

# Alternate Direction Implicit Method

**Solving the Heat Flow Equation in 2D by Applying  
Implicit Equations in One Spatial Direction at a Time**

February 1<sup>st</sup>, 2021

Thomas Steindl<sup>1</sup> & Daylen Thimm<sup>2</sup>

<sup>1</sup>University of Innsbruck, Institute for Astro- and Particle Physics  
**Thomas.Steindl@uibk.ac.at**

<sup>2</sup>**Daylen.Thimm@student.uibk.ac.at**

# Heat Flow Equation in 2D

$$\frac{\partial T(x, y, t)}{\partial t} = \frac{\partial^2 T(x, y, t)}{\partial x^2} + \frac{\partial^2 T(x, y, t)}{\partial y^2}$$

## Explicit Method

**strong time step  
restriction**

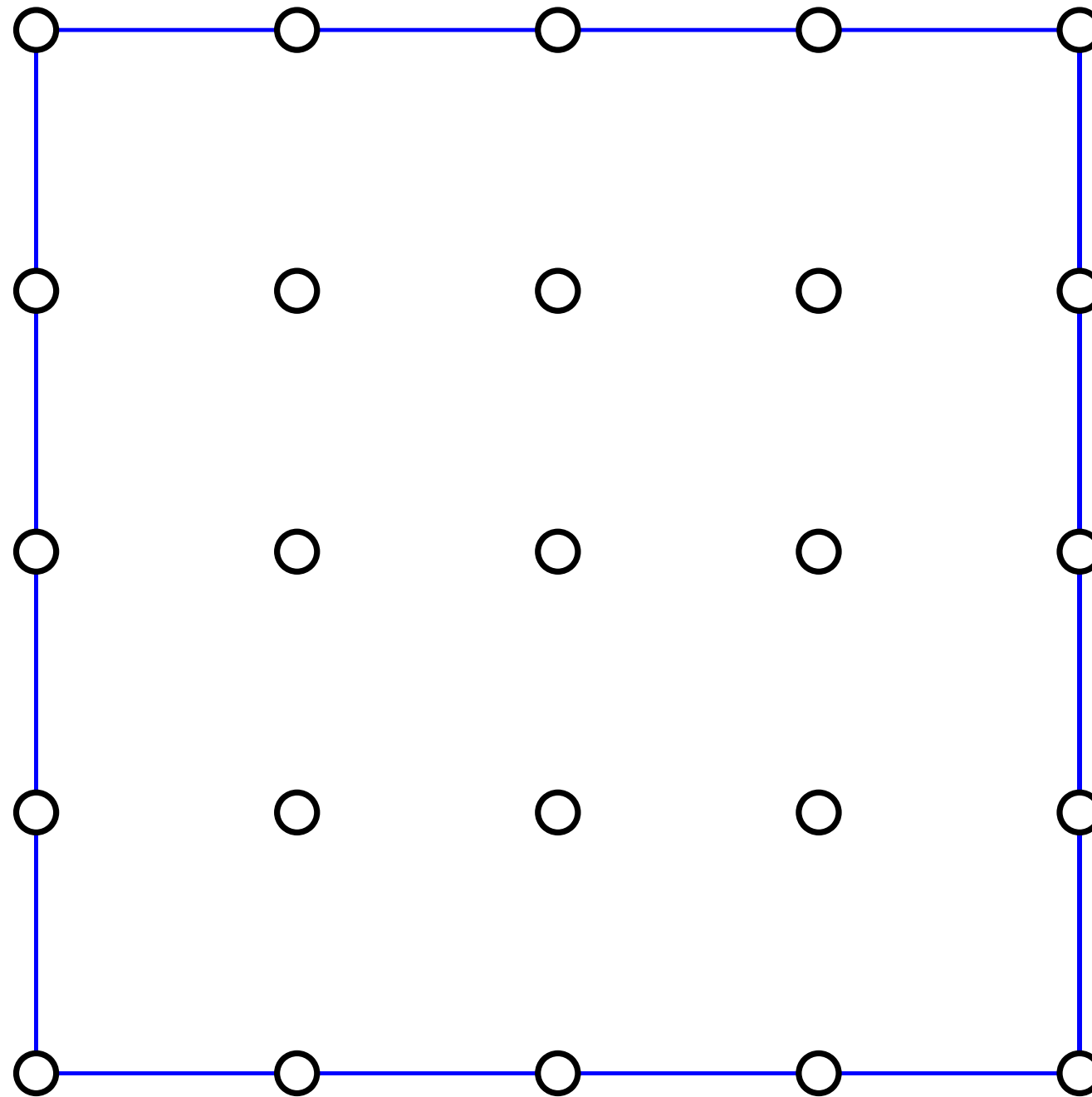
## Implicit Method

**no time step  
restriction**

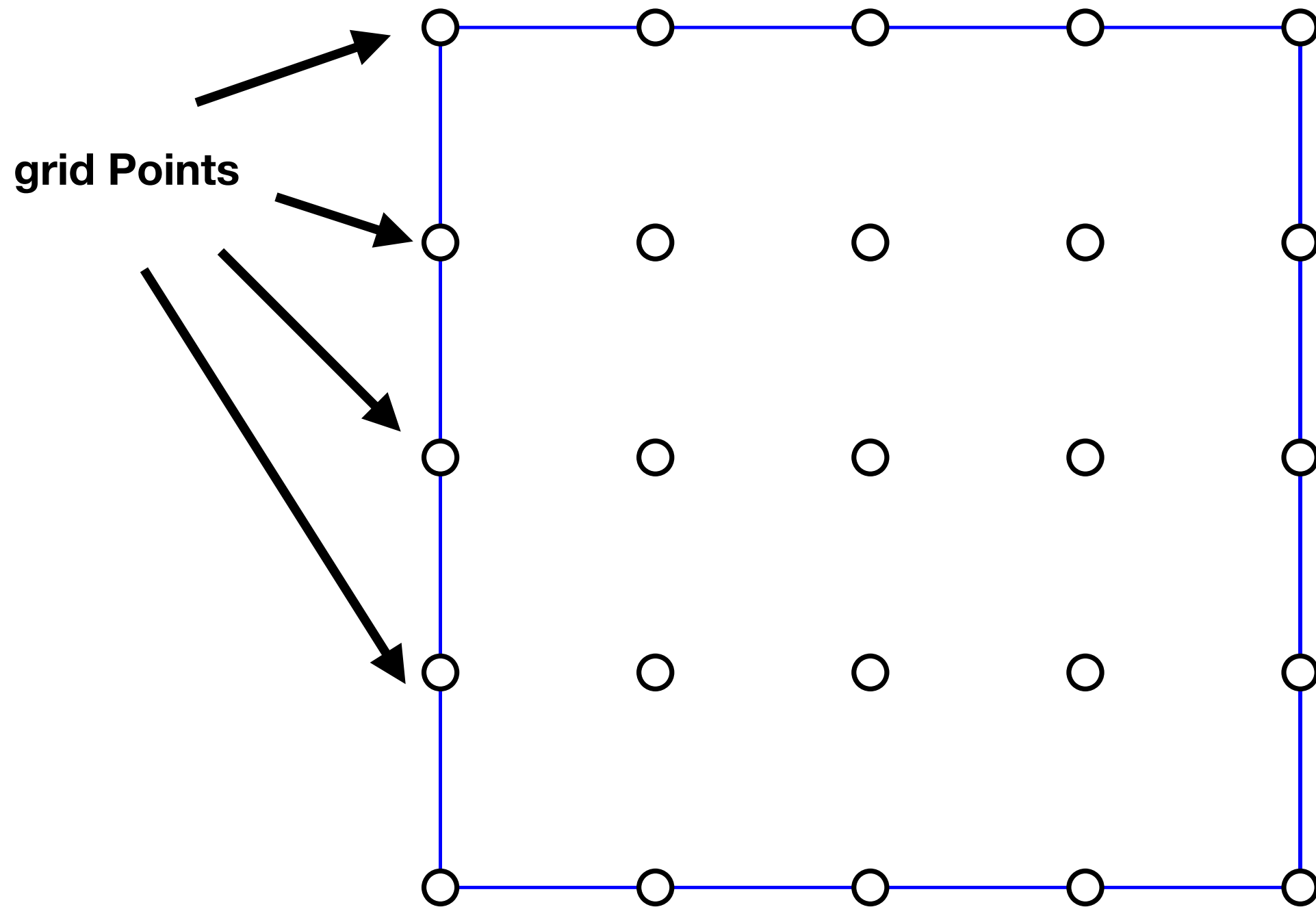
**need to solve  
big SLE**

## Alternate Direction Implicit

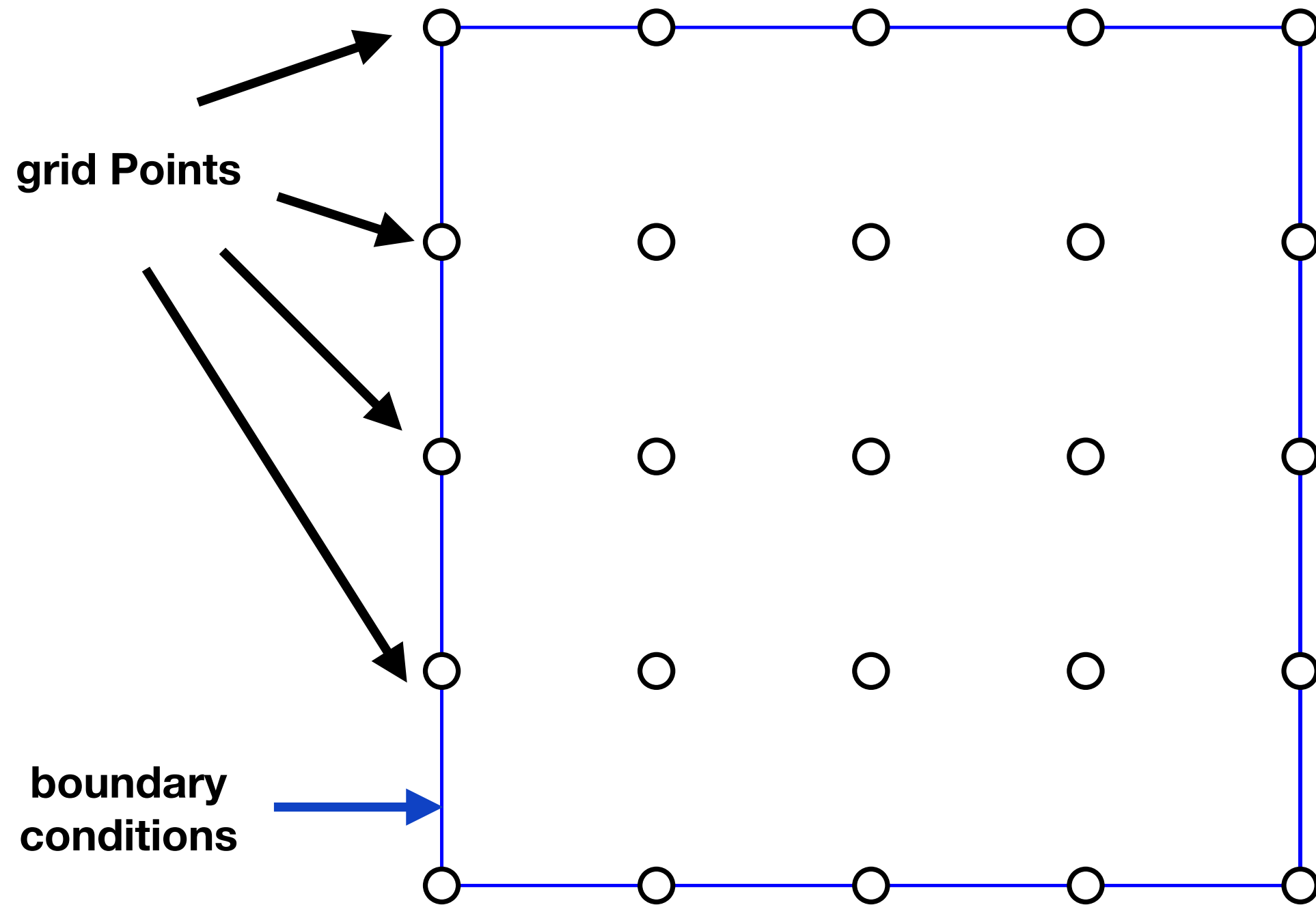
# Heat Flow Equation in 2D



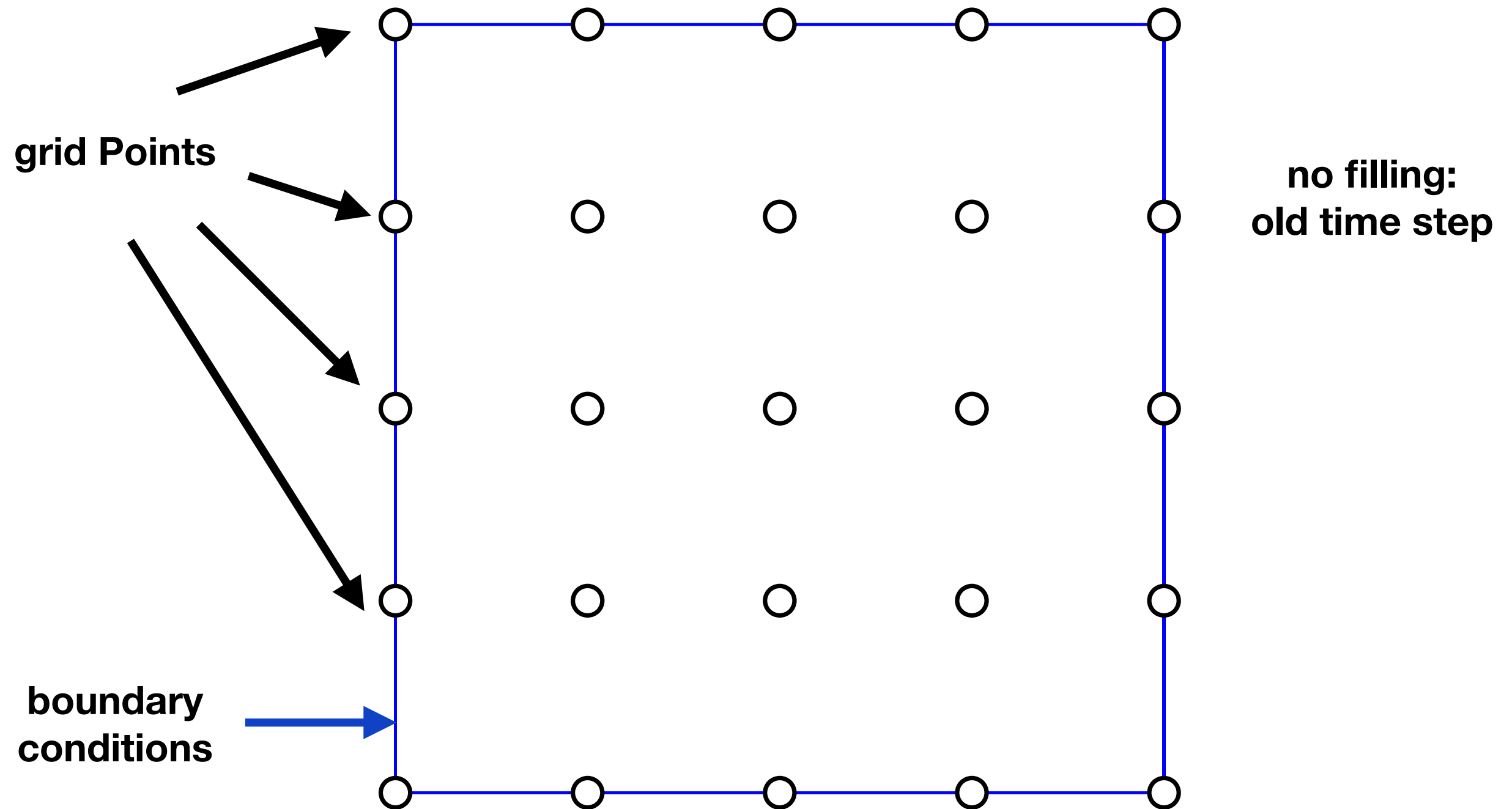
# Heat Flow Equation in 2D



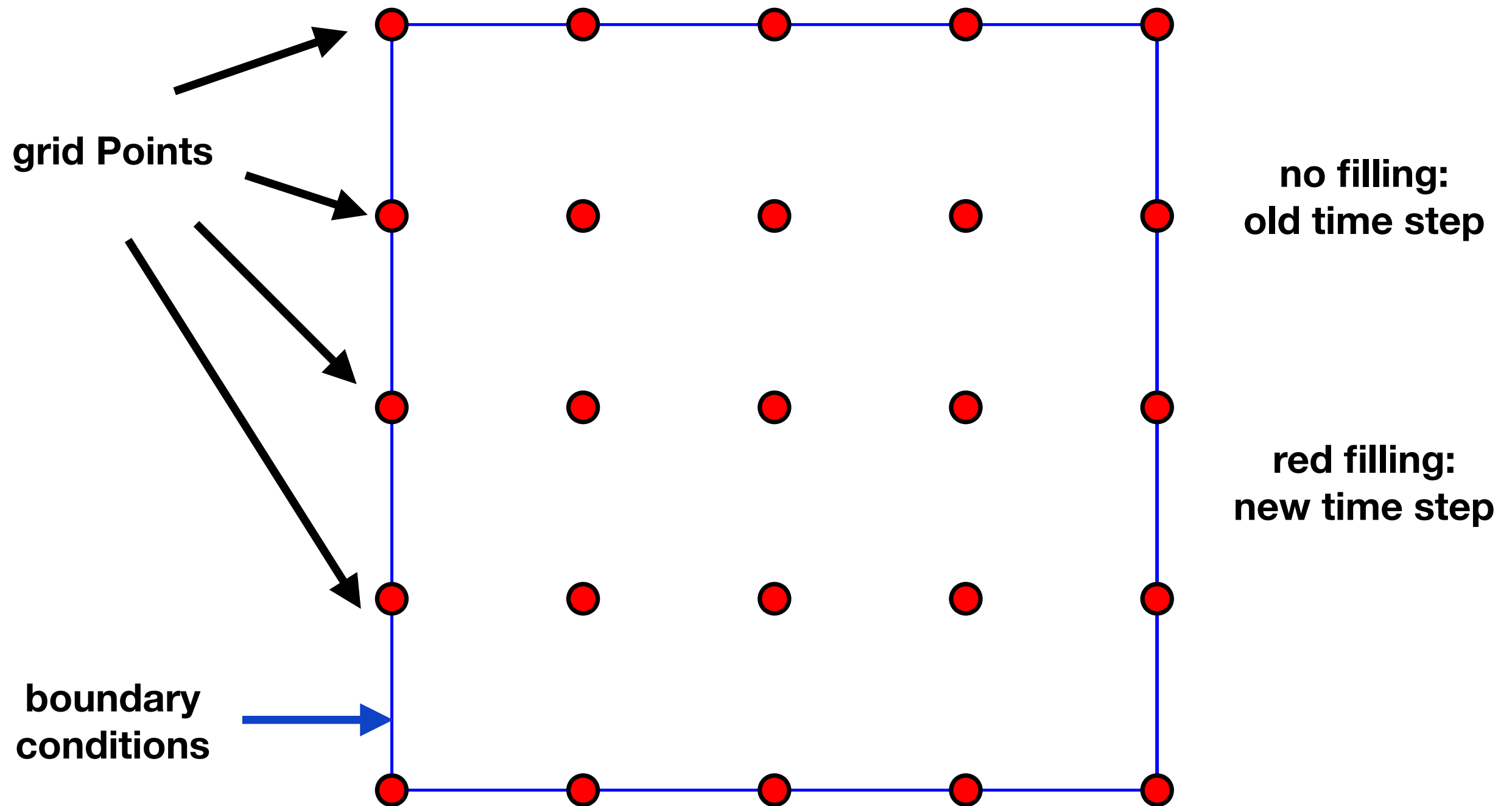
# Heat Flow Equation in 2D



# Heat Flow Equation in 2D

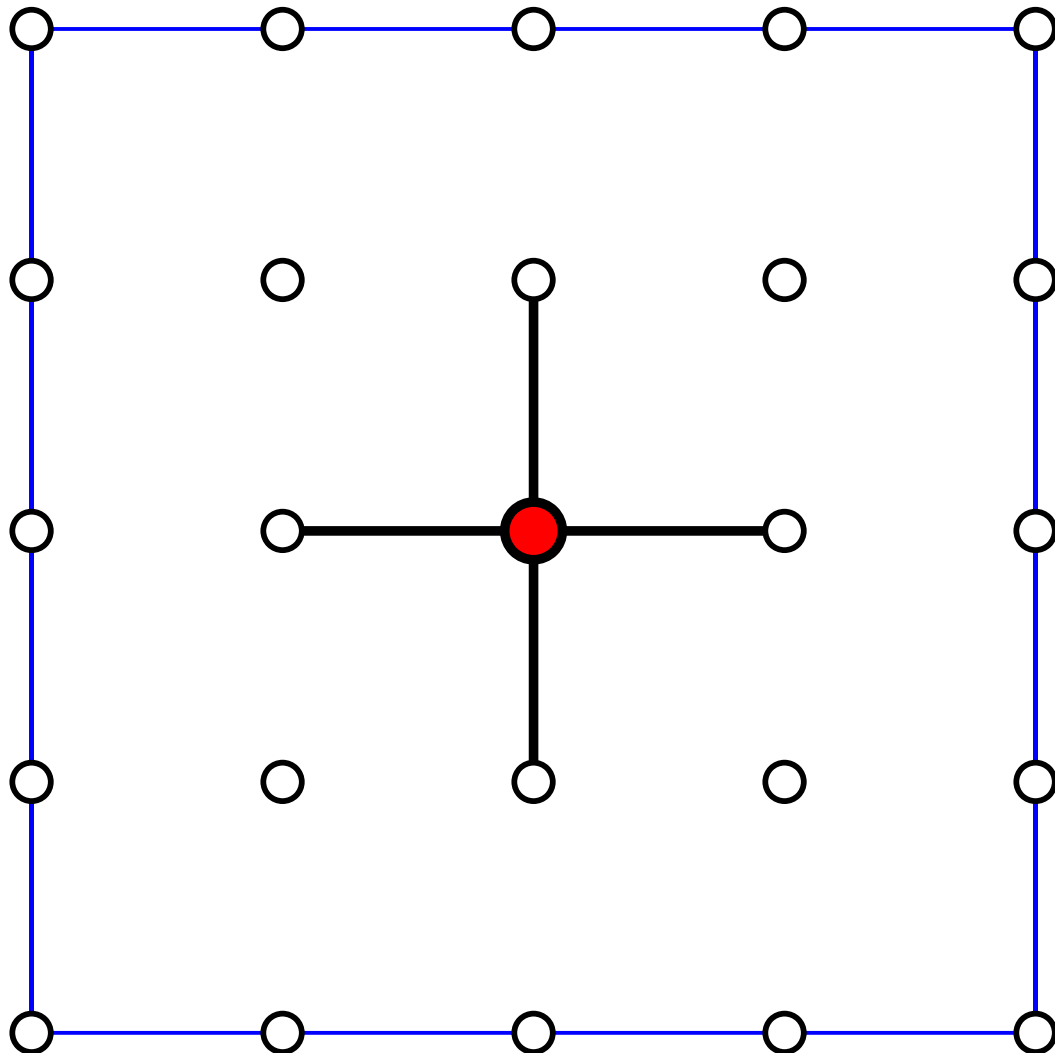


# Heat Flow Equation in 2D



# Explicit Method

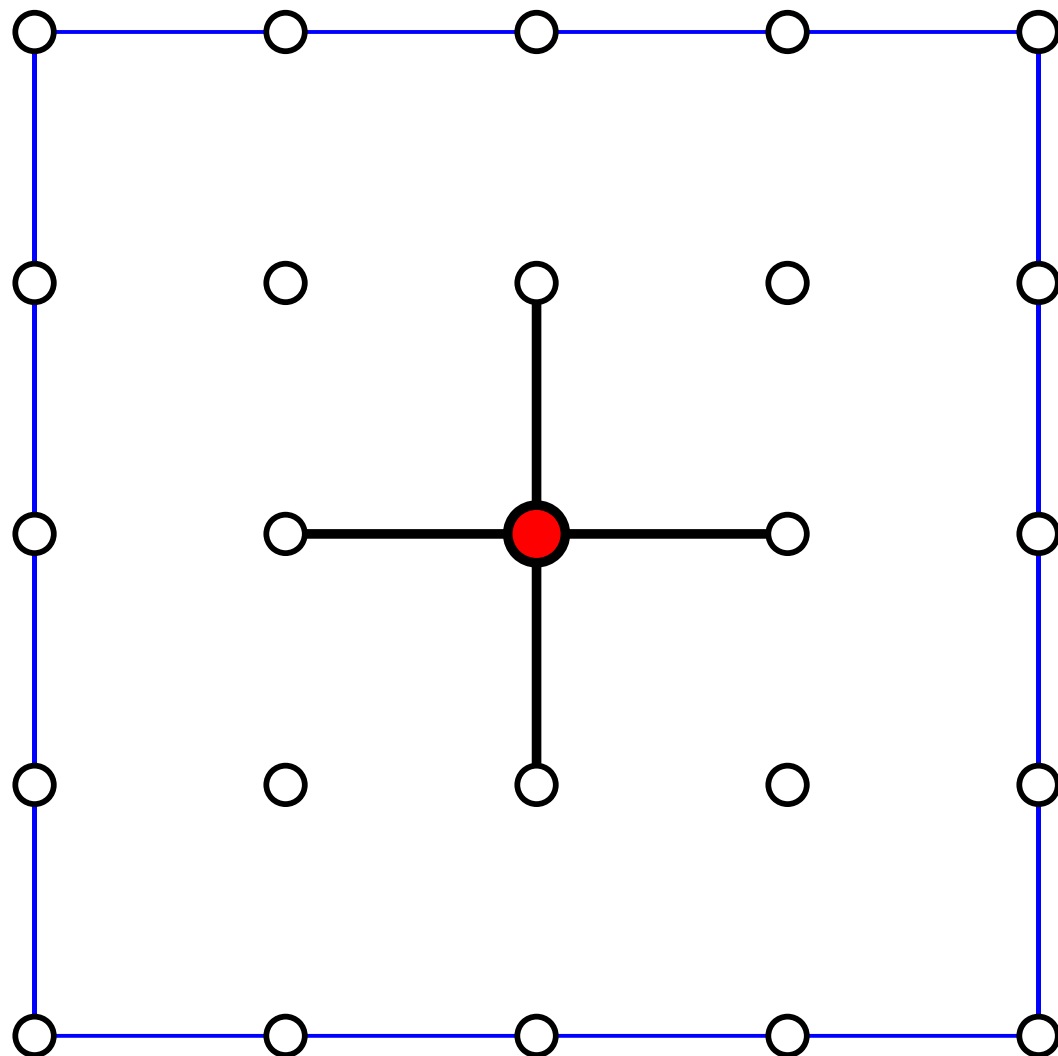
$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \frac{T_{i-1,j}^n - 2T_{i,j}^n + T_{i+1,j}^n}{(\Delta x)^2} + \frac{T_{i,j-1}^n - 2T_{i,j}^n + T_{i,j+1}^n}{(\Delta y)^2}$$





# Explicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \frac{T_{i-1,j}^n - 2T_{i,j}^n + T_{i+1,j}^n}{(\Delta x)^2} + \frac{T_{i,j-1}^n - 2T_{i,j}^n + T_{i,j+1}^n}{(\Delta y)^2}$$

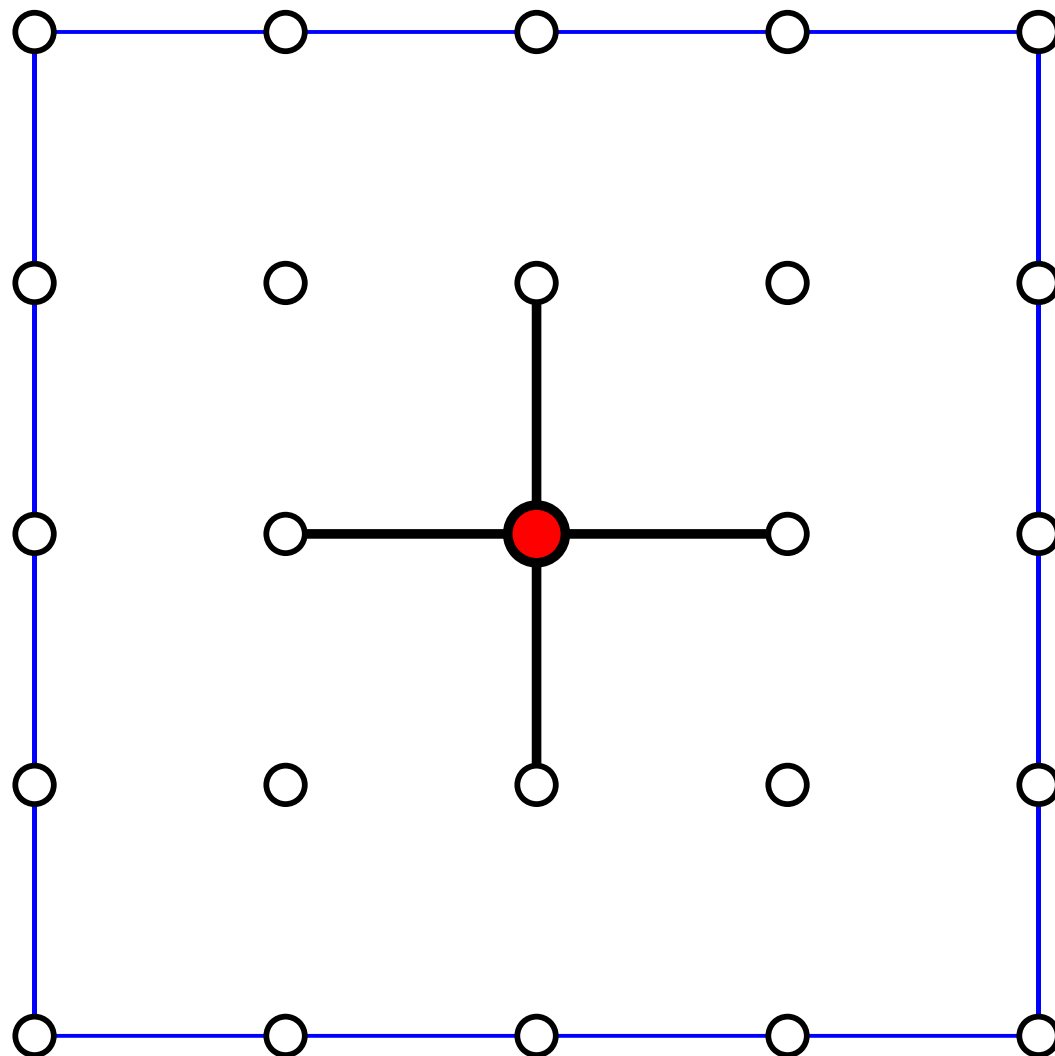


**only stable if**

$$-1 \leq 1 - \frac{4}{\rho} \left( \sin^2 \frac{\beta_p \Delta x}{2} + \sin^2 \frac{\beta_q \Delta y}{2} \right) \leq 1$$

# Explicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \frac{T_{i-1,j}^n - 2T_{i,j}^n + T_{i+1,j}^n}{(\Delta x)^2} + \frac{T_{i,j-1}^n - 2T_{i,j}^n + T_{i,j+1}^n}{(\Delta y)^2}$$



**only stable if**

$$-1 \leq 1 - \frac{4}{\rho} \left( \sin^2 \frac{\beta_p \Delta x}{2} + \sin^2 \frac{\beta_q \Delta y}{2} \right) \leq 1$$

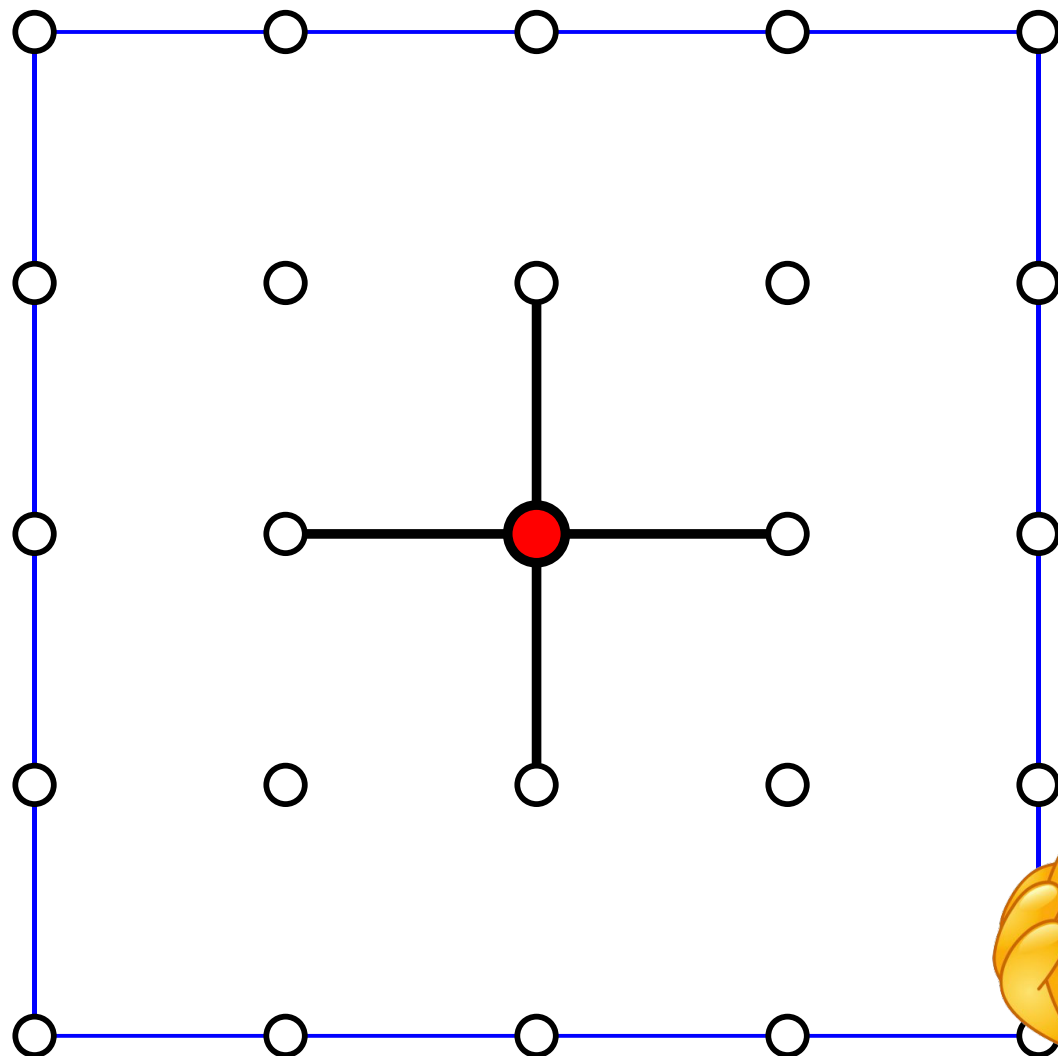
$$\Rightarrow \rho = \frac{(\Delta x)^2}{\Delta t} = \frac{(\Delta y)^2}{\Delta t} \geq 4$$

$$\Rightarrow \Delta t \leq \frac{(\Delta x)^2}{4} = \frac{1}{4N^2}$$

**strong time step  
restriction**

# Explicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \frac{T_{i-1,j}^n - 2T_{i,j}^n + T_{i+1,j}^n}{(\Delta x)^2} + \frac{T_{i,j-1}^n - 2T_{i,j}^n + T_{i,j+1}^n}{(\Delta y)^2}$$

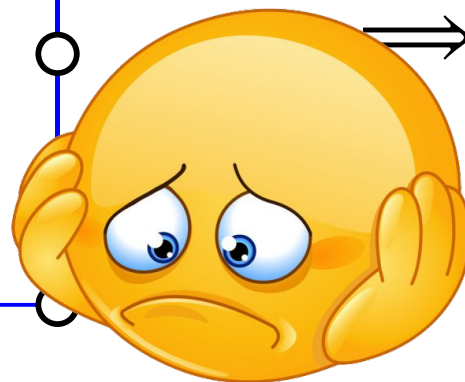


**only stable if**

$$-1 \leq 1 - \frac{4}{\rho} \left( \sin^2 \frac{\beta_p \Delta x}{2} + \sin^2 \frac{\beta_q \Delta y}{2} \right) \leq 1$$

$$\Rightarrow \rho = \frac{(\Delta x)^2}{\Delta t} = \frac{(\Delta y)^2}{\Delta t} \geq 4$$

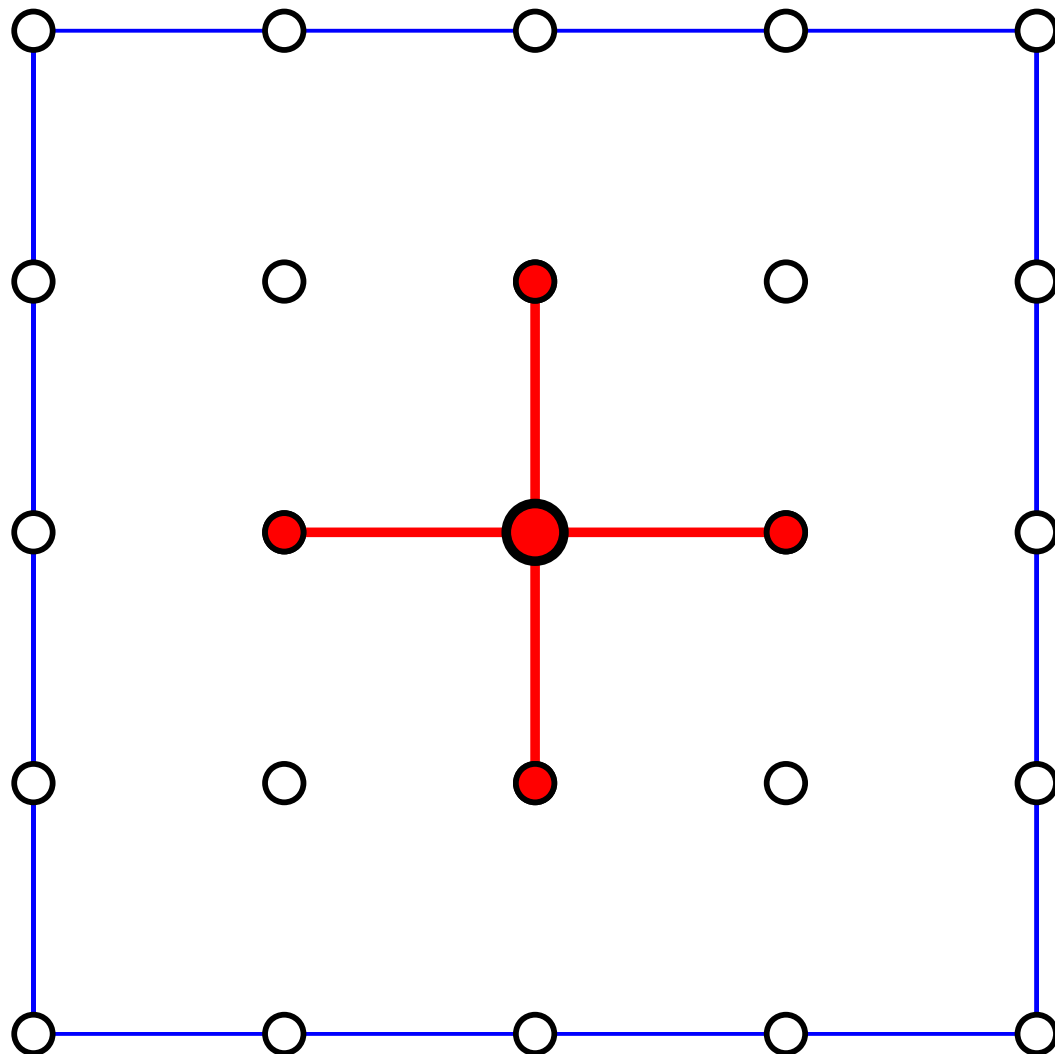
$$\Rightarrow \Delta t \leq \frac{(\Delta x)^2}{4} = \frac{1}{4N^2}$$



**strong time step  
restriction**

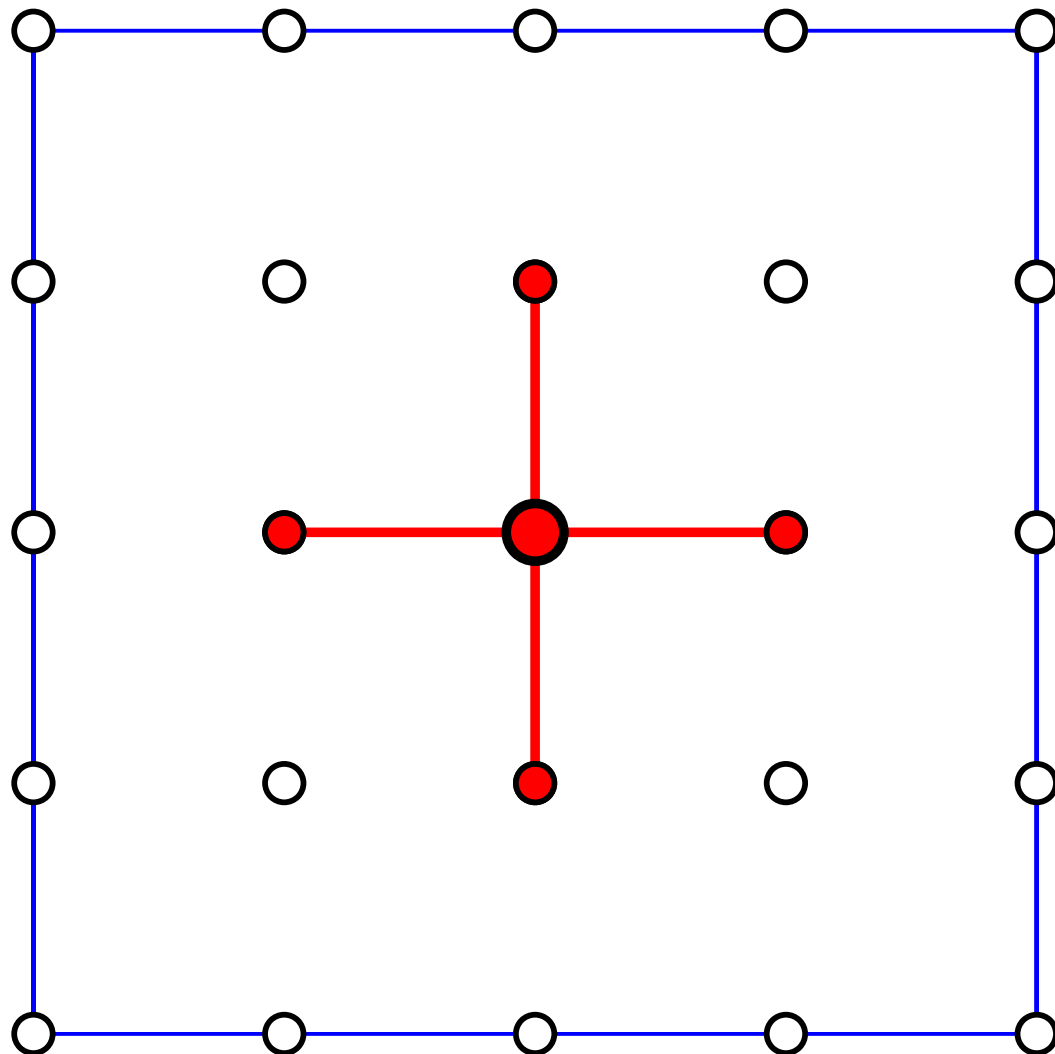
# Implicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{(\Delta y)^2}$$



# Implicit Method

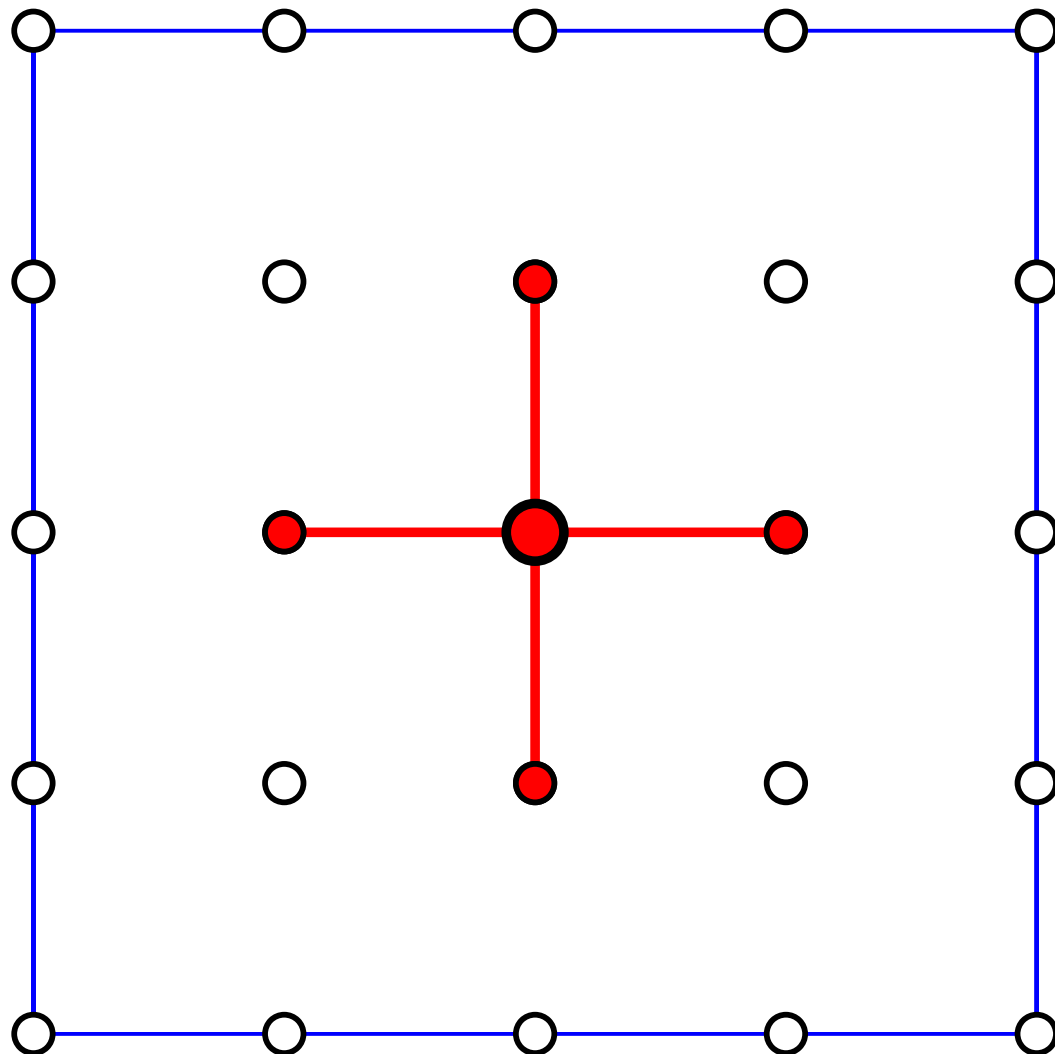
$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{(\Delta y)^2}$$



**stable for all time steps**

# Implicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{(\Delta y)^2}$$



**stable for all time steps**

**system of linear equations**

$$Ax = b \quad A \in \mathbb{R}^{N^2 \times N^2} \quad x, b \in \mathbb{R}^{N^2}$$

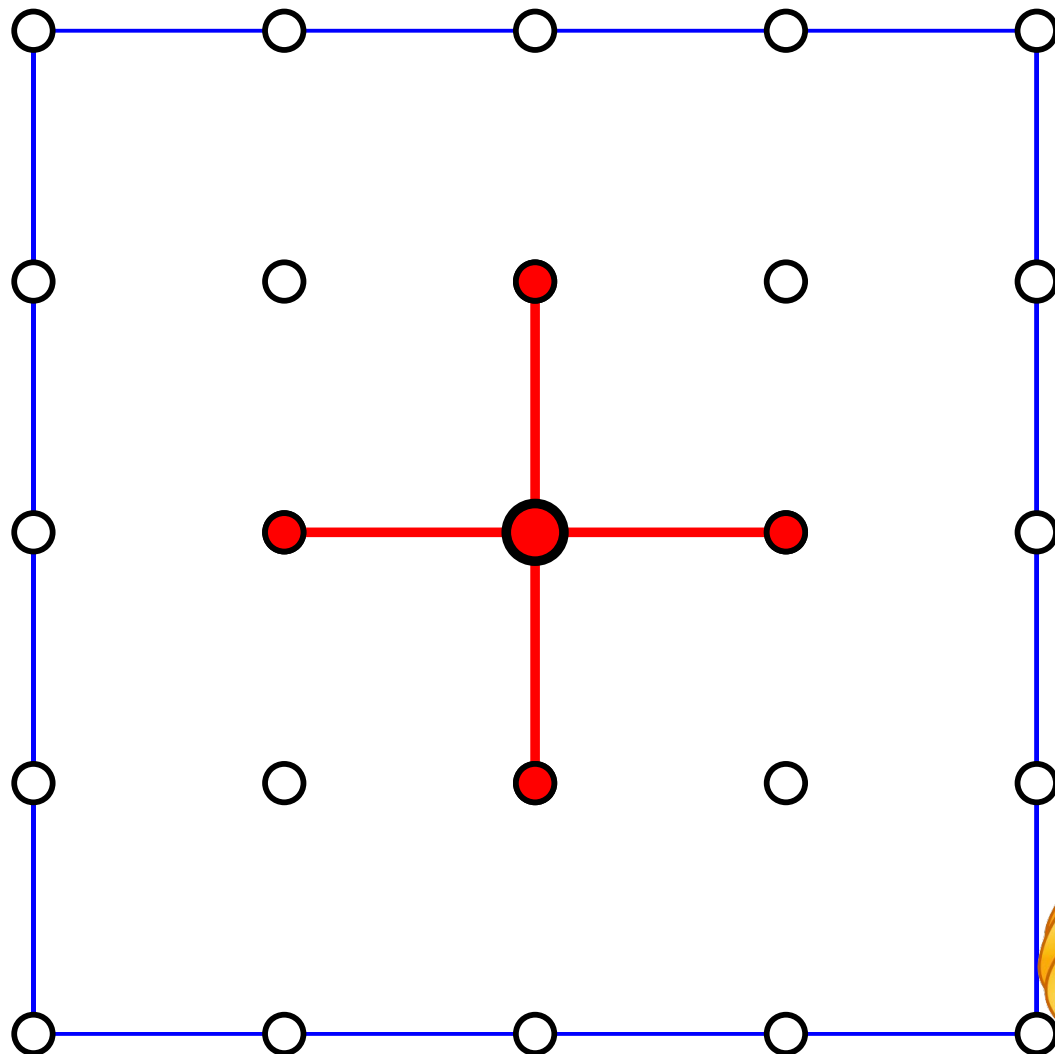
$$T_{i-1,j}^{n+1} + T_{i+1,j}^{n+1} + T_{i,j-1}^{n+1} +$$

$$+ T_{i,j+1}^{n+1} - (4 + \rho)T_{i,j}^{n+1} = -\rho T_{i,j}^n$$

**hard to solve**

# Implicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{(\Delta y)^2}$$



**stable for all time steps**

**system of linear equations**

$$Ax = b \quad A \in \mathbb{R}^{N^2 \times N^2} \quad x, b \in \mathbb{R}^{N^2}$$

$$T_{i-1,j}^{n+1} + T_{i+1,j}^{n+1} + T_{i,j-1}^{n+1} +$$

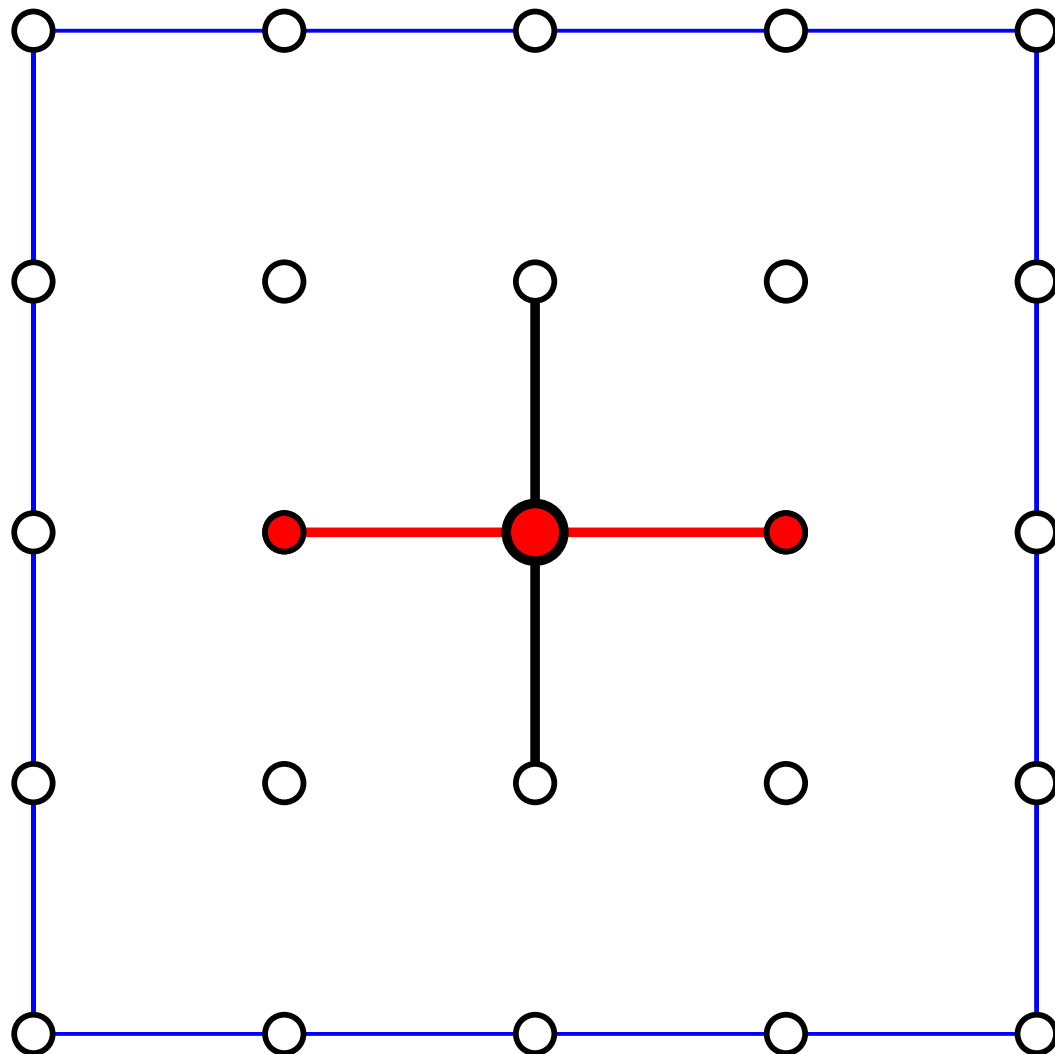
$$+ T_{i,j+1}^{n+1} - (4 + \rho)T_{i,j}^{n+1} = -\rho T_{i-1,j}^n$$



**hard to solve**

# Alternate Direction Implicit Method I

$$\frac{T_{i,j}^{2n+1} - T_{i,j}^{2n}}{\Delta t} = \frac{T_{i-1,j}^{2n+1} - 2T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{2n} - 2T_{i,j}^{2n} + T_{i,j+1}^{2n}}{(\Delta y)^2}$$



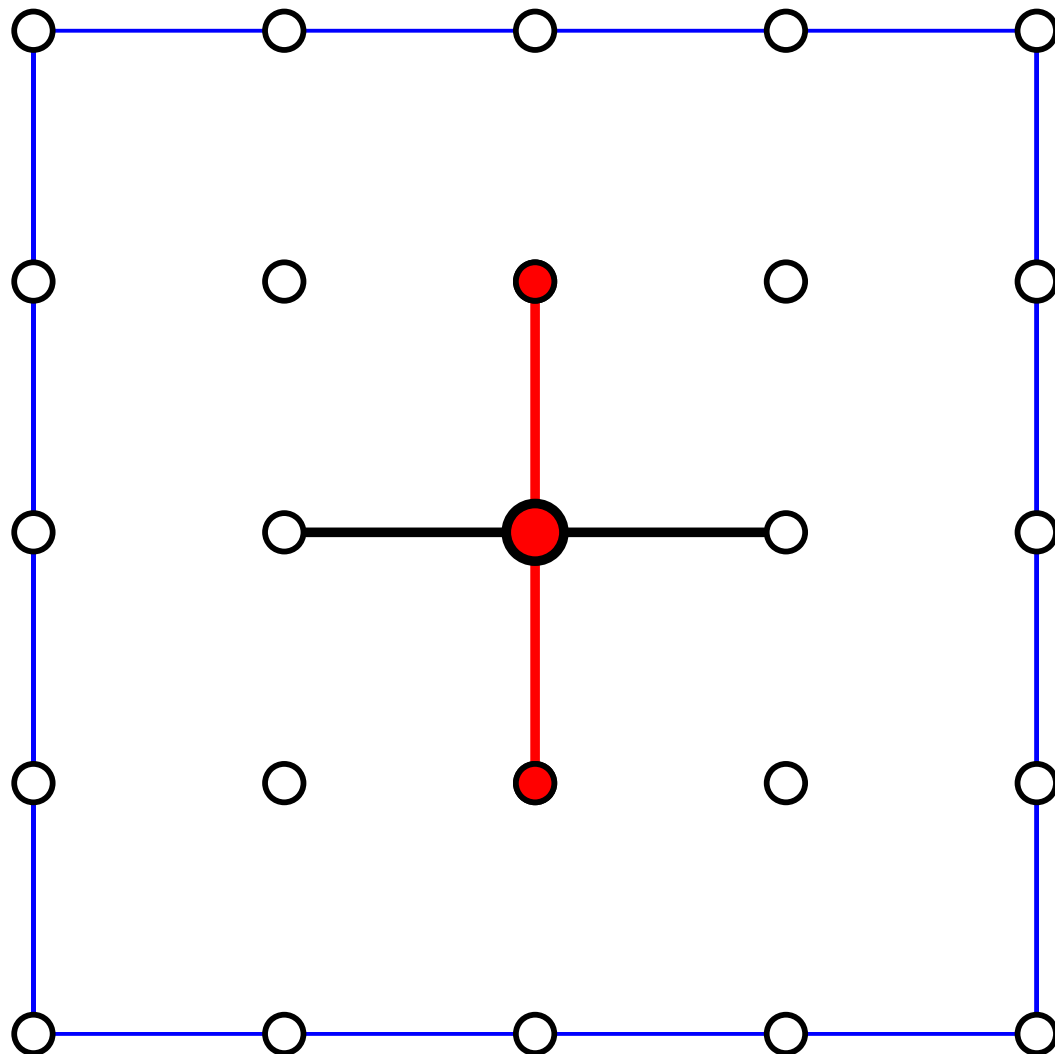
**implicit in x - direction**

**explicit in y - direction**



# Alternate Direction Implicit Method II

$$\frac{T_{i,j}^{2n+2} - T_{i,j}^{2n+1}}{\Delta t} = \frac{T_{i-1,j}^{2n+1} - 2T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{2n+2} - 2T_{i,j}^{2n+2} + T_{i,j+1}^{2n+2}}{(\Delta y)^2}$$



**implicit in y - direction**

**explicit in x - direction**

# Alternate Direction Implicit Method

**N sets of N simultaneous equations**

$$T_{i-1,j}^{2n+1} - (2 + \rho)T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1} = -T_{i,j-1}^{2n} + (2 - \rho)T_{i,j}^{2n} - T_{i,j+1}^{2n}$$

# Alternate Direction Implicit Method

**N sets of N simultaneous equations**

$$T_{i-1,j}^{2n+1} - (2 + \rho)T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1} = -T_{i,j-1}^{2n} + (2 - \rho)T_{i,j}^{2n} - T_{i,j+1}^{2n}$$

**with tridiagonal matrix**

$$\begin{bmatrix} \ddots & & & & \\ \ddots & -(2 + \rho) & 1 & 0 & 0 \\ 0 & 1 & -(2 + \rho) & 1 & 0 \\ 0 & 0 & 1 & -(2 + \rho) & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ T_{i-1,j}^{2n+1} \\ T_{i,j}^{2n+1} \\ T_{i+1,j}^{2n+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ -T_{i-1,j-1}^{2n} + (2 - \rho)T_{i-1,j}^{2n} - T_{i-1,j+1}^{2n} \\ -T_{i,j-1}^{2n} + (2 - \rho)T_{i,j}^{2n} - T_{i,j+1}^{2n} \\ -T_{i+1,j-1}^{2n} + (2 - \rho)T_{i+1,j}^{2n} - T_{i+1,j+1}^{2n} \\ \vdots \end{bmatrix}$$

# Alternate Direction Implicit Method

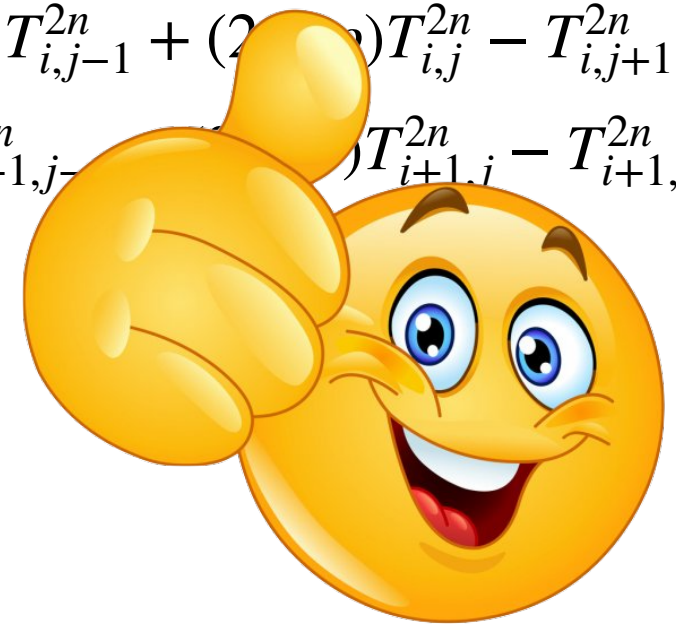
**N sets of N simultaneous equations**

$$T_{i-1,j}^{2n+1} - (2 + \rho)T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1} = -T_{i,j-1}^{2n} + (2 - \rho)T_{i,j}^{2n} - T_{i,j+1}^{2n}$$

**with tridiagonal matrix**

$$\begin{bmatrix} \ddots & & 0 & 0 & 0 \\ \ddots & -(2 + \rho) & 1 & 0 & 0 \\ 0 & 1 & -(2 + \rho) & 1 & 0 \\ 0 & 0 & 1 & -(2 + \rho) & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ T_{i-1,j}^{2n+1} \\ T_{i,j}^{2n+1} \\ T_{i+1,j}^{2n+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ -T_{i-1,j-1}^{2n} + (2 - \rho)T_{i-1,j}^{2n} - T_{i-1,j+1}^{2n} \\ -T_{i,j-1}^{2n} + (2 - \rho)T_{i,j}^{2n} - T_{i,j+1}^{2n} \\ -T_{i+1,j-1}^{2n} + (2 - \rho)T_{i+1,j}^{2n} - T_{i+1,j+1}^{2n} \\ \vdots \end{bmatrix}$$

**use Thomas algorithm**



# Alternate Direction Implicit Method

## Von Neumann stability analysis using the Fourier Ansatz

$$T_{j,k}^n = \sum_{p,q=0}^{N-1} \hat{T}_{p,q}^n e^{ipx_j + iqy_k}$$

**single steps highly unstable**

$$\hat{T}_{p,q}^{2n+1} = \hat{T}_{p,q}^{2n} \underbrace{\frac{\rho - 4 \sin^2 \left( \frac{ph}{2} \right)}{\rho + 4 \sin^2 \left( \frac{qh}{2} \right)}}_{=: G_{p,q}^{(1)}}$$

$$\hat{T}_{p,q}^{2n+2} = \hat{T}_{p,q}^{2n+1} \underbrace{\frac{\rho - 4 \sin^2 \left( \frac{qh}{2} \right)}{\rho + 4 \sin^2 \left( \frac{ph}{2} \right)}}_{=: G_{p,q}^{(2)}}$$

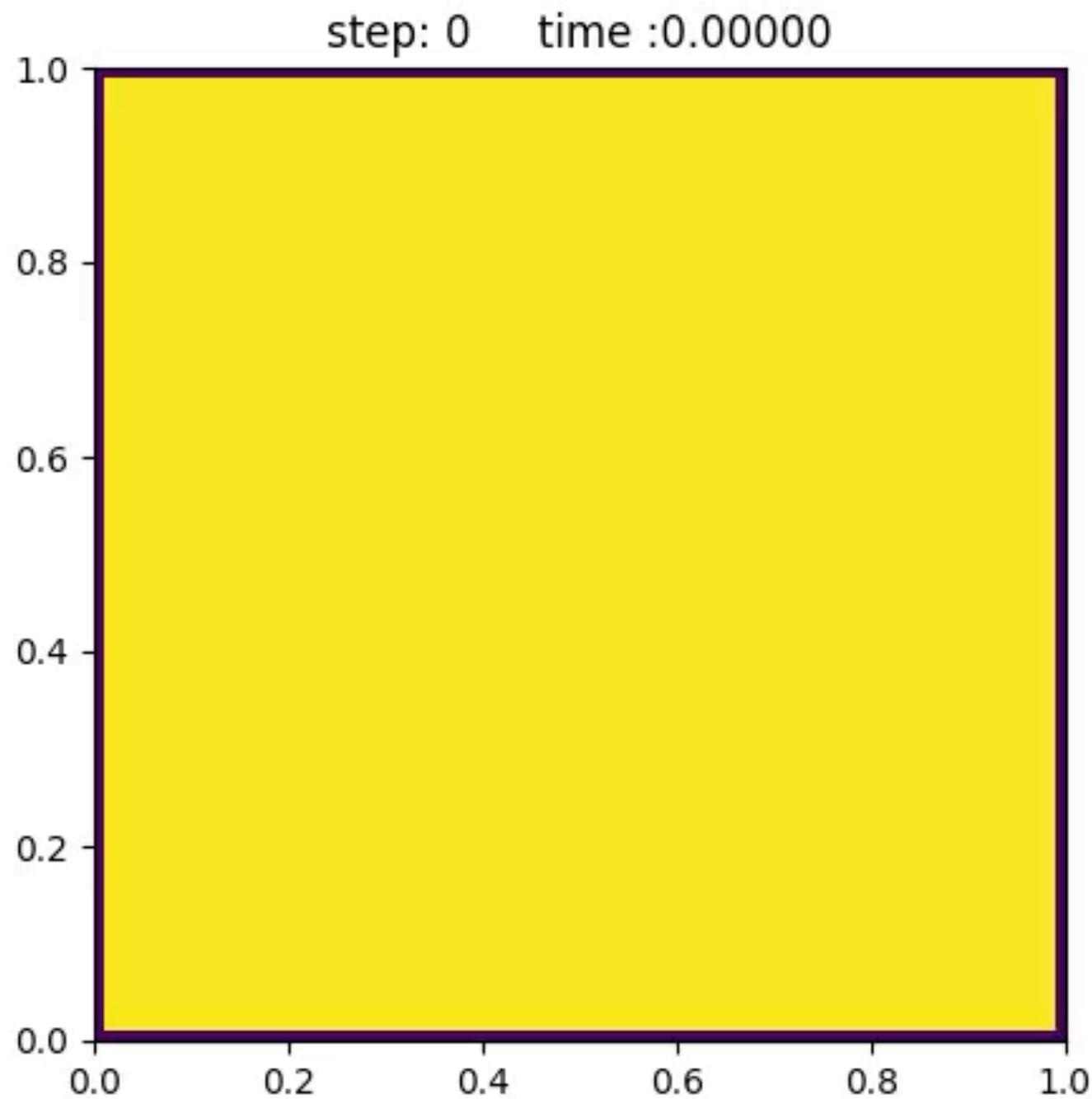
# Alternate Direction Implicit Method

combining the steps

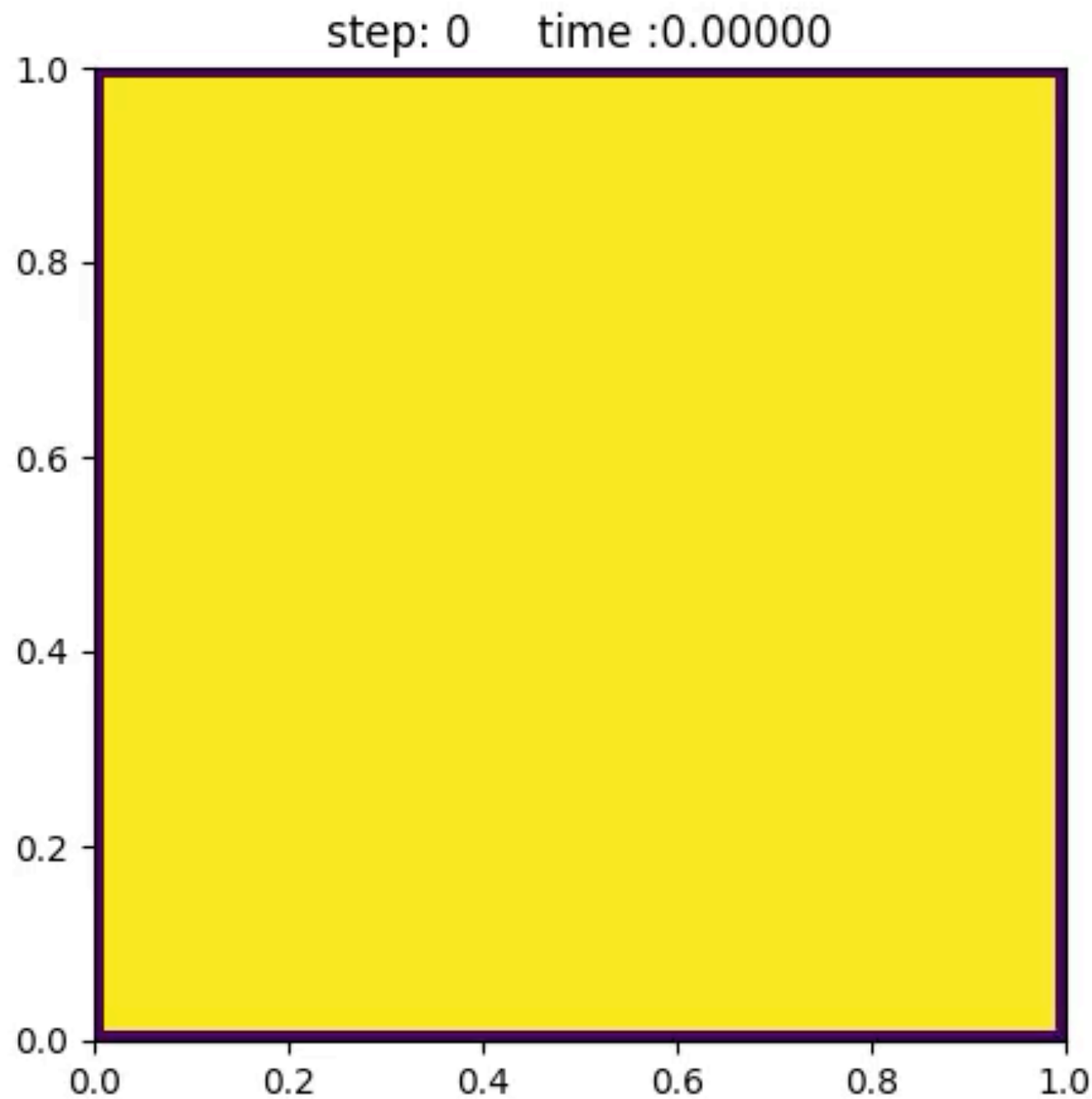
$$\hat{T}_{p,q}^{2n+2} = \hat{T}_{p,q}^{2n} \underbrace{\begin{pmatrix} \rho - 4 \sin^2 \left( \frac{qh}{2} \right) & \\ & \rho + 4 \sin^2 \left( \frac{ph}{2} \right) \end{pmatrix} \begin{pmatrix} \rho - 4 \sin^2 \left( \frac{ph}{2} \right) & \\ & \rho + 4 \sin^2 \left( \frac{qh}{2} \right) \end{pmatrix}}_{=: G_{p,q}}$$

**Stable for all time steps**

# Alternate Direction Implicit Method



# Alternate Direction Implicit Method





# Analytic Solution

$$\frac{\partial T(x, y, t)}{\partial t} = \frac{\partial^2 T(x, y, t)}{\partial x^2} + \frac{\partial^2 T(x, y, t)}{\partial y^2}$$

**boundary condition:**  $T(0, y, t) = T(1, y, t) = T(x, 0, t) = T(x, 1, t) = 0$

**Initial condition:**  $T(x, y, 0) = 1$

$$\Rightarrow T(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16}{(2m-1)(2n-1)\pi^2} \sin(n\pi x) \sin(m\pi y) \exp(- (m^2 + n^2)\pi^2 t)$$

# Comparison

compared to analytic solution

