MATHEMATIK

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

# The Variational Collocation Method

based on a publication by Hector Gomez and Laura De Lorenzis


Steffen Plunder

TU Kaiserslautern


March 2, 2017

Seminar: Advanced Finite Element Methods

# Contents

# Preliminaries

We use the following notations, adapted from [Gomez and De Lorenzis, 2016]

- $\Xi = (\xi_1, \ldots, \xi_k)$ denotes a knot vector (including multiple knots).
- $N_{i,p}$ denotes the i.th B-Spline basis function of degree $p$. (NURBS and tensor structures will be discussed later.)

# The Model Problem: Poisson Equation

Consider we aim to solve (once again) the PDE

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \Gamma_D \\ \frac{\partial u}{\partial \mathbf{n}} = h, & \text{on } \Gamma_N. \end{cases}$$

Let $\mathcal{U}$ be a finite dimensional subspace of the solution space such that

$$u = g, \quad \text{on } \partial\Omega, \quad \text{for all } u \in U.$$

# The Basic Idea of Collocation

In addition we also need **collocation points** $x_1, \ldots, x_k, \ldots, x_{k+n}$.

The idea of collocation methods is to find $u \in \mathcal{U}$ such that the **strong form** of the PDE holds at all collocation points, i.e.

$$(-\Delta u)\big|_{x_i} = f\big|_{x_i} \quad \text{for all } i = 1 \ldots k.$$

and

$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{x_i} = h\big|_{x_i}, \quad \text{for } i = k+1, \ldots, k+n.$$

Note: **Only one evaluation of $f$ per collocation point!**

# Collocation with Boundaries

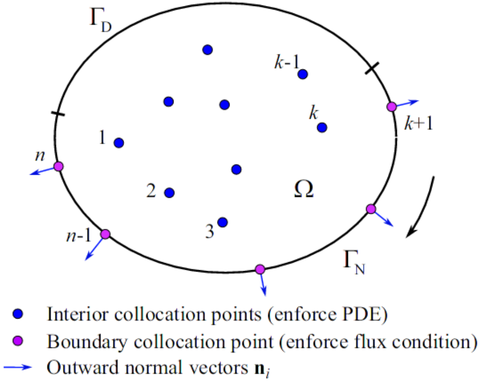



Figure: Source [Schillinger et al., 2013].

# Basic Idea in this Talk

One of the big open questions in collocation methods is the optimal choice of collocation points, since they influence

- uniqueness and existence,
- convergence and stability [Schillinger et al., 2013].

The variational collocation method proposed in [Gomez and De Lorenzis, 2016] combines

- computational efficiency of collocation methods and
- accuracy of isogeometric Galerkin methods.

# Part I: Collocation in Isogeometric Analysis

# Collocation in IGA

For a given subspace of B-Spline functions $\mathcal{U}$, which respect the Dirichlet boundary conditions, the collocation method results in a linear system

$$K\mathbf{d} = \mathbf{F}.$$

- K denotes the system matrix, i.e.

$$K_{ij} = \begin{cases} -\Delta U_i(x_j), & \text{for } i = 1, \dots, k \\ \frac{\partial U_i}{\partial \mathbf{n}}(x_j), & \text{for } i = k+1, \dots, k+n. \end{cases}$$

- **d** contains coefficients such that $u_h = \sum d_i u_i$ is the numerical solution.
- The load vector is given as $F_i = f(x_i)$ resp. $F_i = h(x_i)$ on the boundary.

# Which collocation points shall we use?

Possible choices are

- Orthogonal collocation: Gauss collocation points,
- Greville abscissae: $x_i = \frac{1}{p}\left(\xi_i + \cdots + \xi_{i+p-1}\right)$,
- Cauchy-Galerkin points: Second part of this talk ...

The choice of collocation points leads to different stability and convergence properties!

# Existence and Uniqueness!

Before we can talk about convergence and stability, the existence and uniqueness of the linear system

$$\mathrm{K}\,\mathbf{d} = \mathbf{F}$$

has to be mentioned!

Of cause these properties rely on the choice of collocation points! Our goal is to get a square system, i.e.

$$\dim \mathcal{U} = \left|\{x_1, \ldots, x_{k+n}\}\right| = k + n.$$

# Collocation at Greville Points

- Uniqueness: Schoenberg-Whitney Conditions ($\xi_i < x_i < \xi_{i+p+1}$)
- For engineering applications the points have been proven their efficiency. But unstable behaviour can be constructed by a specific choice of the PDE.
- Boundary conditions:
  - Dirichlet boundary conditions are already satisfied by the choice of $\mathcal{U}$. Greville points on $\Gamma_D$ are simply ignored.
  - For Greville points onto the Neumann boundary, we need to use the appropriate right hand side.

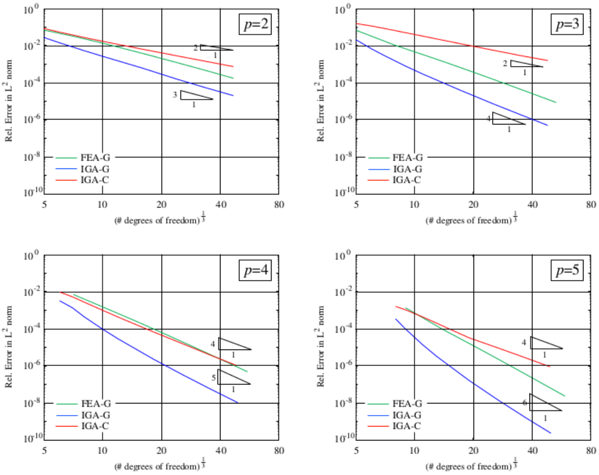



Figure: Comparison between collocation and the Galerkin method for 3D smooth Poisson problem. [Cottrell et al., 2009].

# Computational efficiency

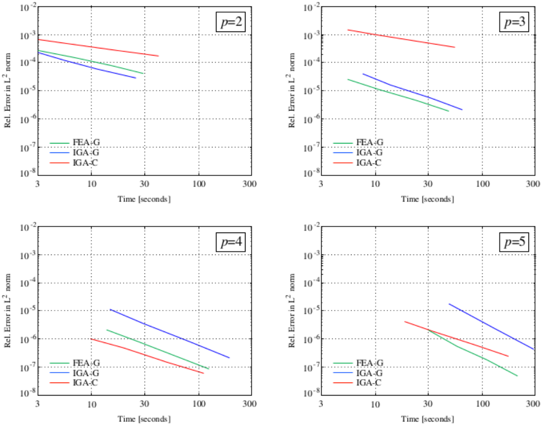



Figure: Smooth 3D Poisson. Time include assembly, preconditioning and iterative solution. [Cottrell et al., 2009]

# Part II: Variational Collocation in Isogeometric Analysis

# Variational Methods

Galerkin methods are based on the **weak formulation** of the PDE. For given finite dimensional subspaces

$$\mathcal{U}^h \subset \{u \in \mathcal{H}^1 \mid u = g \text{ on } \partial\Omega\},$$

$$\mathcal{W}^h \subset \{u \in \mathcal{H}^1 \mid u = 0 \text{ on } \partial\Omega\}.$$

The Galerkin solution $u_h \in \mathcal{U}^h$ is defined by

$$\int_\Omega \nabla\omega \cdot \nabla u_h \,\mathrm{d}x = \int_\Omega \omega \cdot f \,\mathrm{d}x, \quad \text{for all } \omega \in \mathcal{W}^h.$$

*Note: From now on we assume to have only Dirichlet boundary conditions.*

A link between collocation and finite elements...

## Theorem (Cauchy, 1821)

*Let $\Omega \subseteq \mathbb{R}^d$ be measurable. We consider functions*

$$R : \Omega \to \mathbb{R}, \quad \omega : \Omega \to \mathbb{R}.$$

*If $R$ is continuous and $\omega \neq 0$ non negative in $\Omega$, then there exists*

$$\tau \in \Omega,$$

*such that*

$$\int_\Omega \omega(\mathbf{x}) R(\mathbf{x})\,\mathrm{d}\mathbf{x} = R(\tau) \int_\Omega \omega(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

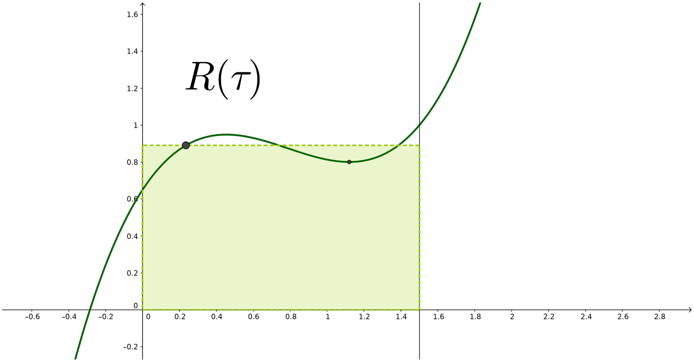



Figure: The picture to remember the proof.

# The Idea of variational collocation

The solution $u_h \in \mathcal{U}^h$ of the Galerkin method is characterised by

$$\int_\Omega \nabla\omega \cdot \nabla u_h \,\mathrm{d}x = \int_\Omega \omega \cdot f \,\mathrm{d}x, \quad \text{for all } \omega \in \mathcal{W}^h.$$

**If all $\omega$ and $u_h$ are sufficiently smooth**, restriction onto the support of $\omega$ and partial integration yields again the strong formulation

$$\int_{\operatorname{supp}\omega} \omega \cdot (\Delta u + f) \,\mathrm{d}x = 0, \quad \text{for all } \omega \in \mathcal{W}^h.$$

If we require $u_h \in \mathcal{C}^2(\Omega)$, we can apply Cauchy's Theorem for $R = \Delta u + f$, hence for each $\omega \in \mathcal{W}^h$ there exists a point $\tau_\omega$, such that

$$(\Delta u + f)|_{\tau_\omega} \cdot \int_{\operatorname{supp}\omega} \omega \,\mathrm{d}x = 0, \quad \text{for all } \omega \in \mathcal{W}^h.$$

# The Idea of Variational Collocation

$$(\Delta u + f)|_{\tau_\omega} \cdot \int_{\operatorname{supp}\omega} \omega \,\mathrm{d}x = 0, \quad \text{for all } \omega \in \mathcal{W}^h.$$

By definition of the spline basis functions, the last integral is non-zero and we get

$$(\Delta u + f)|_{\tau_\omega} = 0, \quad \text{for all } \omega \in \mathcal{W}^h.$$

This implies that we can compute the Galerkin approximation by a collocation method, **if the last conditions uniquely characterises a function $u_h$!** If these requirements are meet, we call these collocation points,

**Cauchy-Galerkin Points**.

# Necessary Assumptions

- $\mathcal{U}^h \subset \mathcal{C}^2(\Omega)$
- Existence of disjoint collocation points $\tau_\omega \in \operatorname{supp}\omega$ for all $\omega \in \mathcal{W}^h$ which satisfy the Cauchy Theorem.

# Existence of Cauchy-Galerkin Points

## Theorem ( [Gomez and De Lorenzis, 2016])

*If the knot vector Xi satisfies $\xi_i < \xi_{i+p+1}$ for all $i \in 1, \ldots, n$ (this ensures that the multiplicity is at most $p+1$), and integrable function $R : [a, b] \to \mathbb{R}$ is orthogonal to the Spline space, i.e.*

$$\int_\Omega \omega \cdot R \,\mathrm{d}x = 0, \quad \text{for all } \omega \in \mathcal{S}_{p,\Xi}.$$

*Then there exists points $\tau_1, \ldots, \tau_{n+1}$ in $[a, b]$ strictly increasing with*

$$\xi_i < \tau_i < \xi_{i+p}$$

*so that $R$ is orthogonal to $\mathcal{S}_{0,\Xi}$, that is*

$$\int_{\xi_i}^{\xi_{i+p}} R \,\mathrm{d}x = 0, \quad \text{for all } i \in 1, \ldots, n.$$

# Finding Cauchy-Galerkin Points

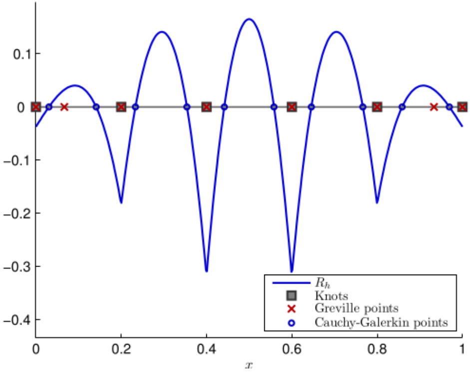
Since the proofs are non constructive, we still need to find these points!
Strategies to estimate these points

- Taylor series (of cause).
- Superconvergence points.

**The exact positions are not know a priori!**

# Finding Cauchy-Galerkin Points

If $u_h$ is the solution of the Galerkin method, the Cauchy-Galerkin points are locate at the roots of the residue$R_h = \Delta u_h - f$.



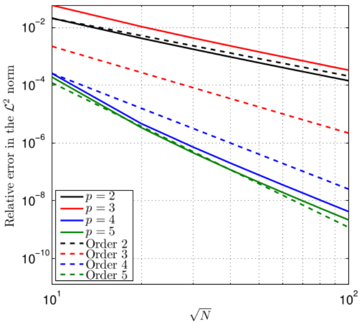
# Superconvergence Points

Like seen before, we are interested in the roots of $R_h = \Delta u_h - \Delta u$.
To estimate these the already developed field of superconvergence points can be used.
These points satisfy

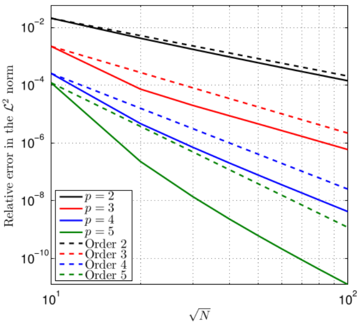
$$R_h(x) \in \mathcal{O}(h^{q+\sigma}),$$

where $q$ denotes the order of convergence and $\sigma$ represents the additional order of convergence in this point.
In the literature on superconvergence (for example [Wahlbin, 2006]) also special points where quantities like the residue etc.

# Convergence analysis




(a) Greville points, $\mathcal{L}^2$ norm.




(b) CG points, $\mathcal{L}^2$ norm.

Figure: Numerical solution of a linear elasticity problem.

# Positions of the CG Points for the Laplace Equation

It is already known [?] that for even degree the knots and midpoints are superconvergence points.
For odd degree [Gomez and De Lorenzis, 2016] compute for maximal smoothness, that CG points are located between the knots at the relative positions

CG points for different interpolation degrees.

| $p$ | Zeros of $e''$ |
|---|---|
| 3 | $\pm\frac{1}{\sqrt{3}}$ |
| 5 | $\pm\frac{\sqrt{225-30\sqrt{30}}}{15}$ |
| 7 | $\pm 0.5049185675126533$ |

# Summary

- Cauchy-Galerkin fix the discrepancy between even and odd degrees.
- The order of convergence is still less than for Galerkin-Methods, since CG points can only be estimated a priori.

# End

Thanks for your attention!

# References

- Cottrell, J., Hughes, T., and Bazilevs, Y. (2009).
  *Isogeometric Analysis: Toward Integration of CAD and FEA*.
  Wiley.
- Gomez, H. and De Lorenzis, L. (2016).
  The variational collocation method.
  *Computer Methods in Applied Mechanics and Engineering*, 309:152–181.
- Schillinger, D., Evans, J. A., Reali, A., Scott, M. A., and Hughes, T. J. (2013).
  Isogeometric collocation: Cost comparison with galerkin methods and extension to adaptive hierarchical {NURBS} discretizations.
  *Computer Methods in Applied Mechanics and Engineering*, 267:170 – 232.
- Wahlbin, L. (2006).
  *Superconvergence in Galerkin finite element methods*.
  Springer.