45	9236.81	8544.69	1476.11
Fars	3621	16,389.85	356	8369.26	7229.3	961.17
Boshehr	2970	12,208.22	227	6645.29	5578.29	468.87
Shomal-e-kerman	2019	18,611	380	5125.95	4778.5	684.3
Jonob-e-kerman	2708	18,158.96	368	6828.3	6352.11	821.24
Gilan	2890	27,176	621	6697.23	6043.37	1564.04
Mazandaran	3331	24,348.22	537.83	8345.28	7642.44	1646.3
Garb-e-mazandaran	1691	9143.15	232	3970.7	3676.43	811.37
Golestan	2021	14,066	327.86	5064.46	4652.29	988.64
Hormozgan	3750	22,194	447	9156.68	8303.15	1010.62
Yazd	1765	17,066	349	4491	4197	578

We have used our proposed procedure to calculate the OSS. Toward this end, we should estimate the global optimal solution of model (18). To do this, we first solved model (20) to calculate the profitability scores to each of the companies. The last column of Table 2

Table 4 The ACE measures and corresponding OSS points to companies

Company	ACE_j	x1	x2	x3	y1	y2	y3
Tabriz	1	1686	8283	572	3816	3520	873
Azarbayejansharghi	0.1205	2076.84	9251.92	622.42	4535.46	4189.05	1019.25
Azarbayejangharbi	0.4637	2356.77	9856.85	650.48	5058.47	4656.07	1101.88
Ardebil	0.631	1394.89	6859.02	247.21	3150.2	2717.73	480.09
Ostan Esfahan	0.4838	5030.28	13,171.06	695.92	10,267.33	8780.29	1279.04
Isfahan	0.8463	2652.91	9896.03	625.88	5663.9	5068.15	1040.03
Chaharmahal-o-B.	0.6934	1084.49	6859.59	169.18	2094.04	1732.25	315
Markazi	0.5152	2281.22	7982.96	399.84	4994.95	4220.86	613.98
Hamedan	0.4858	1556.25	7528.53	398.43	3520.54	3138.14	662.91
Lorestan	0.512	1460.33	7135.29	312.47	3309.85	2901.81	561.09
Alborz	0.9637	2989.73	10,702.83	666.78	6286.35	5640.86	1159.07
Tehran	1	10,756	30,768	1742	20,512	19,046	4267
Ostan Tehran	0.6363	6362.21	17,231.46	936.28	12,653.34	11,163.82	1965.9
Ghom	1	1406	6859	250	3188	2753	486
Mashhad	0.9107	3125.17	11,715.74	745.44	6476.94	5965.14	1377.99
Khorasanrazavi	0.4014	3848.66	11,385.8	646.88	7992.95	6913.75	1121.12
Khorasanjonobi	0.4671	1110.98	6859.54	175.84	2184.18	1816.35	329.09
Khorasanshomali	0.7739	1077.15	6859.6	167.34	2069.07	1708.94	311.09
Ahwaz	1	4228	9263	419	8957	7145	502
Khozestan	0.8981	8317.28	22,734.21	1247.76	16,195.31	14,600.05	2860.48
Kohkiloyeh-o-B.	0.8568	934.49	6859.86	131.48	1583.65	1256.01	235.21
Zanjan	0.5946	1432.74	6881.78	251.6	3242.67	2794.62	486.15
Ghazvin	0.7846	2252.62	7700.93	331.78	4924.4	4104.36	530.05
Semnan	0.6855	1280.02	6859.23	218.33	2759.36	2353.04	419
Sistan-o-B.	0.3181	2358.74	8271.78	461.84	5163.87	4403.75	686.85
Kermanshah	0.4901	1544.8	7564.89	409.62	3499.31	3133.21	677.84
Kurdestan	0.5565	1461.73	7142.45	314.09	3313	2905.67	563.03
Illam	1	857	6860	112	1320	1010	194
Shiraz	0.4874	2705.72	9496.59	584.77	5807.13	5086.16	928.02
Fars	0.4004	3564.76	9986.04	547.37	7530.74	6332.78	841.97
Boshehr	0.6887	2610.92	7885.44	322.16	5651.21	4628.27	492.83
Shomal-e-kerman	0.4832	1915.66	7468.98	325.79	4238.14	3595.32	546.05
Jonob-e-kerman	0.3482	2640.94	7938.6	331.06	5713.88	4682.7	501.97
Gilan	0.4823	2706.95	10,813.98	703.7	5695.35	5267.65	1255.04
Mazandaran	0.5102	2816.48	10,585.8	672.67	5940.33	5386.94	1171.87
Garb-e-mazandaran	0.7257	1417.6	6918.02	263.35	3214.03	2784.79	502.04
Golestan	0.5745	1514.51	7410.86	374.79	3431.38	3050.24	635.98
Hormozgan	0.557	3979.72	9363.45	451.67	8437.85	6817.72	586.98
Yazd	0.5803	2285.69	7875.97	371.58	4998.89	4196.86	577.98

Table 5 The ARE measures and corresponding OSS points to companies

Company	ARE_j	x1	x2	x3	y1	y2	y3
Tabriz	1	1686	8283	572	3816	3520	873
Azarbayejansharghi	1.2233	1731	16,838.12	344.12	4386.18	4098.4	569.48
Azarbayejangharbi	1.1718	2330.08	15,291.21	380.96	5841.47	5386.99	1122
Ardebil	1.0008	852.97	10,953.87	218.1	1679.63	1552.46	349.57
Ostan Esfahan	1.1551	5069.27	30,167.62	512.03	11,670.36	9987.58	1059.51
Isfahan	1.0294	2498.09	13,004.45	295.01	5964.23	5198.61	752.66
Chaharmahal-o-B.	1.0896	971	8584.53	158	1837.78	1588.23	287.15
Markazi	1.2042	2214.95	13,177.37	314.99	5526.14	5084.99	936.64
Hamedan	1.5474	2118	15,957.31	368.96	5334.62	4940.37	917.83
Lorestan	1.4034	1797.05	11,360.89	221.01	4253.31	3840.52	474.37
Alborz	1	2659	12,047	349	6286	5188	1159
Tehran	1	10,756	30,768	1742	20,512	19,046	4267
Ostan Tehran	1.2471	7592.19	29,964.53	720.02	16,309.69	13,926.72	1494.52
Ghom	1	1406	6859	250	3188	2753	486
Mashhad	1	2596	14,456	396	6477	5947	1378
Khorasanrazavi	1	2998	39,082	572	7613	6913	1121
Khorasanjonobi	1.1342	907.99	9713.36	189	1781.72	1611.32	329.64
Khorasanshomali	1	699	9922	196	1205	1106	311
Ahwaz	1	4228	9263	419	8957	7145	502
Khozestan	1	7322	28,928	507	16,195	13,450	914
Kohkiloyeh-o-B.	1.4664	1082.99	8005.93	174	2162.18	1856.67	326.03
Zanjan	1.0647	1362.95	11,526.76	231.99	3234.15	3004.73	453.63
Ghazvin	1	1750	11,648	236	4412	4104	530
Semnan	1.0627	1240.98	9571.14	183	2721.64	2447.77	367.92
Sistan-o-B.	1.289	2460.05	14,883	388.31	6152.09	5660.7	1247.12
Kermanshah	1.5597	1922.01	13,557.53	296	4842.43	4493.76	710.64
Kurdestan	1.1825	1273.02	11,543.83	232	2960.74	2749.73	436.26
Illam	1	857	6860	112	1320	1010	194
Shiraz	1.5212	3680.14	22,827.86	530.02	8616.15	7754.82	1440.03
Fars	1.1747	3620.96	17,495.26	356	8435.83	7345.65	851.62
Boshehr	1	2970	12,874	227	5486	4628	389
Shomal-e-kerman	1.2951	2018.93	16,268.46	363.36	5097.87	4731.75	822.46
Jonob-e-kerman	1.3042	2708	14,751.46	368	6636.18	6015.55	1163.49
Gilan	1.3855	2890.07	27,176.7	500.37	7209.22	6582.39	1278.74
Mazandaran	1.375	3331.05	24,582.39	557.11	7902.83	7243.49	1493.13
Garb-e-mazandaran	1.7872	1691.01	10,033.07	232	4011.61	3608.39	478.9
Golestan	1.5603	2021.04	14,066.3	318.43	5085.58	4711.53	811.01
Hormozgan	2.3839	3749.9	21,344.55	446.99	8894.11	7825.94	1233.51
Yazd	1	1765	17,066	349	4491	4197	578

Table 6 The ACRE measure and corresponding OSS points

Company	ACRE	k	x1	x2	x3	y1	y2	y3
Tabriz	1.0086	7	1686.0000	8150.4144	261.4907	3937.7224	3624.5378	833.6387
Azarbayjanshaghi	9.7316	84	1731.0000	9648.4397	264.2406	4318.2831	3964.8111	918.8742
Azarbayjangharbi	2.3531	16	2330.0000	12,580.0707	356.7958	5727.6372	5261.8350	1217.2568
Ardebil	2.7960	24	853.0000	4814.2102	131.4360	2124.4199	1949.7810	452.9989
Ostan Esfahan	1.9094	12	5069.0000	21,655.9797	512.0000	11,621.6620	9718.0139	1415.6423
Esfahan	1.0400	7	2343.7911	10,062.1606	295.0000	5473.9878	4506.2899	924.7320
Chaharmahalbakhtiari	3.9957	20	971.0000	5116.2827	143.5177	2390.2238	2172.6640	479.9454
Markazi	2.0869	3	2215.0000	5566.5346	285.9208	4470.2609	3829.7363	557.0855
Hamedan	3.2491	15	2118.0000	11,227.5623	314.1191	5221.2304	4751.1704	1055.1359
Lorestan	3.1324	16	1657.9960	7776.8452	221.0000	3949.7801	3324.7803	736.3607
Alborz	0.9679	6	2659.0000	11,347.9613	349.0000	6162.1971	5111.0515	1097.7644
Tehran	0.9651	3	10,756.0000	30,768.0000	1742.0000	20,150.9888	18,710.7904	4191.9008
Ostantehran	1.4993	9	6182.1803	27,503.3643	720.0000	14,398.5666	11,999.9434	2237.0462
Ghom	1.2077	6	1406.0000	5864.2268	183.3820	3288.9096	2871.2956	506.5733
Mashhad	0.9761	8	2596.0000	14,456.0000	396.0000	6399.2760	5875.6360	1361.4640
Khorasan razavi	2.0695	15	2998.0000	15,275.3958	434.8683	7321.7987	6615.4380	1418.2475
Khorasan jonobi	6.1118	32	908.0000	4947.5142	138.8865	2241.8425	2059.1901	476.5962
Khorasan shomali	3.7119	24	699.0000	3849.0169	106.7780	1734.5742	1592.9541	368.8910
Ahvaz	1	0	4228.0000	9263.0000	419.0000	8957.0000	7145.0000	502.0000
Khozestan	0.9758	7	7012.7369	26,443.8863	507.0000	15,444.0249	12,776.4713	871.0221
Kohkiloyeh-o-boyerahmad	4.3983	12	10,756.0000	30,768.0000	1742.0000	10,920.5888	10,140.0904	2271.7508
Zanjan	2.0327	3	1363.0000	3408.7183	173.1315	2762.8604	2355.5337	331.2074
Ghazvin	1.1598	2	1750.0000	4445.0816	212.6053	3613.9163	3044.4336	393.1652
Semnan	2.2226	4	1241.0000	3298.8756	157.7744	2553.4762	2181.5959	314.3056
Sistan- o-balochestan	3.4937	12	2460.0000	8589.2723	307.3874	5497.1341	4716.9457	731.1774
kermanshah	3.5915	18	1922.0000	10,163.8144	295.0600	4678.3542	4299.4651	993.5384
Kordestan	2.9234	20	1273.0000	7216.9511	196.8155	3168.5248	2907.6504	676.1532

Table 6 (continued)

Company	ACRE	k	x1	x2	x3	y1	y2	y3
Ilam	4.0438	12	857.0000	3257.3770	111.0022	1946.1290	1690.6271	286.0220
Shiraz	2.9696	11	3680.0000	16,086.4663	491.6474	8690.4703	7646.4392	1415.8832
Fars	2.3866	16	3621.0000	16,364.6936	356.0000	8367.7411	7226.6514	963.6593
Boshehr	1.5068	9	2944.3580	11,476.8350	227.0000	6538.7722	5414.3039	455.0917
Shomalekerman	2.5060	4	2019.0000	5246.2269	264.9052	4085.3625	3520.3546	534.6630
jonobekerman	3.2502	3	2708.0000	6643.1271	304.6218	5679.0839	4684.3223	502.1417
Gilan	2.6689	17	2890.0000	16,110.4865	441.2029	7209.5002	6619.3448	1534.1174
mazandaran	2.5844	14	3331.0000	18,100.5641	501.0252	8260.8457	7551.0011	1713.4529
Karb e mazandaran	3.3929	15	1678.5614	8174.6280	232.0000	4040.0660	3461.5112	783.7545
Golestan	2.9645	14	2021.0000	10,598.8800	297.9218	4969.3479	4513.2096	992.8457
Hormozgan	3.5592	2	3750.0000	8818.2914	390.6104	7959.7954	6437.4792	554.2388
Yazd	1.5356	3	1765.0000	4554.2359	220.5825	3625.2705	3079.4797	424.0980

Table 7 ACRE measure changes for company Tabriz for different k

k	0	1	2	3	4	5	6	7	8	9	10
$R_j^{(*)(CR)}$	0	0.5063	0.6962	0.8930	0.9564	0.9673	0.9840	1.0087	1.0016	0.9145	0.8214

shows the results of model (20). We then, proceed to second step by solving model (23) with $\sigma = 0.05$ for different k. The new ACRE measures along with the corresponding OSS points are given in Table 6.

In the following, five different companies are considered to be analyzed: Tabriz, Azarbayjansharghi, Tehran, Hormozgan and Yazd. Tabriz and Tehran were technical, cost and revenue efficient. Azarbayjansharghi was technically efficient but, it was inefficient in both, cost and revenue aspects. Hormozgan is inefficient in all aspects and finally, Yazd is cost inefficient, while, it is technical and revenue efficient. Consider the first company (Tabriz) that was efficient in three different aspects. The OSS occurs in $k=7$ with $R_1^{(*)(RC)} \cong 1.0086$. The optimal scale size to this company is defined as $(x_1^*, x_2^*, x_3^*, y_1^*, y_2^*, y_3^*) = (1686,\ 8150.4144,\ 261.4907,\ 3937.7224,\ 3624.5378,\ 833.6387)$. Clearly, this new point is located on BCC frontier. Comparing these new values to inputs and outputs to the original one, we find that to achieve an optimal size, Tabriz should reduce its inputs as $(x_1^*, x_2^*, x_3^*) = (0,\ 132.5856,\ 310.8093)$ and increase its outputs as $(y_1^*, y_2^*, y_3^*) = (121.7224,\ 104.5378,\ 39.3613)$ to be in optimal size.

Now, consider the second company (Azarbayjansharghi). An OSS to Azarbayjansharghi occurs in $k = 84$ with $R_2^{(*)(RC)} = 9.7316$. An optimal size to this company is calculated as $(x_1^*, x_2^*, x_3^*, y_1^*, y_2^*, y_3^*) = (1731,\ 9648.4397,\ 264.2406,\ 4318.2831,\ 3964.8111,\ 918.8742)$. An interesting point is that the new value to the third output is $y_3^* = 918.9030$ that means that we should reduce this output from 1019 to 918.8742.

To provide a better comprehension of how the procedure of calculating ACRE works, we analyze a specific company, more deeply. Consider the first company (Tabriz). After solving model (19), we get $PR_{39}^{Max} = 1.4063$. With $\sigma = 0.05$, model (23) is solved for $k = 0,\ 1,\ 2,\ \ldots,\ 10$ and results are given in Table 7 and Fig. 1. It can be seen that the maximum value to $ACRE_{39} = 1.0087$ at $k = 7$. To achieve a more accurate estimation, we can choose a smaller value to σ in the neighborhood of $ACRE_{39} - k\sigma$.

5 Conclusions

Cost minimization and revenue maximization are two important issues in the context of management and economics. In this sense, decision makers are interested to adjust (increase or decrease) their operational volume in order to achieve an optimal size. An interesting and most frequently studied subject in the context of DEA is determining an optimal scale size to each DMU. MPSSs are defined to maximize the outputs produced 'per unit' of the inputs. It seemed to be a MPSS had the optimal size. However, this is not correct and as Cesaroni and Giovannola (2015) stated, OSS and MPSS of a specific DMU do not need to be coincided. The previous definition of OSS in Cesaroni and Giovannola (2015) was cost-based and in this paper, we first redefined it in revenue-orientation. In the following, we have proposed an alternative efficiency measure based on costs of inputs and revenues of outputs and in this sense, it provides a more powerful OSS. It has been shown that the ACRE measure is the ratio of profitability efficiency measure to ray average productivity. Then, a numerical method is proposed to calculate an approximation to the optimal solution of the proposed model when the underlying technology set is convex and continuous. This numerical method is useful when we deal with large-scale problems. The real applicability of the approach has been illustrated by a real example on Iranian electricity distribution companies.

References

Amirteimoori, A., & Emrouznejad, A. (2011). Flexible measures in production process: A DEA-based approach. *RAIRO Operations Research, 45,* 63–74.

An, Q., Yang, M., Chu, J., Jie, Wu, & Zhu, Q. (2017). Efficiency evaluation of an interactive system by data envelopment analysis approach. *Computers & Industrial Engineering, 103,* 17–25.

Assani, S., Jiang, J., Cao, R., & Yang, F. (2018). Most productive scale size decomposition for multi-stage systems in data envelopment analysis. *Computers & Industrial Engineering, 120,* 279–287.

Banker, R. D. (1984). Estimating most productive scale size using data envelopment analysis. *European Journal of Operational Research, 17,* 35–44.

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Models for the estimation of technical and scale inefficiencies in data envelopment analysis. *Management Science, 30,* 1078–1092.

Baumol, W. J. (1977). On the proper cost tests for natural monopoly in a multiproduct industry. *American Economic Review, 67,* 809–822.

Cesaroni, G., & Giovannola, D. (2015). Average–cost efficiency and optimal scale sizes in non-parametric analysis. *European Journal of Operational Research, 242,* 121–133.

Charnes, A., & Cooper, W. W. (1962). Programming with linear fractional functions. *Naval Research Logistics Quarterly, 9,* 181–186.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research, 2,* 429–444.

Cooper, W. W., Seiford, L. M., & Zhu, J. (2004). *Handbook of data envelopment analysis*. Norwell, MA: Kluwer Academic Publishers.

Emrouznejad, A., & Yang, G. (2018). A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016. *Socio-Economic Planning Sciences, 61,* 4–8.

Erbetta, F., & Rappuoli, L. (2008). Optimal scale in the Italian gas distribution industry using data envelopment analysis. *Omega, 36*(2), 325–336.

Lee, C. Y. (2016). Most productive scale size versus demand fulfillment: A solution to the capacity dilemma. *European Journal of Operational Research, 248*(3), 954–962.

Ouellette, P., Quesnel, J. P., & Vigeant, S. (2012). Measuring returns to scale in DEA models when the firm is regulated. *European Journal of Operational Research, 220*(2), 571–576.

Podinovski, V. V. (2017). Returns to scale in convex production technologies. *European Journal of Operational Research, 258,* 970–982.

Sahoo, B. K., Khoveyni, M., Eslami, R., & Chaudhury, P. (2016). Returns to scale and most productive scale size in DEA with negative data. *European Journal of Operational Research, 255*(2), 545–558.

Zhu, Q., Wu, J., & Song, M. (2018). Efficiency evaluation based on data envelopment analysis in the big data context. *Computers & Operations Research, 98,* 291–300.