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Diverting Networks with Odd Paths

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Abstract

The problem of Shortest Odd Path is to find a path from one vertex to another in a graph, where the number of edges in the path must be odd. Although the problem might seem like merely a curiosity, its utility is that many more useful problems are easier to solve if we know how to solve Shortest Odd Path.

One of these problems is Network Diversion: given a graph, two vertices and a marked edge, compute the cheapest set of edges to delete from the graph such that all paths from one vertex to another must pass through the marked edge. Many of its variants are NP-complete, but its complexity on planar graphs remains an open problem.

We implement an efficient algorithm based on [Der85] to solve Shortest Odd Path on undirected graphs, and use that to implement the first-ever efficient algorithm for Network Diversion on planar graphs.

Contents

Chapter 1

Introduction

One of the most well-known, well-studied and well-understood algorithmic problems is to find the Shortest Path in a graph. The problem is simple: given a graph and two vertices, find the shortest sequence of edges to go from one vertex to the other. Yet, the applications are almost limitless: to find the fastest route home, to find the cheapest airline tickets to Kuala Lumpur, to solve a Rubik's Cube in the fewest moves, to determine the best-case running time of an algorithm, or to move an enemy in a video game.

This thesis, however, is about a curious little variant, called the Shortest Odd Path. Now we consider only paths consisting of an odd number of edges. If you were to step through the graph, and start walking with your right foot, then an odd path is one where you would also end up on your right foot. The applications of this variant are not remotely as obvious. It rarely, if ever, matters whether a path has odd or even length in any of the examples mentioned above, and it is difficult to come up with example problems where it does matter.

The reason we care is because many other more useful problems are much easier to solve if we already have an algorithm to determine the shortest odd path in a graph. Consider for example the problem of the Shortest Bottleneck Path: find the shortest path from one vertex to another, except that we are also given a 'bottleneck' edge with the requirement that the path has to go through the bottleneck. Imagine doing a road-trip in Norway, but for the complete road-trip experience you also really want to drive through Norway's longest tunnel, and preferably without using the same roads more than once. Coming up with an algorithm for this is not as simple as it sounds, but we will show that it is much easier if we know how to solve Shortest Odd Path.

An even more directly useful problem to solve is the one of Network Diversion: given two vertices and a marked edge in a graph, find the cheapest set of edges to delete from the graph such that all paths from one vertex to another must pass through the marked edge. This one

has more immediate practical applications: imagine you are a military commander in a war, you know that the enemy wants to move their troops and supplies, and you are very prepared to ambush them if they cross a certain bridge. Now what is the fastest or cheapest way to destroy bridges to funnel the enemy through the ambush?

Solving Network Diversion efficiently is no simple task, and many of its variants have been proved to be NP-complete. Whether there exists an algorithm to solve Network Diversion in polynomial time on undirected *planar graphs* is for now an open problem, and it turns out the answer is yes: it is possible if you already have an efficient algorithm to solve Shortest Odd Path. This is the topic of our thesis. We develop and implement an efficient algorithm to solve Shortest Odd Path, and then use that to implement the first-ever efficient algorithm for Network Diversion on planar graphs.

Overview of the contents

We start this thesis with some preliminaries, mainly around graph theory, in Chapter 2. Then we warm up our problem-solving skills in Chapter 3, where we solve a much easier variant of Shortest Odd Path, called Shortest Odd Walk. In Chapter 4 we reach the star of the thesis: our algorithm for Shortest Odd Path. With this star in hand we can head to Chapter 5 to solve Network Diversion for planar graphs.

Most of the algorithms discussed here have also been implemented and tested in practice [Sim24b], and Chapter 6 presents the codebase. In addition, the source code for this thesis itself can be found at [Sim24a].

The reader may visit the chapters and source code in any order they like, though we would like to suggest a chronological order if nothing else.

Chapter 2

Preliminaries

2.1 Graphs

In the study of algorithms, we often use graphs as an abstract structure to represent the fundamental algorithmic problem without distractions. For example, when you want to find the fastest route to walk to the study hall, or if you want the cheapest combination of flights to take you to Kuala Lumpur, then both questions are really the same problem. If we remove all the details that are unnecessary to solve the problem, like the names of the airports and whether we are walking or flying, then we end up with a graph. This section defines various concepts related to graphs, and Section 2.2 will formalize the underlying problem of this example as well as some other graph problems. Later, Section 2.3 defines the concept of *planar* graphs.

Definition 2.1.1 (Graph)**.** A *graph* $G := (V, E, \text{from}, \text{to})$ is given by

- V, a collection of *vertices*
- E, a collection of *edges*
- $\text{from} : E \to V$, a mapping from each edge to its source vertex
- $\text{to} : E \to V$, a mapping from each edge to its target vertex

We also define the convenience function $\text{reverse} : E \to E$. For an edge $e \in E$, $\text{reverse}(e)$ is the edge going in the opposite direction, where $\text{from}(e) = \text{to}(\text{reverse}(e))$, and $\text{to}(e) = \text{from}(\text{reverse}(e))$.

This definition of a graph is a little unusual. A more common definition is to instead define $G := (V, E)$, where the edges is a subset of the cartesian product of the vertices: $E \subseteq V \times V$. The problem with this definition is that we cannot consider graphs with parallel edges, which we will need to later in the thesis.

If we are working with multiple graphs at once, say two graphs G and H, then writing just V is ambiguous. In such cases we instead denote $V(G)$ and $V(H)$ as G's and H's vertices, respectively. The same goes for $E(G)$ and $E(H)$ for their edges.

Definition 2.1.2 (Weighted graph)**.** A *weighted graph* $G := (V, E, \text{from}, \text{to}, \text{weight})$ is a graph, where $\text{weight} : E \to \mathbb{R}$ is the *weight* of each edge.

If a graph is not weighted, we often treat it as if all edges have unit weight, a weight of 1. Algorithms intended for weighted graphs will therefore often work on unweighted graphs as well. Although weights in the general case can be negative, all the algorithms presented in this thesis are designed for graphs of non-negative weights, and the reader may assume that $\text{weight} : E \to \mathbb{R}_{\geq 0}$ unless otherwise stated.

Definition 2.1.3 (Directed and undirected graphs)**.** Let G be a graph. G is said to be an *undirected graph* if each edge has an opposite: $\forall e \in E \; \exists e' \in E : \text{reverse}(e) = e'$. If G is not undirected, we say that G is a *directed graph*.

Most of the algorithms presented in this thesis are designed for undirected graphs. The reader may assume that all graphs in the thesis are undirected unless otherwise stated.

Definition 2.1.4 (Neighbourhood)**.** Let G be a graph, and let $u \in V$ be a vertex in the graph. The *neighbourhood* of u, denoted as $N(u)$, is defined as the vertices in G that are reachable from u using just a single edge: $N(u) := \{to(e) \mid e \in E, \; \text{from}(e) = u\}$.

In code, it is usually more useful to consider neighbourhoods in terms of edges. We will therefore denote $G[u]$ as the edges in G that start in u: $G[u] := \{e \mid e \in E, \; \text{from}(e) = u\}$. We also denote $\deg(u) := |N(u)|$ as the size of u's neighbourhood, often referred to as the *degree* of u.

Slett? Omskriv?

Definition 2.1.5 (Simple graph)**.** Let G be a graph. G is said to be a *simple* graph if for each pair of vertices $u, v \in V$, there exists *at most* one edge e such that $\text{from}(e) = u$ and $\text{to}(e) = v$. If two or more edges have the same endpoints, we say the edges are *parallel* to each other, and that the graph has *parallel* edges and is thus not simple.

Definition 2.1.6 (Walk)**.** A *walk* $P := [e_1, e_2, ..., e_k]$ in a graph G, for $e_i \in E$, is a sequence of edges where each edge ends where the next one starts: $\forall i \in \{1, 2, .., k-1\} : \text{to}(e_i) = \text{from}(e_{i+1})$. If $s := \text{from}(e_1)$ and $t := \text{to}(e_k)$, we say that P is an *s-t-walk* in G.

Another way to denote a walk is to give a sequence of vertices in the order they are visited: $[u_1, u_2, ..., u_n]$, for $u_i \in V$. This works as long as the graph is simple, if there are multiple edges from u_i to u_{i+1}, then the walk is ambiguous.

Definition 2.1.7 (Path)**.** A *path* $P := [e_1, e_2, ..., e_k]$ in a graph G is a walk with the extra requirement that each vertex is used at most once: $\forall i, j \in \{1, 2, .., k\} : j \neq i + 1 \rightarrow \text{to}(e_i) \neq \text{from}(e_j)$. If $s := \text{from}(e_1)$ and $t := \text{to}(e_k)$, we say that P is path from s to t, or an *s-t-path* in G.

Note that in some literature, a walk is referred to as a path, and a path is referred to as a *simple* path. In this thesis, when we refer to paths they are always simple, meaning that they never repeat any vertices. If any vertices are repeated, we will refer to it as a walk.

Definition 2.1.8 (Cycle)**.** A *cycle* in a graph is a walk that starts and ends in the same vertex. If it does not repeat any vertices except in the last vertex, then we call it a *simple cycle*.

Definition 2.1.9 (The cost of a walk)**.** Let $P := [e_1, e_2, ..., e_k]$ be a walk in a weighted graph G. The *cost* of P is defined as the sum of the weights of its edges: $\sum_{i=1}^{k} \text{weight}(e_i)$. In a collection of walks, we say that the *shortest* walk is the *cheapest* one, the one with the lowest cost. Likewise for the longest and most expensive walk.

Note that in some literature, *shortest* may instead mean *fewest edges*, and the term *length* could refer to both the number of edges and the cost. For that ambiguous reason, we will from now on avoid the word *length*, and *shortest* will always mean *cheapest*. If a graph is unweighted, we pretend that all the edges have a unit weight of 1, and in that case the cost is the same as the number of edges.

Definition 2.1.10 (Cut)**.** Let $G = (V, E, \text{from}, \text{to})$ be a connected and undirected graph. A *cut* $C \subseteq E$ of G is a subset of edges such that $(V, E \setminus C, \text{from}, \text{to})$ is a disconnected graph of exactly two components. If two vertices $s, t \in V$ end up in separate components after the cut, we denote C as an *s-t-cut* in G. See Figure 2.1 for an example of an *s-t*-cut.

(a) A connected graph with an *s-t*-cut marked in red

(b) The disconnected graph of exactly two components, after deleting all the edges in the cut

Figure 2.1: A graph before and after an *s-t*-cut of edges has been deleted

2.2 Graph problems

Now that we know what a graph is, we are ready to formalize the underlying problem of the example we started with in this chapter. Both problems can be represented by the quest to

find the shortest path from one vertex to another in an abstract graph. From a computational perspective, it does not matter whether the edges are roads or flights, or whether the vertices are crossroads or airports. Vertices and edges can represent whatever we want them to. We call the problem Shortest Path, and it is defined as:

Shortest Path
Input: a graph G, two vertices $s, t \in V$
Output: an s-t-path in G of minimum cost

This thesis will focus on a curious variant of the Shortest Path problem, called Shortest Odd Path:

Shortest Odd Path
Input: a graph G, two vertices $s, t \in V$
Output: an s-t-path in G of minimum cost, that uses an odd number of edges

We will also give an algorithm in Chapter 3 for the less restrictive variation called Shortest Odd Walk:

Shortest Odd Walk
Input: a graph G, two vertices $s, t \in V$
Output: an s-t-walk in G of minimum cost, that uses an odd number of edges.

Dijkstra's Algorithm

Both our algorithms for Shortest Odd Path and Shortest Odd Walk borrow ideas from the famous Dijkstra's Algorithm. The algorithm solves Shortest Path on graphs with non-negative weights, and handles both directed and undirected graphs. We show it here for reference.

Code Listing 2.1: Dijkstra's Algorithm for Shortest Path

```
fn dijkstras_shortest_path(graph, s, t) {
    for u in V(graph) {
        dist[u] = ∞;
        done[u] = false;
    }
    dist[s] = 0;
    queue = priority_queue((0, s));

    while queue is not empty {
        (dist_u, u) = queue.pop();
        if not done[u] {
            done[u] = true;
            for edge in graph[u] {
                dist_v = dist_u + weight(edge);
                if dist_v < dist[v] {
                    dist[v] = dist_v;
                    queue.push((dist_v, v));
                }
```

```
19                 }
20             }
21             if done[t] {
22                 break;
23             }
24         }
25
26         return dist[t];
27     }
```

2.3 Planarity

A fascinating and important class of graphs that we will focus on in Chapter 5 are *planar graphs*. We will give the most important definitions and facts about planar graphs here, and refer the reader to [Nis88] if they wish to read more.

2.3.1 Planar embeddings

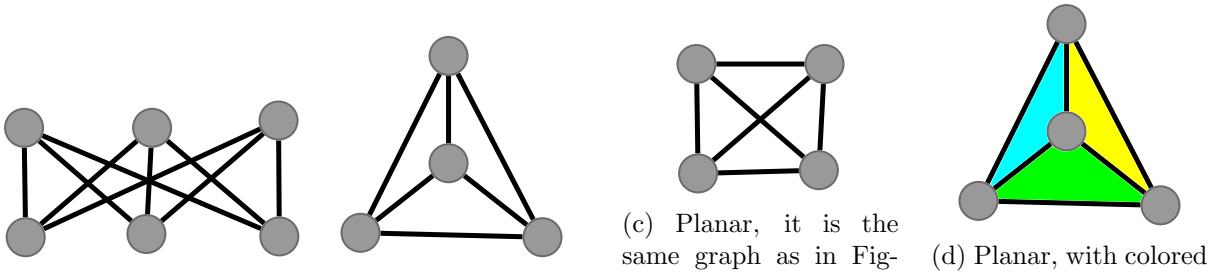
Definition 2.3.1 (Embedding)**.** Let G be a graph. An *embedding* of G is a drawing of G on the plane $\mathbb{R}^2$, with points representing vertices and curves representing edges between their endpoints' respective points, such that none of the edges intersect each other except in their endpoints.

Definition 2.3.2 (Planar graph)**.** We say that a graph G is a *planar graph* if there exists a planar embedding of G. A planar graph along with a specific planar embedding is called a *plane graph*.

Many real-life graphs, especially those based on physical structures, are either planar or almost so. Two edges crossing often entails an inefficiency or extra cost: like having to build a bridge over a road instead of joining the two roads in a crossroad. Many algorithmic problems are much easier to solve in planar graphs, and they are common enough in practical use that the restriction is not too restrictive to be useful.

Definition 2.3.3 (Straight-line embedding)**.** Let G be a graph. A *straight-line embedding* of G is a planar embedding of G where each edge can be drawn as a line segment between its endpoint vertices and still not cross any other edge. In a straight line embedding we can forgo the mappings of the edges altogether and consider the mapping of vertices only. Such embeddings always exist: if G is planar then there is a straight-line embedding of G.

See Figure 2.2b, Figure 2.2c and Figure 2.2d for an example of a planar graph. Figure 2.2b and Figure 2.2d also show planar embeddings of the graph. Figure 2.2a shows a graph that is not planar, since no planar embeddings of the graph exist. Note that in all these examples we have



(a) Not planar

(b) Planar

(c) Planar, it is the same graph as in Figure 2.2b

(d) Planar, with colored faces.

Figure 2.2: Examples of planar and non-planar graphs

drawn all the edges as straight line segments, but that is not necessary: as long as a curve does not cross any other curves it can be as curved as we want.

It is generally complicated to determine whether a given graph is planar in practice, and to compute an appropriate embedding if it is. For all the algorithms we have implemented in this paper, if they take a planar graph as input, we have for simplicity assumed that we are also given a planar embedding of the graph. Furthermore, since all planar graphs also have straight-line embeddings, we have assumed that the given embeddings are straight-line embeddings. Our theoretical results hold for planar graphs in general, but in practice these assumptions make implementing the algorithms less tedious.

2.3.2 Duality

The next topic is easy to visualize and understand, but challenging to formalize. Imagine loading a drawing of a planar graph like the one in Figure 2.2b into an image editing program, and using the fill tool to cover each region in a different color, like in Figure 2.2d. Each such region is called a *face* of the graph, including the region 'outside' the graph called the *outer* face. Two faces are adjacent if they are separated by just a single edge: if we were to delete the edge our fill tool would give both the same color. We will now formalize this concept.

Definition 2.3.4 (Face)**.** Let G be a plane graph. A *face* of G is a region in the embedding bounded by a cycle that contains no other vertices or edges. Equivalently, we can define faces as the connected components that remain in $\mathbb{R}^2$ after we delete all vertices and edges from our embedding.

Note that different embeddings of the same planar graph may yield different faces. When we refer to a face of a graph, it is always in relation to a certain embedding of the graph.

Definition 2.3.5 (Duality of planar graphs)**.** Let G be a plane graph. The *dual graph* of G, denoted as $G^\star$, is the graph where

- The vertices represent faces of G

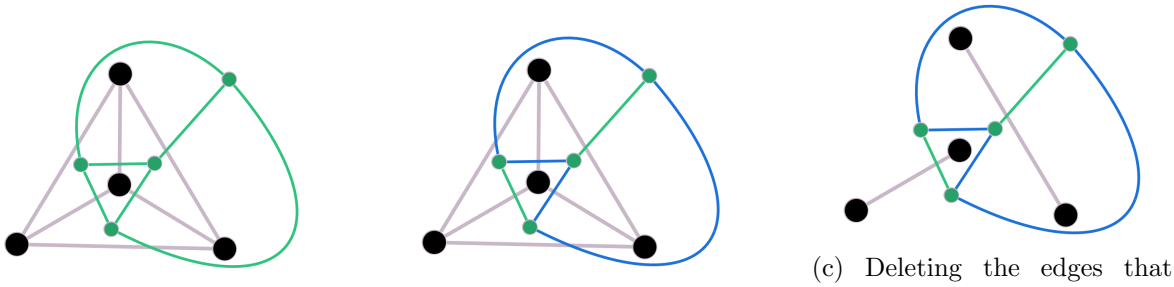
- There is an edge between two faces if they are adjacent in G.

Each edge in $e \in E(G)$ will always have a face on either side, possibly the same face, and thus have a corresponding edge $e^\star \in E(G^\star)$ in the dual graph. We can therefore define two convenience functions $\text{left}, \text{right} : E(G) \to V(G^\star)$ to get the left and right faces of an edge, respectively. If G is weighted, we usually set the weights of $E(G^\star)$ according to their counterparts: $\text{weight}(e^\star) := \text{weight}(e)$. See Figure 2.3a for an example of a dual graph.

Note that the dual graph is also planar, and the dual of the dual is the original graph[1]. We could then for example let e be an edge in the dual graph, and then refer to its real counterpart as $e^\star$. It would not be wrong, but it could possibly lead to confusion. Furthermore, we do not need that fact for any of the results in this thesis. We will therefore give variable names like G, u and e for the graphs, vertices and edges that we are 'working on', and use variable names like $G^\star$, $u^\star$ and $e^\star$ for their dual equivalents only in intermediary computations before arriving at results for our original graph.

Fact 2.3.1 (Cycle–cut duality)**.** Let $G := (V, E, from, to)$ be a connected planar graph, and let $C^\star$ be a simple cycle in $G^\star$. Then $C^\star$ will always correspond to a minimal cut in G. If we define $C := \{e \mid e^\star \in E(G^\star)\}$ as the edges in $E(G)$ that correspond to edges in $C^\star$, then $(V, E \setminus C, \text{from}, \text{to})$ is an disconnected graph of exactly two components. If the cycle $C^\star$ is not simple, then we still end up with an disconnected graph, but we may have more than just two components.

See Figure 2.3 for an example. In Figure 2.3b we have found a simple cycle in the dual graph, and if we delete the corresponding edges in the original graph we end up with the disconnected graph in Figure 2.3c.



(a) A planar graph with its dual colored in green

(b) A simple cycle in the dual, colored in blue

(c) Deleting the edges that cross the dual's cycle cuts the graph in two

Figure 2.3: A simple cycle in a dual graph always corresponds to a cut in the original graph.

Theorem 2.3.1 (The relation between the number of vertices, edges and faces)**.** Let $G := (V, E, \text{from}, \text{to})$ be a connected planar graph, where $n := |V|, m := |E|$, and f is the number of faces in any embedding of G.
Claim: $n + f - m = 2$.

[1]Strictly speaking, there is an embedding of $G^\star$ such that $G^{\star\star} \cong G$.

Proof. Let $H \sqsubseteq G$ be any non-empty connected subgraph that does not have any cycles, and let $n_H := |V(H)|$. Since it does not have any cycles, we have that:

- The outside face must be the only face: $f_H := 1$.
- Each edge must connect a 'new' vertex to the rest of the graph, except the first edge which connects two new vertices. Therefore the number of edges is one less than the number of vertices: $m_H := n_H - 1$.

We now have that $n_H + f_H - m_H = n_H + 1 - (n_H - 1) = 2$, so the equality holds for this subgraph.

Now we can iteratively add either an edge alone or both a vertex and an edge to H until we have G. If we add just an edge, we increase both m_H and f_H by 1, and the equality still holds. If we add a new vertex with a new edge, we increase both n_H and m_h by one, and the equality still holds.

Therefore, the equality $n + f - m = 2$ holds for any connected planar graph G. □

Corollary 2.3.1. The number of faces is fixed. A graph may have different faces depending on the embedding, but the number of faces is always the same.

Corollary 2.3.2. Since all faces (except possibly the outer face) are bounded by at least three edges, and all edges touch at most two faces, we can show that if $n \geq 3$, then $m \leq 3n - 6$.

Chapter 3

Shortest Odd Walk

Before we start on the main topic of this thesis, we want to discuss a closely related problem:

Shortest Odd Walk
Input: A weighted graph G, two vertices $s, t \in V$
Output: the shortest s-t-walk in G that uses an odd number of edges

The difference is simple: a walk may use the same vertices multiple times, whereas a path can not. A naïve attempt at solving Shortest Odd Path will often accidentally use the same vertices multiple times, and then be an odd walk instead. Therefore, we want to present an algorithm to solve Shortest Odd Walk first, and explain why it does not solve Shortest Odd Path.

3.1 Intuition

Our algorithm will take inspiration from Dijkstra's algorithm for Shortest Path, and assume that all the edges have non-negative weights. Remember, in Dijkstra's algorithm we have an array to keep the tentative best distance to each vertex. In this algorithm, we will keep two such arrays, one for the best distance using an odd walk, and one for the best distance using an even walk. Each vertex can be scanned at most twice: once when we have found the definitive best odd walk and want to find potential improvements to the even walks of its neighbours, and similar when we find the best odd walk.

3.2 Pseudocode

Here is the pseudocode of the algorithm. To see the code implemented in Rust, see the GitHub repository [Sim24b].

Code Listing 3.1: Shortest Odd Walk

```
def shortest_odd_walk(graph, s, t) {
    for u in 0..n {
        even_dist[u] = ∞
        odd_dist[u] = ∞
        even_done[u] = false;
        odd_done[u] = false;
    }
    even_dist[s] = 0

    queue = priority_queue([(0, true, s)]);
    while queue is not empty {
        (dist_u, even, u) = queue.pop()
        if even {
            if even_done[u] continue;
            even_done[u] = true;

            for edge in graph[u] {
                v = to(edge);
                dist_v = dist_u + weight(edge);
                if dist_v < odd_dist[v] {
                    odd_dist[v] = dist_v;
                    queue.push((dist_v, false, v));
                }
            }
        }
        else {
            if odd_done[u] continue;
            odd_done[u] = true;

            for edge in graph[u] {
                v = to(edge);
                dist_v = dist_u + weight(edge);
                if dist_v < even_dist[v] {
                    even_dist[v] = dist_v;
                    queue.push((dist_v, true, v));
                }
            }
        }
        if odd_dist[t] < ∞ {
            return odd_dist[y];
        }
    }
    return None;
}
```

In the pseudocode we show how to find the best odd walk from the source vertex to the target vertex. If we instead want to find the best odd or even walks to all vertices, we can simply remove the if-clause around the target, and return the arrays instead.

3.3 Analysis

Consider Figure 3.1. There are no odd paths from s to t, but we have an infinite amount of odd walks by utilizing the cycles $[a, b, c]$ or $[a, c, b]$ an odd number of times to offset the parity. Our

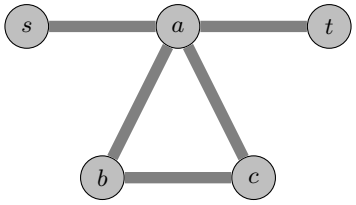


Figure 3.1: No odd s-t-path exist, yet we still have many odd s-t-walks.

algorithm would perhaps first find an odd walk to a, then an even walk to b, then an odd walk to c, then an even walk to a, and lastly an odd walk to t. This is one of the two odd s-t-walks of minimum cost. However, a is visited twice in the walk, once for each parity, and the resulting walk is not a path. Therefore, this algorithm cannot be used to solve Shortest Odd Path.

The main limitation of the algorithm is that the edges in the input graph must have either non-negative weights or no weights at all. Otherwise we cannot guarantee that `even_dist[u]` and `odd_dist[u]` have their final, correct values when we scan a vertex u. Note that unlike most other algorithms shown in this thesis, this algorithm does not require the input graph to be undirected, it may also be directed.

Theorem 3.3.1. Let (G, s, t) be an instance of Shortest Odd Walk, let $n := |V|$ and let $m := |E|$.
Claim: the algorithm runs in time at most $O(m \cdot \log m)$, or $O(m \cdot \log n)$ if the graph is simple.

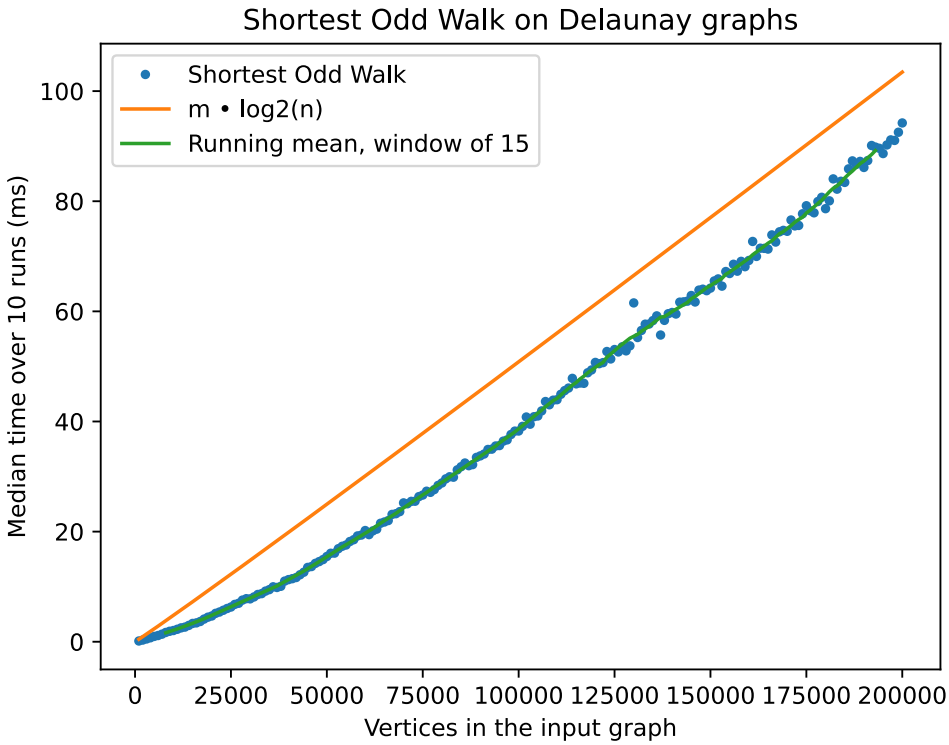
Proof. Because of our `odd_done` and `even_done` arrays, we can guarantee that each vertex is scanned at most twice, once for each parity. For each scan, we loop through each of the neighbours in linear time, and consider putting them in the queue. The total cost of the scans is therefore at most $O(m)$. A vertex may be put into the queue many times before it is scanned, in the worst case once for each of its neighbours. That means that we put vertices in the queue at most $O(m)$ times, for a total cost of $O(m)$, and removing all of them takes a total of $O(m \cdot \log m)$.

The algorithm runs in time at most $O(m) + O(m \cdot \log m) = O(m \cdot \log m)$, which shows the first part of the claim.

If the graph is simple we may simplify the complexity further: $O(m \cdot \log m) \subseteq (m \cdot \log n^2) = O(m \cdot 2 \cdot \log n) = O(m \cdot \log n)$, which shows the second part of the claim. □

To test how well the algorithm scales, we generate 200 Delaunay graphs of sizes 1000, 2000, 3000 and so on until 200 000. We explain how these graphs are generated in Section 6.4.1. For each graph, we have estimated a source and target with the maximum shortest path between them, and run our Shortest Odd Walk algorithm. We take the mediam running time of 10 runs, and plot the results below.

We have also tried to find constants to convert the $O(m \cdot \log n)$ theoretical complexity into a comparable function, and plotted it next to the real practical results.



As we can see, the algorithm easily solves Shortest Odd Walk on graphs of 200000+ vertices in less than 100ms. These results are excellent. Despite having a similiar complexity to the other algorithms in this thesis, this algorithm is still by far the fastest in practice.

Though we discovered it independently, the algorithm is not particularly groundbreaking or in any way creative. Therefore, we do not expect it to be original. It is, however, quite fast, and we are happy with that. The main reason we include it in this thesis is because of its pedagogical value in introducing our main topic: Shortest Odd Path.

Her burde vi finne en eksisterende algoritme og referere til den.

Chapter 4

Shortest Odd Path

Now that we have tried out some algorithms for Shortest Odd Walk, we are finally ready to add the restriction that each vertex is used at most once, and thus solve Shortest Odd Path. The algorithm we are about to present is based on Derigs' algorithm [Der85], though with some improvements.

Improvements? Simplifications? Modifications? Improved presentation?

4.1 Reduction to Shortest Alternating Path

Consider first another related problem:

Shortest Alternating Path
Input: a weighted graph $G := (V, E, \text{from}, \text{to}, \text{weight})$, two vertices $s, t \in V$, and a set $F \subseteq E$
Output: an s-t-path in G of minimum cost, where every other edge used is in F

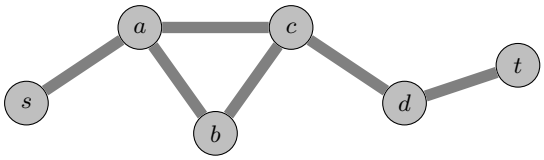
Derigs observed that Shortest Odd Path can be reduced to a special case of Shortest Alternating Path, by constructing what we will refer to as a *mirror graph*.

Dobbelsjekk dette, men Shortest Alternating Path er antagelig NP-komplett og vi burde nevne det før vi sier at vi skal løse et spesialtilfelle.

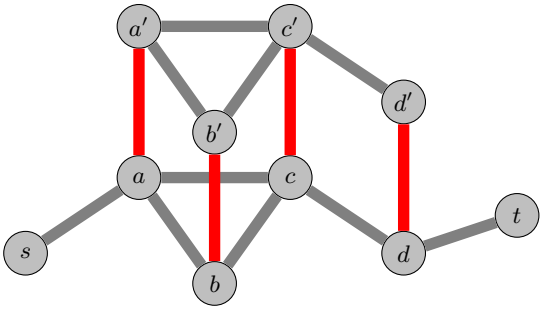
Definition 4.1.1 (Mirror graph)**.** Let G be a graph, and $s, t \in V(G)$ be two vertices. We construct a supergraph $M \supset G$, by adding an extra copy of everything in the graph not directly related to s and t. For each vertex $u \in V(G) \setminus \{s, t\}$, we add a 'mirror' vertex u', and a connecting 'mirror' edge of weight 0 between u and u'. For each edge $e \in E(G) \setminus (G[s] \cup G[t])$ from u to v we add an edge e' of the same weight between the 'mirror' copies of its endpoints, from u' to v'. Now M is the mirror graph of G with respect to s and t.

See Figure 4.1 for an example. If G is the graph in Figure 4.1a, then the mirror graph could look like Figure 4.1b. The part of the mirror graph M that is also present in G is referred to as the 'real' side of the graph, while the new vertices are on the 'mirror' side. Vertices and edges from the mirror side are usually labeled with an $'$. For convenience, we define the function mirror : $V(M) \setminus \{s,t\} \to V(M) \setminus \{s,t\}$ to go from a vertex to its counterpart on the other side of the mirror. Furthermore, in an abuse of notation, we also define mirror : $E(M) \setminus (G[s] \cup G[t]) \to E(M) \setminus (G[s] \cup G[t])$ as the same but for edges.

The edges in M that 'cross' the mirror by going between vertices and their counterparts form a matching in M. We refer to these edges as *the edges in the matching*, or sometimes simply as just *the matching*. Note that with the exception of s and t, this is an almost perfect matching in M.



(a) The input graph G



(b) The mirror graph M of G, with the matching marked in red

Figure 4.1: Reduction from Shortest Odd Path to Shortest Alternating Path.

Our reduction from Shortest Odd Path to Shortest Alternating Path follows:

1. Let (G, s, t) be an instance of Shortest Odd Path.
2. Construct M as the mirror graph of G, and let F be the matching in M. Now (M, s, t, F) is an instance of Shortest Alternating Path.
3. Let P' be the shortest alternating path of (M, s, t, F), if one exists. If none exist, then we do not have any odd s-t-paths in G either and are already done.
4. Construct P by filtering out the edges in the matching from P', and for each edge $e' \in E(M)$ from the mirror side of M we replace it by the corresponding edge $\text{mirror}(e') \in E(G)$ from the real side.
5. Now P is the shortest odd s-t-path in G.

For example, if our input G for Shortest Odd Path is Figure 4.1a, then M and F could look like Figure 4.1b. The only alternating s-t-path is $P' := [(s,a),(a,a'),(a',b'),(b',b),(b,c),(c,c'),(c',d'),(d',d),(d,t)]$. When we translate it to a path in G, we end up with $P := [(s,a),(a,b),(b,c),(c,d),(d,t)]$, which is the shortest odd s-t-path in G.

Now that we have two copies of most vertices in M, we run the risk of accidentally using the same vertex multiple times and ending up with a walk rather than a path in G, like with our algorithm in Chapter 3. The key to note here is that F is an (almost) perfect matching, and when we step on a vertex u we have to cross the mirror and step on mirror(u) next. We will never visit u, go somewhere else, and then later come back to visit mirror(u). So both copies must be used directly after each other, and when we translate the path in to M to a path in G the two copies are effectively merged to just a single step in the path. Therefore, vertices are never repeated and we always end up with a path.

To see why the reduction necessarily yields an *odd* path, simply observe that for each step we take in the graph, we have to go to the other side of the mirror. If we take another step, we get back to the same side again. It is only when we reach the target vertex t that we can stop and not have to go to the other side. Therefore, to reach a neighbour of t, we must have used an even number of edges from the matching and an even number of edges not in the matching. When we take the last step to reach t we have used an odd number of edges and thus found an odd path. If this alternating s-t-path in M is the shortest such path, then the corresponding path in G must also be the *shortest* odd s-t-path in G.

The interested reader may see [Der85] for more details on this reduction.

Here we should mention that SAP is NP-complete, and who proved it, and it does not matter to us.

Ball and Derigs [BD83] have shown how to efficiently solve Shortest Alternating Path. In their algorithms, subgraphs are shrunk into pseudonodes whenever possible, to make the graph smaller. The drawback is that certain pseudonodes must later be expanded again, which is the most complicated and expensive part of their algorithms. In our case, however, we have a special case of Shortest Alternating Path. The set F is, with the exception of s and t, a perfect matching of M, and we will therefore never have to expand pseudonodes after shrinking them. The curious reader may visit [BD83] for more on these algorithms and why our almost-perfect matching is a simpler case.

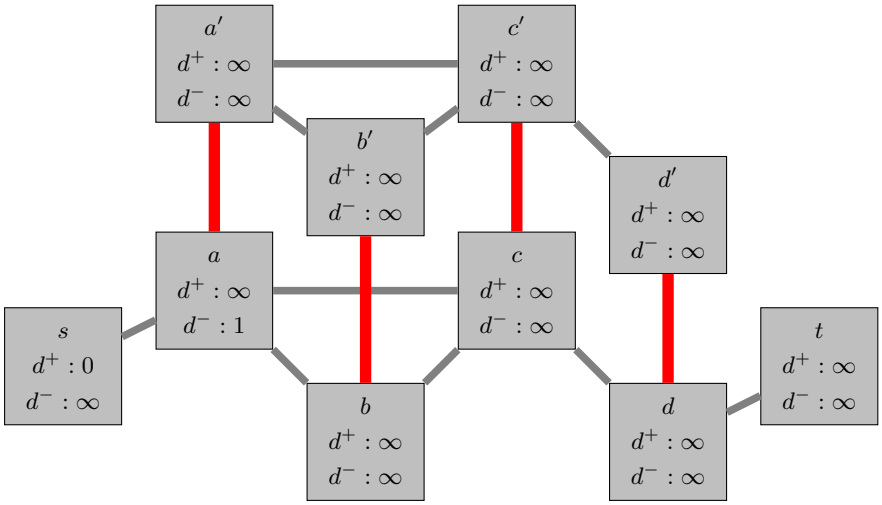
4.2 The idea for our Shortest Alternating Path algorithm

We will explain the general idea of our algorithm by following an example, and solve for the graph in Figure 4.1a. First we construct the mirror graph like explained in Section 4.1, to produce the graph in Figure 4.1b. Then we initialize an empty priority queue of vertices and edges to be scanned. For each vertex $u \in V(M)$, we denote

- d_u^+ := the length of the shortest alternating s-u-path ending on a matched edge
- d_u^- := the length of the shortest alternating s-u-path ending on a non-matched edge
- pred_u := the last edge used to find u's most recent value for d_u^-

Initally these are either ∞ or undefined, except for the source vertex s, where we can set $d_s^+ := 0$. Then, for each edge $e \in N(s)$, we can set $d^-_{\text{to}(e)} := \text{weight}(e)$, $pred_{to(e)} := e$, and add $\text{to}(e)$ to our priority queue with priority $2 \cdot \text{weight}(e)$.

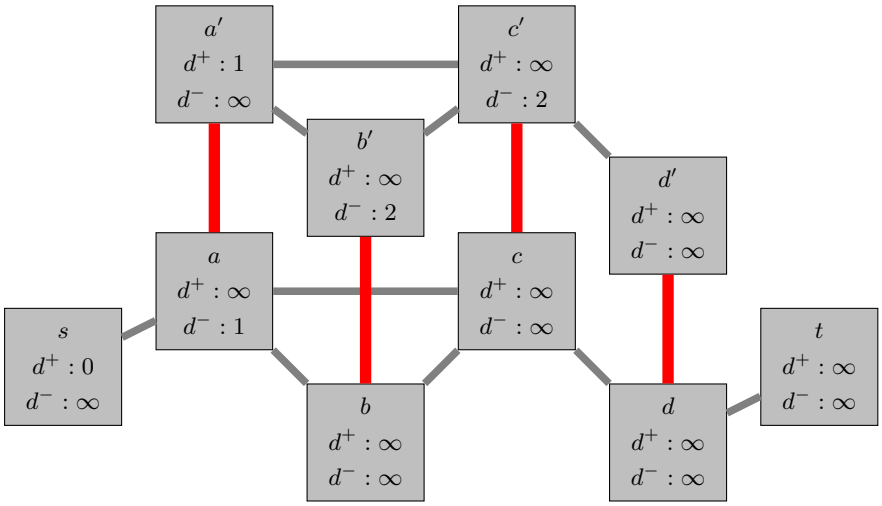
We visualize it in the diagram below.



Queue:

- Vertex(2,a)

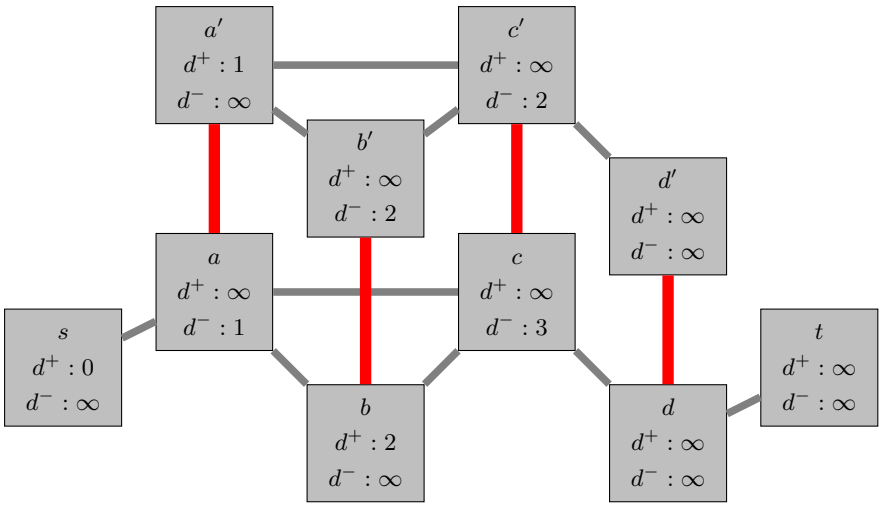
The first and only vertex in the queue is a. We pop it, set $d_{a'}^+ := d_a^-$, and 'scan' a'. By that, we mean to look at each neighbour $e \in G[a']$, and see if our new value $d_{a'}^+ + \text{weight}(e)$ is better than the previous value $d^-_{\text{to}(edge)}$. That is the case for both b' and c', so we update their values and add them to the queue. Their priorities in the queue is equal to twice their d^- values, which is $2 \cdot 2 = 4$ for both of them.



Queue:

- ~~Vertex(2,a)~~
- Vertex(4,b')
- Vertex(4,c')

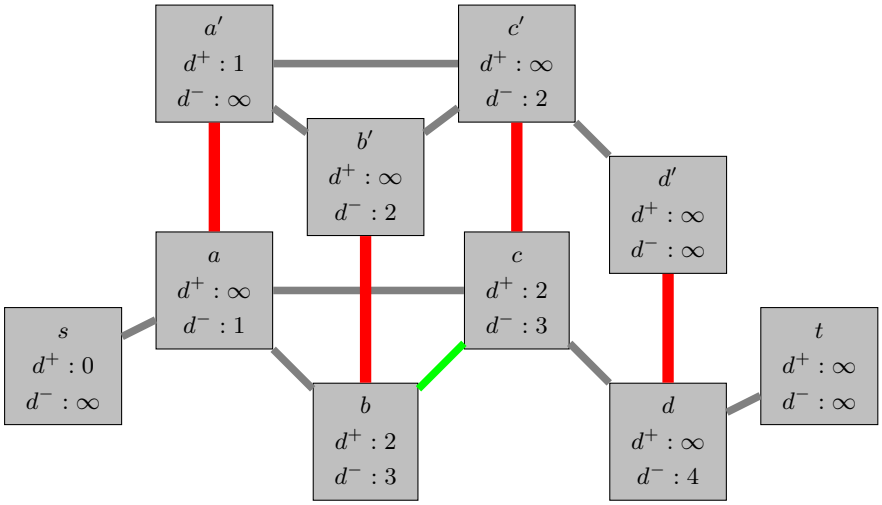
The next vertex in the queue is b', so we set $d_b^+ := d_{b'}^-$ and scan b':



Queue:

- ~~Vertex(4,b')~~
- Vertex(4,c')
- Vertex(6,c)

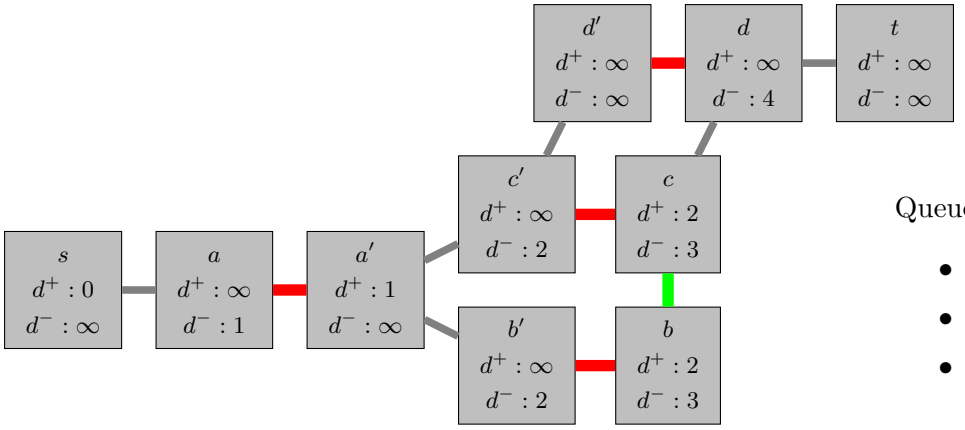
Now c' is the next in the queue, we set $d_c^+ := d^- c'$ and scan c'. This is where the interesting part happens: now we have set both d^+ and d^- for b and c, and that means that we have found an odd cycle in the graph. The edge between them, e, is called the *blossom edge*, and is marked in green. We add e to the queue, with the priority $d_c^+ + d_b^+ + \text{weight}(e)$.



Queue:

- ~~Vertex(4,c')~~
- Blossom(5,(b,c))
- Vertex(6,c)
- Vertex(6,d)

Next up is to scan this blossom edge, and compute its corresponding odd cycle by backtracking from c and b until they meet at a'. To visualize the cycle, we like to 'stretch out' the graph a little, and draw it like below. Note that some of the edges are omitted for clarity. Now we can see that the cycle consists of $[a', c', c, b, b', a']$. We call the set $\mathbb{B} := \{c', c, b, b'\}$ a *blossom*, and a' the *base* of the blossom, inspired by the famous Blossom algorithm by [Edm65].

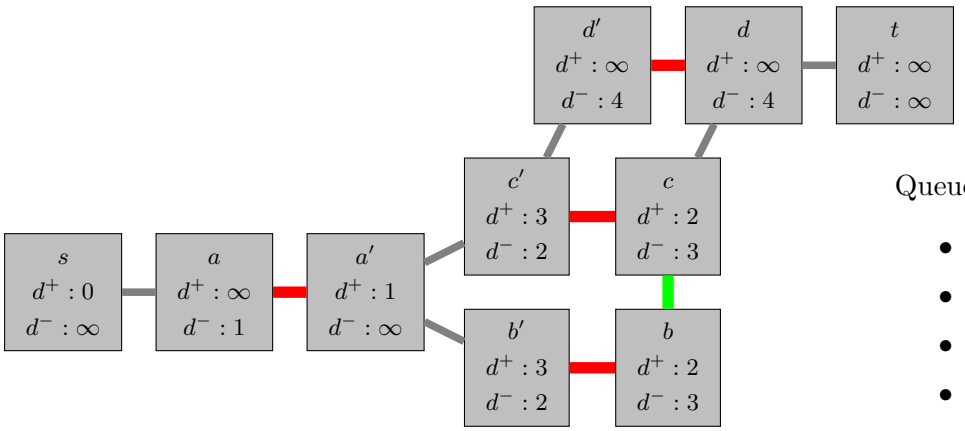


The first reason why we care about this blossom is because now we can immediately set the final, optimal d^- and d^+ for all the vertices in the blossom. That is because we now have two alternating paths to each vertex, one goes around the cycle while the other takes the shortcut. One of these ends up on a matched edge, and the other on a normal edge. Furthermore, both of these are the optimal paths and can be used to set final values for d^+ and d^-. For example, to go from s to c', we can either go through $[s, a, a', c']$ with a cost of d^-_c, or go along $[s, a, a', b', b, c, c']$ with a cost of d^+_c.

More specifically, for each $u \in \mathbb{B}$:

- If $d^+_u = \infty$, we set $d^+_u = d^-_{\text{mirror}(u)}$.
- If we can improve d^-_u by coming from its neighbour in the blossom, we do so.

After all these values have been set, we immediately scan all the vertices in $\mathbb{B}$ that just received values for d^+. In this example we scan c' and b', and discover d'. Unfortunately, since this is a very small blossom we don't have any vertices that receive new values for d^-.

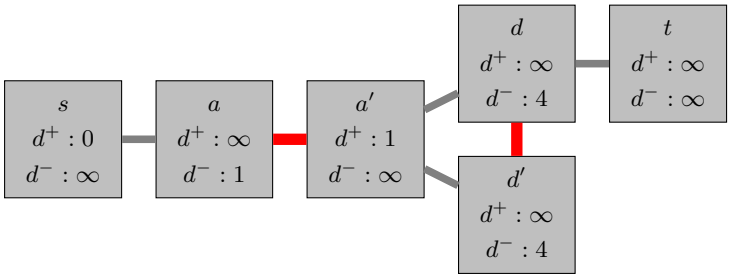


The second reason we compute the blossom is that we no longer care much about the individual vertices in $\mathbb{B}$, and can shrink it into just the base a'. We will still scan vertices like c from the queue as before, but whenever we are backtracking to compute blossoms we can skip the verties

in $\mathbb{B}$ entirely and go straight to the base a' instead. In this example, in a few iterations the algorithm will find either (d', a') or (d, a') as a blossom edge, with just $\{d, d'\}$ as its blossom and a' as the base here as well. If we didn't contract the previous blossom, this new blossom would instead consist of $\{c', c, b, b', d, d'\}$, but we are already completely done with most of those vertices and computing all of it again would be a waste. Therefore we shrink them.

If the reader is familiar with the more general Shortest Alternating Path algorithm [BD83] or the original blossom algorithm [Edm65], and worry that such pseudonodes often have to be expanded again, then remember that in our case the set F is an almost-perfect matching and those cases never happen.

trenger vi si dette? Droppe det? Formulere det annerledes?



Queue:

- Vertex(6,c)
- Vertex(6,d)
- Vertex(8,d')

Let us now skip a few steps, until t eventually reaches the front of the queue. At that point we have that $d_t^- = 5$, and that is also the cost of the shortest odd path in our original input graph. To compute the exact path we can backtrack from t to s and then translate that path as described in Step 4 of our reduction in Section 4.1. We end up with the path $[(s, a), (a, b), (b, c), (c, d), (d, t)]$, which is the shortest odd s-t-path in the graph.

Would be nice with a figure of a larger blossom, to better show what the structure is, and how those values are set.

4.3 Pseudocode

Here we give a detailed pseudocode of the algorithm, along with explanations of some of the finer details should the interested reader consider implementing the algorithm on their own. For all of the code in one piece without comments or alternatives, see Appendix A. And of course, to see the code implemented in Rust, see the GitHub repository [Sim24b].

4.3.1 Initialization

First we initialize the arrays we need, with appropriate default values for all vertices. The mirror graph is constructed as described in Definition 4.1.1.

Code Listing 4.1: Initialization

```
fn init(input_graph, s, t) {
    graph = create_mirror_graph(input_graph);

    for u in 0..n {
        d_plus[u] = ∞;
        d_minus[u] = ∞;
        pred[u] = null;
        completed[u] = false;
        basis[u] = u;
        in_current_blossom[u] = false;
    }
    d_plus[s] = 0;
    completed[s] = true;

    for edge in graph[s] {
        priority_queue.push(Vertex(weight(edge), to(edge)));
        d_minus[to(edge)] = weight(edge);
        pred[to(edge)] = e;
    }
}
```

The main function ties it all together. The `control` function includes the main loop and does most of the work, until there is nothing more to do. Then we can either find the shortest odd path by backtracking, or conclude that no odd paths exist.

Code Listing 4.2: Main

```
fn main(input_graph, s, t){
    init(input_graph, s, t);

    control();

    if d_minus[t] == ∞ {
        return None;
    }
    cost = d_minus[t];
    path = backtrack();

    return Some(cost, path);
}
```

Here is how to backtrack once we know we have the shortest odd path to t. Each of the edges from the mirror side must be replaced by their equivalents on the real side of the input graph. Note that unlike when backtracking blossoms, here we do not consider the base of the vertices. Here we pretend we never shrunk the blossoms into pseudonodes, so that we find the entire path.

Code Listing 4.3: Backtracking

```
fn backtrack() {
    current_edge = pred[t];
    path = [current_edge];
    while from(current_edge) != s {
        current_edge = pred[mirror(from(current_edge))];
        if from(current_edge) < input_graph.n() {
            path.push(current_edge);
        }
        else {
            path.push(shift_edge_by(current_edge, -input_graph.n()));
        }
    }
    return path;
}
```

4.3.2 The control loop

This is the main loop of the algorithm. Each iteration of the outer loop will either scan a vertex, handle a blossom edge, or conclude that we are done. Each vertex can be put on the queue many times, but we only want scan it once, so we discard those that have already been scanned. Each blossom consists of many edges, each of which can be put on the queue, and those too we want to handle only once. If the two endpoints of a blossom edge already have the same basis, then we know they have already been computed as part of the same blossom and the edge may safely be discared.

Code Listing 4.4: Control, the main loop

```
fn control() -> bool {
    loop {
        while ! priority_queue.is_empty() {
            match priority_queue.top() {
                Vertex(_, u) => {
                    if completed[u] {
                        priority_queue.pop();
                    }
                    else {
                        break;
                    }
                },
                Blossom(_, edge) => {
                    if base_of(from(edge)) == base_of(to(edge)) {
                        priority_queue.pop();
                    }
                    else {
                        break;
                    }
                }
            }
        }

        if priority_queue.is_empty() {
            // No odd s-t-paths exist :(
            return;
        }
        match priority_queue.pop() {
            Vertex(_, u) => {
                if u == t {
                    // We have found a shortest odd s-t-path :)
                    return;
                }
                d_plus[u] = d_minus[mirror(u)];
                scan(mirror(u));
            }
            Blossom(_, edge) => {
                blossom(e);
            }
        }
    }
}
```

Code Listing 4.5: Scan

```
fn scan(u) {
    completed[u] = true;
    dist_u = d_plus[u];
    for edge in graph[u] {
        v = to(edge);
        new_dist_v = dist_u + weight(edge);

        if ! completed[v] {
            if new_dist_v < d_minus[v] {
```

```
10                  d_minus[v] = new_dist_v;
11                  pred[v] = edge;
12                  priority_queue.push(Vertex(new_dist_v, v));
13              }
14          }
15          else if d_plus[v] < ∞ and base_of(u) != base_of(v) {
16              priority = d_plus[u] + d_plus[v] + weight(edge);
17              priority_queue.push(Blossom(priority, edge));
18              if new_dist_v < d_minus[v] {
19                  d_minus[v] = new_dist_v;
20                  pred[v] = e;
21              }
22          }
23      }
24  }
```

4.3.3 Backtracking a blossom edge

When we compute a blossom edge e, we need to compute the vertices and edges that make up the blossom. We do this by creating two paths, one from from(e) and one from to(e), and backtrack towards the source while alternating between matched and non-matched edges until the paths meet up at a common ancestor b. Then we set b as the base, and the two paths make up our blossom.

The naïve way would be to backtrack both paths individually all the way to the source, and only then see where they start to overlap. That would run in time linear to all the vertices in the graph and is a waste of time. A slightly better idea is to backtrack one path all the way to the source, and then backtrack the other only until it reaches a vertex in the other's path. That is better, but this too would run in linear time even if we somehow know beforehand which path needs the fewest edges.

Instead, we iteratively backtrack both paths at the same time, and mark each vertex when added to a path. Whenever one path reaches a vertex that is already marked by the other, we mark that vertex as the base and delete the vertices in the other path that came after it. Now we can compute the vertices in the blossom in time linear to the number of vertices in the blossom, rather than the entire graph.

Implementing this may sound difficult, tedious and error-prone, but is actually way worse. Here are some reasons why this is the most complex part of any algorithm in this thesis, and why any programmer should take particular care when implementing this:

- It is difficult to alternate through matched and non-matched edges, while simultaneously alternating between computing two separate paths.
- The paths may not be of equal length, even if the graph is unweighted.
- The endpoints we start backtracking from may already be the base, if the blossom edge itself is adjacent it.

- The blossom edge should end up in both, either or neither of the paths, depending on where it is in relation to the base.
- The paths may not have any edges at all.
- We have to consider the basis of each vertex found on the paths rather than the vertex itself, because we shrink each blossom into a pseudonode after computing it.
- One path may reach the source vertex before the other path has reached b. Then we have to stop backtracking that path and focus on the other.

Here is our solution. The procedure returns the base and two lists of all the non-matched edges that make up blossom. That usually includes the blossom edge itself, which is part of both paths unless it is adjacent to the base itself.

Code Listing 4.6: Backtrack blossom

```
fn backtrack_blossom(edge) {
    p1 = [ reverse(edge) ];
    p2 = [ edge ];
    u = get_basis(to(edge));
    v = get_basis(from(edge));
    in_current_blossom[u] = true;
    in_current_blossom[v] = true;

    loop {
        if u != s {
            u = get_basis(mirror(u));
            in_current_blossom[u] = true;
            e = pred[u];
            u = get_basis(from(e));
            p1.push(e);

            // Ff true, then u is the base
            if in_current_blossom[u] {
                p1.pop();
                in_current_blossom[u] = false;

                // We remove all the edges in p2 after the base
                while p2 is not empty {
                    e = p2.last();
                    v = get_basis(from(e));
                    in_current_blossom[v] = false;
                    p2.pop();
                    if v == u {
                        break;
                    }
                }
                return (u, p1, p2);
            }
        }
        if v != s {
            // *Here we do the same for the other path*
        }
    }
}
```

The last if-statement closely resembles the first, except with other variables, so we have omitted it here for brevity. See Appendix A for the full version.

4.3.4 Computing blossoms

To compute a blossom, we first have to determine its base and its edges, like discussed in the previous section. Then we can use the edges in the paths to potentially improve values for d^+ and d^-, and to set the new base for all the involved vertices. Afterwards we scan all the vertices that we now gave d^+ values.

Two lists of edges can be processed simultaneously without issues, but we treat them separately here to avoid spending time on concatenating them. Setting blossom values and setting the basis can also be done at the same time, but we split it into to seperate functions to improve readability. However, the scans may only be performed after all the vertices in both list have received their new basis.

Code Listing 4.7: Blossom

```
fn blossom(edge) {
    (b, p1, p2) = backtrack_blossom(edge);

    to_scan1 = set_blossom_values(p1);
    to_scan2 = set_blossom_values(p2);

    set_edge_bases(b, p1);
    set_edge_bases(b, p2);

    for u in to_scan1 {
        scan(u);
    }
    for v in to_scan2 {
        scan(v);
    }
}
```

Code Listing 4.8: Set blossom values

```
fn set_blossom_values(path) {
    to_scan = [];

    for edge in path {
        u = from(edge);
        v = to(edge);
        w = weight(edge);
        in_current_cycle[u] = false;
        in_current_cycle[v] = false;

        // We can set a d_minus
        if d_plus[v] + w < d_minus[u] {
            d_minus[u] = d_plus[v] + w;
            pred[u] = reverse(edge);
        }

        int m = mirror(u);
        // We can set a d_plus, and scan it
        if d_minus[u] < d_plus[m] {
            d_plus[m] = d_minus[u];
            to_scan.push(m);
        }
    }

    return to_scan;
}
```

Code Listing 4.9: Set edge bases

```
    fn set_edge_bases(base, path) {
        for edge in path {
            u = from(edge);
            m = mirror(u);
            set_base(base, u);
            set_base(base, m);
        }
    }
```

4.3.5 Setting the base of blossoms and pseudonodes

When we have found and computed a blossom, we shrink it into a pseudonode by setting the base of all its vertices to the base of the blossom. Whenever we consider a potential blossom edge, we see if the two vertices have the same base, and if they do, they are in fact already in the same pseudonode and the edge can be disregarded. Whenever we set u to have the base b, we also have to see if any other vertices have u as their base and set their bases to b as well. Derigs never specified any data structure to update these bases efficiently.

The naïve solution would be to do something like this:

Code Listing 4.10: Näive basis

```
fn set_base(base, u) {
    basis[u] = base;
    for v in 0..n {
        if basis[v] == u {
            basis[v] = base;
        }
    }
}
fn get_base(u) {
    return basis[u];
}
```

This would search through all vertices in the graph in linear time. We have found two potential improvements to this. The first version is to use an oberserver pattern, where each vertex u keeps a record of the vertices that have u as its base. Initially $dependents[u] = [\,]$ for all of them. Then, when we update u's base to *base*:

Code Listing 4.11: Observer basis

```
fn set_base(base, u) {
    basis[u] = base;
    dependents[base].push(u);
    for v in dependents[u] {
        basis[v] = base;
        dependents[base].push(v);
    }
}
```

Now we go through only the vertices that have u as their base, in time linear to the count of vertices that need to be updated.

The second version is to use a structure resembling UnionFind, where each disjoint set and its representative is a blossom and its base. To update the base of u we simply set the new base and do nothing else. When we require the base of a vertex we recursively query its representative's base and contract the path along the way in the style of UnionFind.

Code Listing 4.12: UF-like basis

```
fn set_base(base, u) {
    basis[u] = base;
}
fn get_base(u) {
    if u != basis[u] {
        basis[u] = get_base(basis[u]);
    }
    return basis[u];
}
```

Now we can update a base in constant time, with the tradeoff of potentially slower queries. Is this faster? Sometimes it is. We benchmark both versions and discuss the results in Section 4.5.1.

4.4 Improvements on Derigs' algorithm

The main idea of our algorithm is the same as the original by Derigs [Der85]. We have, however, made some improving adjustments, and we will discuss these here.

First, the original algorithm used the idea of building up a tree T of alternating edges to mark scanned vertices as done. This is to avoid scanning the same vertex multiple times, and to make sure that a blossom edge is only put into the queue after both its vertices have been scanned. The notation $V(T) := V(T) \cup \{k, l\}$ was used to mark k as done. We had multiple problems with this. To begin, only the matched edges are ever added to the tree, so the disconnected 'tree' would not be a tree at all. Furthermore, unlike how for example Dijkstra's Algorithm builds up an implicit tree of scanned vertices, the vertices in our mirror graph are not at all scanned in the order of distance to the source, so even if we added actual edges to the tree it would still not be a tree. Finally, we found that the notation was misleading and overly complex for what really should be a simple concept. We have replaced this with a boolean array called `completed`, where each vertex u initially has `completed[u] = false` until it has been scanned, at which point we set `completed[u] = true`. It does the job.

Second, we have chosen to utilize sum types to have one priority queue with both vertices and blossom edges in one. The original algorithm used two priority queues that it always had to query together, which was difficult to read, write and debug. We find that combining them into one queue simplifies the code greatly. The way we set their priorities is also different: vertices now have a priority of *twice* its d^-, so that blossom edges can have a priority of the sum of the d^+'s of its two endpoints and its edge weight *without* dividing by two afterwards. Again do we

find this simpler, and we no longer have to convert integer weights to floating points just to prioritize them correctly.

Third, we developed a new data structure to store and update the basis of each vertex. See Section 4.3.5 for a discussion of different structures, and Section 4.5.1 for an empirical analysis of the improvement.

4.5 Analysis

The algorithm does not solve Shortest Odd Path in the absolute general case, but rather has some limitations. These are:

- The input graph must be undirected, otherwise we cannot use blossoms the way we do.
- The edges must have either non-negative weights or no weights at all, otherwise we cannot guarantee that d_u^+ and d_u^- are correct when we scan a vertex u.

Should this be written somewhere else?

Theorem 4.5.1. Let (G, s, t) be an instance of Shortest Odd Path, let $n := |V|$ and let $m := |E|$.
Claim: the algorithm runs in time at most $O(m \cdot \log m)$, or $O(m \cdot \log n)$ of the graph is simple.

Proof. First of all, we construct the mirror graph M with $2n - 2 \in O(n)$ vertices and $2m - \deg(s) - \deg(t) \in O(m)$ edges, in time $O(n + m)$.

With our `completed` array we can guarantee that each vertex is scanned at most once, and the scanning operation just loops through all the neighbours. Therefore, the total cost of all the scans is $O(n + \sum_{u \in V} \deg(u)) = O(n + 2m) = O(n + m)$.

The blossom operation is a little more convoluted. Thanks to the overly complicated code in our `backtrack_blossom` procedure in Section 4.3.3, we can backtrack from a blossom edge and determine the vertices in the blossom in time linear to the size of the blossom. Setting their values for d^+ and d^- can also be done in linear time, and the potential scans have already been accounted for above. The key point here is that we shrink the blossom into a pseudonode afterwards: each vertex can then only be part of such a blossom procedure at most once. Even though we may compute many blossom edges, the total amount of work will therefore never exceed $O(n)$.

Finally, we have the main loop, which iteratively pops vertices and blossom edges from the queue. Each of the $O(n)$ vertices may be put into the priority queue many times, at most once for each of its neighbours. That is a total of $O(m)$ vertices in the queue, for a total cost of

$O(m)$. Though it is unlikely, in the extreme case all edges may be put in the queue as blossom edges as well, again for a total cost of $O(m)$. In total, enqueueing everything costs $O(m)$, and dequeueing everything costs $O(m \cdot \log m)$.

In total, the algorithm runs in time $O(n+m) + O(n+m) + O(n) + O(m) + O(m \cdot \log m) = O(m \cdot \log m)$, which shows the first part of the claim. If the graph is simple, then $O(m \cdot \log m) \subseteq O(m \cdot \log n^2) = O(m \cdot 2 \cdot \log n) = O(m \cdot \log n)$, which shows the second part of the claim. □

A running time of $O(m \cdot \log n)$ means that the algorithm generally performs well on sparse graphs. We chose this algorithm with this running time because in Chapter 5 we will use it on planar graphs, where $m \leq 3n-6$, as shown in Corollary 2.3.2. That means it runs in $O(n \log n)$ on planar graphs.

We should note, however, that there are also other known polynomial algorithms for Odd Shortest Path. In the same paper that Derigs gave the algorithm of this chapter, he also gave another variant [Der85]. The main difference is that we drop the priority queues and use a list instead, and in the control loop instead search through the entire list and pop the element with the lowest priority. That search takes time at most $O(n)$, and can be done at most n times, for a total running time of $O(n^2)$. If the input graphs are dense, then this $O(n^2)$ algorithm may be preferable to our $O(m \log n)$ algorithm.

Nevn de andre variantene, og hvor fort de kan løses. $O(n+m)$ for uvektede grafer er ganske nice.

4.5.1 Testing different data structures for the Basis

As explained in Section 4.3.5, we have developed multiple data structures to keep track of the basis of each vertex. One of them is based on the Observer pattern, the other on the well-known UnionFind structure. We have tested both on 6 different real-life graphs, and we show the results below. We ran our shortest odd path algorithm on each graph 40 times, 20 for each structure, and noted the fastest times for each. We have run a heuristic search to find the pairs of vertices that are the furthest away from each other in each graph, so that the size of the graph accurately matches the size of the problem.

Graph	n	m	Observer	UF	Change
Oldenburg	6105	7035	5.1555 ms	4.6955 ms	-11.277%
San Joaquin County	18263	23874	6.7046 ms	5.7347 ms	-14.212%
Cali Road Network	21048	21693	19.280 ms	19.119 ms	+0.1590%
Musae Github [RAS19]	37700	289003	21.375 ms	19.438 ms	-2.4708%
SF Road Network	174956	223001	93.494 ms	87.570 ms	-6.3364%
Pokec Social Network [TZ12]	1632804	30622565	13.225 s	13.774 s	+4.1543%

Also test on Delaunay graphs

Fjern desimaler og sånt, det er ikke så nøyaktig

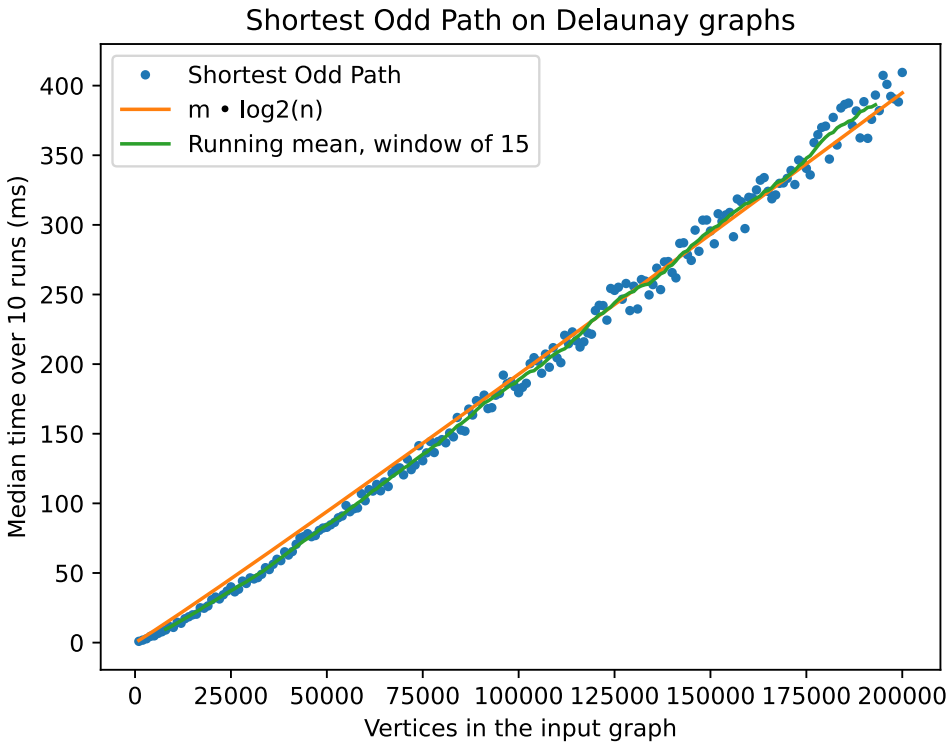
cite the rest of the graphs

The results are a little mixed. The UnionFind-based structure spends 4.2% more time on the huge Pokec Social Network graph. However, it also either outperforms or does as well as the Observer-based structure on all the other graphs, at best shaving off 14.2% of the time spent on the San Joaquin County graph. Since the UnionFind-based structure outperforms the Observer-based structure on average, we have chosen to use that one for the remainder of this thesis.

4.5.2 Running times on Delaunay graphs

To test how well the algorithm scales, we generate 200 Delaunay graphs of sizes 1000, 2000, 3000 and so on until 200 000. We explain how these graphs are generated in Section 6.4.1. For each graph, we have estimated a source and target with the maximum shortest path between them, and run our Shortest Odd Path algorithm. We take the mediam running time of 10 runs, and plot the results below.

The theoretical running time is $O(m \log n)$, and we have tried our best to come up with constants to create a running time function that matches our algorithm. In Delaunay graphs we have that $m \leq 3n$, so we have set $m := 3n$ in this function.



As we can see, the running times grow almost linearly as the inputs grow larger. The slight upwards curve is barely noticable. This is to be expected with a linearithmic theoretical running time.

The focus of this thesis is to create an algorithm that performs well on sparse graphs, especially planar graphs, which is why we consider the running times mainly on sparse graphs. If we instead had implemented an algorithm for denser graphs, or just graphs in general, then testing on graphs with more edges would be more appropriate.

With that said, we are very happy with these results. Being able to solve Shortest Odd Path on graphs of 100000 vertices in a fifth of a second is exactly the kind of performance we were hoping for.

Chapter 5

Network Diversion

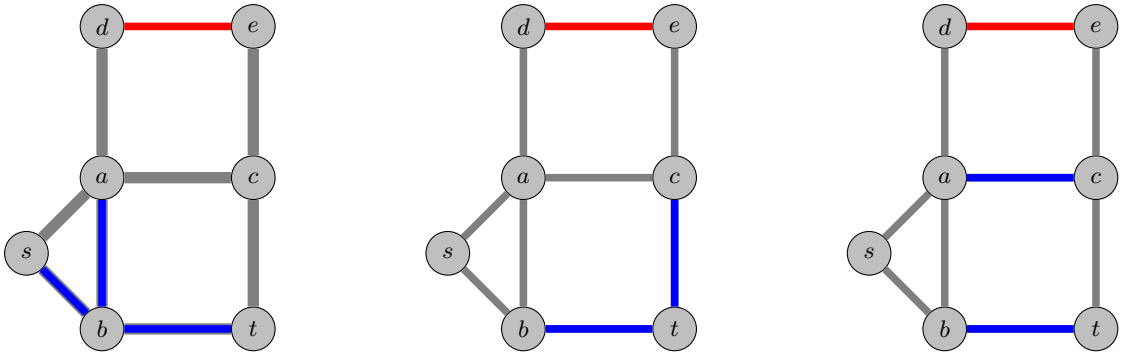
5.1 Introduction to Network Diversion

Now that we have an algorithm for Shortest Odd Path, we will use it to solve a much more useful problem:

Network Diversion
Input: a weighted graph $G := (V, E, \text{from}, \text{to}, \text{weight})$, two vertices $s, t \in V$, and a *diversion edge* $d \in E$
Output: a *diversion set* $D \subseteq E$ of minimum weight such that all s-t-paths in $(V, E \setminus D, \text{from}, \text{to}, \text{weight})$ must go through d

A diversion set may also equivalently be defined as a minimal s-t-cut that includes d. If all edges from the diversion set are deleted except d, then d is the bridge between what would otherwise be two separate components and all s-t-paths must go through d. Network Diversion can then be restated as the quest to find a *minimum* minimal s-t-cut that includes d. Both definitions are equivalent and yield the same optimum results, and being able to switch between formulations of the problem makes it easier to solve them.

See Figure 5.1 for examples. Figure 5.1a and Figure 5.1b show incorrect attempts at diversions, while Figure 5.1c shows a valid diversion.



(a) Invalid, there are still paths that don't go through the diversion edge

(b) Invalid, the diversion set also blocks all paths that use the diversion edge

(c) Valid, the diversion set funnels all traffic through the diversion

Figure 5.1: Valid and invalid diversions, attempting to force all s-t paths to go through the diversion edge in red by deleting the diversion set in blue.

Unlike with Shortest Odd Path, it is much easier to come up with practical applications of Network Diversion. Consider a communications network of machines that communicate offline, where you, a spy, are able to intercept all messages between two specific machines. How can you through outages and other diversions force all traffic in the network to go through where you can intercept the messages? Or for a more direct example: consider a network of roads and bridges, the knowledge that the enemy wants to move troops and supplies from one point to another, and a specific bridge where you are especially prepared to ambush them. How can you with the least amount of artillery destroy bridges as to force the enemy to move through your ambush?

Initially, finding a minimum minimal s-t-cut that includes a specific edge may seem like yet another variation of the well-known minimum s-t-cut problem, of which we have numerous excellent polynomial-time algorithms. Yet, this is considerably harder to solve correctly. If we just use a normal flow algorithm like Edmonds-Karp [EK72], we are very likely to end up with a minimum cut that does *not* include the diversion edge, like in Figure 5.1b. And if we force the flow algorithm to use the diversion edge, we are likely to end up with a cut that is not minimal, a cut where we might as well drop the diversion edge from the set and still have a cut, like in Figure 5.1b.

In fact, [CWN13] have shown that Network Diversion is NP-complete on directed graphs, even without cycles or weights. Whether Network Diversion can be solved in polynomial time on undirected graphs is still an open problem. [CWN13] have found polynomal-time algorithms for the special case where the input graph is s-t-planar, meaning that the graph can be embedded such that s and t are adjacent to the outside face, but whether there is a polynomial-time algorithm for planar graphs in general is still an open problem.

Until now. We will present the first-ever polynomial-time algorithm that solves Network Diversion in undirected planar graphs. It will also work with weighted edges, as long as the

weights are non-negative. Many graphs based on physical structures are planar. In the example of roads and bridges, having two roads cross without a crossroad is usually inefficent and more costful, so such networks are very often planar. The costs associated with cutting an edge or blowing up a bridge is usually non-negative, too. So even if we do not solve Network Diversion in the most general case, solving it for planar graphs of non-negative edges is not far from it in practice.

5.2 Intuition

5.2.1 Bottleneck paths

Before we reveal the algorithm for Network Diversion, we will first look at a curious little problem that we call Shortest Bottleneck Path:

Shortest Bottleneck Path
Input: a graph G, two vertices $s, t \in V$, and a 'bottleneck' edge $b \in E$
Output: the shortest s-t-path in G that goes through the bottleneck b

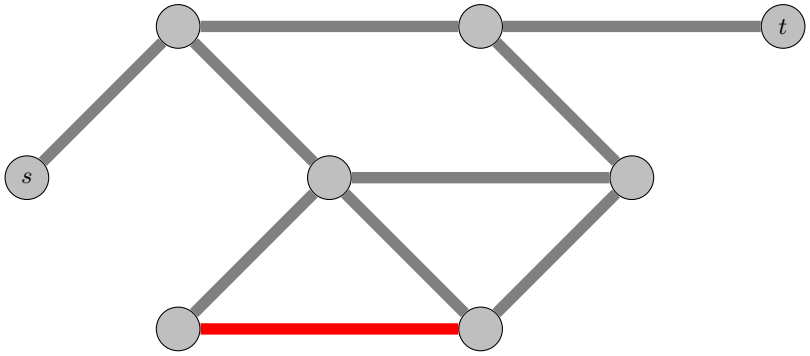
Velg et annet begrep. Shortest Detour Path, Shortest Checkpoint Path, eller noe fra det 2013-papiret.

There is no obvious way to solve Shortest Bottleneck Path. One might attempt to find the shortest paths from s to from(b) and from to(b) to t, but those two paths might overlap and reuse the same vertices, and therefore would their concatenation not necessarily be a path but merely a walk instead.

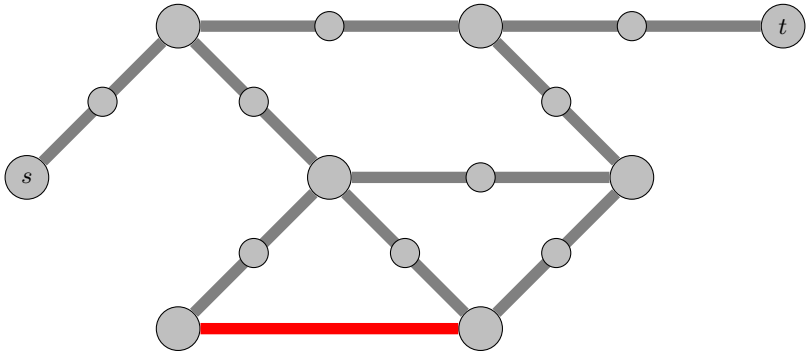
Instead we create a new graph H, by subdividing all edges in G *except* b, like seen in Figure 5.2. The key point to see here is that any odd s-t-path in H must necessarily go through the bottleneck, otherwise it would not be odd. We can visualize it by 'stepping through' the edges in H. If we start on our right leg, then in the beginning every time we reach a vertex that is also in G, we reach it by stepping on our left leg. That continues until we use the bottleneck edge, and from then on we step on all vertices from G using our right leg. If we require that we must end at t on our right leg, then the path must be odd, and any odd path must go through the bottleneck. Therefore we can simply run our Shortest Odd Path algorithm on H, and if such a path exists we can reverse the subdivision of the edges in the path and the result is the Shortest Bottleneck Path in G.

If we extend the problem to have multiple bottleneck edges, where we have to go through all of them in any order, then our idea will not work[1]. The problem is that we have no way of knowing whether we have used the marked edges 1, or 3, or 5, etc. times, because in all of them we hit vertices from G using our right leg. We can, however, use this idea to find paths that use a certain set of edges an odd amount of times. As it turns out, that is exactly what we need to solve Network Diversion.

[1]This is actually a good thing, because otherwise we would have solved the Traveling Salesman Problem in polynomial time and complexity theory as we know it would break down.



(a) An instance of Shortest Bottleneck Path, bottleneck marked in red.



(b) All edges except the bottleneck have been subdivided, to create an instance of Shortest Odd Path.

Figure 5.2: Shortest Bottleneck Path reduced to Shortest Odd Path by subdividing edges.

5.2.2 From a dual path to a real diversion

Remember, we want to find a minimum minimal s-t-cut in G that includes the diversion edge d.

Instead of looking for a minimal cut in G, let us instead look for a simple cycle in the dual graph $G^\star$, as is explained to be equivalent in Fact 2.3.1. We can do this by finding a path in $G^\star$ from and to the left and right faces of d, without using $d^\star$ itself, and then adding $d^\star$ at the end to complete the cycle. If the path found is also the shortest such path, then it corresponds to the *minimum* minimal cut in G that uses d, though it is not necessarily an s-t-cut.

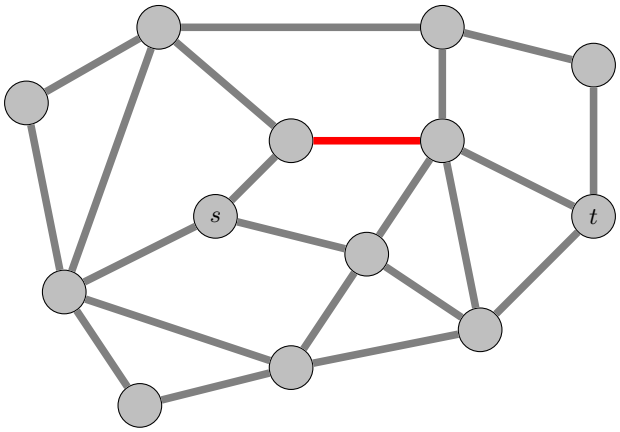
To force s and t to end up in different components after the cut, we need some additional details. First we find any s-t-path in G, not necessarily the shortest path. Then we subdivide all the edges in the dual graph *except* those who cross edges on the found s-t-path. Now we can look for the shortest *odd* path from and two the left and right faces of d in the subdivided dual graph, and add $d^\star$ at the end to make it a cycle. Like before, this corresponds to a minimum minimal cut in G that uses d, but now it must also cross the edges in the found s-t-path an odd number of times, like explained in Section 5.2.1.

This cycle, and the found s-t-path, can be interpreted as Jordan curves in our embedding. By the Jordan Curve Theorem, the curve of the cycle divides the plane into an 'inside' and an 'outside'. Since the curve of the s-t-path crosses the curve of the cycle an odd number of times, exactly one of its endpoints must be on the inside, and the other on the outside. The endpoints are s and t, meaning that s and t end up in different components. It follows that this cut is an s-t-cut in G, specifically a minimum minimal cut in s-t-cut in G that uses d.

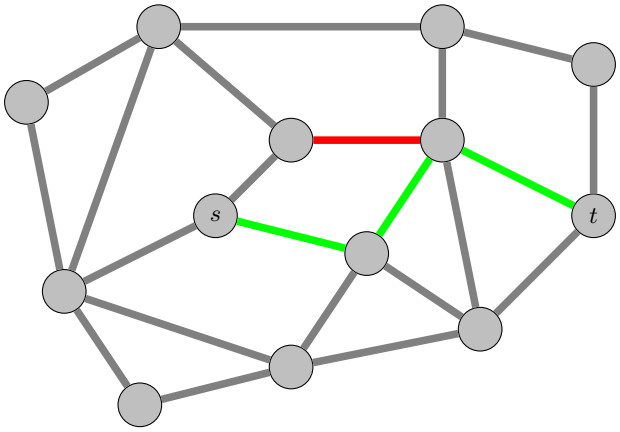
This is the main idea for our algorithm. Note that we did not come up with this idea ourselves, but have to thank [Dra24] for his as for now unpublished work on the subject.

5.2.3 Example

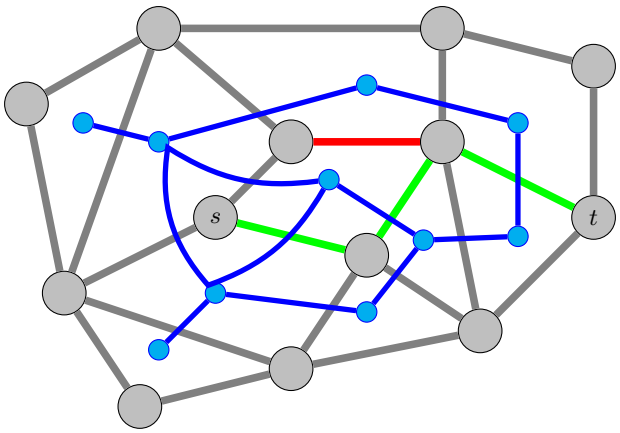
We will explain the algorithm by following an example. We want to find the minimum minimal s-t-cut that includes the diversion edge marked in red.



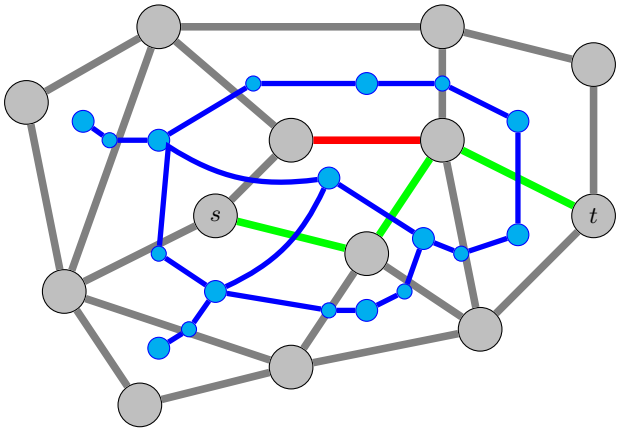
First we find any s-t-path that does not use the diversion edge. It does not necessarily have to be the shortest path. We have marked such a path in green below.



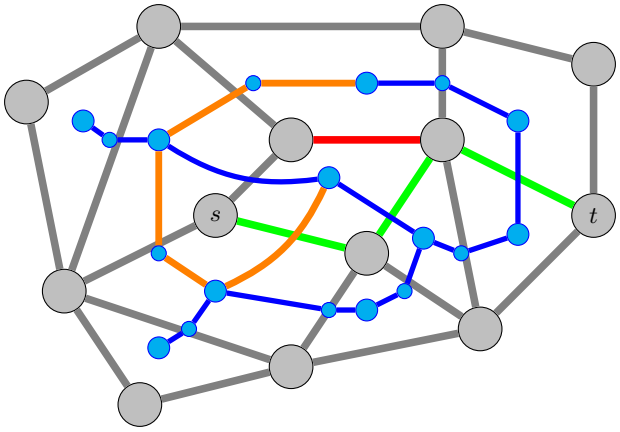
Next up is to compute the dual graph. We delete the dual edge that crosses the diversion edge, and color the rest in blue here. Note that we have omitted the outside face and its edges in this visualizion, otherwise we would have a much too cluttered illustration.



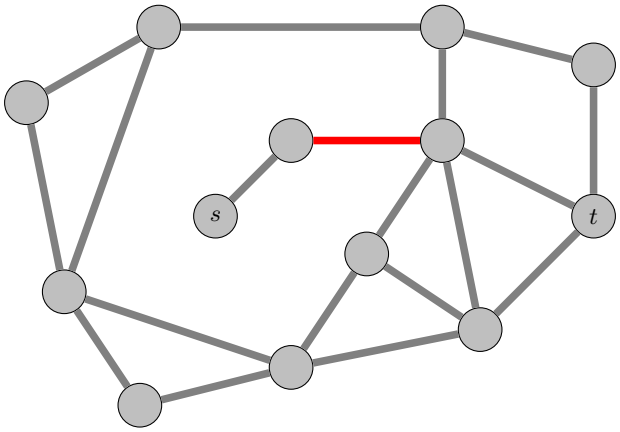
Now we subdivide all the edges in the dual graph except those who cross the path in green.



Last up is to find the shortest odd path in the subdivided dual graph from and to the regions to the left and right of the diversion edge, using our newfound favorite algorithm. If we find such a path, we know that it must cross s-t-path in green an odd number of times. If we add the dual equivalent of the diversion edge to the path to create a cycle, then we know that this cycle goes around either s or t, but not both. We illustrate one of the shortest odd paths in below, this time in orange in an attempt to not run out of colors.



With this, we finally have our diversion set. Simply delete the edges in the original graph that crosses the odd path we found and colored in orange. We end up with a graph where all s-t-paths must pass through the diversion edge. The problem is solved.



5.3 Pseudocode

Here comes the pseudocode for our Network Diversion algorithm.

Code Listing 5.1: Main

```
fn network_diversion(graph, s, t, d) {
    graph.delete_edge(d);
    path = shortest_path(graph, s, t);
    graph.add_edge(d);

    match path {
        None => {
            // No s-t-paths exist without d anyway,
            //  so no diversion is needed.
            return Some(0, []);
        }
        Some(p) {
            p* = [ e* for e in p ];
            dual = subdivide_edges_except(graph*, p*);
```

```
15
16              match shortest_odd_path(dual, left(d), right(d)) {
17                  // There are no s-t-paths that go through d,
18                  //  and therefore no way to divert the network.
19                  None => return None;
20                  Some(cost, odd_path) {
21                      diversion = [e for e* in un_subdivide_edges(odd_path)];
22                      return Some(cost, diversion);
23                  }
24              }
25          }
26      }
27  }
```

5.4 Analysis

Theorem 5.4.1. Let (G, s, t, d) be an instance of Network Diversion, and let $n := |V|$. Claim: our algorithm runs in time $O(n \log n)$.

Proof. We find first a shortest s-t-path in G that does not use d, in time $O(n + m)$.

Then we subdivide all the edges in $G^\star$ except those found in the path, in time $O(n + m)$. This new graph has size $n' \leq 2n \in O(n)$ and $m' \leq 2m \in O(m)$.

Next up is to find an odd path in the subdivided graph, in time $O(m' \log n') = O(m \log n)$.

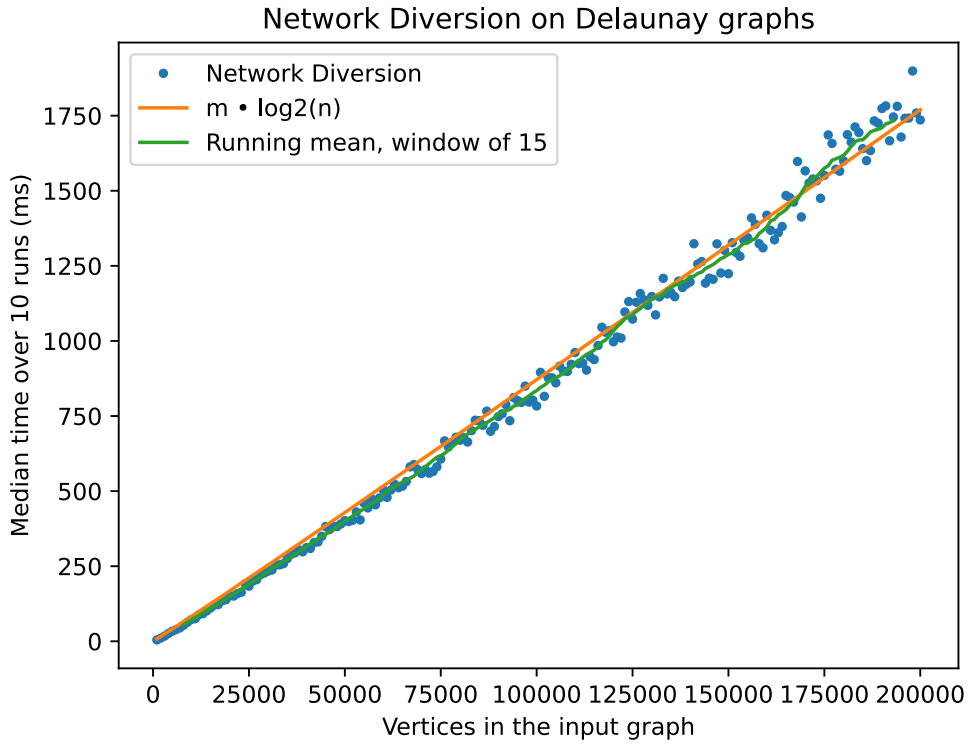
Lastly, if we are interested in the specific set of edges in the diversion and not just the cost, we un-subdivide the odd path in time $O(n') = O(n)$.

In total, we have a running time of $O(n + m) + O(m \log n) + O(n) = O(m \log n)$. Since G is planar we have that $m \in O(n)$, and we can simplify the complexity to just $O(n \log n)$, which completes the proof. □

Note that here we have assumed that the dual graph $G^\star$ has already been computed prior to starting the algorithm. If we have a straight-line embedding of G we can compute $G^\star$ in $O(n + m)$, which would not change the overall running time. However, if we do not have such an embedding the total running time might be considerably more.

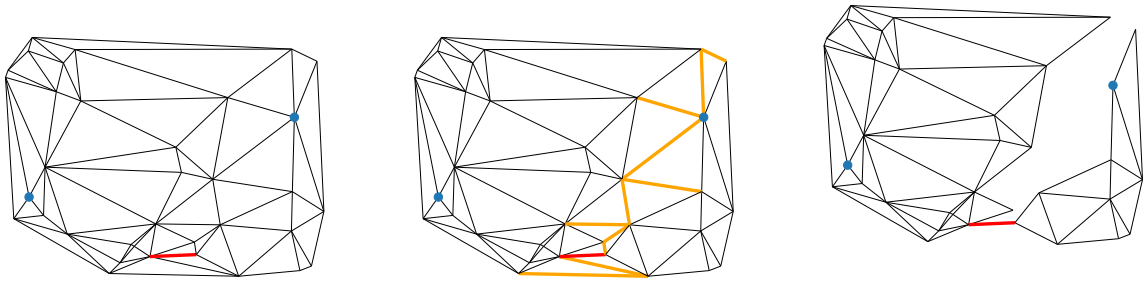
We compare the theoretical and practical running times on Delaunay graphs, like we did in Section 4.5.2. For each graph, we have estimated a source and target vertex of maximum distance between each other, and picked three edges in the graph as diversion edges. We pick whichever diversion edge leads to the worst running time over 10 runs, and plot the median over those 10 runs.

Here too have we tried to create a function out of the theoretical running time of $O(n \log n)$, this time with different constants. We have set $m := 3n$ in the plot, since the graphs are planar.



As we can see, the running times grow just barely more than linearly compared to the input size. This is not surprising considering the linearithmic theoretical running time. The algorithm easily solves Network Diversion on planar graphs of 100000 vertices in less than a second. Now compare that to the existing algorithms that will need more than a second to solve for more than 30 vertices, and it is clear why a polynomial running time matters so much.

See Figure 5.3 for an yet another example of what a diversion set may look like, this time on a Delaunay graph of 35 vertices.



(a) s and t are marked in blue, the diversion edge in red

(b) One possible diversion set marked in orange

(c) With diversion set removed, all s-t-paths must go through the diversion edge

Figure 5.3: Example of a solution for Network Diversion

Chapter 6

The Codebase

The crux of this thesis is not about the theory we have presented in the other chapters, but rather to see how well these algorithms work in practice, if at all. We have implemented most of the algorithms mentioned in the paper, and in this chapter we will present the codebase. Of course, we invite the reader to explore the repository on their own: [Sim24b].

6.1 Functionality

Our library is written in Rust. We chose that language because of performance, but also because its type system and strict compiler helps massively in reducing the developer overhead during development. Being able to fearlessly perform large refactors, and to use sum-types rather than just product types to describe the data, has had a huge impact on the quality of our code.

With that, we have successfully implemented the following algorithms:

- Shortest Odd Walk on graphs of non-negative weights, as described in Chapter 3.
- Shortest Odd Path on undirected graphs of non-negative weights, as described in Chapter 4.
- Network Diversion on undirected planar graphs of non-negative weights, as described in Chapter 5.
- Shortest Path on unweighted graphs, using a classic breadth-first search. It also runs on weighted graphs, through happily ignoring the weights.
- Shortest Path on graphs of non-negative weights, using Dijkstra's Algorithm as shown in Section 2.2.
- Shortest Bottleneck Path on undirected graphs of non-negative weights, as described in Section 5.2.1.

All of them are generic with respect to edge weights, and can handle any type of number-like weights such as `i32`, `u64` and `f64`. When we subdivide edges in Network Diversion and Shortest Bottleneck Path we use a neat little trick to be able to split a weight into two weights without potentially accidentally rounding down an integer: we set one weight to zero and the other to the original weight. Any path that uses one of those edges will have to use the other afterwards, and their sum will then be the same as the original weight prior to the split.

6.2 Data structures

To perform these algorithms, we have implemented a few different data structures. The first is a basic structure for undirected graphs, using a vector of vectors to represent the graph with adjacency lists. It is generic both in the type of its edge weights, but also on the type of its edges. The 'basic' edge has the basic edge methods like `.to()` and `.weight()`, but we also have a struct for planar edges, where we have the methods `.left()` and `.right()`. The methods work like described in Section 2.1 and Section 2.3.

We also have a data structure for planar graphs. They consist of two undirected graphs of planar edges: the 'real' graph and its dual. Each of them has edges that know which faces are to their left and right in the other graph. Even though our algorithm for Network Diversion will work any planar graphs, parsing them and finding an embedding is rough. We therefore assume that we are the given coordinates of each of the vertices, and that they form a straight-line embedding. From there we can compute the dual. Whether the given coordinates form true planar embeddings is usually not check, since the code to verify that runs in the painfully slow $O(m^2)$ and has been disabled by default.

Another limitation is that we can only handle simple planar graphs: if we have parallel edges, then the way we compute the dual will not work. If the input graph is not simple, we have to somehow combine the parallel edges until it is. Depending on the usecase we have many reasonable strategies for combining them. If the edges represent bridges to be blown up with artillery, then the combined edge should probably be the sum of the weights of its components, the sum of the artillery needed to destroy the edges between those two vertices. If we in another case just need to cut any of the edges between them, then we may want to just take the cheapest one, or perhaps we are forced to take the most expensive one. Rather than making assumptions about the usecases, we have implemented five strategies for combining parallel edges, to cover as many usecases as possible:

- Take the first edge
- Take the last edge
- Keep the highest weight

- Keep the lowest weight
- Sum all weights

We hope that these are enough. If not, then our framework can easily be extended with more strategies.

As discussed in Section 4.3.5 and benchmarked in Section 4.5.1, we have implemented two different data structures to keep track of the basis. They are called ObserverBase and UnionFindBase. Both are implemented using a common trait and can be switched out interchangably in the Shortest Odd Path algorithm. UnionFindBase usually performs the best, and has been set as the default. Since ObserverBase can outperform the other in certain cases, we have kept both. The 'näive' basis we mentioned was too inefficient for anything but a temporary prototype and has long since been deleted.

6.3 Testing

The repository includes a multiple large test suites, to ensure the correctness of everything we have implemented. First of all, each data structure comes with its own set of unit tests. Secondly, we have a total of 17 hand-crafted graphs of various shapes and sizes, and for each them we have many queries for the different graph problems. We have found the optimal solutions manually, and confirmed that the expected answers match the actual answers provided by the algorithms. Lastly, we also have numerous problem-specific assertions in place, like asserting that the output of Odd Shortest Path really is a path, or that the output of Network Diversion indeed cuts the graph in two except for the diversion edge.

These tests have been immensively helpful in the development of our algorithms. Whenever we modified anything, we could instantly verify the validity with just the press of a hotkey and its subsequent run of the test suites. Though we do not present many formal proofs for the algorithms in this thesis, we like to think of the tests as informal proofs by empirical analysis.

6.4 Benchmarking

Write about the 'real-life' graphs we benchmarked

Write about marking benches

6.4.1 Delaunay triangulations

To compare the theoretical and practical runtimes of our algorithms, we needed a collection of graphs of easily scalable sizes. Furthermore, as to also be used for benching Network Diversion, the graphs had to be planar graphs with a built-in planar embedding.

Our solution was this: for a given integer n, generate n random points in the plane, to be the vertices in our graph. Then, using the `scipy` library in Python, compute a Delaunay triangulation of the points. Each of the triangles in the triangulation consists of three points, of which we added three edges with random weights. Extra care had to be taken not to add the same edge multiple times, once for each triangle. The result is a straight-line embedding of a graph of size n, where each face is a triangle in the triangulation, and the dual graph is a Voronoi diagram of the set of points. We will refer to a graph generated like this as a *Delaunay graph*.

Using this technique, we generated 200 Delaunay graphs of sizes 1000, 2000, 3000 and so on until 200000. Then we used a few heuristic searches to estimate some of the pairs of vertices that were the farthest away from each other, as inputs for our Shortest Odd Walk and Shortest Odd Path algorithms. After that we picked some of the worst diversion edges farthest from these pairs, as inputs for our Network Diversion. The intention is that each graph gets the worst-case or almost worst-case queries, so that the size of the problem roughly matches the size of the graph.

The interested reader may visit the GitHub repository [Sim24b] to see the graphs and the Python scripts used to generate them.

Chapter 7

Conclusion

The main topic of this thesis is to solve Shortest Odd Path on undirected graphs. We have given a detailed explanation and pseudocode of Derigs' [Der85] algorithm, with improvements on both its presentation and performance. Afterwards, we used it to give the first-ever efficient algorithm for Network Diversion on planar graphs with non-negative edges. Although requiring planarity and non-negative weights may seem very restrictive, in practical use many graphs fit the criteria.

We have also given algorithms to solve some minor problems like Shortest Odd Walk and Shortest Bottleneck Path. We have successfully implemented all of these algorithms in Rust, and tested them thoroughly. All algorithms have been benchmarked to show that their theoretical running times matches them in practice. In particular, we show that we can solve Shortest Odd Path and Network Diversion on sparse graphs of 200000 vertices in 0.5s and 1.8s, respectively.

Husk å oppdatere disse tallene etter benkinger er kjørt på nytt

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Appendix A

The full, uninterrupted pseudocode for Shortest Odd Path

Here is the full pseudocode in one big code block. For explanations and discussion of different variants, see Section 4.3.

Code Listing A.1: Algorithm for Shortest Odd Path

```
fn main(input_graph, s, t){
    init(input_graph, s, t);

    control();

    if d_minus[t] == ∞ {
        return None;
    }
    cost = d_minus[t];
    path = backtrack();

    return Some(cost, path);
}

fn init(input_graph, s, t) {
    graph = create_mirror_graph(input_graph);

    for u in 0..n {
        d_plus[u] = ∞;
        d_minus[u] = ∞;
        pred[u] = null;
        completed[u] = false;
        basis[u] = u;
        in_current_blossom[u] = false;
    }
    d_plus[s] = 0;
    completed[s] = true;

    for edge in graph[s] {
        priority_queue.push(Vertex(weight(edge), to(edge)));
        d_minus[to(edge)] = weight(edge);
        pred[to(edge)] = e;
    }
}

fn backtrack() {
    current_edge = pred[t];
    path = [current_edge];
    while from(current_edge) != s {
        current_edge = pred[mirror(from(current_edge))];
        if from(current_edge) < input_graph.n() {
            path.push(current_edge);
        }
        else {
            path.push(shift_edge_by(current_edge, -input_graph.n()));
        }
```

```
    }
    return path;
}

fn control() -> bool {
    loop {
        while ! priority_queue.is_empty() {
            match priority_queue.top() {
                Vertex(_, u) => {
                    if completed[u] {
                        priority_queue.pop();
                    }
                    else {
                        break;
                    }
                },
                Blossom(_, edge) => {
                    if base_of(from(edge)) == base_of(to(edge)) {
                        priority_queue.pop();
                    }
                    else {
                        break;
                    }
                }
            }
        }

        if priority_queue.is_empty() {
            // No odd s-t-paths exist :(
            return;
        }
        match priority_queue.pop() {
            Vertex(delta, u) => {
                if u == t {
                    // We have found a shortest odd s-t-path :)
                    return;
                }
                d_plus[u] = d_minus[mirror(u)];
                scan(mirror(u));
            }
            Blossom(delta, edge) => {
                blossom(e);
            }
        }
    }
}

fn scan(u) {
    completed[u] = true;
    dist_u = d_plus[u];
    for edge in graph[u] {
        v = to(edge);
        new_dist_v = dist_u + weight(edge);

        if ! completed[v] {
            if new_dist_v < d_minus[v] {
                d_minus[v] = new_dist_v;
                pred[v] = edge;
                priority_queue.push(Vertex(new_dist_v, v));
            }
        }
        else if d_plus[v] < ∞ and base_of(u) != base_of(v) {
            priority = d_plus[u] + d_plus[v] + weight(edge);
            priority_queue.push(Blossom(priority, edge));
            if new_dist_v < d_minus[v] {
                d_minus[v] = new_dist_v;
                pred[v] = e;
            }
        }
    }
}

fn backtrack_blossom(edge) {
    p1 = [ reverse(edge) ];
    p2 = [ edge ];
```

```
    u = get_basis(to(edge));
    v = get_basis(from(edge));
    in_current_blossom[u] = true;
    in_current_blossom[v] = true;

    loop {
        if u != s {
            u = get_basis(mirror(u));
            in_current_blossom[u] = true;
            e = pred[u];
            u = get_basis(from(e));
            p1.push(e);

            // Ff true, then u is the base
            if in_current_blossom[u] {
                p1.pop();
                in_current_blossom[u] = false;

                // We remove all the edges in p2 after the base
                while p2 is not empty {
                    e = p2.last();
                    v = get_basis(from(e));
                    in_current_blossom[v] = false;
                    p2.pop();
                    if v == u {
                        break;
                    }
                }
                return (u, p1, p2);
            }
        }
        if v != s {
            v = get_basis(mirror(v));
            in_current_blossom[v] = true;
            e = pred[v];
            v = get_basis(from(e));
            p2.push(e);

            if in_current_blossom[v] {
                p2.pop();
                in_current_blossom[v] = false;

                while p1 is not empty {
                    e = p1.last();
                    u = get_basis(from(e));
                    in_current_blossom[u] = false;
                    p1.pop();
                    if u == v {
                        break;
                    }
                }
                return (v, p1, p2);
            }
        }
    }
}

fn blossom(edge) {
    (b, p1, p2) = backtrack_blossom(edge);

    to_scan1 = set_blossom_values(p1);
    to_scan2 = set_blossom_values(p2);

    set_edge_bases(b, p1);
    set_edge_bases(b, p2);

    for u in to_scan1 {
        scan(u);
    }
    for v in to_scan2 {
        scan(v);
    }
}

fn set_blossom_values(path) {
```

```
    to_scan = [];

    for edge in path {
        u = from(edge);
        v = to(edge);
        w = weight(edge);
        in_current_cycle[u] = false;
        in_current_cycle[v] = false;

        // We can set a d_minus
        if d_plus[v] + w < d_minus[u] {
            d_minus[u] = d_plus[v] + w;
            pred[u] = reverse(edge);
        }

        int m = mirror(u);
        // We can set a d_plus, and scan it
        if d_minus[u] < d_plus[m] {
            d_plus[m] = d_minus[u];
            to_scan.push(m);
        }
    }

    return to_scan;
}

fn set_base(base, u) {
    basis[u] = base;
}
fn get_base(u) {
    if u != basis[u] {
        basis[u] = get_base(basis[u]);
    }
    return basis[u];
}
```