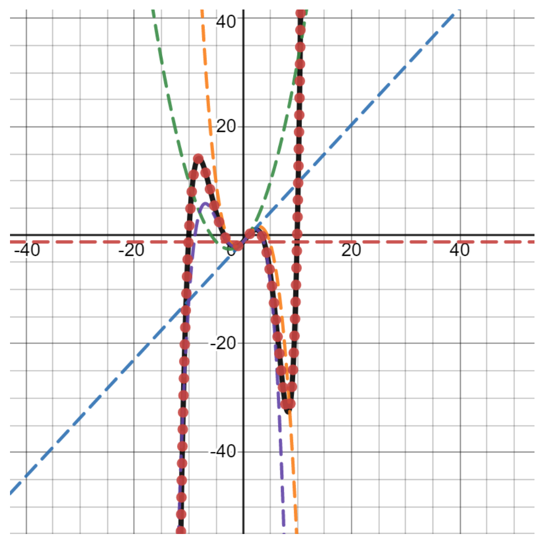
Mathamus's Guide to the Taylor & Maclaurin Series

Maclaurin Series

This series is used to approximate a function *f(x)*. We first take a point and move the axis to set its x value to zero. Then we use the function below:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

- $\sum_{n=0}^{\infty}$ states we will set n as every whole number from 0 to ∞ and add each term together
- $f^{(n)}(0)$ means take the derivative of $f(0)$ n times. (when $n = 3$, $f^{(n)}(0) = f'''(0)$)
- $n!$ means to take the factorial for every value of n
- x^n means to find x to power of n



Using this information, we get:

- $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$
- $f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + ...$
- $f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + ...$

Black line: Original function
Red dashed line (n = 0): First term of the Maclaurin series, where $f(x) = f(0)$
Blue dashed line (n→1): First and second term of the Maclaurin series, where $f(x) = f(0) + f'(0)x$
Green dashed line (n→2): First, second, and third term of the Maclaurin series, where $f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2}$
Orange dashed line (n→3): First to the fourth term of the Maclaurin series
Purple dashed line (n→4): First to the fifth term of the Maclaurin series
Red dotted line (n→∞): An infinite amount of terms of the Maclaurin series

This data demonstrates that increasing n and adding more terms to the Maclaurin series results in an very precise approximation of the original function.

Taylor Series

This series is very similar to the Maclaurin series, except that the x value is not set at zero. Instead we take the starting value of x and factor that into the equation.

In the Taylor series, we change $f^{(n)}(0)$ to $f^{(n)}(a)$ and x^n to $(x - a)^n$, where a is the starting value of x. This is the main difference between the Taylor and Maclaurin series.

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

- a is the starting x value (which value of x we want to approximate the curve from)
- $\sum_{n=0}^{\infty}$ states we will set n as every whole number from 0 to ∞ and add each term together
- $f^{(n)}(0)$ means take the derivative of $f(0)$ n times. (when $n = 3$, $f^{(n)}(0) = f'''(0)$)
- $n!$ means to take the factorial for every value of n
- $(x - a)^n$ means to find $x - a$ ($x -$ the initial x value) to each power of n

Using this we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = \frac{f(a)(x-a)^0}{0!} + \frac{f'(a)(x-a)^1}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{6} + \cdots$$

<u>Note: a=5</u>

Black=original function

Red dotted(n=0)=first term in the taylor series, f(x)=f(5)

Blue dotted(n=0,1)=first & second term of taylor series, f(x)=f(5)+f'(5)x

Green dotted(n=0-2)=first to third term of taylor series, f(x)=f(5)+f'(5)x+$\frac{f''(5)(x-a)^2}{2}$

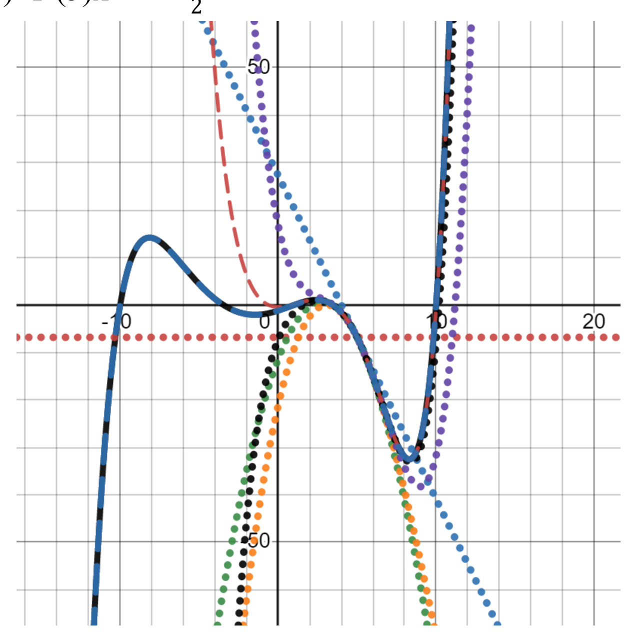
Yellow dotted(n=0-3)=first to fourth term of taylor series

Purple dotted(n=0-4)=first to fifth term of taylor series

Black dotted(n=0-5)=first to sixth term of taylor series

Red dashed(n=0-6)=first to seventh term of taylor series

Blue dashed(n=0-∞)=first to eighth term of taylor series



Other Forms of the Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Set:

$\varepsilon = x - a$

$x = \varepsilon + a$

∴ we get:(note we drop the a for x as we can now take the taylor for a formula, and set any variable to x as our starting variable)

$$\boxed{f(x+\varepsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}\varepsilon^n}$$

Then taking this equation we know

$f^{(n)}(x) = (\frac{d}{dx})^n f(x)$

So subbing in for $f^{(n)}(x)$:

$$f(x+\varepsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}\varepsilon^n = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{d}{dx}\right)^n f(x)\,\varepsilon^n = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\varepsilon\frac{d}{dx}\right)^n f(x)$$

& we also know

$$\sum_{n=0}^{\infty} e^t = \sum_{n=0}^{\infty} \frac{1}{n!}t^n$$

we now know that $e^t = \frac{1}{n!}t^n$.

So treating $\sum_{n=0}^{\infty} \frac{1}{n!}(\varepsilon\frac{d}{dx})^n$ as we would $\sum_{n=0}^{\infty} \frac{1}{n!}(t)^n$ we know $e^{\varepsilon\frac{d}{dx}} = \frac{1}{n!}(\varepsilon\frac{d}{dx})^n$

So subbing that in we get

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\varepsilon\frac{d}{dx}\right)^n f(x) = e^{\varepsilon(\frac{d}{dx})}f(x)$$

Meaning

$$\boxed{f(x+\varepsilon) = e^{\varepsilon\frac{d}{dx}}f(x)}$$

sin(x), cos(x), and Other Cases of Taylor Series

Steps for finding the Taylor Series for each of these cases:

Step 1: Find the 0th to 4th derivation of sin(x).

Step 2: Sub in 0=x in the 0th to the 4th derivative.

Step 3: Use the Theorem to find the first four terms of the Taylor series.

Step 4: Create a sum that represents that series.

f(x) = sin(x) x=0

f(x) = sin(x)	sin(0) = 0
f'(x) = cos(x)	cos(0) = 1
f''(x) = -sin(x)	-sin(0) = 0
f'''(x) = -cos(x)	-cos(0) = -1
f''''(x) = sin(x)	sin(0) = 0

$$P(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

f(x) = cos(x) x=0

f(x) = cos(x)	cos(0) = 1
f'(x) = -sin(x)	-sin(0) = 0
f''(x) = -cos(x)	-cos(0) = -1
f'''(x) = sin(x)	sin(0) = 0
f''''(x) = cos(x)	cos(0) = 1

$$P(x) = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

f(x) = $\frac{1}{1-x}$ x=0

$$f(x) = \frac{1}{1-x} \qquad \frac{1}{1-(0)} = 1$$

$$f'(x) = \frac{1}{(x-1)^2} \qquad \frac{1}{((0)-1)^2} = 1$$

$$f''(x) = \frac{-2}{(x-1)^3} \qquad \frac{-2}{((0)-1)^3} = 2$$

$$f'''(x) = \frac{6}{(x-1)^4} \qquad \frac{6}{((0)-1)^4} = 6$$

$$f''''(x) = \frac{-24}{(x-1)^5} \qquad \frac{-24}{((0)-1)^5} = 24$$

$$P(x) = 1 + x + x^2 + \frac{6}{3!}x^3 + \frac{24}{4!}x^4$$

$$P(x) = 1 + x + x^2 + x^3 + x^4$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

f(x) = e^x x=0

$$f(x) = e^x \qquad e^{(0)} = 1$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$f'''(x) = e^x$$

$$f''''(x) = e^x$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad -\infty < x < \infty$$

Proving Euler's identity: $e^{i\pi} = -1$

Three Taylor expansions of common functions are shown on the right:

Step 1: Taking the expansion for e^{x}, we can replace x with (ix), where i is the imaginary number $\sqrt{-1}$.

Step 2: Using the values of i^{2}, i^{3}, and i^{4}, we can simplify the equation.

Step 3: We can factor out a common factor of i in every other term.

Step 4: The two factored terms in brackets are actually the Taylor expansions of $sin(x)$ and $cos(x)$. This results in: $e^{ix} = cos(x) + isin(x)$

Step 5: The last step is to sub in π for x, leaving us with

$$e^{i\pi} = cos(\pi) + isin(\pi)$$

$$e^{i\pi} = (-1) + i(0)$$

$$e^{i\pi} = (-1)$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\boxed{e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} \dots}$$

$$\boxed{e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} \dots}$$

$$\boxed{e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right)}$$

Find the Maclaurin series at n=0 of f(x). Then write the next three terms in the series.

$$f(x) = \frac{\cos(x)}{2}$$

Find the Taylor series where a=4 of f(x). Then write the next three terms in the series.

$$f(x) = x^3 + x^2 + 3$$

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