Geodesics on Shape Space

Stefanos Pertigkiozoglou

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1 Shape Space

The shape space is proposed in Kendall (1977) as a tool for studying the geometry between shapes. According to Kendall, shape is geometry modulo rotation, translation and scale. In this project we first define the shape space and we show how we can map a scaled, rotated and translated version of a shape to it. Then we derive geodesics that allow us to traverse the space and move from one shape to another.

1.1 Pre-shape space

Suppose that we are representing a shape with an ordered set of k points $x^* \in \mathbb{R}^m$. First we can remove the effect of translation by subtracting by the centroid of the k points $\tilde{x}_i = x_i^* - \bar{x^*}$, where $\bar{x^*} = \frac{1}{k}\sum_{j=1}^{k} x_j^*$.

Then we define the following point coordinates:

$$x_j = \frac{1}{\sqrt{j+j^2}}\left(j\tilde{x}_{j+1} - \sum_{p=1}^{j}\tilde{x}_p\right)$$

This transformation can be obtained by right multiplying $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k)$ with a matrix Q_k where:

$$Q_k(i,j) = \begin{cases} \frac{1}{\sqrt{k}} & \text{if } j = 1 \\ \frac{-1}{\sqrt{(j-1)+(j-1)^2}} & \text{if } j > 1,\ i < j \\ \frac{j-1}{\sqrt{(j-1)+(j-1)^2}} & \text{if } j > 1,\ i = j \\ 0 & \text{if } j > 1,\ i > j \end{cases}$$

An example of such a matrix for $k = 4$ is the following:

$$Q_k = \begin{bmatrix} \frac{1}{\sqrt{4}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{12}} \\ \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{12}} \\ \frac{1}{\sqrt{4}} & 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{12}} \\ \frac{1}{\sqrt{4}} & 0 & 0 & \frac{3}{\sqrt{12}} \end{bmatrix}$$

We can observe that Q_k is orthogonal since (denoting the i^{th} column of Q_k, with $c_i(Q_k)$):

$$c_i(Q_k)^T c_j(Q_k) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Additionally if we add all the rows of Q_k to the top row we can create matrix Q_k^* with top row:

$$Q_k^*(1,j) = \begin{cases} \frac{k}{\sqrt{k}} = \sqrt{k} & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Also the minor of $\mathcal{M}(Q_k^*)_{1,1}$, which is the matrix after we remove the first row and column of Q_k^*, is an upper triangular matrix where the i^{th} diagonal element is $\frac{i}{\sqrt{i+i^2}}$. So overall we have that:

$$\det(Q_k) = \det(Q_k^*) = \sqrt{k}\prod_{i=1}^{k-1}\frac{i}{\sqrt{i+i^2}} = \sqrt{k}\prod_{i=1}^{k-1}\frac{\sqrt{i}}{\sqrt{1+i}} = \sqrt{k}\frac{1}{\sqrt{k}} = 1$$

So Q_k is a real orthogonal matrix and since $\det(Q_k) = 1$ we have $Q_k \in \mathrm{SO}(k)$.

Using Q_k and multiplying to the right of $(\tilde{x_1}, \tilde{x_2} \dots, \tilde{x_k})$ we get:

$$\begin{aligned}(\tilde{x_1}, \tilde{x_2} \dots, \tilde{x_k})Q_k &= (\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\tilde{x}_i, x_1, x_2, \dots, x_{k-1}) \\ &= (0, x_1, x_2, \dots, x_{k-1})\end{aligned}$$

Since $Q_k \in \mathrm{SO}(k)$ we know that $\|\tilde{x}\|_F = \|\tilde{x}Q_k\|_F$.

Finally to remove the scale factor we can divide by the norm $\{\sum_{i=1}^{k-1}\|x_i\|^2\}^{1/2}$. The result will be a matrix:

$$(0, x_1, x_2, \dots, x_{k-1}), \text{ where } \sum_{i=1}^{k-1}\|x_i\|^2 = 1$$

By throwing away the first zero column we arrive at a matrix $(x_1, x_2, \dots, x_{k-1}) \in \mathcal{M}(m, k-1)$, where $\mathcal{M}(m, k-1)$ is the set of real $m \times (k-1)$ matrices.

If we denote with $r : \mathcal{M}(m,k) \to \mathcal{M}(m, k-1)$ the map that removes the first column of a matrix in $\mathcal{M}(m,k)$, we can have an explicit formulation for the mapping of an original shape $x^* = (x_1^*, x_2^*, \dots, x_k^*)$ to the pre-shape $(x_1, x_2, \dots, x_{k-1})$. Specifically:

$$(x_1, x_2, \dots, x_{k-1}) = r\left(\frac{(x^* - \bar{x^*})Q_k}{\|(x^* - \bar{x^*})\|}\right)$$

This process maps all the different translated and scaled versions of the same shape to the same representative matrix $(x_1, x_2, \dots, x_{k-1})$, which Kendall refers to as the pre-shape.

We denote the set containing all the pre-shapes of k points in $\mathbb{R}^m$ with S_m^k:

$$S_m^k = \{(x_1, x_2, \ldots, x_{k-1}) | x_i \in \mathbb{R}^m, \ \sum_{i=1}^{k-1} \|x_i\|^2 = 1\}$$

We can observe that S_m^k is a subset of $\mathbb{R}^{m \times (k-1)}$, with $p \in S_m^k$ iff $p \in \mathbb{R}^{m \times (k-1)}$ and $\|p\|_F = 1$. Thus S_m^k is isomorphic to the unit sphere $S^{m(k-1)-1}$.

1.2 Basic definition of shape space

In the definition of the pre-shape, by substracting by the centroid and normalizing by the norm $\{\sum_{i=1}^{k-1} \|x_i\|^2\}^{1/2}$, we have quotiented out both translation and scale. Our final goal is to arrive to a representation of the shape where rotations are also quotiented out.

A rotation of a given m-dimensional shape corresponds to a left multiplication with a matrix $R \in SO(m)$. So for a given shape $x^* = (x_1^*, x_2^*, \ldots, x_k^*)$ after we scale it by s, rotate it by R, and translate it by T, we have the transformed shape $x^{*\prime}$ where:

$$x^{*\prime} = sRx^* + T$$

We can compute the pre-shape x' of the transformed shape as:

$$\begin{aligned} x' = r\left(\frac{x^{*\prime} - \bar{x}^{*\prime}}{\|x^{*\prime} - \bar{x}^{*\prime}\|}\right) &= r\left(\frac{sRx^* + T - sR\bar{x}^* - T}{\|sRx^* + T - sR\bar{x}^{*\prime} - T\|}\right) \\ &= r\left(\frac{sR(x^* - \bar{x}^*)}{s\|x^* - \bar{x}^{*\prime}\|}\right) \\ &= Rr\left(\frac{(x^* - \bar{x}^*)}{\|x^* - \bar{x}^{*\prime}\|}\right) = Rx \end{aligned}$$

which is the rotated pre-shape of the original shape.

This means that we can arrive at the desired representation for the shapes by taking the pre-shape S_m^k modulo the left action of $\mathrm{SO}(m)$. We denote the shape space with $\Sigma_m^k = S_m^k / \mathrm{SO}(m)$ and the quotient mapping with $\pi : S_m^k \to \Sigma_m^k$. In Figure 1 we show an example of the mapping of the original shape to the pre-shape and then using map π to the shape space Σ_m^k.

1.3 Riemannian metric

As shown in 1.1 the pre-shape space is isomorphic to the sphere $S^{m(k-1)-1}$. As a result a natural choice of a metric is the usual Euclidean inner product. So for two vectors $U =$

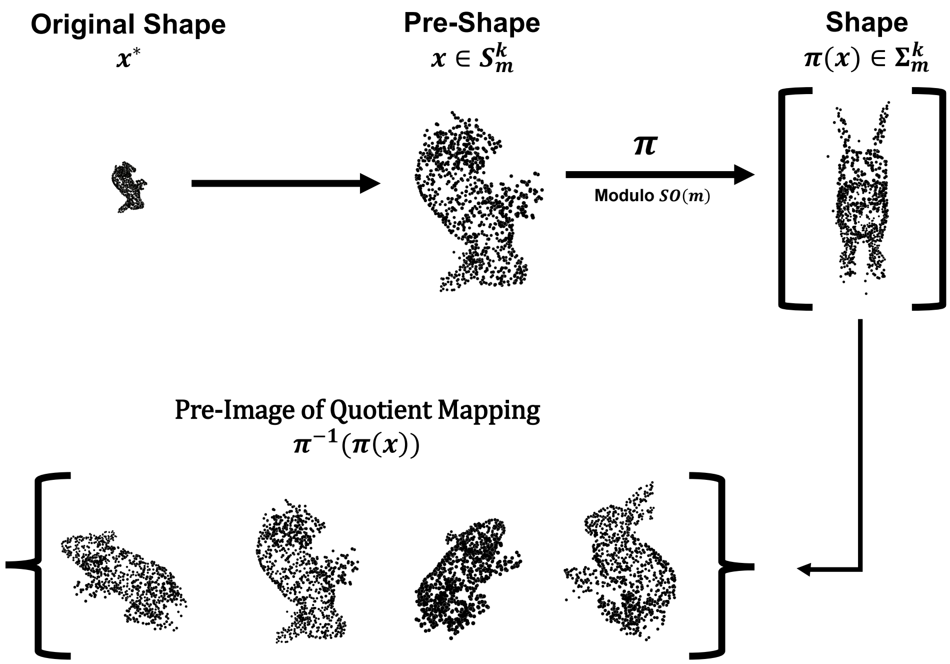

Figure 1: Mapping of a shape represented by a sequence of points to: (1) a pre-shape where scaling and translation is removed, (2) a shape in Σ_m^k where we have quotiented out rotation, translation and scaling.

$(u_1, u_2, \ldots, u_{k-1}) \in \mathcal{M}(m, k-1)$, and $V = (v_1, v_2, \ldots, v_{k-1}) \in \mathcal{M}(m, k-1)$ we define the inner product as:

$$\langle U, V \rangle = \sum_{i=1}^{k-1} u_i^T v_i = \operatorname{tr}\left(U^T V\right) = \operatorname{tr}\left(U V^T\right)$$

For the given metric, at Chapter 6 in Kendall et al. (2008), it is shown that the quotient mapping π is a Riemannian submersion. We will use this fact to find geodesics on the shape space by using geodesics of the pre-shape space.

2 Geodesics of Shape Space

2.1 Geodesics on Pre-Shape

Since the pre-shape S_m^k is isomorphic to the $S^{m(k-1)-1}$ sphere, the tangent space at X is any Z orthogonal to X, so:

$$T_X S_m^k = \{Z | Z \in \mathcal{M}(m, k-1),\ \operatorname{tr}(XZ^T) = 0\}$$

Additionally, for the sphere the geodesics are known to be the arcs of great circles. As a result for $X \in S_m^k$ and any $Z \in S_m^k$ with $\mathrm{tr}(XZ^T) = 0$, the geodesic that connects them is:

$$\gamma_Z(s) = X\cos(s) + Z\sin(s), \quad 0 \le s \le \pi$$

It is easy to see that indeed:

$$\gamma_Z'(0) = -X\sin(0) + Z\cos(0) = Z$$

so Z is the initial velocity of the geodesic

2.2 Vertical and Horizontal Tangent Space

Given a point $X \in S_m^k$ we have that for the quotient mapping $\pi : S_m^k \to \Sigma_m^k$ the fibre at X is:

$$\pi^{-1}(\pi(X)) = \{RX | R \in \mathrm{SO}(m)\}$$

Then we can write the curves $\gamma(s) \in \pi^{-1}(\pi(X))$ with $\gamma(0) = X$ as:

$$\gamma(s) = \gamma_{\mathrm{SO}(m)}(s)X$$

with $\gamma_{\mathrm{SO}(m)}(s)$ being any curve in $\mathrm{SO}(m)$ with $\gamma_{\mathrm{SO}(m)}(0) = I$.
As a result we have that:

$$\gamma'(0) = \gamma_{\mathrm{SO}(m)}'(0)X$$

and since $\gamma_{\mathrm{SO}(m)}'(0) \in T_I\mathrm{SO}(m) = \mathfrak{so}(m)$, we have that $\gamma'(0) = AX$ with $A \in \mathfrak{so}(m)$. This means that the vertical tangent subspace is:

$$\mathrm{Ker}d\pi_X = T_X\left(\pi^{-1}(\pi(X))\right) = \{AX | A \in M_m(\mathbb{R}),\ A^T = -A\}$$

From that we have that the horizontal tangent subspace at X is:

$$\begin{aligned}(\mathrm{Ker}d\pi_X)^\perp &= \{Z | Z \in T_X S_m^k,\ Z \perp \mathrm{Ker}d\pi_X\} \\ &= \{Z | Z \in \mathcal{M}(m, k-1),\ \mathrm{tr}(XZ^T) = 0,\ \mathrm{tr}(AXZ^T) = 0 \text{ for all } A \in \mathfrak{so}(m)\}\end{aligned}$$

For the constraint $\mathrm{tr}(AB^T) = 0$ for all $A \in \mathfrak{so}(m)$, we have that if $a_{i,j}$, $b_{i,j}$ are the respective elements of A, B then:

$$\begin{aligned}\mathrm{tr}(AB^T) = \sum_{i=1}^{m}\sum_{j=1}^{k-1} a_{ij}b_{ij} &= \sum_{i=1}^{m}\sum_{j>i}(a_{ij}b_{ij} - a_{ij}b_{ji}) \\ &= \sum_{i=1}^{m}\sum_{j>1} a_{ij}(b_{ij} - b_{ji}) = 0\end{aligned}$$

and since this must hold for all possible a_{ij} we have that for all b_{ij} it must hold $b_{ij} = b_{ji}$ so B must be symmetric. As a result we arrive at the following formulation for the horizontal tangent space:

$$\begin{aligned}(\text{Ker}d\pi_X)^{\perp} &= \{Z|Z \in \mathcal{M}(m, k-1),\ \text{tr}(XZ^T) = 0,\ \text{tr}(AXZ^T) = 0 \text{ for all } A \in \mathfrak{so}(m)\} \\ &= \{Z|Z \in \mathcal{M}(m, k-1),\ \text{tr}(XZ^T) = 0,\ XZ^T = ZX^T\}\end{aligned}$$

2.3 Geodesics on Shape Space

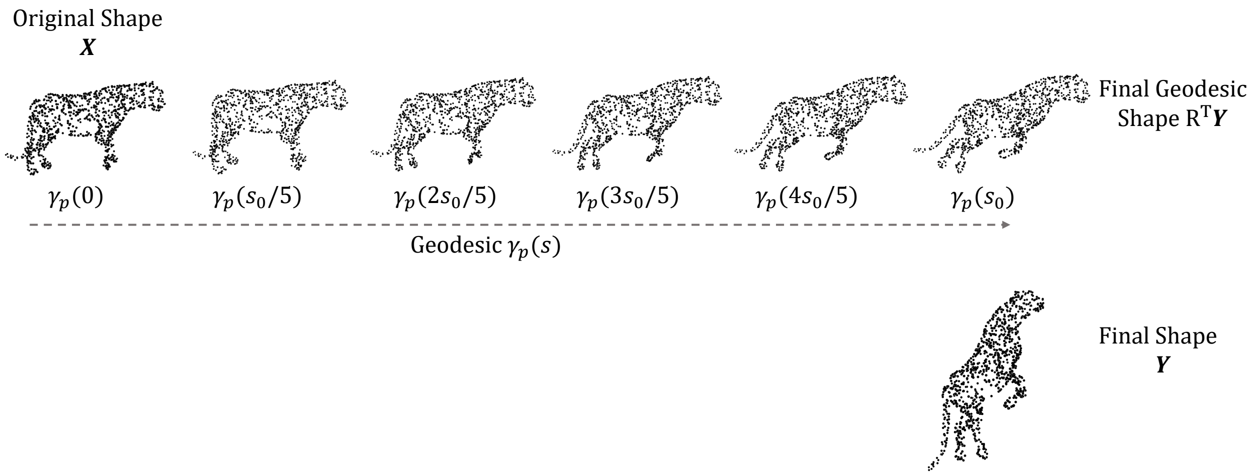

Figure 2: Evolution of the geodesic between the original shape X at $\gamma_P(0)$ and the final shape $R^T Y$ at $\gamma_P(s_0)$

To produce a horizontal geodesic we first introduce the pseudo-singular value decomposition of $YX^T = U\Lambda V$ where $U, V \in \text{SO}(m)$ and $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_m\}$, with $\lambda_1 \geq \ldots \lambda_{m-1} \geq |\lambda_m|$ and $\text{sign}(\lambda_m) = (\det(YX^T))$. Then we can write YX^T as:

$$YX^T = RQ, \quad \text{where } R = UV \text{ and } Q = V^T \Lambda V$$

From that we have that $R^T Y X^T = Q = V^T \Lambda V$ is a symmetric matrix. We can also notice that $R = UV$ is a solution of the Procrustes problem:

$$\min_{R \in \text{SO}(m)} \|R^T Y - X\|_F^2 = \max_{R \in \text{SO}(m)} \text{tr}(R^T Y X^T)$$

Using the above result, given $X, Y \in S_m^k$ such that $\pi(X) \neq \pi(Y)$, we have that $X \neq R^T Y$ for all $R \in \text{SO}(m)$. This means that we can choose $R = UV$ (where U, V are the matrices from the pseudo-singular value decompostion shown above) such that $s_0 = \arccos\left(\text{tr}(R^T Y X^T)\right)$, $0 < s_0 \leq \pi/2$. From that we set an initial velocity Z as:

$$Z = \frac{1}{\sin(s_0)} \left\{R^T Y - X \cos(s_0)\right\}$$

For Z we have that since $R^T Y X^T$ is symmetric and $X, Y \in S_m^k$ so $\operatorname{tr}\left(XX^T\right) = \operatorname{tr}\left(YY^T\right) = 1$:

$$\begin{aligned}
\operatorname{tr}(XZ^T) &= \operatorname{tr}\left(X \frac{1}{\sin(s_0)} \left\{R^T Y - X\cos(s_0)\right\}^T\right) \\
&= \frac{1}{\sin(s_0)} \left(\operatorname{tr}\left(XY^T R\right) - \operatorname{tr}(XX^T)\cos(s_0)\right) \\
&= \frac{1}{\sin(s_0)} \left(\operatorname{tr}\left(R^T Y X^T\right) - \cos(s_0)\right) \\
&= \frac{1}{\sin(s_0)} \left(\operatorname{tr}\left(R^T Y X^T\right) - \operatorname{tr}\left(R^T Y X^T\right)\right) = 0
\end{aligned}$$

additionally $Z \in S_m^k$ since:

$$\begin{aligned}
\operatorname{tr}\left(ZZ^T\right) &= \frac{1}{\sin^2(s_0)} \left(\operatorname{tr}\left(R^T Y Y^T R\right) - \operatorname{tr}\left(R^T Y X^T\right)\cos(s_0) - \operatorname{tr}\left(XY^T R\right)\cos(s_0) + \operatorname{tr}\left(XX^T\right)\cos^2(s_0)\right) \\
&= \frac{1}{\sin^2(s_0)} \left(1 - \cos^2(s_0) - \cos^2(s_0) + \cos^2(s_0)\right) = \frac{\sin^2(s_0)}{\sin^2(s_0)} = 1
\end{aligned}$$

and finally

$$\begin{aligned}
XZ^T &= \frac{1}{\sin(s_0)} \left(XY^T R - XX^T\cos(s_0)\right) \\
&= \frac{1}{\sin(s_0)} \left(R^T Y X^T - XX^T\cos(s_0)\right) = ZX^T
\end{aligned}$$

which means that $Z \in (\operatorname{Ker} d\pi_X)^\perp$.

As a result by taking the geodesic presented in 2.1, setting $Z = \frac{1}{\sin(s_0)}\left\{R^T Y - X\cos(s_0)\right\}$ and constraining $s \in [0, s_0]$ we have:

$$\begin{aligned}
\gamma_P(s) = X\cos(s) + Z\sin(s) &= X\cos(s) + \frac{1}{\sin(s_0)}\left\{R^T Y - X\cos(s_0)\right\}\cos(s) \\
&= \frac{1}{\sin(s_0)}\left(X\cos(s)\sin(s_0) + R^T Y\sin(s) - X\cos(s_0)\sin(s)\right) \\
&= \frac{1}{\sin(s_0)}\left(X\sin(s_0 - s) + R^T Y\sin(s)\right)
\end{aligned}$$

where it is easy to observe that $\gamma_P(s_0) = R^T Y$.

As shown in 2.1, $\gamma_P'(0) = Z \in (\operatorname{Ker} d\pi_X)^\perp$. Since π is a Riemannian submersion, assuming both manifolds are equiped with the Levi-Civita connection, we have that the geodesic γ_P is a horizontal geodesic in S_m^k. Additionally $\gamma_S = \pi \circ \gamma_P$ is a geodesic in Σ_m^k with same length as γ_P which is $s_0 = \arccos\left(\operatorname{tr}(R^T Y X^T)\right)$ [proposition 18.8](Gallier and Quaintance, 2020).

This concludes showing that $\gamma_S = \pi \circ \gamma_P$ is a geodesic between $\pi(X)$ and $\pi(Y) = \pi(R^T Y)$ in shape space, with length $s_0 = \arccos\left(R^T Y X^T\right)$ where $R = UV$ and $YX^T = U\Lambda V$ are matrices from a pseudo-singular value decomposition. In figure 2 we show an example of such a geodesic.

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