

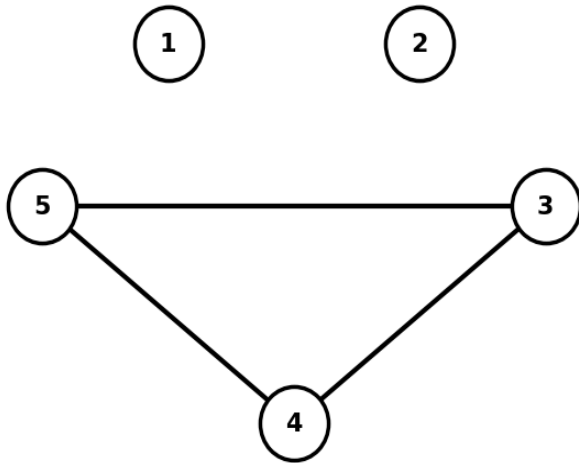
# ***Generalized Threshold Graph LP***

*Baihan Liu*



## Independent $k$ -set question

An *independent  $k$ -set* is a  $k$ -element vertex set  $U$  with no edges between vertices of  $U$ .



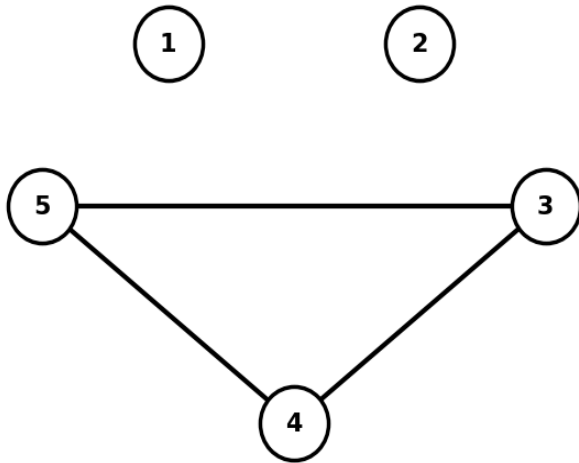
independent 3-sets



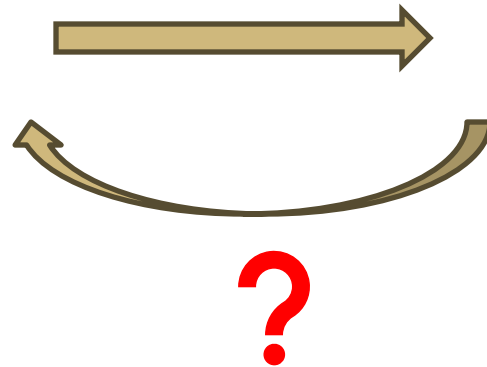
$$S = \{123, 124, 125\}$$

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independent 3-sets



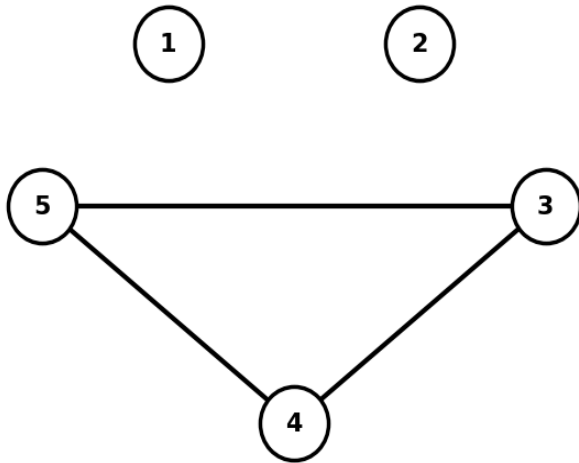
$$S = \{123, 124, 125\}$$

**Question:** Given family of  $k$ -sets ( $k \geq 2$ )  $\mathcal{S}$ ,

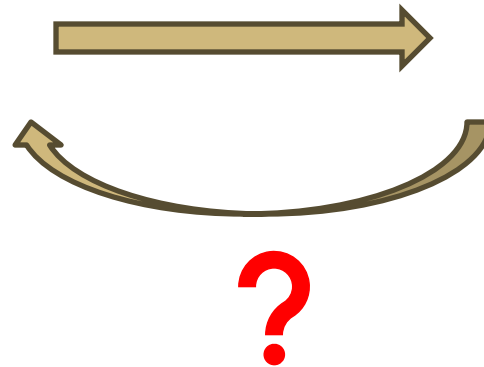
find a graph  $G$  whose independent  $k$ -sets are exactly the sets in  $\mathcal{S}$ .

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independent 3-sets



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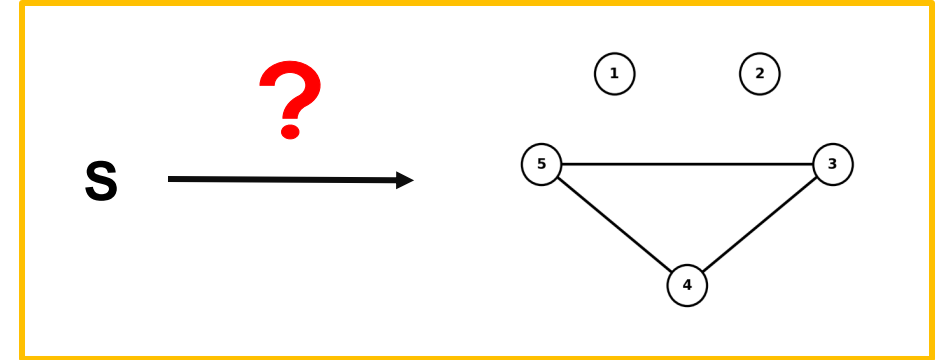
$$V(G) = V = \bigcup_{U \in S} U = \{1, \dots, n\}$$

**Question:** Given family of  $k$ -sets ( $k \geq 2$ )  $\mathcal{S}$ ,

find a graph  $G$  whose independent  $k$ -sets are exactly the sets in  $\mathcal{S}$ .

# Integer program for independent k-set question

Edge variable:  $x_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \text{ is an edge of } G, \\ 0, & \text{otherwise.} \end{cases}$



min 0

s.t.  $x_{ij} = 0$

$$\sum_{\{i,j\} \subseteq E(U)} x_{ij} \geq 1$$

$$x_{ij} \in \{0, 1\}$$

$$\forall U \in S, \forall \{i, j\} \subseteq E(U),$$

$$\forall U \in \binom{V}{k} \setminus S,$$

$$\forall 1 \leq i < j \leq n.$$

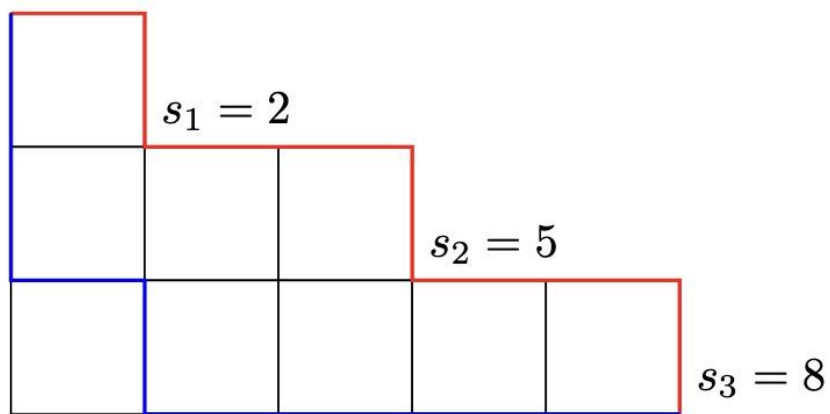
k-sets in S are independent

k-sets not in S are not independent

$$V(G) = V = \bigcup_{U \in S} U = \{1, \dots, n\}$$

## Shifted family (up to relabeling)

A family  $\mathcal{S} \subseteq \binom{V}{k}$  is *shifted* if whenever  $U \in \mathcal{S}$  and  $i < j$  with  $j \in U$ ,  $i \notin U$ , then  $(U \setminus \{j\}) \cup \{i\} \in \mathcal{S}$ .



Red Path:  $(s_1 = 2, s_2 = 5, s_3 = 8)$

Blue Path:  $(u_1 = 1, u_2 = 2, u_3 = 4)$

For example,  $\mathcal{S} = \{123, 124, 125\}$  is shifted.

Family of  $k$ -sets  $\mathcal{S}$ :  $(s_1, s_2, s_3) = 125$ .

$k$ -sets  $U \in \mathcal{S}$ :  $(u_1, u_2, u_3) = 123, 124, \text{ or } 125$ .

$(u_i \leq s_i \text{ for all } i. \ u_1 < \dots < u_k.)$

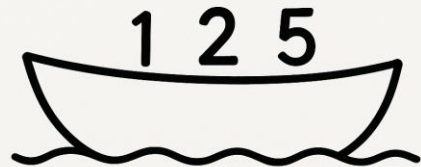


## Shifted $\Leftrightarrow$ $K$ -sum threshold

**Theorem** A family  $\mathcal{S} \subseteq \binom{V}{k}$  is shifted if and only if it is  $k$ -sum-threshold, i.e., there exist weights  $w : V \rightarrow \mathbb{R}$  and a threshold  $t \in \mathbb{R}$  such that

$$\mathcal{S} = \left\{ U \in \binom{V}{k} : \sum_{u \in U} w(u) \leq t \right\}.$$

$$\mathcal{S} = \{123, 124, 125\}$$



## Linear Programming to determine shiftiness of $S$

$$\max \quad \delta$$

$$\sum_{i \in U} w_i \leq t - \delta \quad \forall U \in \mathcal{S},$$

$$\sum_{i \in U} w_i \geq t + \delta \quad \forall U \in \binom{V}{k} \setminus \mathcal{S},$$

$$0 \leq w_i \leq 1 \quad \forall i \in V,$$

$$0 \leq t \leq k,$$

$$\delta \geq 0.$$



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$$0 \leq t \leq k,$$

$$\delta \geq 0.$$

prevent unboundedness

## Linear Programming to determine shiftiness of $\mathcal{S}$

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$$0 \leq t \leq k,$$

$$\delta \geq 0.$$

- $\delta^* > 0$ :  $k$ -sum threshold graph exist.
- infeasible or  $\delta^* = 0$ : no  $k$ -sum threshold graph.

## ***K-sum threshold graph question***

$k = 2$ : threshold graphs.

$\mathcal{S} = \{\text{independent } k\text{-sets of } G\}$  is shifted/ $k$ -sum-threshold  $\iff G$  is a  $k$ -sum-threshold graph.

$$U \subseteq V, |U| = k \text{ is independent in } G \iff \sum_{u \in U} w(u) \leq t.$$



Combinatorial structure defined by **edges**  $\iff$  Linear information on **vertices**

**Question:** Is  $\mathcal{S}$  the family of independent  $k$ -sets of a  $k$ -sum threshold graph  $G$ ?

# Mixed Integer Linear Programming

**Question:** Is  $\mathcal{S}$  the family of independent  $k$ -sets of a  $k$ -sum threshold graph  $G$ ?

max  $\delta$

Independent  $k$ -set Constraints

s.t.  $x_{ij} = 0 \quad \forall U \in \mathcal{S}, \forall \{i, j\} \subseteq E(U),$

$$\sum_{\{i,j\} \subseteq U} x_{ij} \geq 1 \quad \forall U \in \binom{V}{k} \setminus \mathcal{S},$$

$$x_{ij} \in \{0, 1\} \quad \forall 1 \leq i < j \leq n.$$

- $\delta^* > 0$ :  $k$ -sum threshold graph exist.
- infeasible or  $\delta^* = 0$ : no  $k$ -sum threshold graph.

$$\sum_{i \in U} w_i \leq t - \delta \quad \forall U \in \mathcal{S},$$

$$\sum_{i \in U} w_i \geq t + \delta \quad \forall U \in \binom{V}{k} \setminus \mathcal{S},$$

$$0 \leq w_i \leq 1 \quad \forall i \in V,$$

$$0 \leq t \leq k, \quad \text{Shifted Constraints}$$

$$\delta \geq 0.$$

# AMPL Model

```
# ---- Sets & data ----
set V ordered;           # vertices, e.g., 1..n
set KSETS;               # index of all k-sets
set S within KSETS;      # the "independent" family
param inc {KSETS, V} binary; # 1 if vertex v ∈ U, else 0

# Undirected edge index
set E := {i in V, j in V: i < j};

# ---- Variables ----
# Edges (use binary for an actual graph; relax to [0,1] for LP)
var x {E} binary;

# Weights / threshold margin (shifted test)
var w {V} >= 0, <= 1;
var t >= 0, <= 5;
var delta >= 0;
```

```
# ---- Graph side: exact independent k-sets = S ----
# All edges inside every F in S must be zero
s.t. indep_F {U in S}:
    sum { (i,j) in E: inc[U,i] = 1 and inc[U,j] = 1 } x[i,j] = 0;

# Every other k-set must contain at least one edge
s.t. nonindep_U {U in KSETS diff S}:
    sum { (i,j) in E: inc[U,i] = 1 and inc[U,j] = 1 } x[i,j] >= 1;

# ---- Shifted / k-threshold side on the same S ----
s.t. pos {U in S}:
    sum {v in V} inc[U,v] * w[v] <= t - delta;

s.t. neg {U in KSETS diff S}:
    sum {v in V} inc[U,v] * w[v] >= t + delta;

# ---- Objective ----
maximize margin: delta;
```

# AMPL: k-sum threshold graph

```

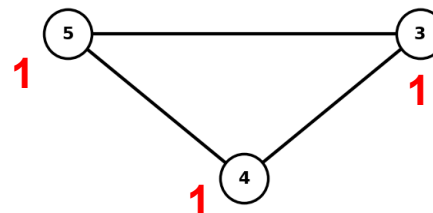
set V := 1 2 3 4 5;

# All 3-sets, each with a name (10 of them)
set KSETS := u123 u124 u125 u134 u135 u145 u234 u235 u245 u345;

# The independent triples S (edit these)
set S := u123 u124 u125;

# Incidence table: rows = KSETS, columns = V
param inc:
    1 2 3 4 5 :=
u123  1 1 1 0 0
u124  1 1 0 1 0
u125  1 1 0 0 1
u134  1 0 1 1 0
u135  1 0 1 0 1
u145  1 0 0 1 1
u234  0 1 1 1 0
u235  0 1 1 0 1
u245  0 1 0 1 1
u345  0 0 1 1 1 ;
    
```

$S = \{123, 124, 125\}$



3-sum threshold graph

```

ampl: display w, t, delta;
w [*] :=
1  0
2  0
3  1
4  1
5  1
;

t = 1.5
delta = 0.5
    
```

```

ampl: display x;
x :=
1 2  0
1 3  0
1 4  0
1 5  0
2 3  0
2 4  0
2 5  0
3 4  1
3 5  1
4 5  1
;
    
```



# AMPL: non-k-sum threshold graph

```
set V := 1 2 3 4 5;

# All 3-sets, each with a name (10 of them)
set KSETS := u123 u124 u125 u134 u135 u145 u234 u235 u245 u345;

# The independent triples S (edit these)
set S := u123 u124 u135;

# Incidence table: rows = KSETS, columns = V
param inc:
    1 2 3 4 5 :=
u123 1 1 1 0 0
u124 1 1 0 1 0
u125 1 1 0 0 1
u134 1 0 1 1 0
u135 1 0 1 0 1
u145 1 0 0 1 1
u234 0 1 1 1 0
u235 0 1 1 0 1
u245 0 1 0 1 1
u345 0 0 1 1 1 ;
```

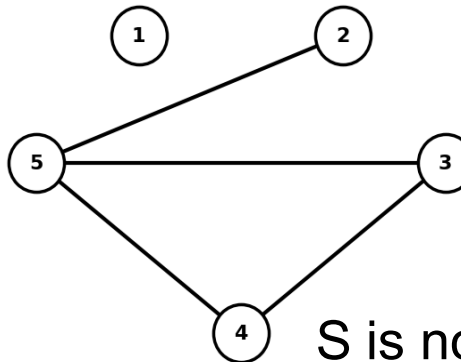
$S = \{123, 124, 135\}$



```
ampl: display w, t, delta;
w [*] :=
1 0
2 0
3 0
4 0
5 0
;
```

```
t = 0
delta = 0
```

```
ampl: display x;
x :=
1 2 0
1 3 0
1 4 0
1 5 0
2 3 0
2 4 0
2 5 1
3 4 1
3 5 0
4 5 1
;
```



S is not shifted, so G is not a 3-sum threshold graph

# MILP complexity and feasible region

Binary vars:  $\Theta(n^2)$ ;  $k$ -set constraints:  $\Theta(\binom{n}{k})$ .

NP-hard? Not a problem.

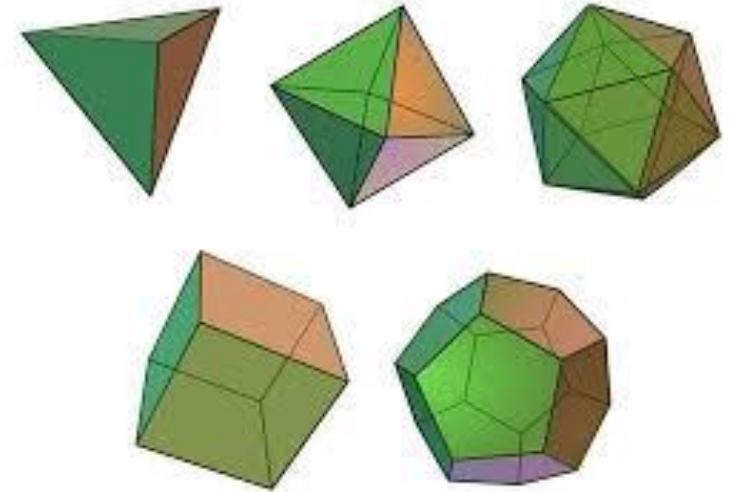
$n \setminus p$	1	2	3	4
1	1			
2	2	12		
3	3	13, 23	123	
4	4	14, 24, 34	124, 134, 234	1234
5	5	15, 25, 45	125, 135, 235, 145, 245, 345	1235, 1245, 1345, 2345
6	6	16, 26, 56	126, 136, 236, 156, 256, 456	1236, 1246, 1346, 2346, 1256, 1356, 2356, 1456, 2456, 3456
7	7	17, 27, 67	127, 137, 237, 167, 267, 567	1237, 1247, 1347, 2347, 1267, 1367, 2367, 1567, 2567, 4567

Table 2: Upper Bound Sequence Representation of Total Cut Complexes in Shifted Complexes

The LP analysis is useful because every feasible solution corresponds to a graph that satisfies the given constraints.

## *Independence Polytopes and Polytopes of degree sequence*

- Independence polytope: size- $k$  independent sets are integer points in the intersection of the independence polytope with an affine hyperplane
- Polytope of degree sequence: threshold graphs are the extreme points.



## ***Bibliography***

- [1] Baihan, & Lei Xue(Supervisor). Shifted Total Cut Complex. 2025. In preparation.
- [2] Bayer, M., Denker, M., Milutinović, M. J., Rowlands, R., Sundaram, S., & Xue, L. (2025). Total cut complexes of graphs. *Discrete & Computational Geometry*, 73(2), 500-527.
- [3] Klivans, C. J. (2003). *Combinatorial properties of shifted complexes* (Doctoral dissertation, Massachusetts Institute of Technology).

*Thank you*

