Generalized Threshold Graph MILP

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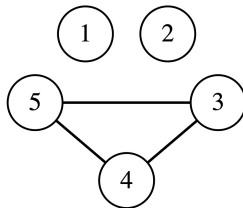
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Independent k-set

Definition

Definition. An independent k-set is a k-element vertex set $U \subseteq V$ with no edges between vertices of U.

Example:

$S = \{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$ is a family of independent 3-sets in the graph on vertex set $V = \{1,2,3,4,5\}$.

* Throughout this talk, all graphs are simple (no loops or multiple edges).

Independent k-set question

We are interesting in the inverse question...

Question

Given a family S of k-sets with $k \geq 2$, does there exist a graph G whose independent k-sets are exactly the sets in S?

Let the vertex set of our graph to be

$$V(G) = V = \bigcup_{U \in S} U = \{1, \ldots, n\}.$$

Additional vertices do not affect whether the graph G exist.

Integer program for independent *k*-set question

Edge variables:

$$x_{ij} = \begin{cases} 1, & \text{if } \{i,j\} \text{ is an edge of } G, \\ 0, & \text{otherwise,} \end{cases} \qquad (1 \le i < j \le n).$$

Integer program:

$$\min 0$$

subject to

$$x_{ij} = 0 \qquad \qquad \forall U \in S, \ \forall \{i,j\} \subseteq U,$$

$$\sum_{\{i,j\} \subseteq U} x_{ij} \ge 1 \qquad \qquad \forall U \in \binom{V}{k} \setminus S,$$

$$x_{ij} \in \{0,1\} \qquad \qquad \forall \, 1 \le i < j \le n.$$

LP relaxation

Edge variables:

$$x_{ij} > 0, \text{if } \{i, j\} \text{ is an edge of } G.$$

$$x_{ij} = 0, \text{otherwise.} \qquad (1 \leq i < j \leq n).$$

LP relaxation program:

$$\min 0$$

subject to

$$x_{ij} = 0 \qquad\qquad \forall U \in S, \ \forall \{i, j\} \subseteq U,$$

$$\sum_{\{i,j\} \subseteq U} x_{ij} \geq 1 \qquad\qquad \forall U \in \binom{V}{k} \setminus S.$$

Equivalent integer program: find complement graph (S is the family of k-cliques)

Edge variables:

$$x_{ij} = \begin{cases} 1, & \text{if } \{i,j\} \text{ is an edge of } G, \\ 0, & \text{otherwise,} \end{cases} \qquad (1 \le i < j \le n).$$

Integer program:

$$\min 0$$

subject to

$$x_{ij} = k \qquad\qquad \forall U \in S, \ \forall \{i,j\} \subseteq U,$$

$$\sum_{\{i,j\} \subseteq U} x_{ij} \le k-1 \qquad\qquad \forall U \in \binom{V}{k} \setminus S,$$

$$x_{ij} \in \{0,1\} \qquad\qquad \forall\, 1 \le i < j \le n.$$

Shifted family (up to relabeling)

Definition

Definition. A family of k-sets S is shifted (up to relabeling) if whenever $U \in S$ and $i < j$ with $i \notin U$, $j \in U$, then

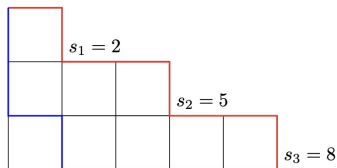$$(U \setminus \{j\}) \cup \{i\} \in S.$$

Example: $S = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$ is shifted (up to relabeling).

Question

Given a family S of k-sets with $k \geq 2$, is it shifted?

Shifted family (up to relabeling)

- The family can be encoded by a nondecreasing sequence $(s_1, \ldots, s_k)$.
- Every $U = (u_1, \ldots, u_k) \in S$ satisfies $u_1 < \cdots < u_k$ and $u_i \leq s_i$ for all i.

Red Path: $(s_1 = 2, s_2 = 5, s_3 = 8)$
Blue Path: $(u_1 = 1, u_2 = 2, u_3 = 4)$

Shifted $\iff$ k-sum-threshold

Theorem

Let $S \subseteq \binom{V}{k}$ with $V = \bigcup_{U \in S} U$. Then the following are equivalent:

- **Shifted (up to relabeling):** whenever $U \in S$ and $i < j$ with $i \notin U$, $j \in U$, also $(U \setminus \{j\}) \cup \{i\} \in S$.

- k-**sum-threshold:** there exist weights $w : V \to \mathbb{R}$ and $t \in \mathbb{R}$ such that

$$S = \Big\{ U \in \binom{V}{k} : \sum_{u \in U} w(u) \leq t \Big\}.$$

Proof: k-sum-threshold $\Rightarrow$ shifted

Assume $\mathcal{S}$ is k-sum-threshold.

There exist weights $w : V \to \mathbb{R}$ and a threshold $t \in \mathbb{R}$ such that

$$\mathcal{S} = \Big\{ U \in \tbinom{V}{k} : \sum_{u \in U} w(u) \leq t \Big\}.$$

Relabel $V = \{1, \ldots, n\}$ so that

$$w(1) \leq w(2) \leq \cdots \leq w(n).$$

Let $U \in \mathcal{S}$ and suppose $i < j$ with $j \in U$ and $i \notin U$. Define

$$U' := (U \setminus \{j\}) \cup \{i\}.$$

Then

$$\sum_{u \in U'} w(u) = \sum_{u \in U} w(u) - w(j) + w(i) \leq \sum_{u \in U} w(u) \leq t.$$

Hence $U' \in \mathcal{S}$, so $\mathcal{S}$ is shifted with respect to $1 < \cdots < n$.

Shifted $\Rightarrow$ k-sum-threshold (setup)

Assume $\mathcal{S} \subseteq \binom{V}{k}$ is shifted (w.r.t. $1 < \cdots < n$).

By the upper bound sequence representation, there is a strictly increasing sequence $(s_1, \ldots, s_k)$ such that, for

$$U = \{u_1 < \cdots < u_k\}, \quad U \in \mathcal{S} \iff u_i \leq s_i \text{ for all } i = 1, \ldots, k.$$

Choose a large base $B > 1$ and define the weights and sum:

$$w(i) := B^i \quad (i = 1, \ldots, n), \qquad \phi(U) := \sum_{u \in U} w(u).$$

Let the "maximal" set be

$$S^* := \{s_1, \ldots, s_k\}, \quad t := \phi(S^*) = \sum_{i=1}^{k} B^{s_i}.$$

Goal: show

$$U \in \mathcal{S} \iff \phi(U) \leq t \iff \sum_{u \in U} w(u) \leq t.$$

Shifted $\Rightarrow$ k-sum-threshold (conclusion)

$$w(i) = B^i, \quad \phi(U) = \sum_{u \in U} B^u, \quad t = \phi(S^*) = \sum_{i=1}^{k} B^{s_i}.$$

1. If $U \in \mathcal{S}$, write $U = \{u_1 < \cdots < u_k\}$. Then $u_i \leq s_i$ for all i, so

$$\phi(U) = \sum_{i=1}^{k} B^{u_i} \leq \sum_{i=1}^{k} B^{s_i} = t.$$

2. If $U \notin \mathcal{S}$, let j be the first index with $u_j > s_j$ (so $u_i \leq s_i$ for all $i < j$). The term B^{u_j} is strictly larger than B^{s_j}, and for sufficiently large B this "first violation" dominates all later terms. Concretely, we can choose B so large that

$$B^{u_j} - B^{s_j} > \sum_{i>j} \left(B^{s_i} - B^{u_i} \right),$$

which implies

$$\phi(U) > \phi(S^*) = t.$$

Thus, for such a choice of B,

$$U \in \mathcal{S} \iff \phi(U) \leq t \iff \sum_{u \in U} w(u) \leq t,$$

so $\mathcal{S}$ is k-sum-threshold.

Linear programming to determine shiftiness of S

Goal: Find a positive margin δ that separate S with its complement.

Variables:

$$w_i \quad (i \in V), \qquad t, \qquad \delta \geq 0.$$

Linear program:

$$\max \delta$$

subject to

$$\sum_{i \in U} w_i \leq t - \delta \qquad\qquad \forall U \in S,$$

$$\sum_{i \in U} w_i \geq t + \delta \qquad\qquad \forall U \in \binom{V}{k} \setminus S,$$

$$0 \leq w_i \leq 1 \qquad\qquad \forall i \in V,$$

$$0 \leq t \leq k,$$

$$\delta \geq 0.$$

Interpreting the LP

- If the optimal value satisfies $\delta^* > 0$, then S is shifted/k-sum threshold.
- If the LP is infeasible or $\delta^* = 0$, then S is not shifted/not k-sum threshold.
- For any k-set $U \in S$, the sum of weights $\sum_{i \in U} w_i$ lies *within* (below) the threshold.
- For any k-set $U \notin S$, the sum of weights $\sum_{i \in U} w_i$ lies *beyond* (above) the threshold.
- We bound w_i and t to prevent unboundedness.

k-sum threshold graph question

What if S is a family of independent k-sets of a graph G and S is also shifted?

Then there exist $w : V(G) \to \mathbb{R}$ and $t \in \mathbb{R}$ such that

$$U \subseteq V(G),\ |U| = k \text{ is independent in } G \iff \sum_{u \in U} w(u) \le t.$$

combinatorial structure defined by edges $\iff$ linear information on vertices

We call graph G is a k-sum-threshold graph.

Question

Given a family S of k-sets with $k \ge 2$, does there exist a k-sum-threshold graph G whose independent k-sets are exactly the sets in S?

Mixed Integer Linear Program

Variables:

$$x_{ij} \in \{0,1\} \ (1 \le i < j \le n), \quad w_i \in [0,1] \ (i \in V), \quad t \in [0,k], \ \delta \ge 0.$$

MILP: max δ, subject to

$$\sum_{\{i,j\} \subseteq E(U)} x_{ij} = 0 \qquad \forall\, U \in S, \qquad\qquad \sum_{i \in U} w_i \le t - \delta \quad \forall\, U \in S,$$

$$\sum_{\{i,j\} \subseteq E(U)} x_{ij} \ge 1 \qquad \forall\, U \in \binom{V}{k} \setminus S, \qquad \sum_{i \in U} w_i \ge t + \delta \quad \forall\, U \in \binom{V}{k} \setminus S,$$

$$x_{ij} \in \{0,1\} \quad \forall\, 1 \le i < j \le n. \qquad 0 \le w_i \le 1 \qquad \forall\, i \in V,$$

$$0 \le t \le k,$$

$$\delta \ge 0.$$

AMPL examples

Example 1: shifted, 3-sum-threshold family

$$S = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$$

- The AMPL model finds feasible weights w, threshold t, and edge variables x.
- $\delta^* \neq 0$. We obtain a 3-sum-threshold graph whose independent 3-sets are exactly the sets in S.

Example 2: non-shifted, not 3-sum-threshold

$$S = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}\}$$

- There exists a graph G with independent 3-sets being S.
- However, $\delta^* = 0$. There is no 3-sum-threshold graph with independent 3-sets exactly S.

MILP complexity and feasible region

- Since there are $\binom{n}{k}$ many k-sets, the formulation has exponentially many constraints, and the resulting MILP is NP-hard to solve in general. It is therefore not very practical for large n and k.

- NP-hard algorithm is still useful. For small values of n and k, we can search over families of independent k-sets that correspond to k-sum-threshold graphs, observe combinatorial patterns, and then formally prove the resulting patterns. In fact, this analysis is being done in my article.

- LP analysis is useful since its feasible solutions correspond to all possible graphs that satisfies the given constraints.

Independence polytopes and degree-sequence polytopes

- **Independence polytope:**

$$P_{\mathrm{ind}}(G) = \mathrm{conv}\{\chi^U : U \subseteq V \text{ independent}\},$$
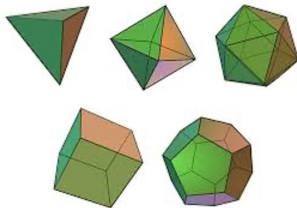
$$\text{where } (\chi^U)_i = \begin{cases} 1, & i \in U, \\ 0, & i \notin U \end{cases}$$

  and independent k-sets are the integer points with $\sum_{i=1}^{n} x_i = k$.

- **Degree-sequence polytope:**

  $P_{\mathrm{deg}}(n) = \mathrm{conv}\{d(G) = (d_1, ..., d_n) : G \text{ is a simple graph on } \{1, \ldots, n\}\},$

  whose extreme points are exactly the degree sequences of threshold graphs.

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Thank you!