V and H Representations of Convex Polyhedra

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December 2, 2025

What Are V and H Representations?

Let P be a polyhedron in $\mathbb{R}^n$. Then we have the following representations:

V-Representation

$$P = \text{conv}\{v_1, v_2, ..., v_k\} + \text{cone}\{e_1, e_2, ..., e_l\}$$

Where $\{v_1, v_2, ..., v_k\}$ is the set of extreme points of P and $\{e_1, e_2, ..., e_l\}$ is the set of extreme rays of P.

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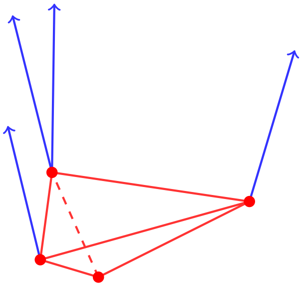
H-Representation

$$Ax \leq b$$

Where A is $m \times n$, x is a $n \times 1$ vector with entries consisting of variables $x_1, x_2 ..., x_n$, and b is a $n \times 1$ vector consisting of scalars.

What Types of Halfplanes are There?

The first sstep is to identify the types of hyperplanes that could form the half spaces. There are two types of edges, which creates three types of facets for our polyhedra:



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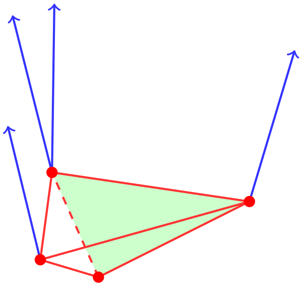
- Bounded edges only



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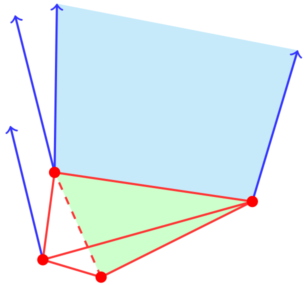
- Bounded and unbounded edges



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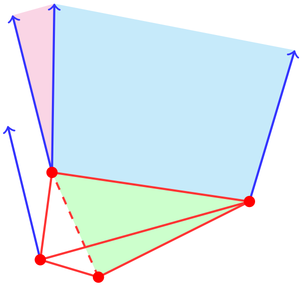
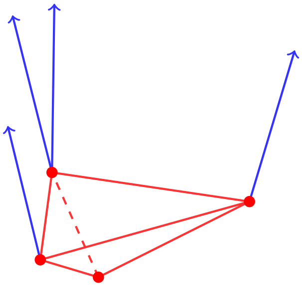
- Bounded edges only
- Bounded and unbounded edges
- Unbounded edges only



Types of Edges on Hyperplanes

In dimension m, hyperplanes have dimension $m - 1$.

We will use $m - 1$ linearly independent vectors that sit in that hyperplane to form a basis.



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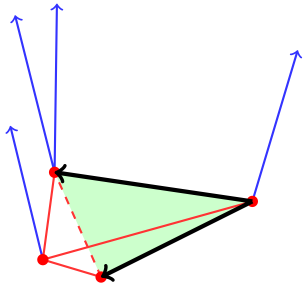
For a facet with only bounded edges, these can consist of the vectors from one vertex to $m-1$ other vertices that lie on a facet.



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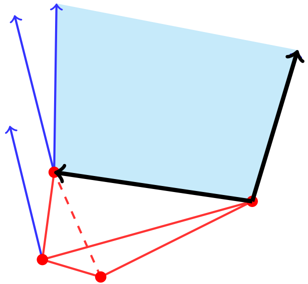
In dimension m, hyperplanes have dimension $m - 1$.

We will use $m - 1$ linearly independent vectors that sit in that hyperplane to form a basis.

For a facet with both bounded and unbounded edges, these can consist of both vectors from one vertex to other vertices and also extreme rays.



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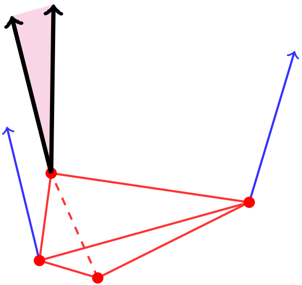

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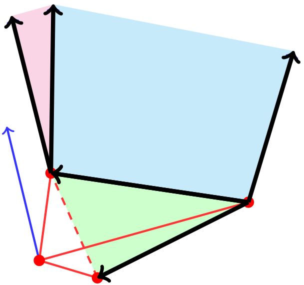
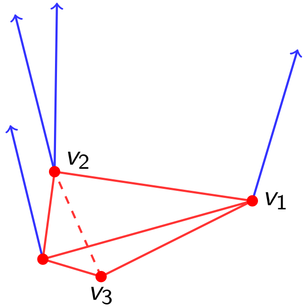


Finalizing the H-Representation

Now test whether the produced hyperplanes are supporting hyperplanes at the testing point. If so, add the half space containing the polyhedra to the H-representation

Removing redundant constraints (which will include duplicates) will finalize the H-representation to its simplest form.



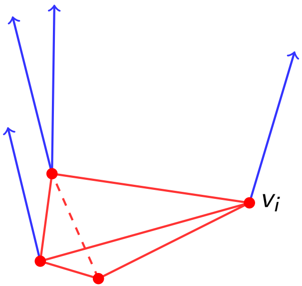
Overview of V to H Representations

To go from a V to H Representation, we will

- For each vertex, create a set containing the vectors from that vertex to the other vertices and the extreme rays.
- For each combination of size dimension minus one from this set, create the hyperplane that goes through the vertex
- Check if the polyhedron sits entirely on one side of the hyperplane
- If so then that half space joins the H representation. Otherwise neither half space is in the H-representation.

Finding Intersections of half spaces

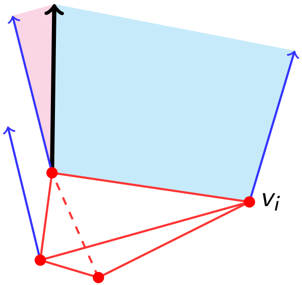
Recall from class that if we are in dimension m, then vertices are the intersection of m hyperplanes.



If x is a feasible solution, then it is a vertex of P.

Finding a Characteristic of Extreme Rays

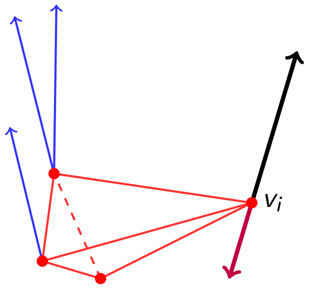
Notice that extreme rays will be parallel the intersection of $m - 1$ hyperplanes.
Then any extreme ray can be represented as a vector that is orthogonal to the normal vectors of the hyperplanes.



Narrowing to Regression Directions

Now to narrow to regression directions, we check if x or $-x$ are not regression directions.

To do so, simply check if $v_i(+|-)x$ is feasible for all vertices v_i. If so, then $(+|-)x$ is a regression direction of P.



Overview of H to V representations

To go from a H to V Representation, we will

- Find all intersections of m hyperplanes in the H representation
- Check if these are in the polyhedron, and if so they are in the V-representation
- Find the the vector orthogonal to $m - 1$ of the hyperplanes normal vectors
- Check if the positive or negative direction of this vector is a regression direction
- Remove any of these vectors that are conic combinations of the others