V and H Representations of Convex Polyhedra

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What Are V and H Representations?

Let P be a polyhedron in $\mathbb{R}^n$. Then we have the following representations:

V-Representation

$$P = \text{conv}\{v_1, v_2, ..., v_k\} + \text{cone}\{e_1, e_2, ..., e_l\}$$

Where $\{v_1, v_2, ..., v_k\}$ is the set of extreme points of P and $\{e_1, e_2, ..., e_l\}$ is the set of extreme rays of P.

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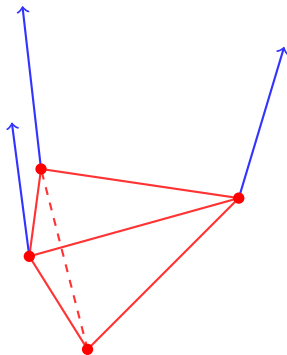
H-Representation

$$Ax \leq b$$

Where A is $m \times n$, x is a $n \times 1$ vector with entries consisting of variables $x_1, x_2..., x_n$, and b is a $n \times 1$ vector consisting of scalars.
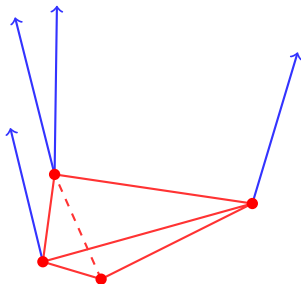
Given: In dimension m, we have

$$P = \text{conv}\{v_1, v_2, ..., v_k\} + \text{cone}\{e_1, e_2, ..., e_l\}$$

We want to find a matrix A and a column vector b such that
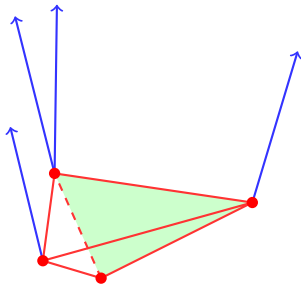$P = \{x : Ax \leq b\}$

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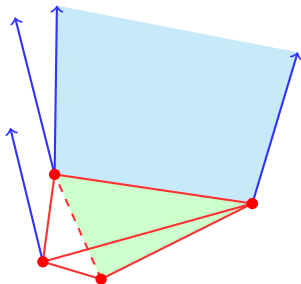
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- Bounded and unbounded edges

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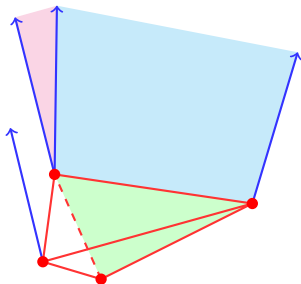
- Bounded edges only
- Bounded and unbounded edges
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Types of Edges on Hyperplanes

In dimension m, hyperplanes have dimension $m - 1$.

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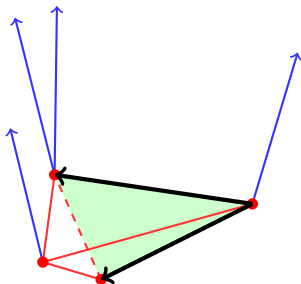In dimension m, hyperplanes have dimension $m - 1$.

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For a facet with only bounded edges, these can consist of the vectors from one vertex to $m - 1$ other vertices that lie on a facet.
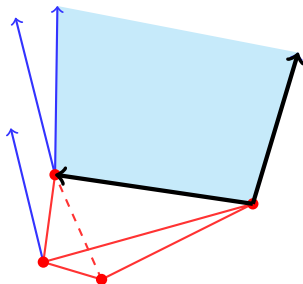
Types of Edges on Hyperplanes

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For a facet with both bounded and unbounded edges, these can consist of both vectors from one vertex to other vertices and also extreme rays.
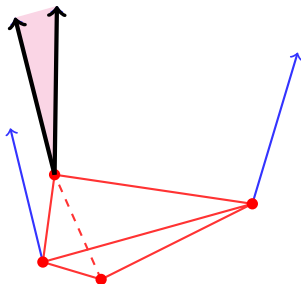
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For a facet with only unbounded edges, these can consist of extreme rays.
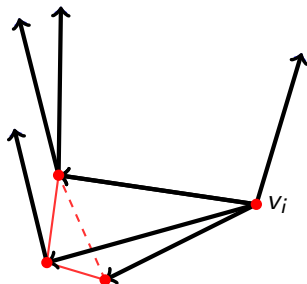
$$P = \text{conv}\{v_1, v_2, ..., v_k\} + \text{cone}\{e_1, e_2, ..., e_l\}$$

For each v_i in the V-representation, we will create a set:

$$G_i = \{\overrightarrow{v_i v_j} : j \in \{1, 2, ..., k\}, j \neq i\} \cup \{e_1, e_2, ..., e_l\}$$

Each linearly independent subset of G_i with size $m - 1$ is a basis for a hyperplane. Now we will find an equation for this hyperplane if it goes through v_i.

Finding the Normals

Each linearly independent subset of G_i with size $m - 1$ is a basis for a hyperplane. Now we will find an equation for this hyperplane if it goes through v_i.

First we will find its normal vector. Let the basis be $\{u_1, u_2, ..., u_{m-1}\}$, and solve:

$$\begin{bmatrix} u_1 \\ \hline u_2 \\ \hline \vdots \\ \hline u_{m-1} \end{bmatrix} \vec{n} = 0_{(m-1) \times 1}$$

Now that we have the normal, we know the equation of the hyperplane with that normal through v_i is

$$\vec{n}x = \vec{n}v_i$$

We can rewrite this as

$$\vec{n}(x - v_i) = 0$$

We can test whether the half spaces formed by these hyperplanes are in the H-representation by checking if they are supporting hyperplanes at v_i. If for every v_j we have

$$\vec{n}(v_j - v_i)(\leq | \geq)0$$

and for every e_j we have
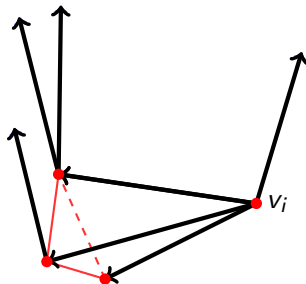
$$\vec{n}(v_i + e_j - v_i)(\leq | \geq)0$$

Then P is in the (negative|positive) half space.

Note that $v_j - v_i = \overrightarrow{v_i v_j}$ and $v_i + e_j - v_i = e_j$. These are exactly the vectors in G_i. Thus if for every $w \in G_i$, we have
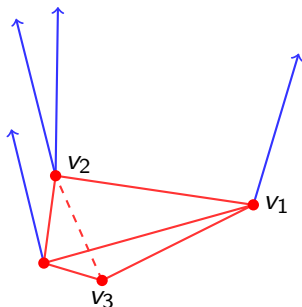
$$\vec{n} w (\leq | \geq) 0$$

then P is in the (negative|positive) half space, which joins the H-representation. Otherwise the hyperplane separates P and neither half space in the H-Representation.

Finalizing the H-Representation

This method will find a H-representation, however it may not be the simplest H-representation.

Removing redundant constraints (which will include duplicates) will finalize the H-representation to its simplest form.

To go from a V to H Representation, we will

For each vertex, create a set containing the vectors from that vertex to the other vertices and the extreme rays.

For each combination of size dimension minus one from this set, create the hyperplane that goes through the vertex

Check if the polyhedron sits entirely on one side of the hyperplane

If so then that half space joins the H representation. Otherwise neither half space is in the H-representation.

H to V Representations

Given a H-Representation $Ax \leq b$, find the V-representation.

We have seen the steps to find the V-representation of a bounded polyhedron in class, however it will be helpful to go through that again, as it is also the first steps to finding the V-representation of unbounded polytopes as well

Recall from class that if we are in dimension m, then vertices are the intersection of m hyperplanes.

Thus to find the intersection of these hyperplanes, choose m rows of A and find the point where each of these rows hold with equality. If these rows are $\{A_{d_1}, A_{d_2}, ..., A_{d_m}\}$, solve
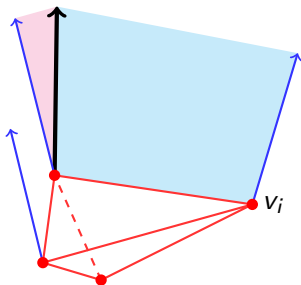
$$\begin{bmatrix} A_{d_1} \\ \hline A_{d_2} \\ \hline \vdots \\ \hline A_{d_m} \end{bmatrix} x = \begin{bmatrix} b_{d_1} \\ b_{d_2} \\ \vdots \\ b_{d_m} \end{bmatrix}$$

If x is a feasible solution, then it is a vertex of P.

Notice that extreme rays will be parallel the intersection of $m - 1$ hyperplanes.

Then any extreme ray can be represented as a vector that is orthogonal to the normal vectors of the hyperplanes.

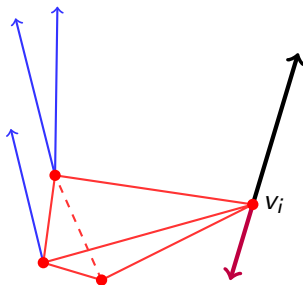Recall that the normal of the hyperplanes is just the rows of A.
Thus solving

$$\begin{bmatrix} A_{d_1} \\ \hline A_{d_2} \\ \hline \vdots \\ \hline A_{d_{m-1}} \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for every combination of rows of A will give the set of possible extreme rays.

Narrowing to Regression Directions

Now to narrow to regression directions, we check if x or $-x$ are not regression directions.

To do so, simply check if $v_i(+|-)x$ is feasible for all vertices v_i. If so, then $(+|-)x$ is a regression direction of P.

Finalizing the H representation

The final step is to remove any regression directions that are conic combinations of other regression directions.

To do so, for each vector e_i, check if it can be written as a positive combination of the other vectors. In other words, see if

$$\left[\begin{array}{c|c|c|c} e_1 & e_2 & \cdots & e_j \end{array}\right] x = e_i$$

has a solution such that $x \geq 0$
If so, remove e_i.

To go from a H to V Representation, we will

Find all intersections of m hyperplanes in the H representation

Check if these are in the polyhedron, and if so they are in the V-representation

Find the the vector orthogonal to $m - 1$ of the hyperplanes normal vectors

Check if the positive or negative direction of this vector is a regression direction

Remove any of these vectors that are conic combinations of the others