Title

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9. Januar 2021

1 Van Neumann Stability Analysis: Explicit Scheme

The explicit scheme for the two dimensional heat equation is

$$\frac{T^{n+1}_{j,k} - T^n_{j,k}}{\Delta t} = \frac{T^n_{j-1,k} - 2T^n_{j,k} + T^n_{j+1,k}}{(\Delta x)^2} + \frac{T^n_{j,k-1} - 2T^n_{j,k} + T^n_{j,k+1}}{(\Delta y)^2} \Leftrightarrow \tag{1}$$

$$\Leftrightarrow T^{n+1}_{j,k} = T^n_{j,k} + \frac{1}{\rho}\left(T^n_{j-1,k} + T^n_{j+1,k} + T^n_{j,k-1} + T^n_{j,k+1} - 4T^n_{j,k}\right) \tag{2}$$

assuming $\Delta x = \Delta y =: h$ and setting $\rho = \frac{h^2}{\Delta t}$. We begin our Van Neumann stability analysis by inserting the Fourier series

$$T^n_{j,k} = \sum_{p,q=0}^{N-1} \widehat{T}^n_{p,q} e^{ipx_j + iqy_k} \tag{3}$$

into the numerical scheme. We obtain

$$\sum_{p,q=0}^{N-1} \widehat{T}^{n+1}_{p,q} e^{i(px_j+qy_k)} = \sum_{p,q=0}^{N-1} (\widehat{T}^n_{p,q} e^{i(px_j+qy_k)} + \frac{1}{\rho}(\widehat{T}^n_{p,q} e^{i(px_{j-1}qy_k)} + \widehat{T}^n_{p,q} e^{i(px_{j+1}+qy_k)} + \\ + \widehat{T}^n_{p,q} e^{i(px_j+qy_{k-1})} + \widehat{T}^n_{p,q} e^{i(px_j+qy_{k+1})} - 4\widehat{T}^n_{p,q} e^{i(px_j+qy_k)})). \tag{4}$$

Using that $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$ and that the discretization in x and y are the same we get

$$\sum_{p,q=0}^{N-1} (\widehat{T}^{n+1}_{p,q} - \widehat{T}^n_{p,q} - \frac{1}{\rho}(\widehat{T}^n_{p,q} e^{-iph} + \widehat{T}^n_{p,q} e^{iph} + \widehat{T}^n_{p,q} e^{-iqh} + \widehat{T}^n_{p,q} e^{iqh} - 4\widehat{T}^n_{p,q}))\, e^{i(px_j+qy_k)} = 0 \tag{5}$$

and because $(e^{i(px_j+qy_k)})_{j,k\in\{0,\dots,N-1\}}$ is a basis of the trigonometrical polynomials of degree N in two variables we have for all $p,q\in\{0,\dots,N-1\}$ that

$$\begin{aligned}
&\widehat{T}^{n+1}_{p,q}-\widehat{T}^{n}_{p,q}-\frac{1}{\rho}\left(\widehat{T}^{n}_{p,q}e^{-iph}+\widehat{T}^{n}_{p,q}e^{iph}+\widehat{T}^{n}_{p,q}e^{-iqh}+\widehat{T}^{n}_{p,q}e^{iqh}-4\widehat{T}^{n}_{p,q}\right)=0 &&\Leftrightarrow\\
\Leftrightarrow&\widehat{T}^{n+1}_{p,q}=\widehat{T}^{n}_{p,q}+\frac{1}{\rho}\left(\widehat{T}^{n}_{p,q}e^{-iph}+\widehat{T}^{n}_{p,q}e^{iph}+\widehat{T}^{n}_{p,q}e^{-iqh}+\widehat{T}^{n}_{p,q}e^{iqh}-4\widehat{T}^{n}_{p,q}\right) &&\Leftrightarrow\\
\Leftrightarrow&\widehat{T}^{n+1}_{p,q}=\widehat{T}^{n}_{p,q}\left(1+\frac{1}{\rho}\left(e^{-iph}+e^{iph}+e^{-qh}+e^{qh}-4\right)\right) &&\Leftrightarrow\\
\Leftrightarrow&\widehat{T}^{n+1}_{p,q}=\widehat{T}^{n}_{p,q}\left(1+\frac{1}{\rho}\left(2\cos(ph)+2\cos(qh)-4\right)\right) &&\Leftrightarrow\\
\Leftrightarrow&\widehat{T}^{n+1}_{p,q}=\widehat{T}^{n}_{p,q}\underbrace{\left(1-\frac{4}{\rho}\left(\sin^2\left(\frac{ph}{2}\right)+\sin^2\left(\frac{qh}{2}\right)\right)\right)}_{=:G_{p,q}},
\end{aligned}$$

where in the last step the identity $\sin^2(\alpha/2)=\frac{1}{2}(1-\cos(\alpha))$ was used. For van Neumann stability we need that $|G_{p,q}|<1$. We therefor want to find conditions on Δt and h such that

$$-1\leq 1-\frac{4}{\rho}\left(\sin^2\left(\frac{ph}{2}\right)+\sin^2\left(\frac{qh}{2}\right)\right)\leq 1. \tag{6}$$

The right hand side inequality is trivial and for the left hand side inequality we need that $\rho\geq 4$. This gives us the CFL condition

$$\frac{h^2}{\Delta t}=\rho\geq 4\Leftrightarrow \Delta t\leq\frac{h^2}{4}=\frac{1}{4N^2}. \tag{7}$$

We conclude that $4N^2t$ time steps are needed to integrate up to a time t.

2 Van Neumann Stability Analysis: Implicit Scheme

The implicit scheme for the two dimensional heat equation is

$$\frac{T^{n+1}_{j,k}-T^{n}_{j,k}}{\Delta t}=\frac{T^{n+1}_{j-1,k}-2T^{n+1}_{j,k}+T^{n+1}_{j+1,k}}{(\Delta x)^2}+\frac{T^{n+1}_{j,k-1}-2T^{n+1}_{j,k}+T^{n+1}_{j,k+1}}{(\Delta y)^2}\Leftrightarrow \tag{8}$$

$$\Leftrightarrow T^{n+1}_{j,k}=T^{n}_{j,k}+\frac{1}{\rho}\left(T^{n+1}_{j-1,k}+T^{n+1}_{j+1,k}+T^{n+1}_{j,k-1}+T^{n+1}_{j,k+1}-4T^{n+1}_{j,k}\right) \tag{9}$$

assuming $\Delta x=\Delta y=:h$ and setting $\rho=\frac{h^2}{\Delta t}$. Again, we begin our Van Neumann stability analysis by inserting the Fourier series

$$T^{n}_{j,k}=\sum_{p,q=0}^{N-1}\widehat{T}^{n}_{p,q}e^{ipx_j+iqy_k} \tag{10}$$

into the numerical stencil. We bring all terms to one side of the equation and again by using $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$ and that the discretization in x and y are the same and also by placing the exponential outside of the brackets we get

$$\sum_{p,q=0}^{N-1} \left(\widehat{T}_{p,q}^{n+1} - \widehat{T}_{p,q}^{n} - \frac{1}{\rho} \left(\widehat{T}_{p,q}^{n+1} e^{-iph} + \widehat{T}_{p,q}^{n+1} e^{iph} + \widehat{T}_{p,q}^{n+1} e^{-iqh} + \right.\right.$$
$$\left.\left. + \widehat{T}_{p,q}^{n+1} e^{iph} - 4\widehat{T}_{p,q}^{n+1} \right) \right) e^{i(px_j + qy_k)} = 0. \tag{11}$$

Again because the occurring exponentials is a basis of the trigonometrical polynomials the exponentials coefficients must be zero. From this we calculate

$$\begin{aligned}
\widehat{T}_{p,q}^{n} &= \widehat{T}_{p,q}^{n+1} - \frac{1}{\rho} \left(\widehat{T}_{p,q}^{n+1} e^{-iph} + \widehat{T}_{p,q}^{n+1} e^{iph} + \widehat{T}_{p,q}^{n+1} e^{-iqh} + \widehat{T}_{p,q}^{n+1} e^{iph} - 4\widehat{T}_{p,q}^{n+1} \right) \\
&= \widehat{T}_{p,q}^{n+1} \left(1 - \frac{1}{\rho} \left(e^{-iph} + e^{iph} + e^{-iqh} + e^{iph} - 4 \right) \right) \\
&= \widehat{T}_{p,q}^{n+1} \left(1 - \frac{1}{\rho} \left(2\cos(ph) + 2\cos(qh) - 4 \right) \right) \\
&= \widehat{T}_{p,q}^{n+1} \left(1 - \frac{4}{\rho} \left(\sin^2\left(\frac{ph}{2} \right) + \sin^2\left(\frac{qh}{2} \right) \right) \right),
\end{aligned} \tag{12}$$

which is equivalent to

$$\widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^{n} \underbrace{\frac{1}{1 - \frac{4}{\rho} \left(\sin^2\left(\frac{ph}{2} \right) + \sin^2\left(\frac{qh}{2} \right) \right)}}_{=:G_{p,q}}. \tag{13}$$

Clearly here $|G_{p,q}| \leq 1$ is always fulfilled. Therefore the implicit scheme is unconditionally stable and hence no CFL condition is existent.

3 Van Neumann Stability Analysis: ADI Scheme

The ADI scheme consists of the two half steps