Title

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1 Van Neumann Stability Analysis: Explicit Scheme

The explicit scheme for the two dimensional heat equation is

$$\frac{T_{j,k}^{n+1} - T_{j,k}^n}{\Delta t} = \frac{T_{j-1,k}^n - 2T_{j,k}^n + T_{j+1,k}^n}{(\Delta x)^2} + \frac{T_{j,k-1}^n - 2T_{j,k}^n + T_{j,k+1}^n}{(\Delta y)^2} \Leftrightarrow \tag{1}$$

$$\Leftrightarrow T_{j,k}^{n+1} = T_{j,k}^n + \frac{1}{\rho}\left(T_{j-1,k}^n + T_{j+1,k}^n + T_{j,k-1}^n + T_{j,k+1}^n - 4T_{j,k}^n\right) \tag{2}$$

assuming $\Delta x = \Delta y =: h$ and setting $\rho = \frac{h^2}{\Delta t}$. We begin our Van Neumann stability analysis by inserting the Fourier series

$$T_{j,k}^n = \sum_{p,q=0}^{N-1} \widehat{T}_{p,q}^n e^{ipx_j + iqy_k} \tag{3}$$

into the numerical scheme. We obtain

$$\sum_{p,q=0}^{N-1} \widehat{T}_{p,q}^{n+1} e^{i(px_j+qy_k)} = \sum_{p,q=0}^{N-1} (\widehat{T}_{p,q}^n e^{i(px_j+qy_k)} + \frac{1}{\rho}(\widehat{T}_{p,q}^n e^{i(px_{j-1}qy_k)} + \widehat{T}_{p,q}^n e^{i(px_{j+1}+qy_k)} +$$
$$+ \widehat{T}_{p,q}^n e^{i(px_j+qy_{k-1})} + \widehat{T}_{p,q}^n e^{i(px_j+qy_{k+1})} - 4\widehat{T}_{p,q}^n e^{i(px_j+qy_k)})). \tag{4}$$

Using that $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$ and that the discretization in x and y are the same we get

$$\sum_{p,q=0}^{N-1} (\widehat{T}_{p,q}^{n+1} - \widehat{T}_{p,q}^n - \frac{1}{\rho}(\widehat{T}_{p,q}^n e^{-iph} + \widehat{T}_{p,q}^n e^{iph} + \widehat{T}_{p,q}^n e^{-iqh} + \widehat{T}_{p,q}^n e^{iqh} - 4\widehat{T}_{p,q}^n))\, e^{i(px_j+qy_k)} = 0 \tag{5}$$

and because $(e^{i(px_j+qy_k)})_{j,k\in\{0,\dots,N-1\}}$ is a basis of the trigonometrical polynomials of degree N in two variables we have for all $p,q \in \{0,\dots,N-1\}$ that

$$\widehat{T}_{p,q}^{n+1} - \widehat{T}_{p,q}^{n} - \frac{1}{\rho}\left(\widehat{T}_{p,q}^{n}e^{-iph} + \widehat{T}_{p,q}^{n}e^{iph} + \widehat{T}_{p,q}^{n}e^{-iqh} + \widehat{T}_{p,q}^{n}e^{iqh} - 4\widehat{T}_{p,q}^{n}\right) = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^{n} + \frac{1}{\rho}\left(\widehat{T}_{p,q}^{n}e^{-iph} + \widehat{T}_{p,q}^{n}e^{iph} + \widehat{T}_{p,q}^{n}e^{-iqh} + \widehat{T}_{p,q}^{n}e^{iqh} - 4\widehat{T}_{p,q}^{n}\right) \quad \Leftrightarrow$$

$$\Leftrightarrow \widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^{n}\left(1 + \frac{1}{\rho}\left(e^{-iph} + e^{iph} + e^{-qh} + e^{qh} - 4\right)\right) \quad \Leftrightarrow$$

$$\Leftrightarrow \widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^{n}\left(1 + \frac{1}{\rho}\left(2\cos(ph) + 2\cos(qh) - 4\right)\right) \quad \Leftrightarrow$$

$$\Leftrightarrow \widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^{n}\underbrace{\left(1 - \frac{4}{\rho}\left(\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right)\right)\right)}_{=:G_{p,q}},$$

where in the last step the identity $\sin^2(\alpha/2) = \frac{1}{2}(1-\cos(\alpha))$ was used. For van Neumann stability we need that $|G_{p,q}| < 1$. We therefor want to find conditions on Δt and h such that

$$-1 \leq 1 - \frac{4}{\rho}\left(\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right)\right) \leq 1. \tag{6}$$

The right hand side inequality is trivial and for the left hand side inequality we need that $\rho \geq 4$. This gives us the CFL condition

$$\frac{h^2}{\Delta t} = \rho \geq 4 \Leftrightarrow \Delta t \leq \frac{h^2}{4} = \frac{1}{4N^2}. \tag{7}$$

We conclude that $4N^2t$ time steps are needed to integrate up to a time t.

2 Van Neumann Stability Analysis: Implicit Scheme

The implicit scheme for the two dimensional heat equation is

$$\frac{T_{j,k}^{n+1} - T_{j,k}^{n}}{\Delta t} = \frac{T_{j-1,k}^{n+1} - 2T_{j,k}^{n+1} + T_{j+1,k}^{n+1}}{(\Delta x)^2} + \frac{T_{j,k-1}^{n+1} - 2T_{j,k}^{n+1} + T_{j,k+1}^{n+1}}{(\Delta y)^2} \Leftrightarrow \tag{8}$$

$$\Leftrightarrow T_{j,k}^{n+1} = T_{j,k}^{n} + \frac{1}{\rho}\left(T_{j-1,k}^{n+1} + T_{j+1,k}^{n+1} + T_{j,k-1}^{n+1} + T_{j,k+1}^{n+1} - 4T_{j,k}^{n+1}\right) \tag{9}$$

assuming $\Delta x = \Delta y =: h$ and setting $\rho = \frac{h^2}{\Delta t}$. Again, we begin our Van Neumann stability analysis by inserting the Fourier series

$$T_{j,k}^{n} = \sum_{p,q=0}^{N-1} \widehat{T}_{p,q}^{n} e^{ipx_j + iqy_k} \tag{10}$$

into the numerical stencil. We bring all terms to one side of the equation and again by using $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$ and that the discretization in x and y are the same and also by placing the exponential outside of the brackets we get

$$\sum_{p,q\,=0}^{N-1} \left(\widehat{T}_{p,q}^{n+1} - \widehat{T}_{p,q}^{n} - \frac{1}{\rho} \left(\widehat{T}_{p,q}^{n+1} e^{-iph} + \widehat{T}_{p,q}^{n+1} e^{iph} + \widehat{T}_{p,q}^{n+1} e^{-iqh} + \right. \right.$$
$$\left. \left. + \widehat{T}_{p,q}^{n+1} e^{iph} - 4\widehat{T}_{p,q}^{n+1} \right) \right) e^{i(px_j + qy_k)} = 0. \tag{11}$$

Again because the occurring exponentials is a basis of the trigonometrical polynomials the exponentials coefficients must be zero. From this we calculate

$$
\begin{aligned}
\widehat{T}_{p,q}^{n} &= \widehat{T}_{p,q}^{n+1} - \frac{1}{\rho} \left(\widehat{T}_{p,q}^{n+1} e^{-iph} + \widehat{T}_{p,q}^{n+1} e^{iph} + \widehat{T}_{p,q}^{n+1} e^{-iqh} + \widehat{T}_{p,q}^{n+1} e^{iph} - 4\widehat{T}_{p,q}^{n+1} \right) \\
&= \widehat{T}_{p,q}^{n+1} \left(1 - \frac{1}{\rho} \left(e^{-iph} + e^{iph} + e^{-iqh} + e^{iph} - 4 \right) \right) \\
&= \widehat{T}_{p,q}^{n+1} \left(1 - \frac{1}{\rho} \left(2\cos(ph) + 2\cos(qh) - 4 \right) \right) \\
&= \widehat{T}_{p,q}^{n+1} \left(1 - \frac{4}{\rho} \left(\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right) \right) \right),
\end{aligned}
\tag{12}
$$

which is equivalent to

$$\widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^{n} \underbrace{\frac{1}{1 - \frac{4}{\rho}\left(\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right)\right)}}_{=:G_{p,q}}. \tag{13}$$

Clearly here $|G_{p,q}| \leq 1$ is always fulfilled. Therefore the implicit scheme is unconditionally stable and hence no CFL condition is existent.

3 Van Neumann Stability Analysis: ADI Scheme

The alternating direction implicit (ADI) method consists of the two half steps

$$\frac{T_{j,k}^{2n+1} - T_{j,k}^{2n}}{\Delta t} = \frac{T_{j-1,k}^{2n+1} - 2T_{j,k}^{2n+1} + T_{j+1,k}^{2n+1}}{(\Delta x)^2} + \frac{T_{j,k-1}^{2n} - 2T_{j,k}^{2n} + T_{j,k+1}^{2n}}{(\Delta y)^2} \tag{14}$$

$$\frac{T_{j,k}^{2n+2} - T_{j,k}^{2n+1}}{\Delta t} = \frac{T_{j-1,k}^{2n+1} - 2T_{j,k}^{2n+1} + T_{j+1,k}^{2n+1}}{(\Delta x)^2} + \frac{T_{j-1,k}^{2n+2} - 2T_{j,k}^{2n+2} + T_{j+1,k}^{2n+2}}{(\Delta y)^2} \tag{15}$$

Due to the fact that these schemes are identical except for an index shift and a transposition of the x and y direction we only consider the first half step for the stability analysis, the second one can then easily be deduced. Using the same strategy as in the

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previous examples we begin by bringing all terms to one side and separating the terms of the $2n$-th step from the ones of the $(2n + 1)$-th step. This leaves us with

$$T_{j,k}^{2n+1} - \frac{1}{\rho}\left(T_{j-1,k}^{2n+1} - 2T_{j,k}^{2n+1} + T_{j,k}^{2n+1}\right) - T_{j,k}^{2n} - \frac{1}{\rho}\left(T_{j-1,k}^{2n} - 2T_{j,k}^{2n} + T_{j,k}^{2n}\right) = 0, \quad (16)$$

assuming $\Delta x = \Delta y =: h$ and setting $\rho = \frac{h^2}{\Delta t}$. Again we start by inserting the Fourier series

$$T_{j,k}^n = \sum_{p,q=0}^{N-1} \widehat{T}_{p,q}^n e^{ipx_j + iqy_k} \quad (17)$$

into the numerical stencil. We bring all terms to one side of the equation and again by using $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$ and that the discretization in x and y are the same and also by placing the exponential outside of the brackets we get

$$\sum_{p,q=0}^{N-1}\left(\widehat{T}_{p,q}^{2n+1} - \frac{1}{\rho}\left(\widehat{T}_{p,q}^{2n+1}e^{-iph} - 2\widehat{T}_{p,q}^{2n+1} + \widehat{T}_{p,q}^{2n+1}e^{iph}\right) - \right.$$
$$\left. -\widehat{T}_{p,q}^{2n+1} - \frac{1}{\rho}\left(\widehat{T}_{p,q}^{2n}e^{-iph} - 2\widehat{T}_{p,q}^{2n} + \widehat{T}_{p,q}^{2n}e^{iph}\right)\right)e^{i(px_j + qy_k)} = 0. \quad (18)$$

Because $\left(e^{i(px_j + qy_k)}\right)_{j,k\in\{0,\ldots,N-1\}}$ is a basis of the trigonometrical polynomials of degree N in two variables we have for all $p, q \in \{0, \ldots, N-1\}$ that the coefficients of the exponentials must be zero. Rearranging the terms gives us that for all $p, q \in \{0, \ldots, N-1\}$ that

$$\widehat{T}_{p,q}^{2n+1}\left(1 - \frac{1}{\rho}\left(e^{-iph} + e^{iph} - 2\right)\right) = \widehat{T}_{p,q}^{2n}\left(1 - \frac{1}{\rho}\left(e^{-iqh} + e^{iqh} - 2\right)\right) \Leftrightarrow \quad (19)$$

$$\Leftrightarrow \widehat{T}_{p,q}^{2n+1} = \widehat{T}_{p,q}^{2n}\frac{1 - \frac{1}{\rho}\left(e^{-iqh} + e^{iqh} - 2\right)}{1 - \frac{1}{\rho}\left(e^{-iph} + e^{iph} - 2\right)} = \widehat{T}_{p,q}^{2n+1} = \widehat{T}_{p,q}^{2n}\frac{1 - \frac{1}{\rho}(2\cos(ph) - 2)}{1 - \frac{1}{\rho}(2\cos(qh) - 2)} \Leftrightarrow \quad (20)$$

$$\Leftrightarrow \widehat{T}_{p,q}^{2n+1} = \widehat{T}_{p,q}^{2n}\frac{1 - \frac{4}{\rho}\left(\sin^2\left(\frac{ph}{2}\right)\right)}{1 - \frac{4}{\rho}\left(\sin^2\left(\frac{qh}{2}\right)\right)} \Leftrightarrow \quad (21)$$

$$\Leftrightarrow \widehat{T}_{p,q}^{2n+1} = \widehat{T}_{p,q}^{2n}\underbrace{\frac{\rho - 4\sin^2\left(\frac{ph}{2}\right)}{\rho - 4\sin^2\left(\frac{qh}{2}\right)}}_{=:G_{p,q}^{(1)}}. \quad (22)$$

By performing an index shift and switching the two space dimensions i.e.

$$\begin{aligned}
2n &\rightarrow 2n+1 \\
2n+1 &\rightarrow 2n+2 \\
p &\rightarrow q \\
q &\rightarrow p
\end{aligned}$$

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we get the analogue equation for the second half step of the ADI method.

$$\widehat{T}_{p,q}^{2n+2} = \widehat{T}_{p,q}^{2n+1} \underbrace{\frac{\rho - 4\sin^2\left(\frac{qh}{2}\right)}{\rho - 4\sin^2\left(\frac{ph}{2}\right)}}_{=:G_{p,q}^{(2)}}. \tag{23}$$