# Generalized Threshold Graph MILP

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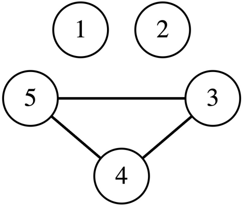
# Independent $k$-set

## Definition

**Definition.** An independent $k$-set is a $k$-element vertex set $U \subseteq V$ with no edges between vertices of $U$.

Example:




$S = \{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$ is a family of independent 3-sets in the graph on vertex set $V = \{1,2,3,4,5\}$.
* Throughout this talk, all graphs are simple (no loops or multiple edges).

# Independent $k$-set question

We are interesting in the inverse question...

**Question**

Given a family $S$ of $k$-sets with $k \geq 2$, does there exist a graph $G$ whose independent $k$-sets are exactly the sets in $S$?

Let the vertex set of our graph to be

$$V(G) = V = \bigcup_{U \in S} U = \{1, \ldots, n\}.$$

Additional vertices do not affect whether the graph G exist.

# Integer program for independent $k$-set question

**Edge variables:**

$$x_{ij} = \begin{cases} 1, & \text{if } \{i,j\} \text{ is an edge of } G, \\ 0, & \text{otherwise,} \end{cases} \qquad (1 \le i < j \le n).$$

**Integer program:**

$$\min 0$$

subject to

$$\begin{aligned} x_{ij} &= 0 && \forall U \in S,\ \forall \{i,j\} \subseteq U, \\ \sum_{\{i,j\} \subseteq U} x_{ij} &\ge 1 && \forall U \in \binom{V}{k} \setminus S, \\ x_{ij} &\in \{0,1\} && \forall\, 1 \le i < j \le n. \end{aligned}$$

# LP relaxation

**Edge variables:**

$$x_{ij} > 0, \text{if } \{i, j\} \text{ is an edge of } G.$$

$$x_{ij} = 0, \text{otherwise.} \qquad (1 \leq i < j \leq n).$$

**LP relaxation program:**

$$\min 0$$

subject to

$$\begin{aligned} x_{ij} &= 0 && \forall U \in S,\ \forall \{i, j\} \subseteq U, \\ \sum_{\{i,j\} \subseteq U} x_{ij} &\geq 1 && \forall U \in \binom{V}{k} \setminus S. \end{aligned}$$

# Equivalent integer program: find complement graph ($S$ is the family of k-cliques)

**Edge variables:**

$$x_{ij} = \begin{cases} 1, & \text{if } \{i,j\} \text{ is an edge of } G, \\ 0, & \text{otherwise,} \end{cases} \qquad (1 \le i < j \le n).$$

**Integer program:**

$$\min 0$$

subject to

$$\begin{aligned} x_{ij} &= k && \forall U \in S,\ \forall \{i,j\} \subseteq U, \\ \sum_{\{i,j\} \subseteq U} x_{ij} &\le k-1 && \forall U \in \binom{V}{k} \setminus S, \\ x_{ij} &\in \{0,1\} && \forall\, 1 \le i < j \le n. \end{aligned}$$

# Shifted family (up to relabeling)

## Definition

**Definition.** A family of $k$-sets $S$ is shifted (up to relabeling) if whenever $U \in S$ and $i < j$ with $i \notin U$, $j \in U$, then

$$(U \setminus \{j\}) \cup \{i\} \in S.$$

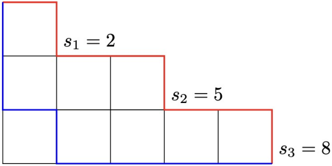
Example: $S = \{\{1,2,3\},\{1,2,4\},\{1,2,5\}\}$ is shifted (up to relabeling).

## Question

Given a family $S$ of $k$-sets with $k \geq 2$, is it shifted?

# Shifted family (up to relabeling)

- The family can be encoded by a nondecreasing sequence $(s_1, \ldots, s_k)$.
- Every $U = (u_1, \ldots, u_k) \in S$ satisfies $u_1 < \cdots < u_k$ and $u_i \leq s_i$ for all $i$.




Red Path: $(s_1 = 2, s_2 = 5, s_3 = 8)$
Blue Path: $(u_1 = 1, u_2 = 2, u_3 = 4)$

# Shifted $\iff$ $k$-sum-threshold

## Theorem

Let $S \subseteq \binom{V}{k}$ with $V = \bigcup_{U \in S} U$. Then the following are equivalent:

- **Shifted (up to relabeling):** whenever $U \in S$ and $i < j$ with $i \notin U$, $j \in U$, also $(U \setminus \{j\}) \cup \{i\} \in S$.
- ***k*-sum-threshold:** there exist weights $w : V \to \mathbb{R}$ and $t \in \mathbb{R}$ such that

$$S = \left\{ U \in \binom{V}{k} : \sum_{u \in U} w(u) \le t \right\}.$$

# Proof: $k$-sum-threshold $\Rightarrow$ shifted

**Assume $\mathcal{S}$ is $k$-sum-threshold.**

There exist weights $w : V \to \mathbb{R}$ and a threshold $t \in \mathbb{R}$ such that

$$\mathcal{S} = \Big\{ U \in \tbinom{V}{k} : \sum_{u \in U} w(u) \le t \Big\}.$$

Relabel $V = \{1, \ldots, n\}$ so that

$$w(1) \le w(2) \le \cdots \le w(n).$$

Let $U \in \mathcal{S}$ and suppose $i < j$ with $j \in U$ and $i \notin U$. Define

$$U' := (U \setminus \{j\}) \cup \{i\}.$$

Then

$$\sum_{u \in U'} w(u) = \sum_{u \in U} w(u) - w(j) + w(i) \le \sum_{u \in U} w(u) \le t.$$

Hence $U' \in \mathcal{S}$, so $\mathcal{S}$ is shifted with respect to $1 < \cdots < n$.

# Shifted $\Rightarrow$ $k$-sum-threshold (I)

Assume $\mathcal{S} \subseteq \binom{V}{k}$ is shifted (w.r.t. $1 < \cdots < n$).

By the upper bound sequence representation, there is a strictly increasing sequence $(s_1, \ldots, s_k)$ such that, for

$$U = \{u_1 < \cdots < u_k\}, \quad U \in \mathcal{S} \iff u_i \leq s_i \text{ for all } i = 1, \ldots, k.$$

Choose a large base $B > 1$ and define the weights and sum:

$$w(i) := B^i \quad (i = 1, \ldots, n), \qquad \phi(U) := \sum_{u \in U} w(u).$$

Let the "maximal" set be

$$S^* := \{s_1, \ldots, s_k\}, \quad t := \phi(S^*) = \sum_{i=1}^{k} B^{s_i}.$$

Goal: show

$$U \in \mathcal{S} \iff \phi(U) \leq t \iff \sum_{u \in U} w(u) \leq t.$$

# Shifted $\Rightarrow$ $k$-sum-threshold (II)

$$w(i) = B^i, \quad \phi(U) = \sum_{u \in U} B^u, \quad t = \phi(S^*) = \sum_{i=1}^{k} B^{s_i}.$$

**1.** If $U \in \mathcal{S}$, write $U = \{u_1 < \cdots < u_k\}$. Then $u_i \le s_i$ for all $i$, so

$$\phi(U) = \sum_{i=1}^{k} B^{u_i} \le \sum_{i=1}^{k} B^{s_i} = t.$$

**2.** If $U \notin \mathcal{S}$, let $j$ be the first index with $u_j > s_j$ (so $u_i \le s_i$ for all $i < j$). The term $B^{u_j}$ is strictly larger than $B^{s_j}$, and for sufficiently large $B$ this "first violation" dominates all later terms. Concretely, we can choose $B$ so large that

$$B^{u_j} - B^{s_j} > \sum_{i > j} \left( B^{s_i} - B^{u_i} \right),$$

which implies

$$\phi(U) > \phi(S^*) = t.$$

Thus, for such a choice of $B$,

$$U \in \mathcal{S} \iff \phi(U) \le t \iff \sum_{u \in U} w(u) \le t,$$

so $\mathcal{S}$ is $k$-sum-threshold.

# Integer program via shifted definition

Given $V = \{1, \ldots, n\}$, fixed $k$, and a family $S \subseteq \binom{V}{k}$.

**Variables:**

$$p_{i\ell} \in \{0,1\} \quad (i \in V,\ \ell = 1, \ldots, n), \qquad \mathrm{pos}_i \in \mathbb{Z} \quad (i \in V).$$

**Objective:**

$$\min\ 0 \quad \text{(feasibility problem)}.$$

**Permutation constraints:**

$$\sum_{\ell=1}^{n} p_{i\ell} = 1 \quad \forall i \in V, \qquad \sum_{i=1}^{n} p_{i\ell} = 1 \quad \forall \ell = 1, \ldots, n.$$

# Integer program via shifted definition

**Position constraints:**

$$\mathrm{pos}_i = \sum_{\ell=1}^{n} \ell \, p_{i\ell} \quad \forall i \in V.$$

**Shifted constraints:**
For all $U \in S$ and all pairs $(i, j)$ with $j \in U$, $i \notin U$:

- Let $U' = (U \setminus \{j\}) \cup \{i\}$.
- If $U' \notin S$, impose

$$\mathrm{pos}_i \ \geq \ \mathrm{pos}_j + 1,$$

i.e.

$$\sum_{\ell=1}^{n} \ell \, p_{i\ell} \ \geq \ \sum_{\ell=1}^{n} \ell \, p_{j\ell} + 1.$$

# Linear programming via $k$-sum threshold definition

**Goal:** Find a positive margin $\delta$ that separate $S$ with its complement.
**Variables:**

$$w_i \quad (i \in V), \qquad t, \qquad \delta \geq 0.$$

**Linear program:**

$$\max \ \delta$$

subject to

$$\begin{aligned}
&\sum_{i \in U} w_i \leq t - \delta && \forall U \in S, \\
&\sum_{i \in U} w_i \geq t + \delta && \forall U \in \binom{V}{k} \setminus S, \\
&0 \leq w_i \leq 1 && \forall i \in V, \\
&0 \leq t \leq k, && \\
&\delta \geq 0. &&
\end{aligned}$$

# Interpreting the LP

- If the optimal value satisfies $\delta^* > 0$, then $S$ is shifted/$k$-sum threshold.
- If the LP is infeasible or $\delta^* = 0$, then $S$ is not shifted/not $k$-sum threshold.
- For any $k$-set $U \in S$, the sum of weights $\sum_{i \in U} w_i$ lies *within* (below) the threshold.
- For any $k$-set $U \notin S$, the sum of weights $\sum_{i \in U} w_i$ lies *beyond* (above) the threshold.
- We bound $w_i$ and $t$ to prevent unboundedness.

# Comparison of the two formulations

**Comparison (variables and types)**

- **IP via shifted definition (permutation-based):**
  - Number of variables: $n^2 + n$.
  - Types: $n^2$ *binary* variables $p_{i\ell}$ and $n$ *integer* variables $\text{pos}_i$.
  - Integer Program(IP); much more combinatorial and typically harder to solve.
- **LP via $k$-sum-threshold:**
  - Number of variables: $n + 2$.
  - Types: all variables are *continuous* (real-valued).
  - Complexity driven mainly by the number of constraints over all $k$-sets.
- **Summary:**
  - IP is larger ($O(n^2)$ variables) and uses integrality to encode the permutation explicitly.
  - LP is small ($O(n)$ variables) and purely continuous.

# $k$-sum threshold graph question

What if $S$ is a family of independent $k$-sets of a graph $G$ and $S$ is also shifted?

Then there exist $w : V(G) \to \mathbb{R}$ and $t \in \mathbb{R}$ such that

$$U \subseteq V(G),\ |U| = k \text{ is independent in } G \quad \iff \quad \sum_{u \in U} w(u) \leq t.$$

combinatorial structure defined by edges $\iff$ linear information on vertices

We call graph $G$ is a $k$-sum-threshold graph.

**Question**

Given a family $S$ of $k$-sets with $k \geq 2$, does there exist a $k$-sum-threshold graph $G$ whose independent $k$-sets are exactly the sets in $S$?

# Mixed Integer Linear Program

**Variables:**

$$x_{ij} \in \{0,1\} \ (1 \le i < j \le n), \quad w_i \in [0,1] \ (i \in V), \quad t \in [0,k], \ \delta \ge 0.$$

**MILP:** max $\delta$, subject to

$$\begin{aligned}
\sum_{\{i,j\} \subseteq E(U)} x_{ij} &= 0 && \forall\, U \in S, \\
\sum_{\{i,j\} \subseteq E(U)} x_{ij} &\ge 1 && \forall\, U \in \binom{V}{k} \setminus S, \\
x_{ij} &\in \{0,1\} && \forall\, 1 \le i < j \le n.
\end{aligned}
\qquad
\begin{aligned}
\sum_{i \in U} w_i &\le t - \delta && \forall\, U \in S, \\
\sum_{i \in U} w_i &\ge t + \delta && \forall\, U \in \binom{V}{k} \setminus S, \\
0 \le w_i &\le 1 && \forall\, i \in V, \\
0 \le t &\le k, \\
\delta &\ge 0.
\end{aligned}$$

# AMPL examples

**Example 1: shifted, $3$-sum-threshold family**

$$S = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$$

- The AMPL model finds feasible weights $w$, threshold $t$, and edge variables $x$.
- $\delta^* \neq 0$. We obtain a 3-sum-threshold graph whose independent 3-sets are exactly the sets in $S$.

**Example 2: non-shifted, not $3$-sum-threshold**

$$S = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}\}$$

- There exists a graph $G$ with independent 3-sets being $S$.
- However, $\delta^* = 0$. There is no 3-sum-threshold graph with independent 3-sets exactly $S$.

# MILP complexity and feasible region

- Since there are $\binom{n}{k}$ many $k$-sets, the formulation has exponentially many constraints, and the resulting MILP is NP-hard to solve in general. It is therefore not very practical for large $n$ and $k$. For example, $n = 50$ and $k = 5$.
- NP-hard algorithm is still useful. For small values of $n$ and $k$, we can search over families of independent $k$-sets that correspond to $k$-sum-threshold graphs, observe combinatorial patterns, and then formally prove the resulting patterns. In fact, this analysis is being done in my article.
- LP analysis is useful since its feasible solutions correspond to all possible graphs that satisfies the given constraints.

# Independence polytopes and degree-sequence polytopes

- **Independence polytope:**

$$P_{\mathrm{ind}}(G) = \mathrm{conv}\{\chi^U : U \subseteq V \text{ independent}\},$$

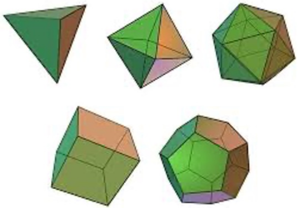
$$\text{where } (\chi^U)_i = \begin{cases} 1, & i \in U, \\ 0, & i \notin U \end{cases}$$

and independent $k$-sets are the integer points with $\sum_{i=1}^{n} x_i = k$.

- **Degree-sequence polytope:**

$$P_{\mathrm{deg}}(n) = \mathrm{conv}\{d(G) = (d_1, ..., d_n) : G \text{ is a simple graph on } \{1, \dots, n\}\},$$

whose extreme points are exactly the degree sequences of threshold graphs.



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# Thank you!