Generalized Threshold Graph LP

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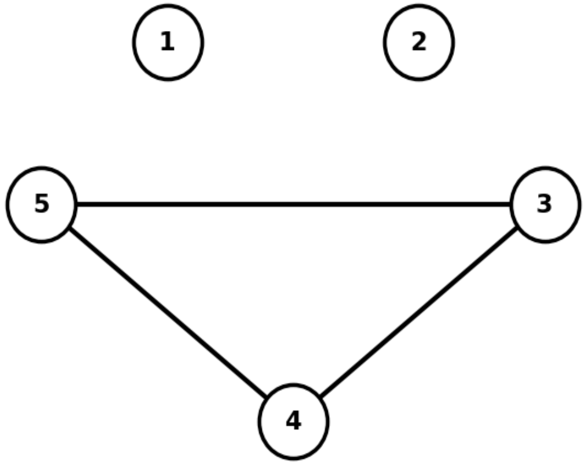
Independent k-set question

An *independent* k*-set* is a k-element vertex set U with no edges between vertices of U.



independent 3-sets

S = {123, 124, 125}

?

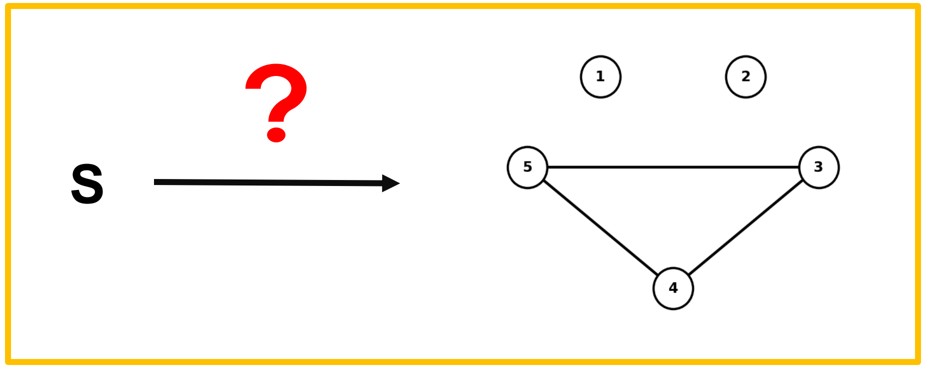
$$V(G) = V = \bigcup_{U \in S} U = \{1, \ldots, n\}$$

Given family of k-subsets S $(k \geq 2)$,

find a graph G whose independent k-sets are exactly the sets in S.

Integer program for independent k-set question

Edge variable: $x_{ij} = \begin{cases} 1, & \text{if } \{i,j\} \text{ is an edge of } G, \\ 0, & \text{otherwise.} \end{cases}$



Sparsest: $\min \sum_{1 \le i < j \le n} x_{ij}$

Densest: $\max \sum_{1 \le i < j \le n} x_{ij}$

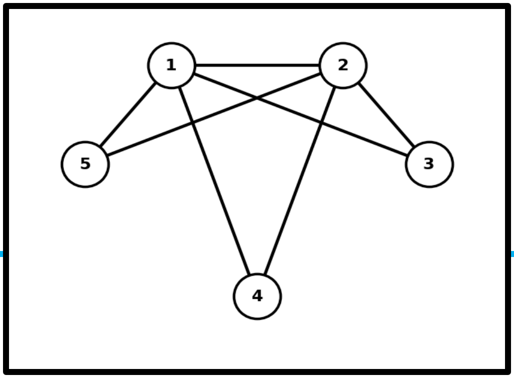
$$
\begin{aligned}
\min \quad & 0 & \\
\text{s.t.} \quad & x_{ij} = 0 & \forall U \in S,\ \forall \{i,j\} \subseteq E(U), \\
& \sum_{\{i,j\} \subseteq E(U)} x_{ij} \ge 1 & \forall U \in \binom{V}{k} \setminus S, \\
& x_{ij} \in \{0,1\} & \forall 1 \le i < j \le n.
\end{aligned}
$$

k-sets in S are independent

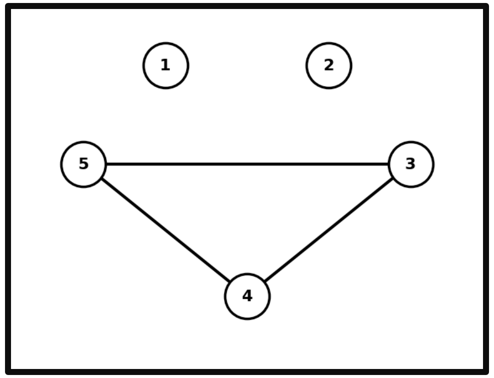
k-sets not in S are not independent

$$V(G) = V = \bigcup_{U \in S} U = \{1, \dots, n\}$$

Equivalent question: find complement graph of G



Complement graph



$$
\begin{aligned}
\min \quad & 0 \\
\text{s.t.} \quad & x_{ij} = k && \forall U \in S,\ \forall \{i,j\} \subseteq E(U), \\
& \sum_{\{i,j\} \subseteq E(U)} x_{ij} \le k - 1 && \forall U \in \binom{V}{k} \setminus S, \\
& x_{ij} \in \{0,1\} && \forall 1 \le i < j \le n.
\end{aligned}
$$

k-sets in S are k-cliques

k-sets not in S are not k-cliques

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LP relaxation program for independent k-set question

Edge variable:

$$x_{ij} \begin{cases} > 0, & \text{if } \{i,j\} \text{ is an edge of } G, \\ = 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \min \quad & 0 \\ \text{s.t.} \quad & x_{ij} = 0 && \forall U \in S,\ \forall \{i,j\} \subseteq E(U), \\ & \sum_{\{i,j\} \subseteq E(U)} x_{ij} \ \geq 1 && \forall U \in \binom{V}{k} \setminus S, \end{aligned}$$

Why don't we need additional vertices for V(G)?

$$V(G) = V = \bigcup_{U \in S} U = \{1, \ldots, n\}$$

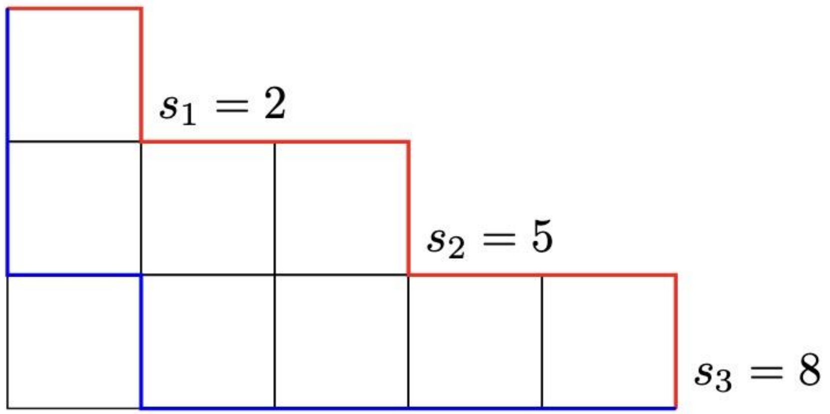
From a theorem that we proved in article [1], for k greater than 2, additional vertices do not affect whether the graph G exists.

Theorem 4.4. *Let Δ be a pure simplicial complex and σ be the intersection of all facets of Δ. If Δ is a total cut complex of a graph G, then for any additional vertex v in G, it could only form edges with vertices of σ.*

- ***Case 1:*** *If $\dim(\Delta) = n - 1$, it can form a total k-cut complex by adding k vertices/vertex.*
- ***Case 2:*** *If $\dim(\Delta) = n - 2$, it can form a total k-cut complex by adding $k - 1$ vertices/vertex.*
- ***Case 3:*** *If $\dim(\Delta) \leq n - 3$, if Δ isn't a total cut complex of a graph with same set of vertices, it cannot form a total cut complex of a graph with more vertices. In other words, if Δ is a total cut complex, it is a total k-cut complex of a graph G with the same vertex set V, where $k = n - \dim(\Delta) - 1$.*

Shifted family (up to relabeling)

A family $\mathcal{S} \subseteq \binom{V}{k}$ is *shifted* if whenever $U \in \mathcal{S}$ and $i < j$ with $j \in U$, $i \notin U$, then $(U \setminus \{j\}) \cup \{i\} \in \mathcal{S}$.



Red Path: $(s_1 = 2, s_2 = 5, s_3 = 8)$

Blue Path: $(u_1 = 1, u_2 = 2, u_3 = 4)$

For example, $\mathcal{S} = \{123, 124, 125\}$ is shifted.

Family of k-sets $\mathcal{S}$: $(s_1, s_2, s_3) = 125$.

k-sets $U \in \mathcal{S}$: $(u_1, u_2, u_3) = 123,\ 124,\ 125$.

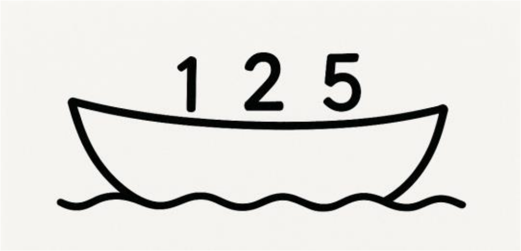
$u_i \leq s_i$ for all i. $u_1 < ... < u_k$.

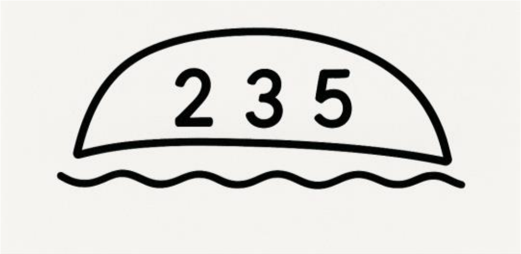
Shifted <=> K-sum threshold

Theorem A family $\mathcal{S} \subseteq \binom{V}{k}$ is shifted if and only if it is k-**sum-threshold**, i.e., there exist weights $w : V \to \mathbb{R}$ and a threshold $t \in \mathbb{R}$ such that

$$\mathcal{S} = \left\{ U \in \binom{V}{k} : \sum_{u \in U} w(u) \leq t \right\}.$$

S = {123, 124, 125}





Shifted <=> K-sum threshold

Theorem 2. *Let $V = \{1, \ldots, n\}$ and fix k. A family $\mathcal{S} \subseteq \binom{V}{k}$ is shifted (w.r.t. $1 < \cdots < n$) if and only if it is k-sum-threshold, i.e., there exist weights $w : V \to \mathbb{R}$ and a threshold $t \in \mathbb{R}$ such that*

$$\mathcal{S} = \Big\{ U \in \tbinom{V}{k} : \sum_{u \in U} w(u) \le t \Big\}.$$

Proof. **(k-sum-threshold $\Rightarrow$ shifted).** Assume $\mathcal{S}$ is k-sum-threshold, so

$$\mathcal{S} = \Big\{ U : \sum_{u \in U} w(u) \le t \Big\}$$

for some w and t. Relabel V so that $w(1) \le w(2) \le \cdots \le w(n)$. If $U \in \mathcal{S}$ and $i < j$ with $j \in U$, $i \notin U$, let $U' = (U \setminus \{j\}) \cup \{i\}$. Then

$$\sum_{u \in U'} w(u) = \sum_{u \in U} w(u) - w(j) + w(i) \le \sum_{u \in U} w(u) \le t,$$

so $U' \in \mathcal{S}$ and $\mathcal{S}$ is shifted.

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Shifted <=> K-sum threshold

(Shifted $\Rightarrow$ k-sum-threshold). Assume $\mathcal{S}$ is shifted. By the upper bound sequence representation, there is a strictly increasing sequence $(s_1, \ldots, s_k)$ such that

$$U = \{u_1 < \cdots < u_k\} \in \mathcal{S} \quad \Longleftrightarrow \quad u_i \leq s_i \text{ for all } i.$$

Choose a large base $B > 1$ and define

$$w(i) := B^i \quad (i = 1, \ldots, n), \qquad \phi(U) := \sum_{u \in U} w(u).$$

Let $S^* = \{s_1, \ldots, s_k\}$ and set

$$t := \phi(S^*) = \sum_{i=1}^{k} B^{s_i}.$$

(i) If $U \in \mathcal{S}$, then $u_i \leq s_i$ for all i, so $B^{u_i} \leq B^{s_i}$ for each i and hence

$$\phi(U) = \sum_{i=1}^{k} B^{u_i} \leq \sum_{i=1}^{k} B^{s_i} = t.$$

(ii) If $U \notin \mathcal{S}$, then there is a first index j with $u_j > s_j$, while $u_i \leq s_i$ for all $i < j$. Because the weights $w(i) = B^i$ grow very fast in i, the single larger entry $u_j > s_j$ makes the sum $\phi(U)$ strictly bigger than $\phi(S^*)$ once B is chosen large enough. In other words, for large B we have $\phi(U) > t$ whenever $U \notin \mathcal{S}$.

Thus

$$U \in \mathcal{S} \iff \phi(U) \leq t \iff \sum_{u \in U} w(u) \leq t,$$

so $\mathcal{S}$ is k-sum-threshold. $\square$

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Linear Programming to determine shiftiness

- $\delta^* > 0$: k-sum threshold graph exist.
- infeasible or $\delta^* = 0$: no k-sum threshold graph.

$$\max \quad \delta$$

$$\left.\begin{aligned} &\sum_{i \in U} w_i \le t - \delta && \forall U \in \mathcal{S}, \\ &\sum_{i \in U} w_i \ge t + \delta && \forall U \in \binom{V}{k} \setminus \mathcal{S}, \end{aligned}\right\} \quad \mathcal{S} = \left\{ U \in \binom{V}{k} : \sum_{u \in U} w(u) \le t \right\}.$$

$$\left.\begin{aligned} &0 \le w_i \le 1 && \forall i \in V, \\ &0 \le t \le k, \\ &\delta \ge 0. \end{aligned}\right\} \quad \text{prevent unboundedness}$$

K-sum threshold graph question

$k = 2$: threshold graphs.

$$\mathcal{S} = \{\text{independent } k\text{-sets of } G\} \text{ is shifted}/k\text{-sum-threshold} \iff G \text{ is a } k\text{-sum-threshold graph.}$$

$$U \subseteq V,\ |U| = k \text{ is independent in } G \iff \sum_{u \in U} w(u) \leq t.$$

Combinatorial structure defined by edges $\Longleftrightarrow$ **Linear information on vertices**

Question: Is $\mathcal{S}$ the family of independent k-sets of a k-sum threshold graph G?

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Mixed Integer Linear Programming

Question: Is $\mathcal{S}$ the family of independent k-sets of a k-sum threshold graph G?

$$\max \quad \delta$$

Independent k-set Constraints

$$
\begin{aligned}
\text{s.t.} \quad & x_{ij} = 0 && \forall U \in S,\ \forall \{i,j\} \subseteq E(U), \\
& \sum_{\{i,j\} \subseteq U} x_{ij} \ \geq\ 1 && \forall U \in \binom{V}{k} \setminus S, \\
& x_{ij} \in \{0,1\} && \forall 1 \leq i < j \leq n.
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{i \in U} w_i \ \leq\ t - \delta && \forall U \in \mathcal{S}, \\
& \sum_{i \in U} w_i \ \geq\ t + \delta && \forall U \in \tbinom{V}{k} \setminus \mathcal{S}, \\
& 0 \leq w_i \leq 1 && \forall i \in V, \\
& 0 \leq t \leq k, \\
& \delta \geq 0.
\end{aligned}
$$

Shifted Constraints

- $\delta^* > 0$: k-sum threshold graph exist.
- infeasible or $\delta^* = 0$: no k-sum threshold graph.

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AMPL Code

```
# ---- Sets & data ----
set V ordered;                # vertices, e.g., 1..n
set KSETS;                    # index of all k-sets
set S within KSETS;           # the "independent" family
param inc {KSETS, V} binary;  # 1 if vertex v ∈ U, else 0

# Undirected edge index
set E := {i in V, j in V: i < j};

# ---- Variables ----
# Edges (use binary for an actual graph; relax to [0,1] for LP)
var x {E} binary;

# Weights / threshold margin (shifted test)
var w {V} >= 0, <= 1;
var t >= 0, <= 5;
var delta >= 0;
```

```
# ---- Graph side: exact independent k-sets = S ----
# All edges inside every F in S must be zero
s.t. indep_F {U in S}:
    sum { (i,j) in E: inc[U,i] = 1 and inc[U,j] = 1 } x[i,j] = 0;

# Every other k-set must contain at least one edge
s.t. nonindep_U {U in KSETS diff S}:
    sum { (i,j) in E: inc[U,i] = 1 and inc[U,j] = 1 } x[i,j] >= 1;

# ---- Shifted / k-threshold side on the same S ----
s.t. pos {U in S}:
    sum {v in V} inc[U,v] * w[v] <= t - delta;

s.t. neg {U in KSETS diff S}:
    sum {v in V} inc[U,v] * w[v] >= t + delta;

# ---- Objective ----
maximize margin: delta;
```

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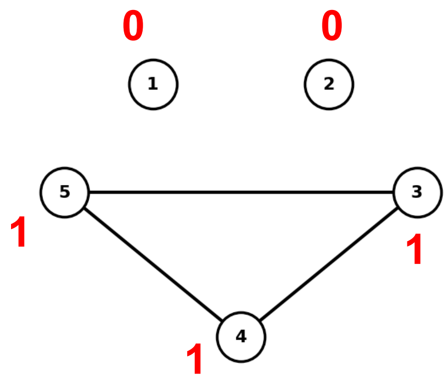
AMPL: k-sum threshold graph

```
set V := 1 2 3 4 5;

# All 3-sets, each with a name (10 of them)
set KSETS := u123 u124 u125 u134 u135 u145 u234 u235 u245 u345;

# The independent triples S (edit these)
set S := u123 u124 u125;

# Incidence table: rows = KSETS, columns = V
param inc:
        1  2  3  4  5 :=
 u123   1  1  1  0  0
 u124   1  1  0  1  0
 u125   1  1  0  0  1
 u134   1  0  1  1  0
 u135   1  0  1  0  1
 u145   1  0  0  1  1
 u234   0  1  1  1  0
 u235   0  1  1  0  1
 u245   0  1  0  1  1
 u345   0  0  1  1  1 ;
```

```
ampl: display w, t, delta;
w [*] :=
1  0
2  0
3  1
4  1
5  1
;

t = 1.5
delta = 0.5
```

```
ampl: display x;
x :=
1 2   0
1 3   0
1 4   0
1 5   0
2 3   0
2 4   0
2 5   0
3 4   1
3 5   1
4 5   1
;
```

S = {123, 124, 125}



3-sum threshold graph

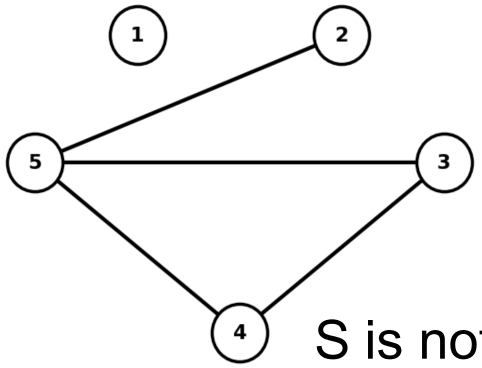
AMPL: non-k-sum threshold graph

```
set V := 1 2 3 4 5;

# All 3-sets, each with a name (10 of them)
set KSETS := u123 u124 u125 u134 u135 u145 u234 u235 u245 u345;

# The independent triples S (edit these)
set S := u123 u124 u135;

# Incidence table: rows = KSETS, columns = V
param inc:
        1  2  3  4  5 :=
 u123   1  1  1  0  0
 u124   1  1  0  1  0
 u125   1  1  0  0  1
 u134   1  0  1  1  0
 u135   1  0  1  0  1
 u145   1  0  0  1  1
 u234   0  1  1  1  0
 u235   0  1  1  0  1
 u245   0  1  0  1  1
 u345   0  0  1  1  1 ;
```

```
ampl: display w, t, delta;
w [*] :=
1  0
2  0
3  0
4  0
5  0
;

t = 0
delta = 0
```

```
ampl: display x;
x :=
1 2    0
1 3    0
1 4    0
1 5    0
2 3    0
2 4    0
2 5    1
3 4    1
3 5    0
4 5    1
;
```

S = {123, 124, 135}



S is not shifted, so G is not a 3-sum threshold graph

MILP Complexity and feasible region

Binary vars: $\Theta(n^2)$; k-set constraints: $\Theta(\binom{n}{k})$.

The MILP is NP-hard and only practical for small n and k, but this is enough for our purposes: we use it to run examples, spot patterns, and then conjecture and prove the combinatorial structure. In fact, in article[1] we used Python experiments to discover a combinatorial description of the independent k-sets of k-sum-threshold graphs and then proved it formally.

The LP analysis is useful because every feasible solution corresponds to a graph that satisfies the given constraints.

$n\backslash p$	1	2	3	4
1	1			
2	2	12		
3	3	13, 23	123	
4	4	14, 24, 34	124, 134, 234	1234
5	5	15, 25, 45	125, 135, 235, 145, 245, 345	1235, 1245, 1345, 2345
6	6	16, 26, 56	126, 136, 236, 156, 256, 456	1236, 1246, 1346, 2346, 1256, 1356, 2356, 1456, 2456, 3456
7	7	17, 27, 67	127, 137, 237, 167, 267, 567	1237, 1247, 1347, 2347, 1267, 1367, 2367, 1567, 2567, 4567

Table 2: Upper Bound Sequence Representation of Total Cut Complexes in Shifted Complexes

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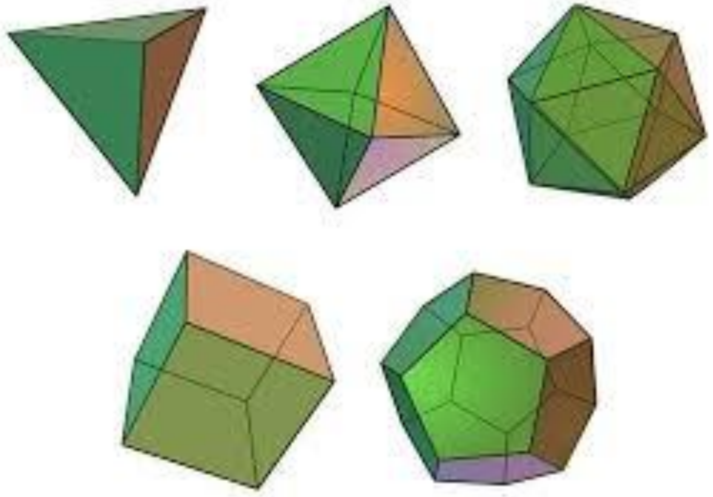
Independence Polytopes and Polytopes of degree sequence

- Independence polytope: size-k independent sets are integer points in the intersection of the independence polytope with an affine hyperplane

- Polytope of degree sequence: threshold graphs are the extreme points.

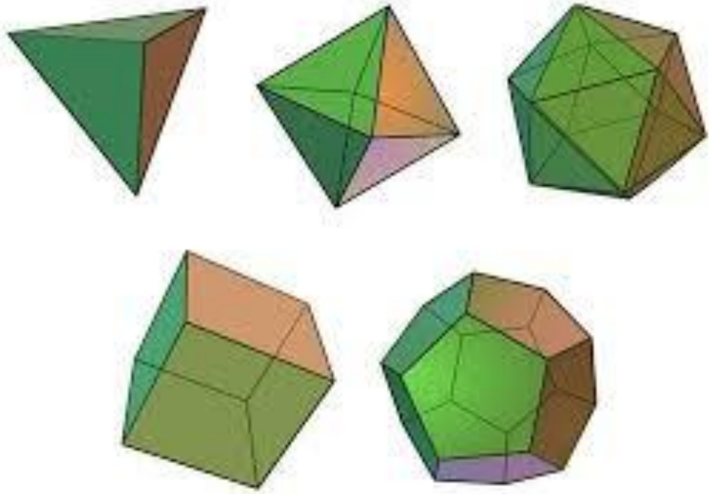


Independence Polytopes and Polytopes of degree sequence

- Independence polytope:

$$P_{\text{ind}}(G) = \text{conv}\{\chi^I : I \text{ independent}\}.$$

$$\sum_i \chi_i^I = |I| \;\Rightarrow\; \{0,1\}^V \cap P_{\text{ind}}(G) \cap \{\sum_i x_i = k\} = \{\text{independent sets of size } k\}.$$



- Degree–sequence polytope:

$$P_{\text{deg}} = \text{conv}\{d(G) : G \text{ on } n \text{ vertices}\}.$$

$$\text{extreme points of } P_{\text{deg}} \iff \text{threshold graphs(2-sum threshold graph)}.$$

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Bibliography

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[2] Bayer, M., Denker, M., Milutinović, M. J., Rowlands, R., Sundaram, S., & Xue, L. (2025). Total cut complexes of graphs. *Discrete & Computational Geometry*, *73*(2), 500-527.

[3] Klivans, C. J. (2003). *Combinatorial properties of shifted complexes* (Doctoral dissertation, Massachusetts Institute of Technology).

Thank you