

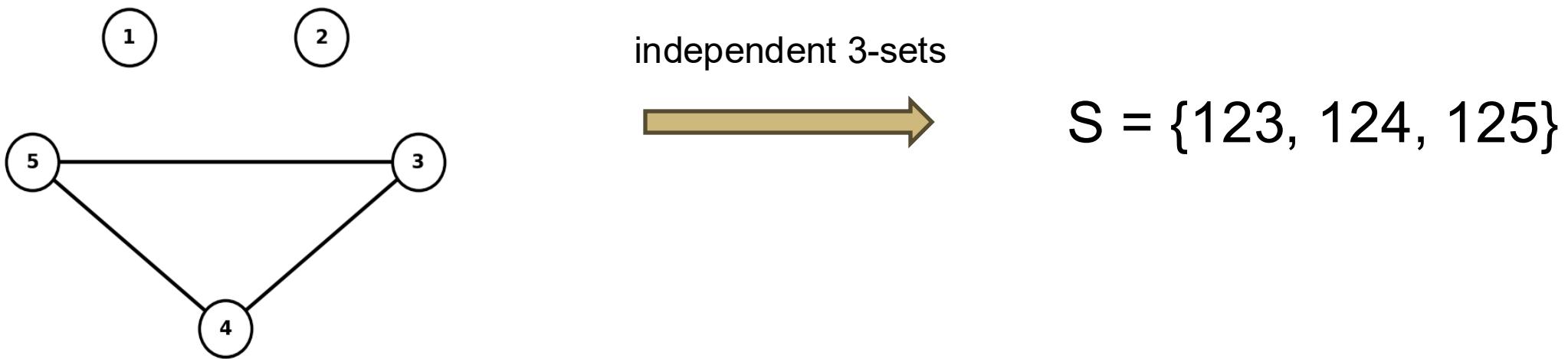
Generalized Threshold Graph LP

Baihan Liu



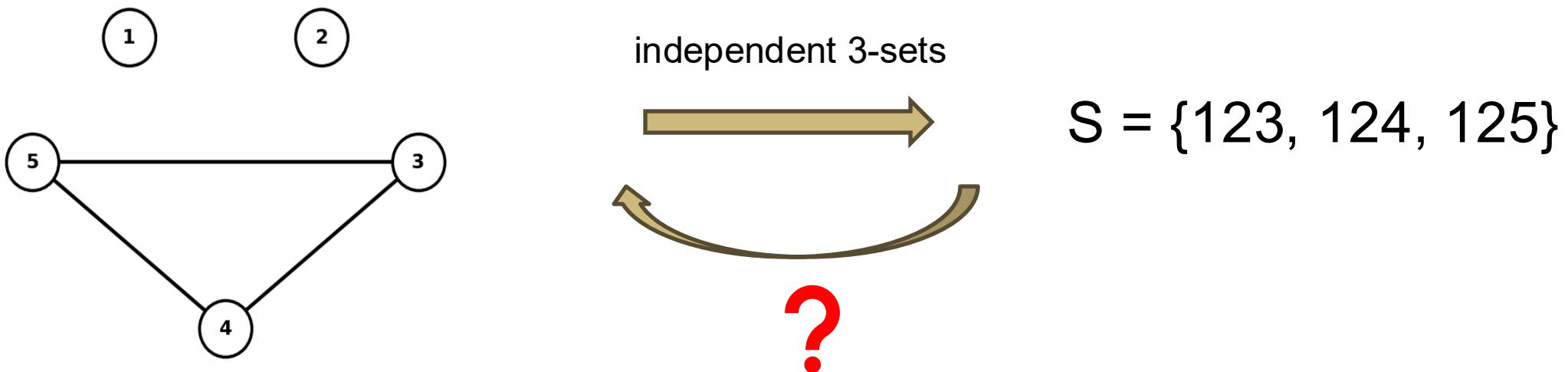
Independent k-set question

An *independent k-set* is a k -element vertex set U with no edges between vertices of U .



Independent k-set question

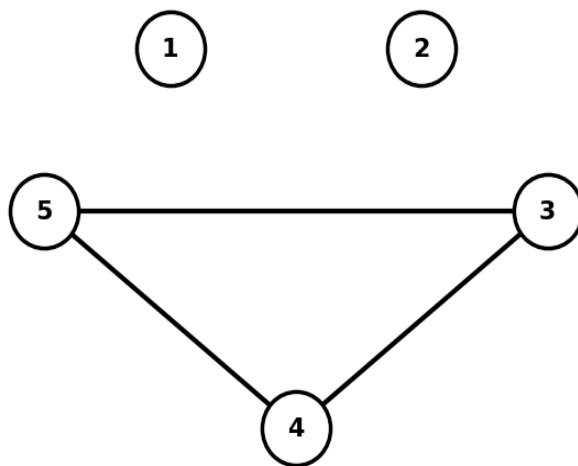
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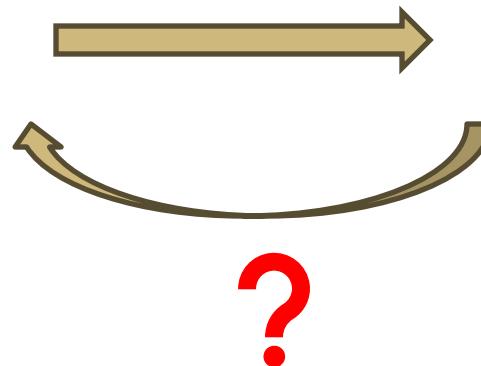
Question: Given family of k -sets ($k \geq 2$) \mathcal{S} ,
find a graph G whose independent k -sets are exactly the sets in \mathcal{S} .

Independent k-set question

An *independent k-set* is a k -element vertex set U with no edges between vertices of U .



independent 3-sets



$$S = \{123, 124, 125\}$$

$$V(G) = V = \bigcup_{U \in S} U = \{1, \dots, n\}$$

Question: Given family of k -sets ($k \geq 2$) S ,
find a graph G whose independent k -sets are exactly the sets in S .

Integer program for independent k-set question

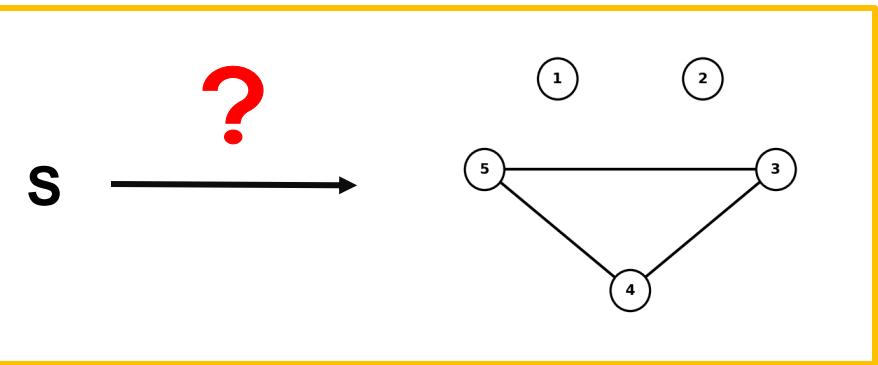
Edge variable: $x_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \text{ is an edge of } G, \\ 0, & \text{otherwise.} \end{cases}$

$$\min 0$$

$$\text{s.t. } x_{ij} = 0$$

$$\sum_{\{i,j\} \subseteq E(U)} x_{ij} \geq 1 \quad \forall U \in \binom{V}{k} \setminus S,$$

$$x_{ij} \in \{0, 1\} \quad \forall 1 \leq i < j \leq n.$$



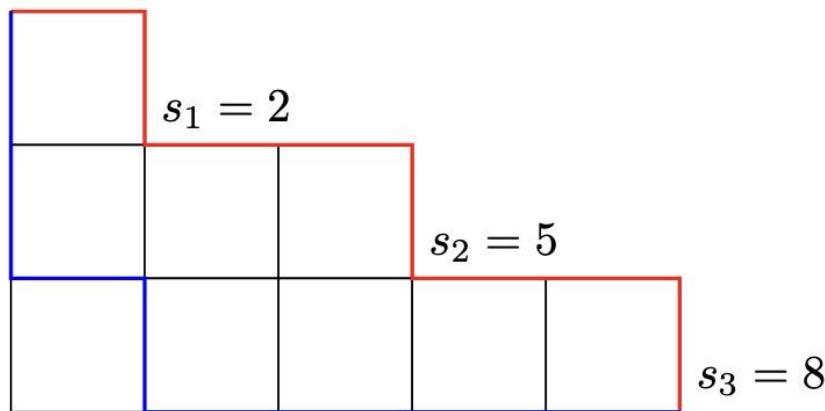
k-sets in S are independent

k-sets not in S are not independent

$$V(G) = V = \bigcup_{U \in S} U = \{1, \dots, n\}$$

Shifted family (up to relabeling)

A family $\mathcal{S} \subseteq \binom{V}{k}$ is *shifted* if whenever $U \in \mathcal{S}$ and $i < j$ with $j \in U$, $i \notin U$, then $(U \setminus \{j\}) \cup \{i\} \in \mathcal{S}$.



For example, $\mathcal{S} = \{123, 124, 125\}$ is shifted.

Family of k-sets \mathcal{S} : $(s_1, s_2, s_3) = 125$.

k-sets $U \in \mathcal{S}$: $(u_1, u_2, u_3) = 123, 124$, or 125 .

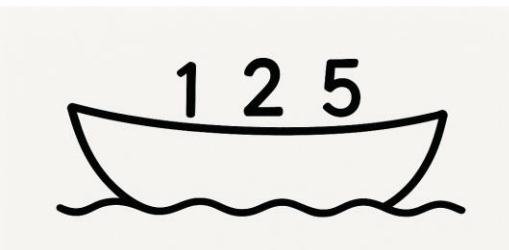
$(u_i \leq s_i \text{ for all } i. u_1 < \dots < u_k.)$

Shifted <=> K-sum threshold

Theorem A family $\mathcal{S} \subseteq \binom{V}{k}$ is shifted if and only if it is ***k-sum-threshold***, i.e., there exist weights $w : V \rightarrow \mathbb{R}$ and a threshold $t \in \mathbb{R}$ such that

$$\mathcal{S} = \left\{ U \in \binom{V}{k} : \sum_{u \in U} w(u) \leq t \right\}.$$

$$S = \{123, 124, 125\}$$



Linear Programming to determine shiftiness of S

$$\max \quad \delta$$

$$\sum_{i \in U} w_i \leq t - \delta \quad \forall U \in \mathcal{S},$$

$$\sum_{i \in U} w_i \geq t + \delta \quad \forall U \in \binom{V}{k} \setminus \mathcal{S},$$

$$0 \leq w_i \leq 1 \quad \forall i \in V,$$

$$0 \leq t \leq k,$$

$$\delta \geq 0.$$

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$$0 \leq t \leq k,$$

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prevent unboundedness

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$$0 \leq t \leq k,$$

$$\delta \geq 0.$$

- $\delta^* > 0$: k -sum threshold graph exist.
- infeasible or $\delta^* = 0$: no k -sum threshold graph.

K-sum threshold graph question

$k = 2$: threshold graphs.

$\mathcal{S} = \{\text{independent } k\text{-sets of } G\}$ is shifted/ k -sum-threshold $\iff G$ is a k -sum-threshold graph.

$$U \subseteq V, |U| = k \text{ is independent in } G \iff \sum_{u \in U} w(u) \leq t.$$



Combinatorial structure defined by edges \longleftrightarrow Linear information on vertices

Question: Is \mathcal{S} the family of independent k -sets of a k -sum threshold graph G ?

Mixed Integer Linear Programming

Question: Is \mathcal{S} the family of independent k-sets of a k -sum threshold graph G ?

$$\max \quad \delta$$

Independent k-set Constraints

$$\text{s.t.} \quad x_{ij} = 0 \quad \forall U \in S, \quad \forall \{i, j\} \subseteq E(U),$$

$$\sum_{\{i,j\} \subseteq U} x_{ij} \geq 1 \quad \forall U \in \binom{V}{k} \setminus S,$$

$$x_{ij} \in \{0, 1\} \quad \forall 1 \leq i < j \leq n.$$

- $\delta^* > 0$: k -sum threshold graph exist.

- infeasible or $\delta^* = 0$: no k -sum threshold graph.

$$\sum_{i \in U} w_i \leq t - \delta \quad \forall U \in \mathcal{S},$$

$$\sum_{i \in U} w_i \geq t + \delta \quad \forall U \in \binom{V}{k} \setminus \mathcal{S},$$

$$0 \leq w_i \leq 1 \quad \forall i \in V,$$

$0 \leq t \leq k$, Shifted Constraints

$$\delta \geq 0.$$

AMPL Model

```
# ---- Sets & data ----
set V ordered;                      # vertices, e.g., 1..n
set KSETS;                           # index of all k-sets
set S within KSETS;                 # the "independent" family
param inc {KSETS, V} binary;         # 1 if vertex v ∈ U, else 0

# Undirected edge index
set E := {i in V, j in V: i < j};

# ---- Variables ----
# Edges (use binary for an actual graph; relax to [0,1] for LP)
var x {E} binary;

# Weights / threshold margin (shifted test)
var w {V} >= 0, <= 1;
var t >= 0, <= 5;
var delta >= 0;

# ---- Graph side: exact independent k-sets = S ----
# All edges inside every F in S must be zero
s.t. indep_F {U in S}:
      sum { (i,j) in E: inc[U,i] = 1 and inc[U,j] = 1 } x[i,j] = 0;

# Every other k-set must contain at least one edge
s.t. nonindep_U {U in KSETS diff S}:
      sum { (i,j) in E: inc[U,i] = 1 and inc[U,j] = 1 } x[i,j] >= 1;

# ---- Shifted / k-threshold side on the same S ----
s.t. pos {U in S}:
      sum {v in V} inc[U,v] * w[v] <= t - delta;

s.t. neg {U in KSETS diff S}:
      sum {v in V} inc[U,v] * w[v] >= t + delta;

# ---- Objective ----
maximize margin: delta;
```

AMPL: *k*-sum threshold graph

```

set V := 1 2 3 4 5;
# All 3-sets, each with a name (10 of them)
set KSETS := u123 u124 u125 u134 u135 u145 u234 u235 u245 u345;
# The independent triples S (edit these)
set S := u123 u124 u125;

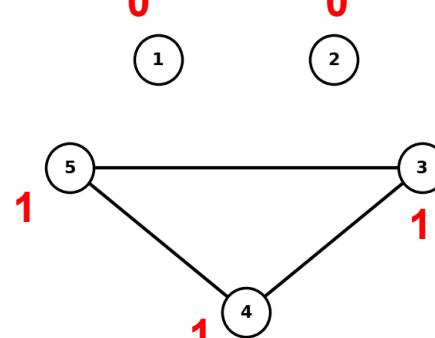
# Incidence table: rows = KSETS, columns = V
param inc:
    1 2 3 4 5 :=
u123 1 1 1 0 0
u124 1 1 0 1 0
u125 1 1 0 0 1
u134 1 0 1 1 0
u135 1 0 1 0 1
u145 1 0 0 1 1
u234 0 1 1 1 0
u235 0 1 1 0 1
u245 0 1 0 1 1
u345 0 0 1 1 1 ;

```

$$S = \{123, 124, 125\}$$



<pre> ampl: display w, t, delta; w [*] := 1 0 2 0 3 1 4 1 5 1 ; t = 1.5 delta = 0.5 </pre>	<pre> ampl: display x; x := 1 2 0 1 3 0 1 4 0 1 5 0 2 3 0 2 4 0 2 5 0 3 4 1 3 5 1 4 5 1 ; </pre>
--	--



3-sum threshold graph

AMPL: non-k-sum threshold graph

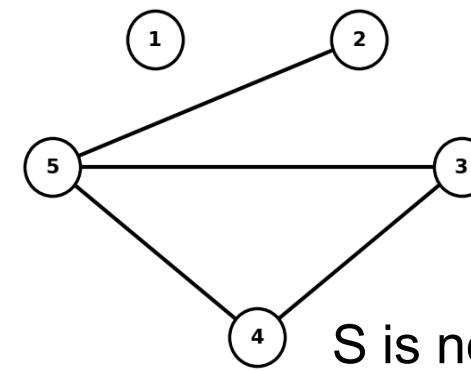
```
set V := 1 2 3 4 5;  
  
# All 3-sets, each with a name (10 of them)  
set KSETS := u123 u124 u125 u134 u135 u145 u234 u235 u245 u345;  
  
# The independent triples S (edit these)  
set S := u123 u124 u135;  
  
# Incidence table: rows = KSETS, columns = V  
param inc:  
    1 2 3 4 5 :=  
u123  1 1 1 0 0  
u124  1 1 0 1 0  
u125  1 1 0 0 1  
u134  1 0 1 1 0  
u135  1 0 1 0 1  
u145  1 0 0 1 1  
u234  0 1 1 1 0  
u235  0 1 1 0 1  
u245  0 1 0 1 1  
u345  0 0 1 1 1 ;
```

$S = \{123, 124, 135\}$



```
ampl: display w, t, delta;  
w [*] :=  
1 0  
2 0  
3 0  
4 0  
5 0  
;  
t = 0  
delta = 0
```

```
ampl: display x;  
x :=  
1 2 0  
1 3 0  
1 4 0  
1 5 0  
2 3 0  
2 4 0  
2 5 1  
3 4 1  
3 5 0  
4 5 1  
;
```



S is not shifted, so G is not a 3-sum threshold graph

MILP complexity and feasible region

Binary vars: $\Theta(n^2)$; k -set constraints: $\Theta(\binom{n}{k})$.

NP-hard? Not a problem.

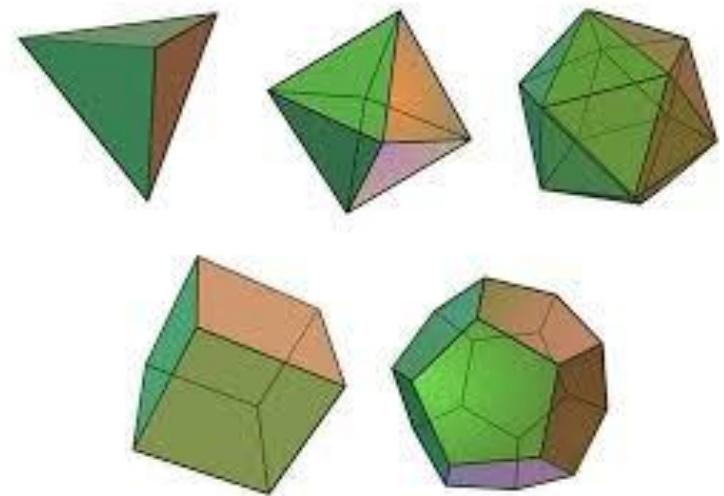
$n \setminus p$	1	2	3	4
1	1			
2	2	12		
3	3	13, 23	123	
4	4	14, 24, 34	124, 134, 234	1234
5	5	15, 25, 45	125, 135, 235, 145, 245, 345	1235, 1245, 1345, 2345
6	6	16, 26, 56	126, 136, 236, 156, 256, 456	1236, 1246, 1346, 2346, 1256, 1356, 2356, 1456, 2456, 3456
7	7	17, 27, 67	127, 137, 237, 167, 267, 567	1237, 1247, 1347, 2347, 1267, 1367, 2367, 1567, 2567, 4567

Table 2: Upper Bound Sequence Representation of Total Cut Complexes in Shifted Complexes

The LP analysis is useful because every feasible solution corresponds to a graph that satisfies the given constraints.

Independence Polytopes and Polytopes of degree sequence

- Independence polytope: size- k independent sets are integer points in the intersection of the independence polytope with an affine hyperplane
- Polytope of degree sequence: threshold graphs are the extreme points.



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- [3] Klivans, C. J. (2003). *Combinatorial properties of shifted complexes* (Doctoral dissertation, Massachusetts Institute of Technology).

Thank you

