Generalized Threshold Graph MILP

Baihan Liu

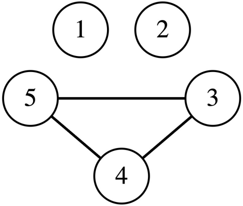
University of Colorado Denver

Independent k-set

Definition

Definition. An independent k-set is a k-element vertex set $U \subseteq V$ with no edges between vertices of U.

Example:



$S = \{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$ is a family of independent 3-sets in the graph on vertex set $V = \{1,2,3,4,5\}$.

* Throughout this talk, all graphs are simple (no loops or multiple edges).

Independent k-set question

We are interesting in the inverse question...

Question

Given a family S of k-sets with $k \geq 2$, does there exist a graph G whose independent k-sets are exactly the sets in S?

Let the vertex set of our graph to be

$$V(G) = V = \bigcup_{U \in S} U = \{1, \ldots, n\}.$$

Additional vertices do not affect whether the graph G exist.

Integer program for independent k-set question

Edge variables:

$$x_{ij} = \begin{cases} 1, & \text{if } \{i,j\} \text{ is an edge of } G, \\ 0, & \text{otherwise,} \end{cases} \qquad (1 \leq i < j \leq n).$$

Integer program:

$$\min 0$$

subject to

$$\begin{aligned} x_{ij} &= 0 && \forall U \in S,\ \forall \{i,j\} \subseteq U, \\ \sum_{\{i,j\} \subseteq U} x_{ij} &\geq 1 && \forall U \in \binom{V}{k} \setminus S, \\ x_{ij} &\in \{0,1\} && \forall\, 1 \leq i < j \leq n. \end{aligned}$$

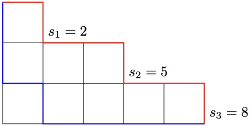
Shifted family (up to relabeling)

Definition

Definition. A family of k-sets S is shifted (up to relabeling) if whenever $U \in S$ and $i < j$ with $i \notin U$, $j \in U$, then

$$(U \setminus \{j\}) \cup \{i\} \in S.$$

Example: $S = \{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$ is shifted (up to relabeling).



Red Path: $(s_1 = 2, s_2 = 5, s_3 = 8)$
Blue Path: $(u_1 = 1, u_2 = 2, u_3 = 4)$

Question

Given a family S of k-sets with $k \geq 2$, is it shifted?

Shifted $\iff$ k-sum-threshold

Theorem

Let $S \subseteq \binom{V}{k}$ with $V = \bigcup_{U \in S} U$. Then the following are equivalent:

- **Shifted (up to relabeling):** whenever $U \in S$ and $i < j$ with $i \notin U$, $j \in U$, also $(U \setminus \{j\}) \cup \{i\} \in S$.
- ***k*-sum-threshold:** there exist weights $w : V \to \mathbb{R}$ and $t \in \mathbb{R}$ such that

$$S = \left\{ U \in \binom{V}{k} : \sum_{u \in U} w(u) \leq t \right\}.$$

Linear programming to determine shiftiness of S

Goal: Find a positive margin δ that separate S with its complement.
Variables:

$$w_i \quad (i \in V), \qquad t, \qquad \delta \geq 0.$$

Linear program:

$$\max \ \delta$$

subject to

$$\begin{aligned}
\sum_{i \in U} w_i &\leq t - \delta && \forall U \in S, \\
\sum_{i \in U} w_i &\geq t + \delta && \forall U \in \binom{V}{k} \setminus S, \\
0 \leq w_i &\leq 1 && \forall i \in V, \\
0 \leq t &\leq k, \\
\delta &\geq 0.
\end{aligned}$$

k-sum threshold graph question

What if S is a family of independent k-sets of a graph G and S is also shifted?

Then there exist $w : V(G) \to \mathbb{R}$ and $t \in \mathbb{R}$ such that

$$U \subseteq V(G),\ |U| = k \text{ is independent in } G \quad \Longleftrightarrow \quad \sum_{u \in U} w(u) \leq t.$$

combinatorial structure defined by edges $\Longleftrightarrow$ linear information on vertices

We call graph G is a k-sum-threshold graph.

Question

Given a family S of k-sets with $k \geq 2$, does there exist a k-sum-threshold graph G whose independent k-sets are exactly the sets in S?

Mixed Integer Linear Program

Variables:

$$x_{ij} \in \{0,1\}\ (1 \le i < j \le n), \quad w_i \in [0,1]\ (i \in V), \quad t \in [0,k],\ \delta \ge 0.$$

MILP: max δ, subject to

$$\begin{aligned}
\sum_{\{i,j\} \subseteq E(U)} x_{ij} &= 0 && \forall\, U \in S, \\
\sum_{\{i,j\} \subseteq E(U)} x_{ij} &\ge 1 && \forall\, U \in \binom{V}{k} \setminus S, \\
x_{ij} &\in \{0,1\} && \forall\, 1 \le i < j \le n.
\end{aligned}
\qquad
\begin{aligned}
\sum_{i \in U} w_i &\le t - \delta && \forall\, U \in S, \\
\sum_{i \in U} w_i &\ge t + \delta && \forall\, U \in \binom{V}{k} \setminus S, \\
0 \le w_i &\le 1 && \forall\, i \in V, \\
0 \le t &\le k, \\
\delta &\ge 0.
\end{aligned}$$

AMPL examples

Example 1: shifted, 3-sum-threshold family

$$S = \{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$$

- The AMPL model finds feasible weights w, threshold t, and edge variables x.
- $\delta^* \neq 0$. We obtain a 3-sum-threshold graph whose independent 3-sets are exactly the sets in S.

Example 2: non-shifted, not 3-sum-threshold

$$S = \{\{1,2,3\},\{1,2,4\},\{1,3,5\}\}$$

- There exists a graph G with independent 3-sets being S.
- However, $\delta^* = 0$. There is no 3-sum-threshold graph with independent 3-sets exactly S.

- **Independence polytope:**

$$P_{\mathrm{ind}}(G) = \mathrm{conv}\{\chi^U : U \subseteq V \text{ independent}\},$$

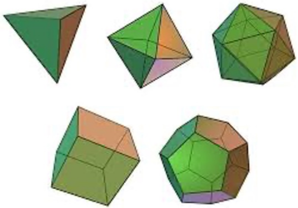
$$\text{where } (\chi^U)_i = \begin{cases} 1, & i \in U, \\ 0, & i \notin U \end{cases}$$

and independent k-sets are the integer points with $\sum_{i=1}^{n} x_i = k$.

- **Degree-sequence polytope:**

$$P_{\mathrm{deg}}(n) = \mathrm{conv}\{d(G) = (d_1, ..., d_n) : G \text{ is a simple graph on } \{1, \ldots, n\}\},$$

whose extreme points are exactly the degree sequences of threshold graphs.



Bibliography

B. Liu and L. Xue.
Shifted Total Cut Complex.
In preparation, 2025.

M. Bayer, M. Denker, M. J. Milutinović, R. Rowlands, S. Sundaram, and L. Xue.
Total cut complexes of graphs.
Discrete & Computational Geometry, 73(2):500–527, 2025.

C. J. Klivans.
Combinatorial properties of shifted complexes.
PhD thesis, MIT, 2003.

Thank you!