universität innsbruck

Alternate Direction Implicit Method

Solving the Heat Flow Equation in 2D by Applying Implicit Equations in One Spatial Direction at a Time

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Heat Flow Equation in 2D

$$\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial^2 T(x,y,t)}{(\partial x)^2} + \frac{\partial^2 T(x,y,t)}{(\partial y)^2}$$

Explicit Method

strong time step restriction

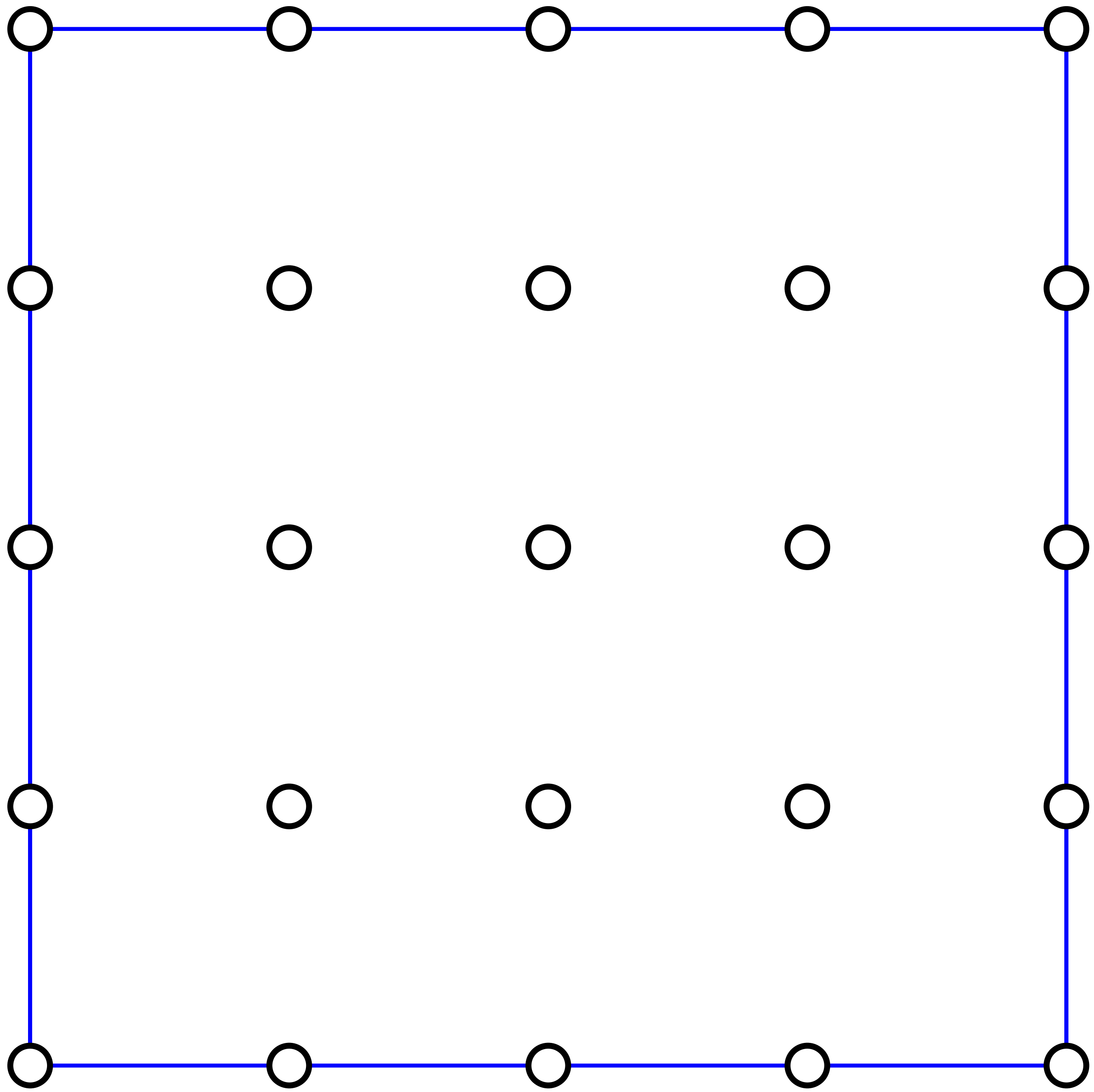
Implicit Method

no time step restriction

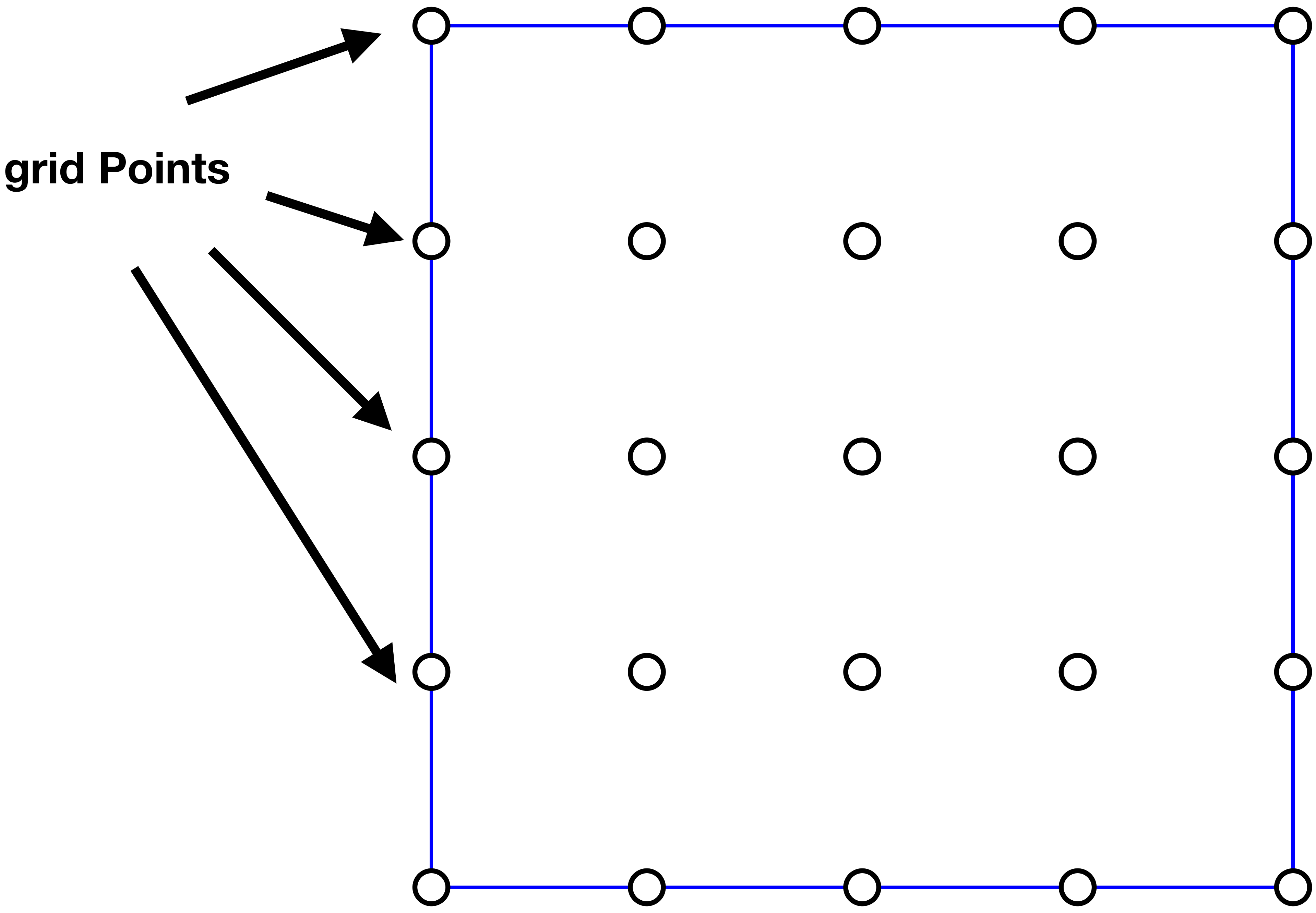
need to solve big SLE

Alternate Direction Implicit

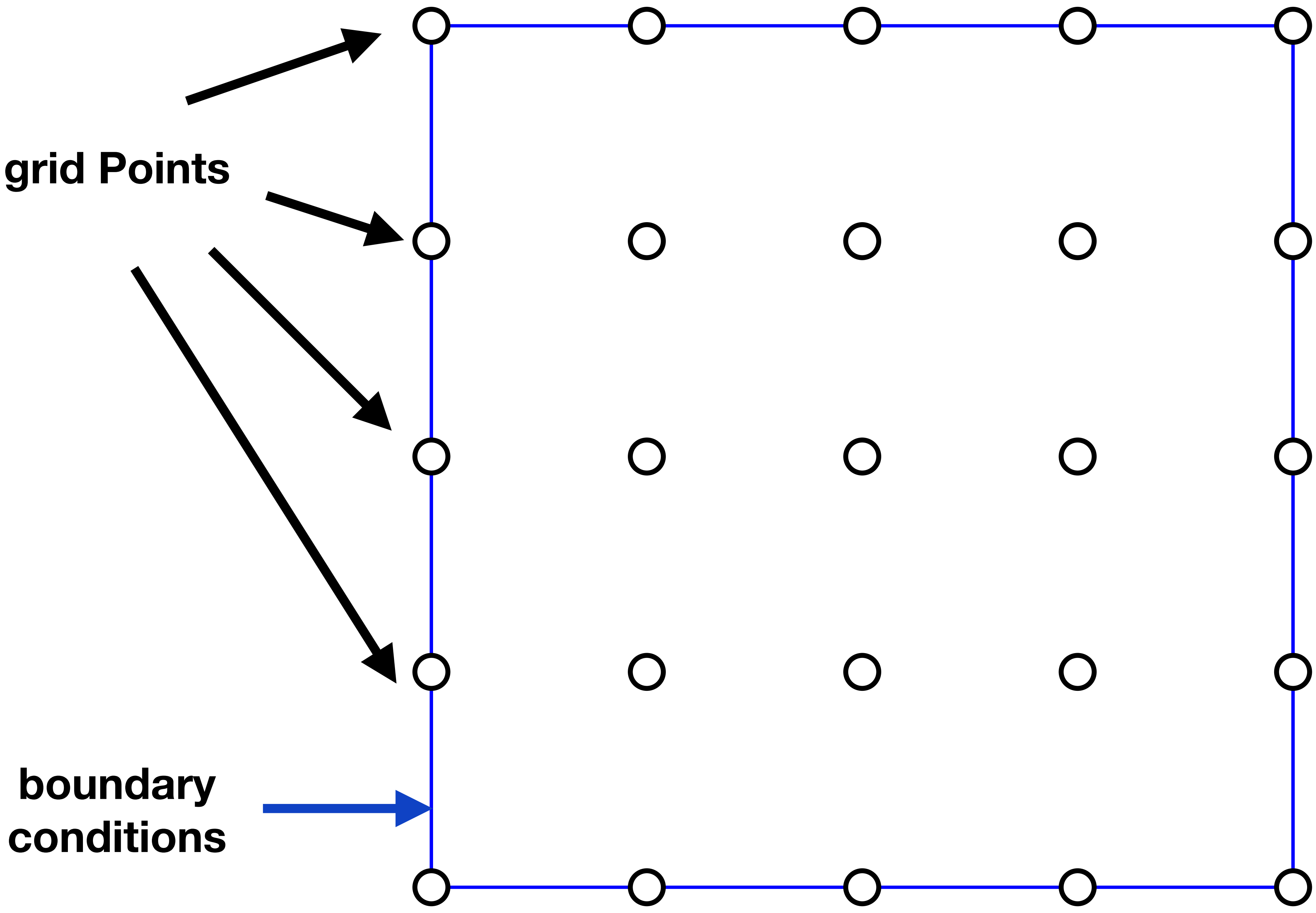
Heat Flow Equation in 2D



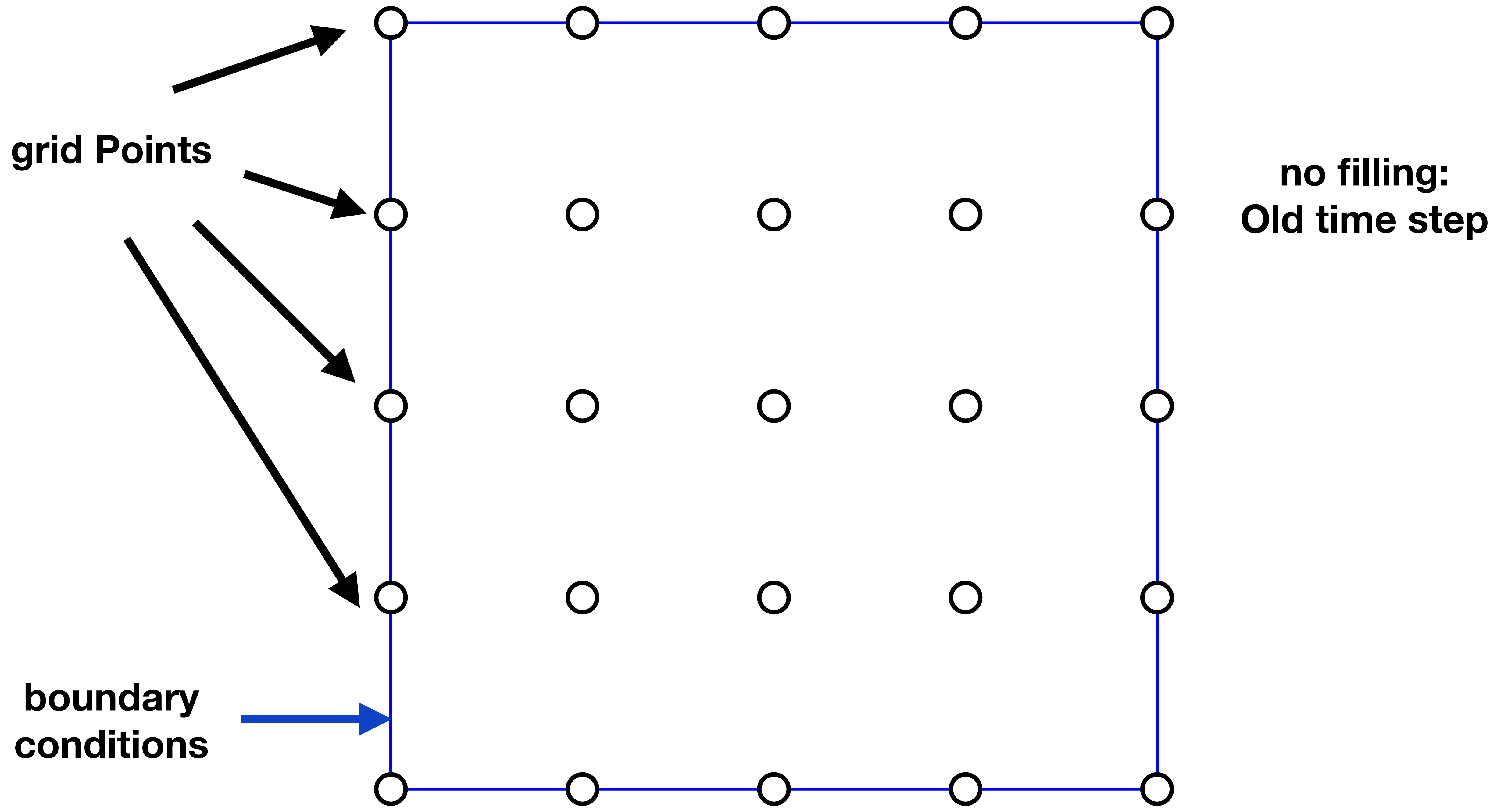
Heat Flow Equation in 2D



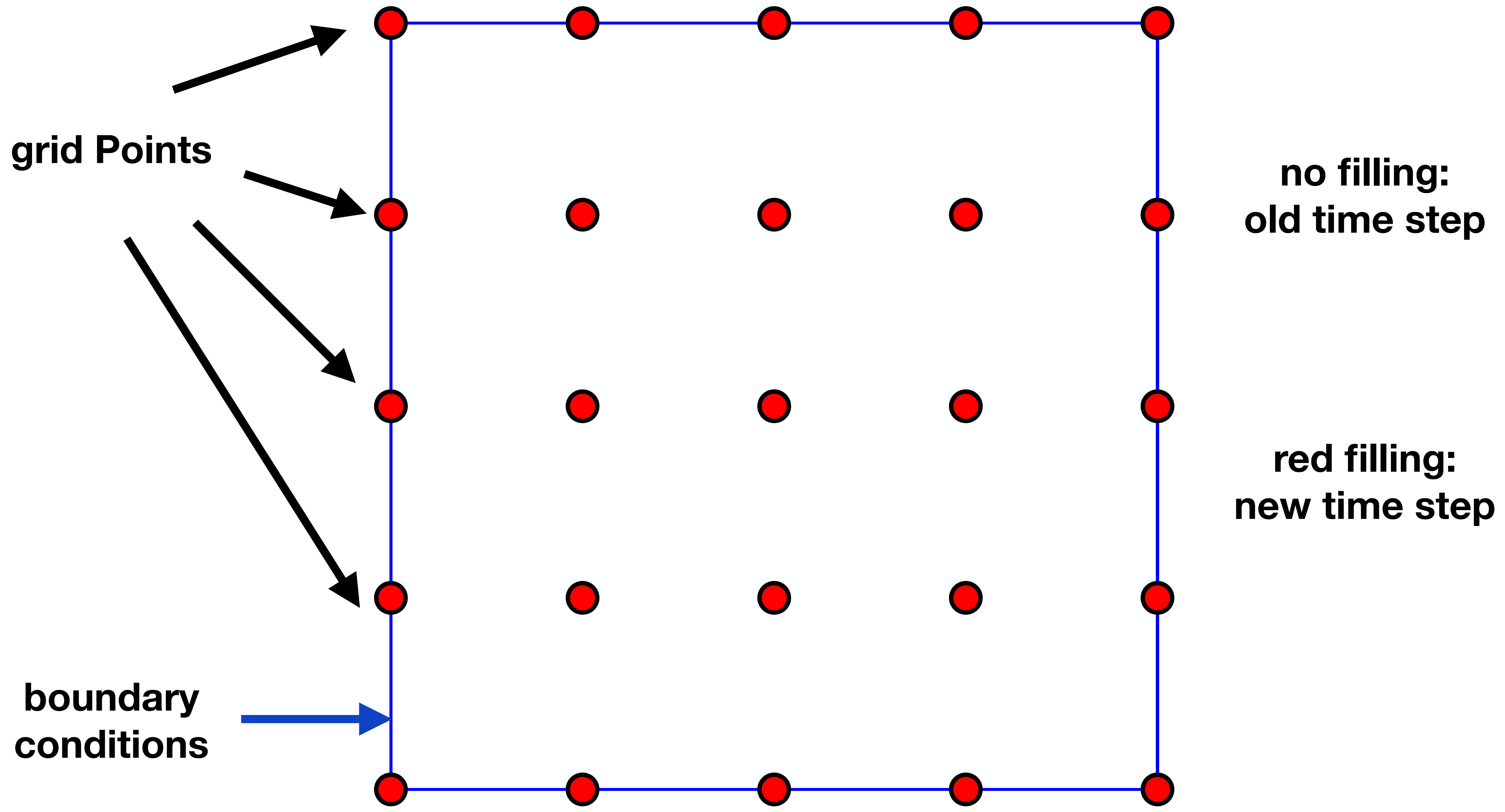
Heat Flow Equation in 2D



Heat Flow Equation in 2D

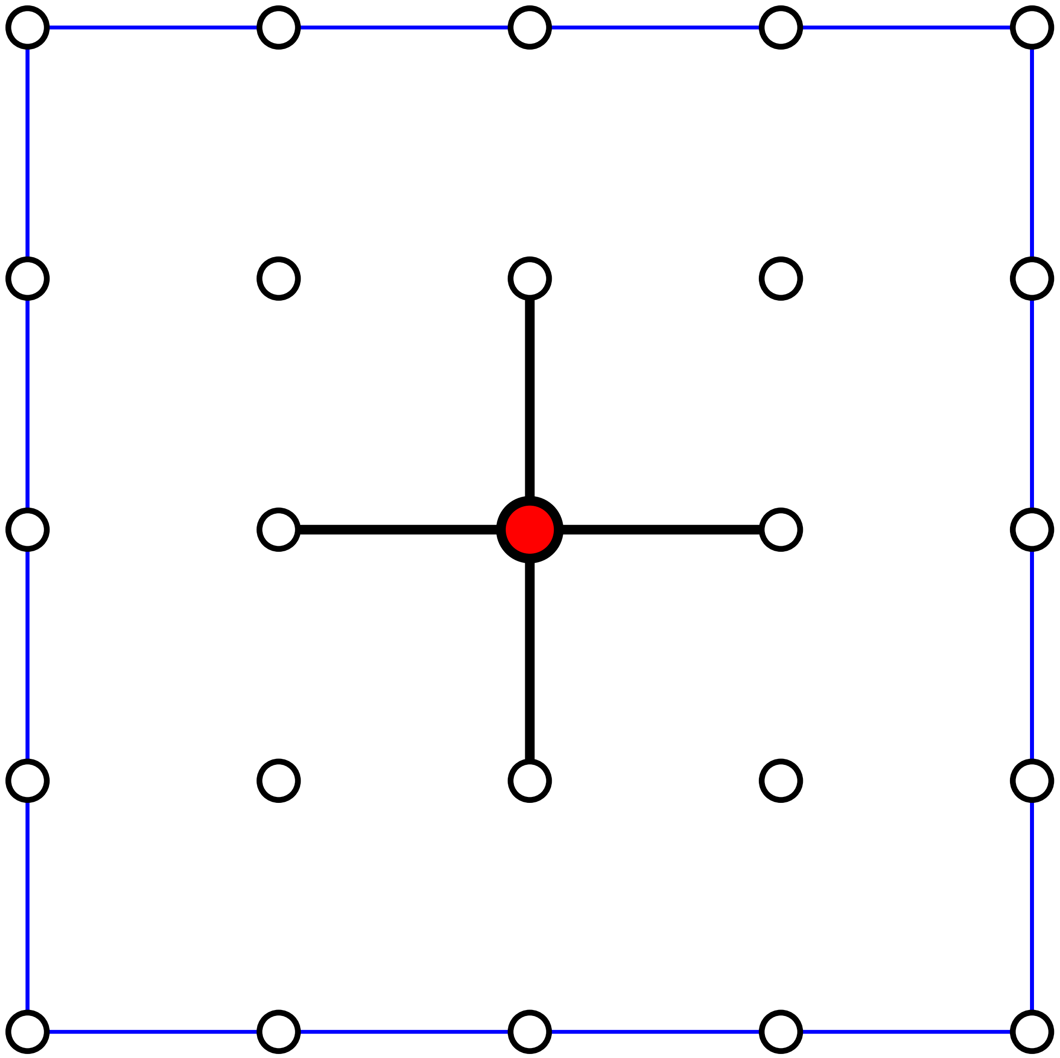


Heat Flow Equation in 2D



Explicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j,n}}{\Delta t} = \frac{T_{i-1,j}^{n} - 2T_{i,j}^{n} + T_{i+1,j}^{n}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n} - 2T_{i,j}^{n} + T_{i,j+1}^{n}}{(\Delta y)^2}$$



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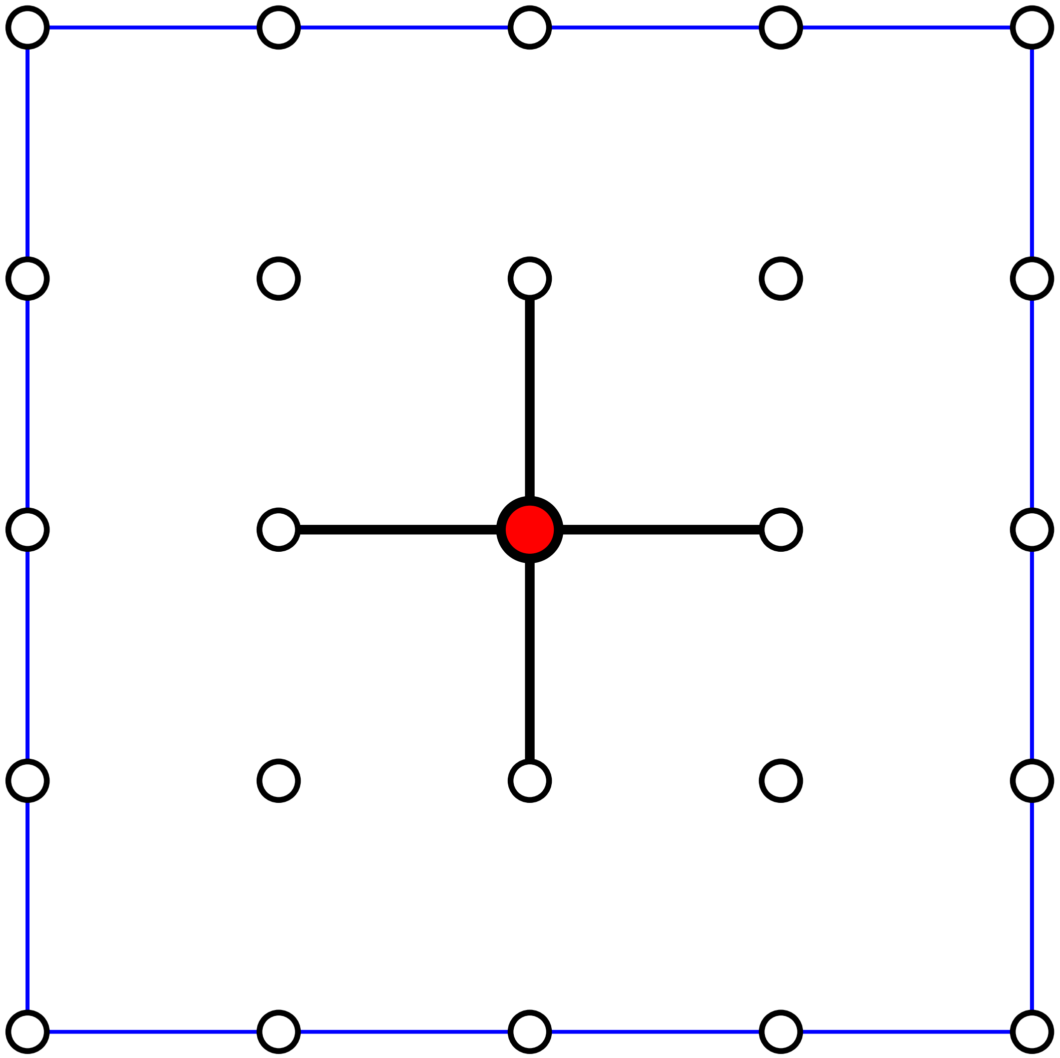

only stable if

$$-1 \leq 1 - \frac{4}{\rho}\left(\sin^2\frac{\beta_p \Delta x}{2} + \frac{\beta_q \Delta y}{2}\right) \leq 1$$

Explicit Method

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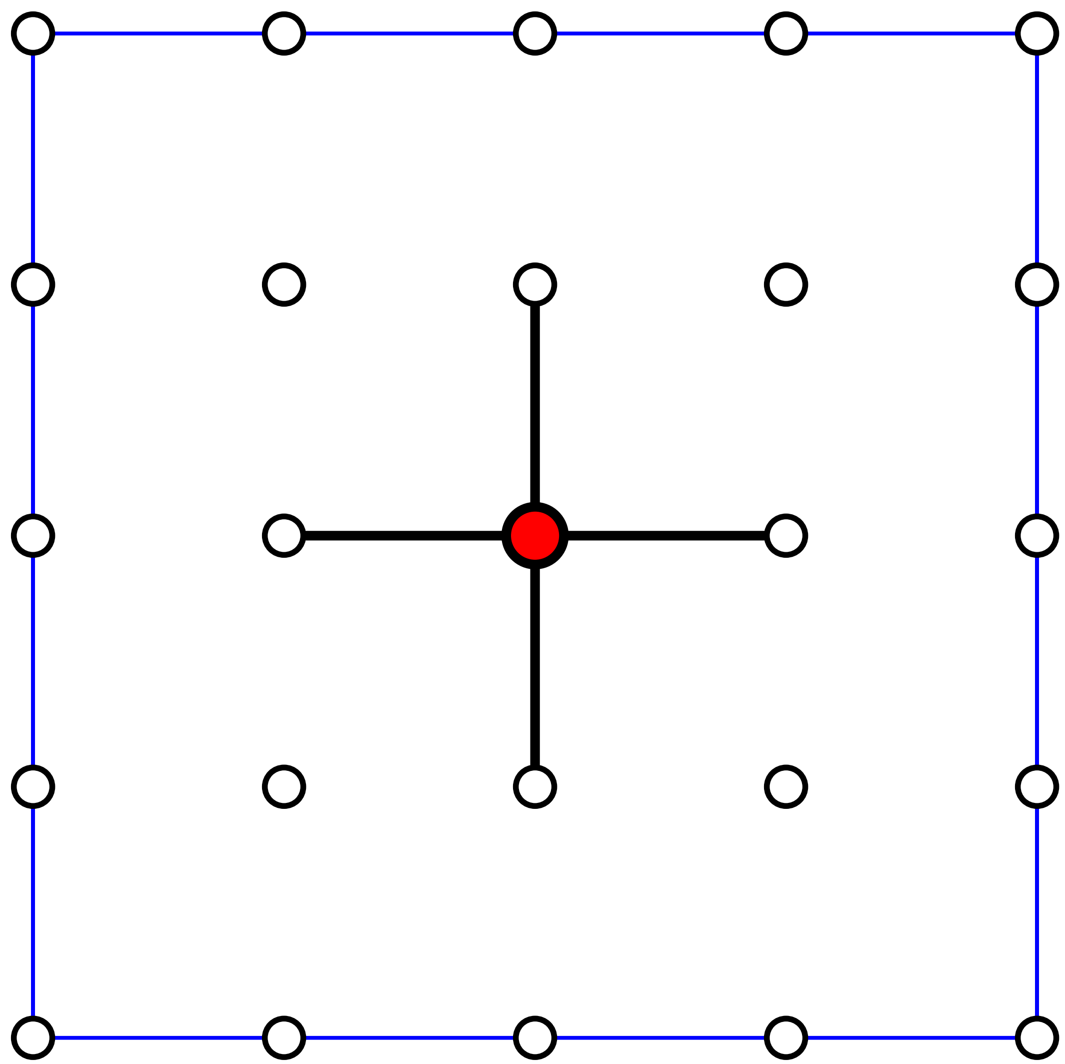


only stable if

$$-1 \leq 1 - \frac{4}{\rho}\left(\sin^2\frac{\beta_p \Delta x}{2} + \frac{\beta_q \Delta y}{2}\right) \leq 1$$

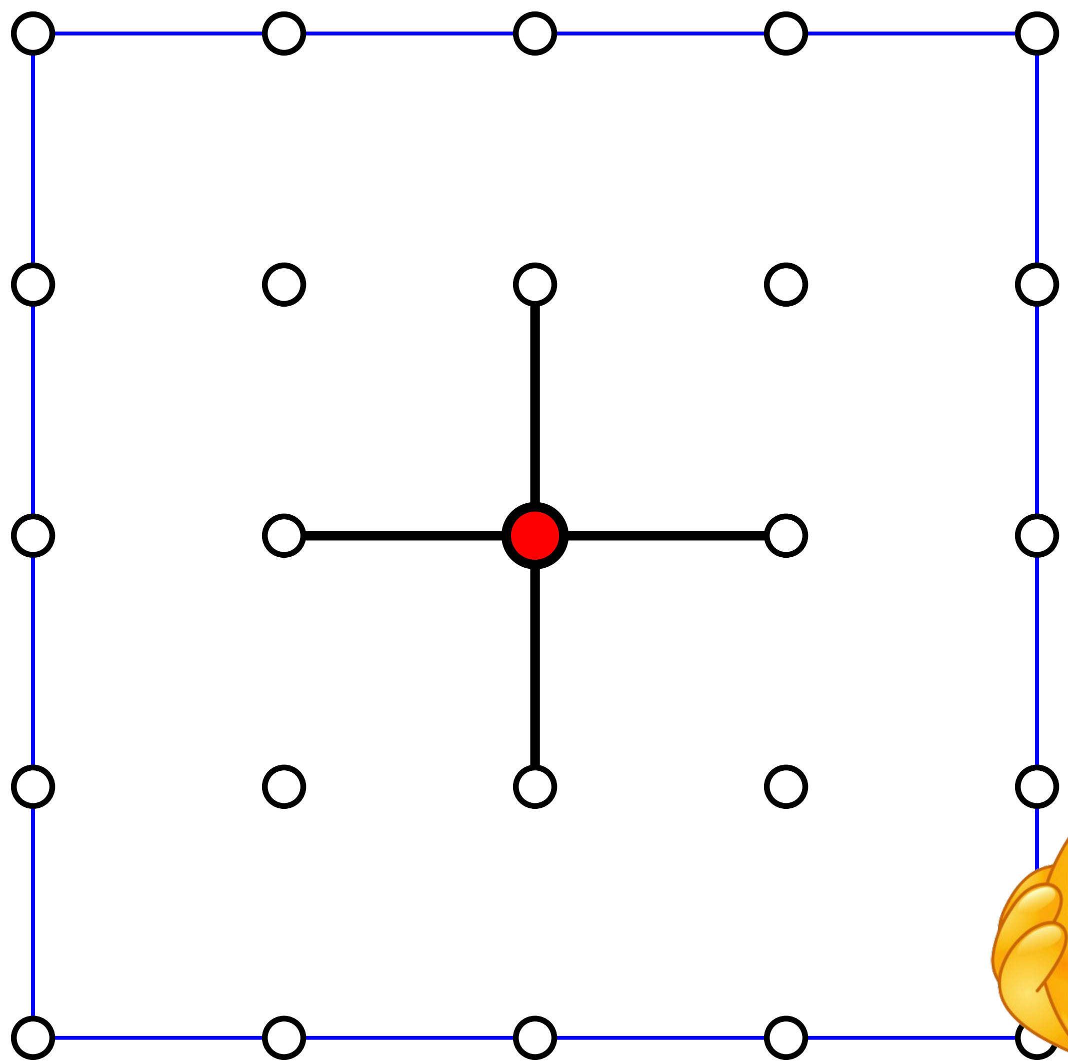
$$\implies \rho = \frac{(\Delta x)^2}{\Delta t} = \frac{(\Delta y)^2}{\Delta t} \geq 4$$

$$\implies \Delta t \leq \frac{(\Delta x)^2}{4} = \frac{1}{4N^2}$$

strong time step restriction

Explicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j,n}}{\Delta t} = \frac{T_{i-1,j}^{n} - 2T_{i,j}^{n} + T_{i+1,j}^{n}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n} - 2T_{i,j}^{n} + T_{i,j+1}^{n}}{(\Delta y)^2}$$



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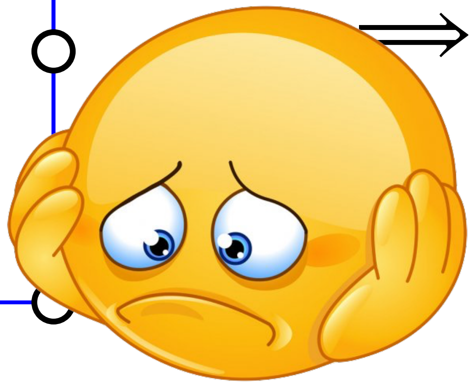
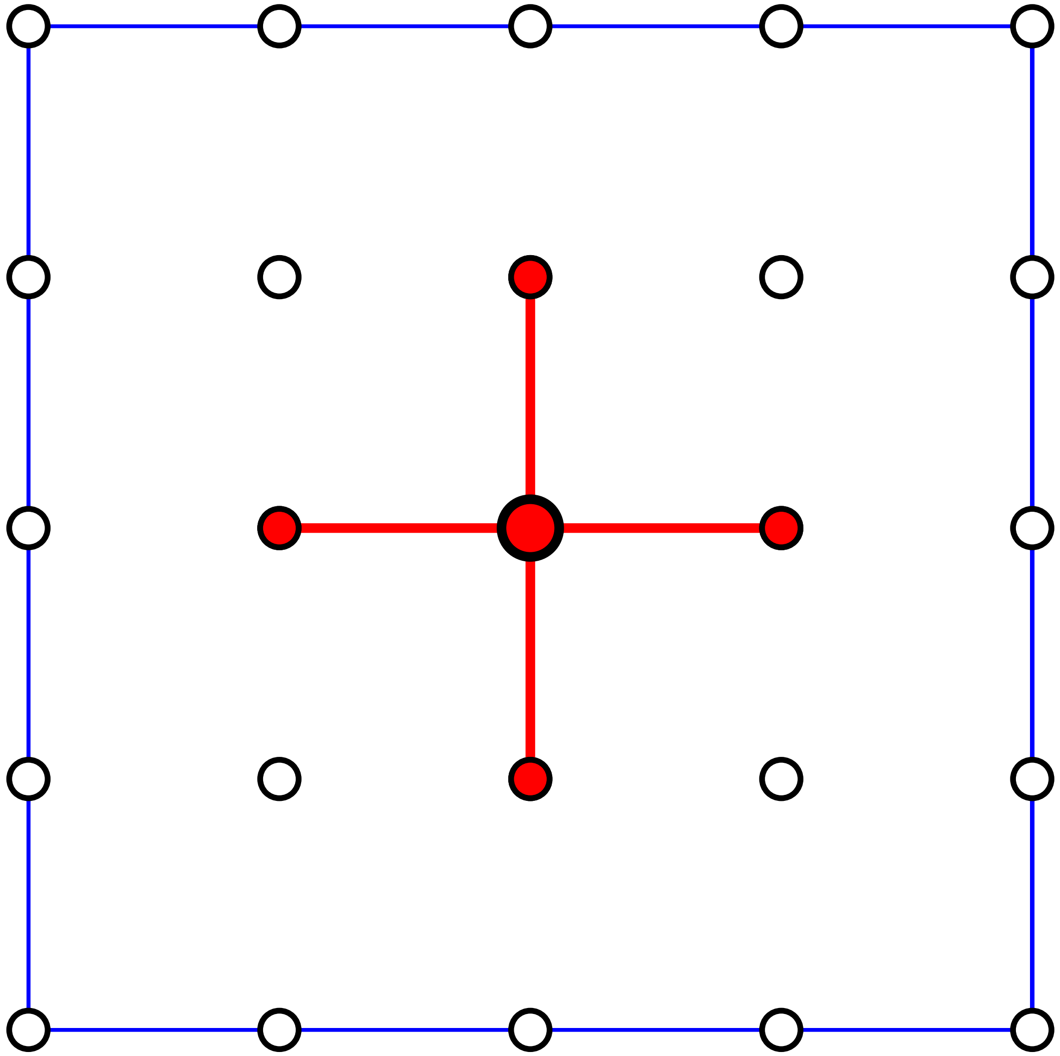


strong time step restriction

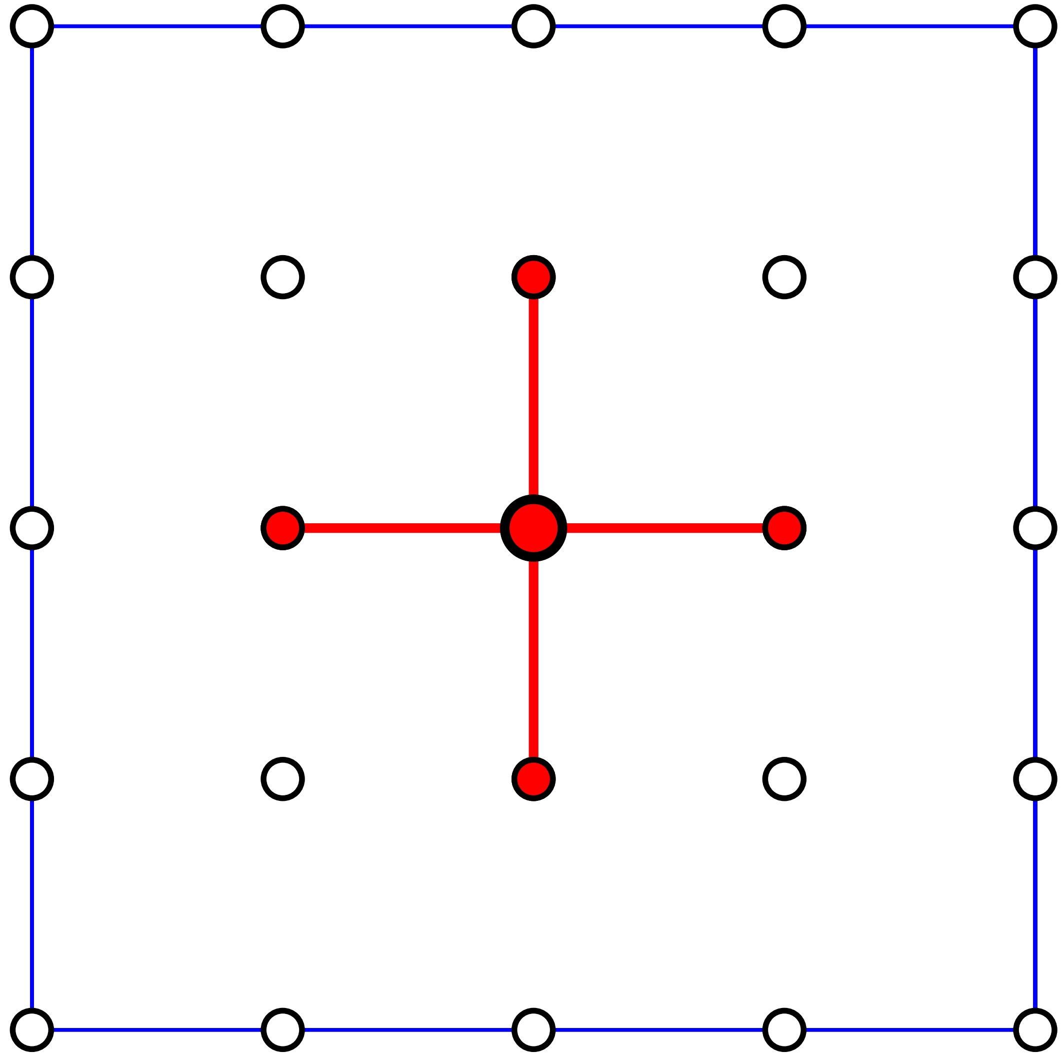
Implicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{(\Delta y)^2}$$



Implicit Method

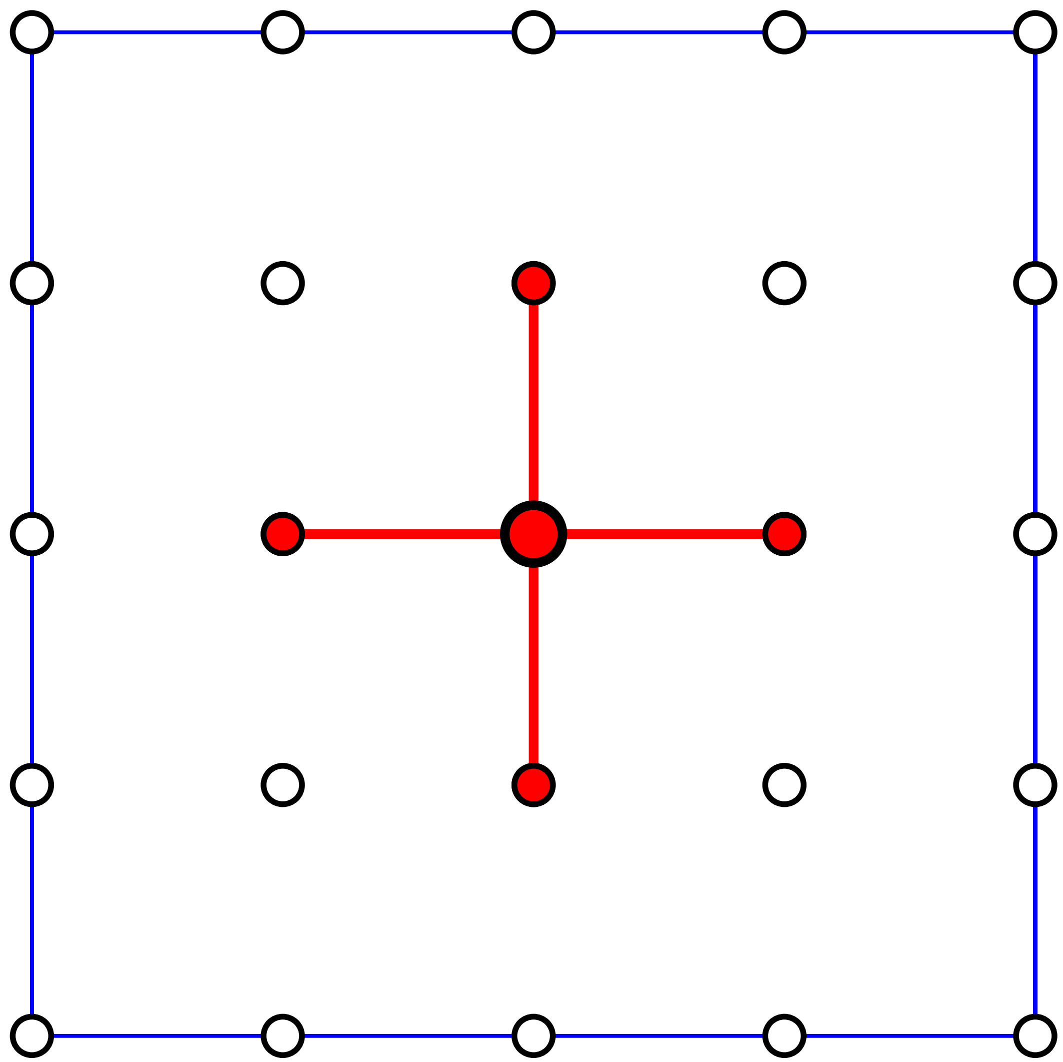
$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{(\Delta y)^2}$$



stable for all time steps

Implicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{(\Delta y)^2}$$



stable for all time steps

system of linear equations

$$Ax = b \qquad A \in \mathbb{R}^{N^2 \times N^2} \qquad x, b \in \mathbb{R}^{N^2}$$

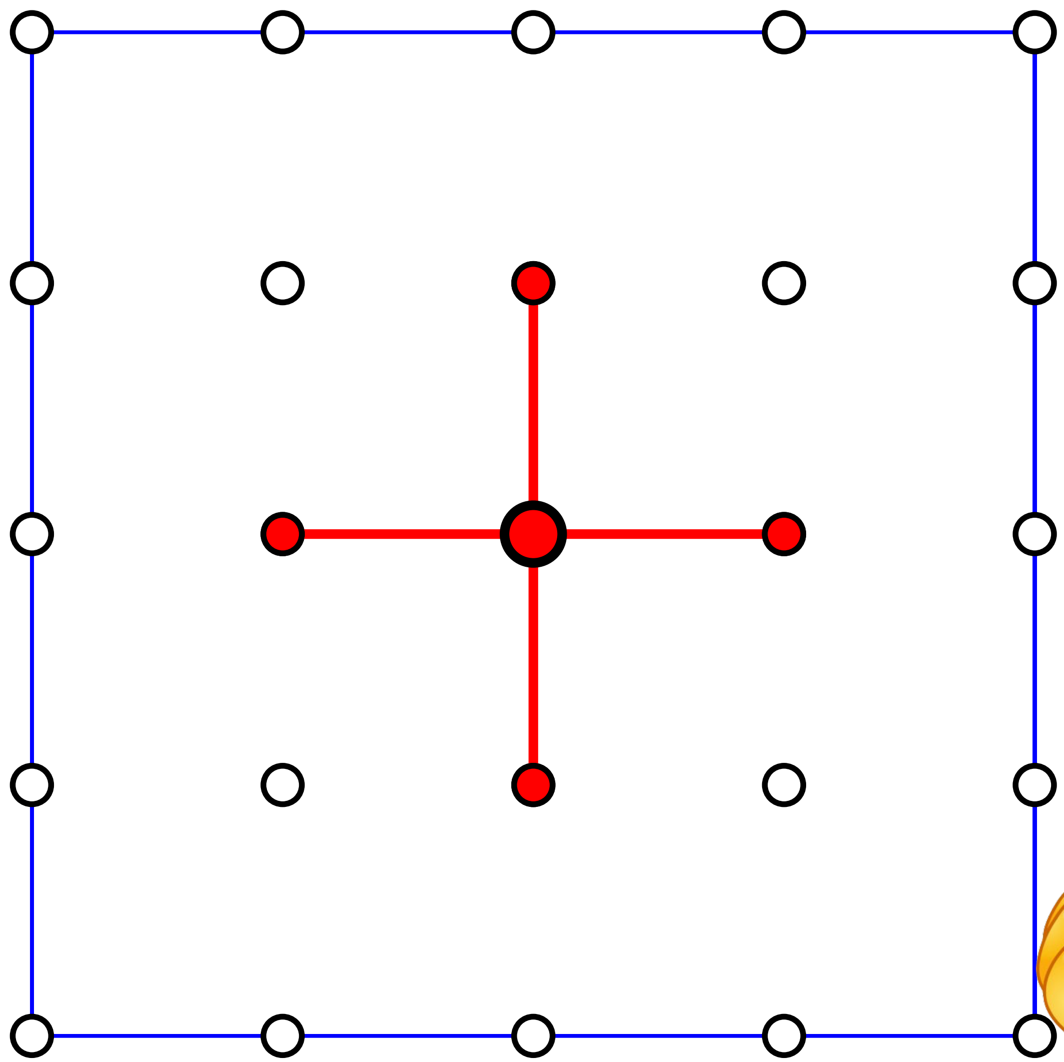
$$T_{i-1,j}^{n+1} + T_{i+1,j}^{n+1} + T_{i,j-1}^{n+1} +$$

$$+ T_{i,j+1}^{n+1} - (4+\rho) T_{i,j}^{n+1} = -\rho T_{i-1,j}^{n}$$

hard to solve

Implicit Method

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{(\Delta y)^2}$$



stable for all time steps

system of linear equations

$$Ax = b \qquad A \in \mathbb{R}^{N^2 \times N^2} \qquad x, b \in \mathbb{R}^{N^2}$$

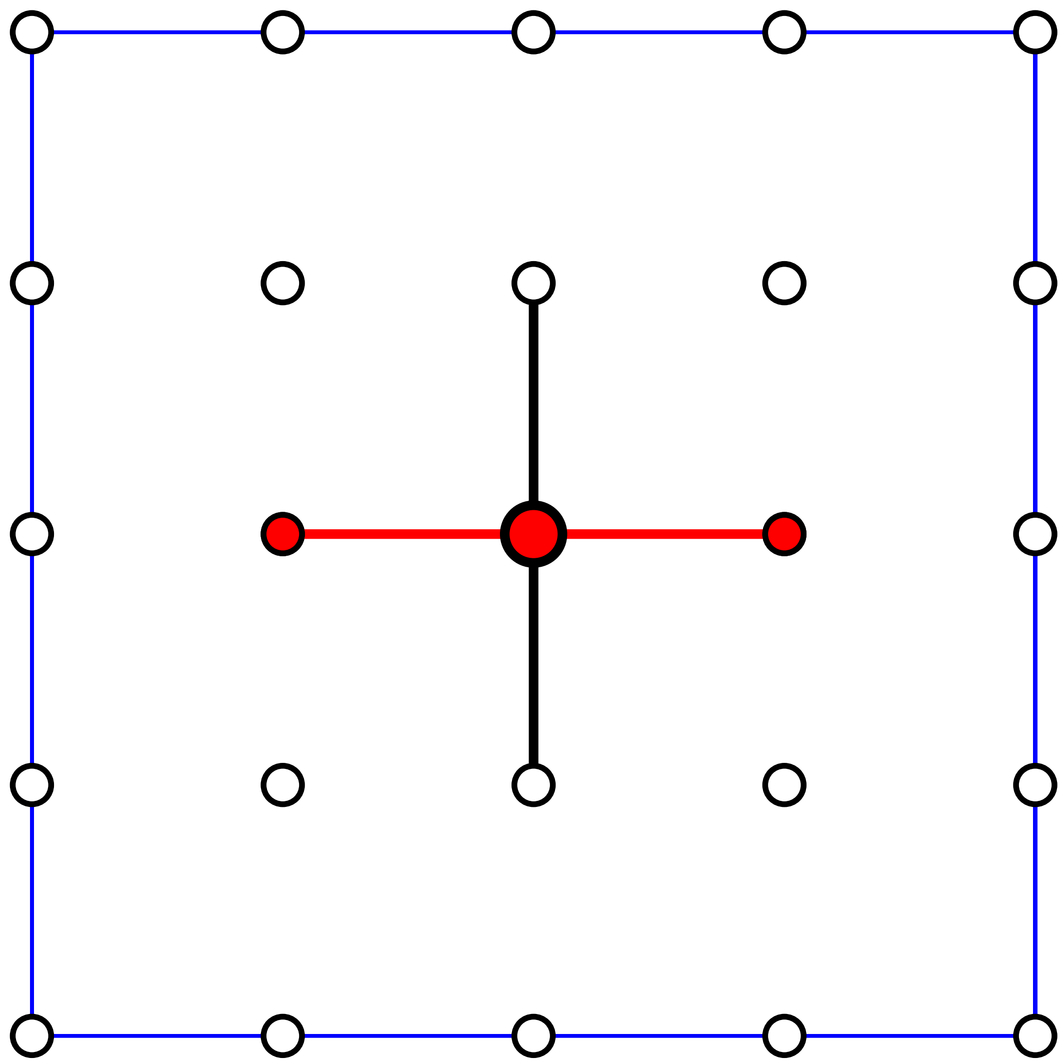
$$T_{i-1,j}^{n+1} + T_{i+1,j}^{n+1} + T_{i,j-1}^{n+1} +$$

$$+ T_{i,j+1}^{n+1} - (4+\rho) T_{i,j}^{n+1} = -\rho T_{i-1,j}^{n}$$

hard to solve

Alternate Direction Implicit Method I

$$\frac{T_{i,j}^{2n+1} - T_{i,j}^{2n}}{\Delta t} = \frac{T_{i-1,j}^{2n+1} - 2T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{2n} - 2T_{i,j}^{2n} + T_{i,j+1}^{2n}}{(\Delta y)^2}$$

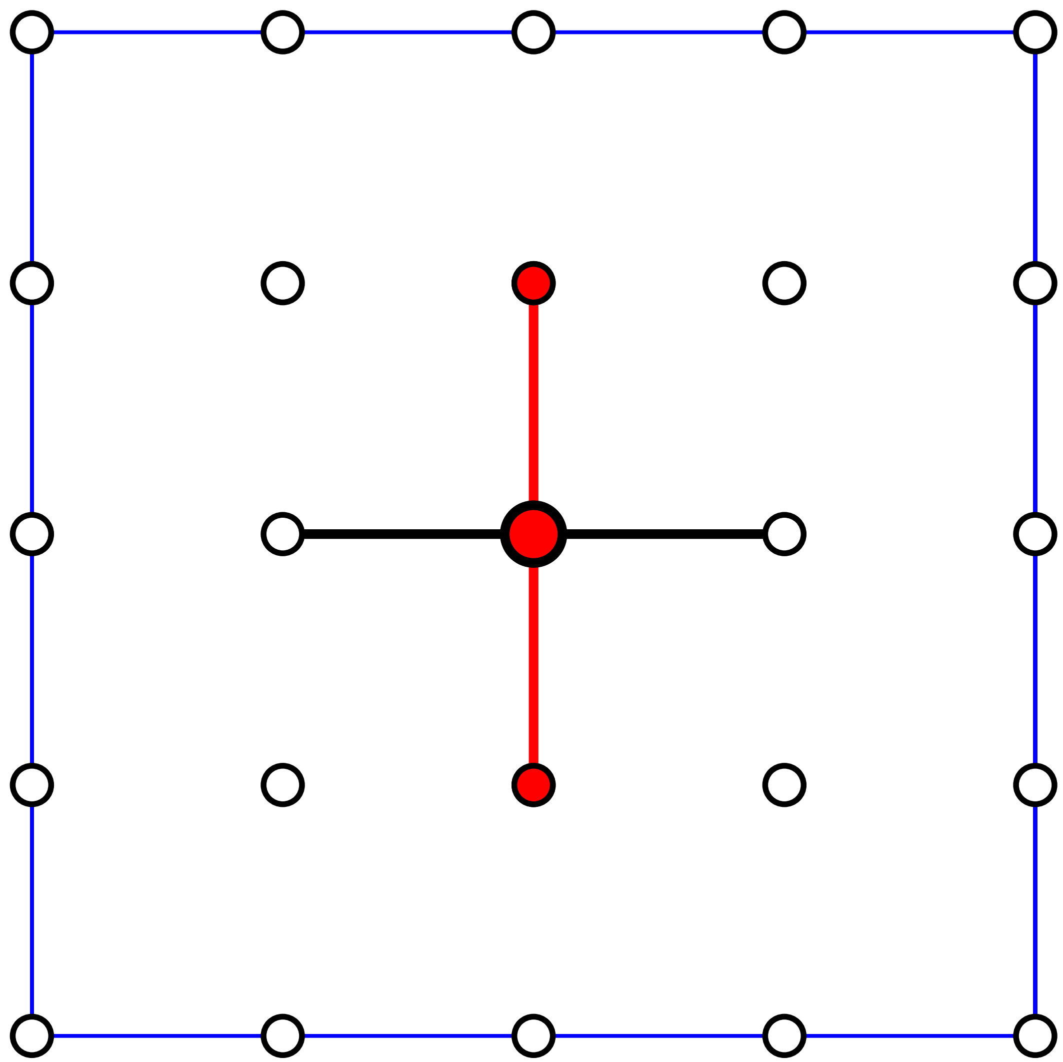


implicit in x - direction

explicit in y - direction

Alternate Direction Implicit Method II

$$\frac{T_{i,j}^{2n+2} - T_{i,j}^{2n+1}}{\Delta t} = \frac{T_{i-1,j}^{2n+1} - 2T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1}^{2n+2} - 2T_{i,j}^{2n+2} + T_{i,j+1}^{2n+2}}{(\Delta y)^2}$$



implicit in y - direction

explicit in x - direction

Alternate Direction Implicit Method

N sets of N simultaneous equations

$$T_{i-1,j}^{2n+1} - (2+\rho)T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1} = -\,T_{i,j-1}^{2n} + (2-\rho)T_{i,j}^{2n} - T_{i,j+1}^{2n}$$

Alternate Direction Implicit Method

N sets of N simultaneous equations

$$T_{i-1,j}^{2n+1} - (2+\rho)T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1} = -\,T_{i,j-1}^{2n} + (2-\rho)T_{i,j}^{2n} - T_{i,j+1}^{2n}$$

with tridiagonal matrix

$$\begin{bmatrix} \ddots & \ddots & 0 & 0 & 0 \\ \ddots & -(2+\rho) & 1 & 0 & 0 \\ 0 & 1 & -(2+\rho) & 1 & 0 \\ 0 & 0 & 1 & -(2+\rho) & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ T_{i-1,j}^{2n+1} \\ T_{i,j}^{2n+1} \\ T_{i+1,j}^{2n+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ -T_{i-1,j-1}^{2n} + (2-\rho)T_{i-1,j}^{2n} - T_{i-1,j+1}^{2n} \\ -T_{i,j-1}^{2n} + (2-\rho)T_{i,j}^{2n} - T_{i,j+1}^{2n} \\ -T_{i+1,j-1}^{2n} + (2-\rho)T_{i+1,j}^{2n} - T_{i+1,j+1}^{2n} \\ \vdots \end{bmatrix}$$

Alternate Direction Implicit Method

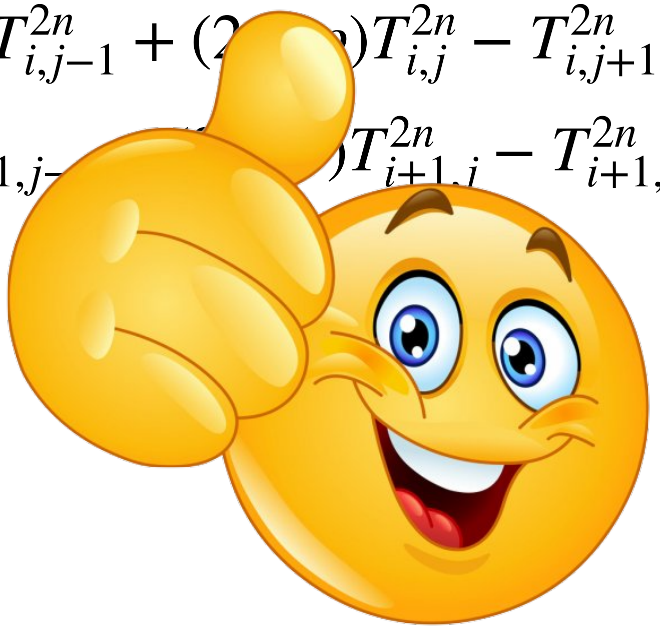
N sets of N simultaneous equations

$$T_{i-1,j}^{2n+1} - (2+\rho)T_{i,j}^{2n+1} + T_{i+1,j}^{2n+1} = -T_{i,j-1}^{2n} + (2-\rho)T_{i,j}^{2n} - T_{i,j+1}^{2n}$$

with tridiagonal matrix

$$\begin{bmatrix} \ddots & \ddots & 0 & 0 & 0 \\ \ddots & -(2+\rho) & 1 & 0 & 0 \\ 0 & 1 & -(2+\rho) & 1 & 0 \\ 0 & 0 & 1 & -(2+\rho) & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ T_{i-1,j}^{2n+1} \\ T_{i,j}^{2n+1} \\ T_{i+1,j}^{2n+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ -T_{i-1,j-1}^{2n} + (2-\rho)T_{i-1,j}^{2n} - T_{i-1,j+1}^{2n} \\ -T_{i,j-1}^{2n} + (2-\rho)T_{i,j}^{2n} - T_{i,j+1}^{2n} \\ -T_{i+1,j-1}^{2n} + (2-\rho)T_{i+1,j}^{2n} - T_{i+1,j+1}^{2n} \\ \vdots \end{bmatrix}$$

use Thomas algorithm



Alternate Direction Implicit Method

Van Neuman stability analysis using the Fourier Ansatz

$$T_{j,k}^{n} = \sum_{p,q=0}^{N-1} \hat{T}_{p,q}^{n} e^{ipx_j + iqy_k}$$

single steps highly unstable

$$\hat{T}_{p,q}^{2n+1} = \hat{T}_{p,q}^{2n} \underbrace{\frac{\rho - 4\sin^2\left(\frac{ph}{2}\right)}{\rho + 4\sin^2\left(\frac{qh}{2}\right)}}_{=:G_{p,q}^{(1)}} \qquad \hat{T}_{p,q}^{2n+2} = \hat{T}_{p,q}^{2n+1} \underbrace{\frac{\rho - 4\sin^2\left(\frac{qh}{2}\right)}{\rho + 4\sin^2\left(\frac{ph}{2}\right)}}_{=:G_{p,q}^{(2)}}$$

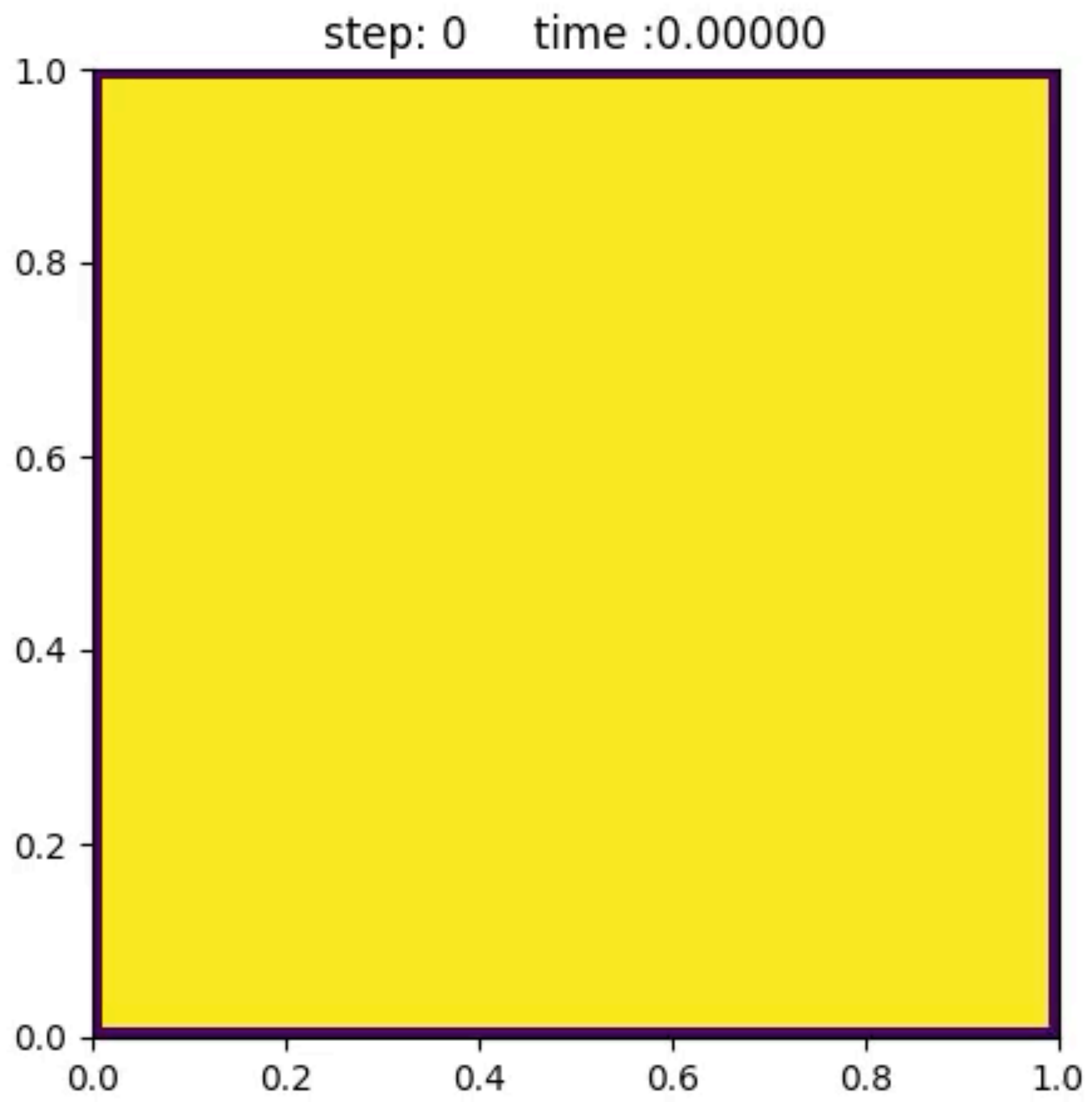
Alternate Direction Implicit Method

combining the steps

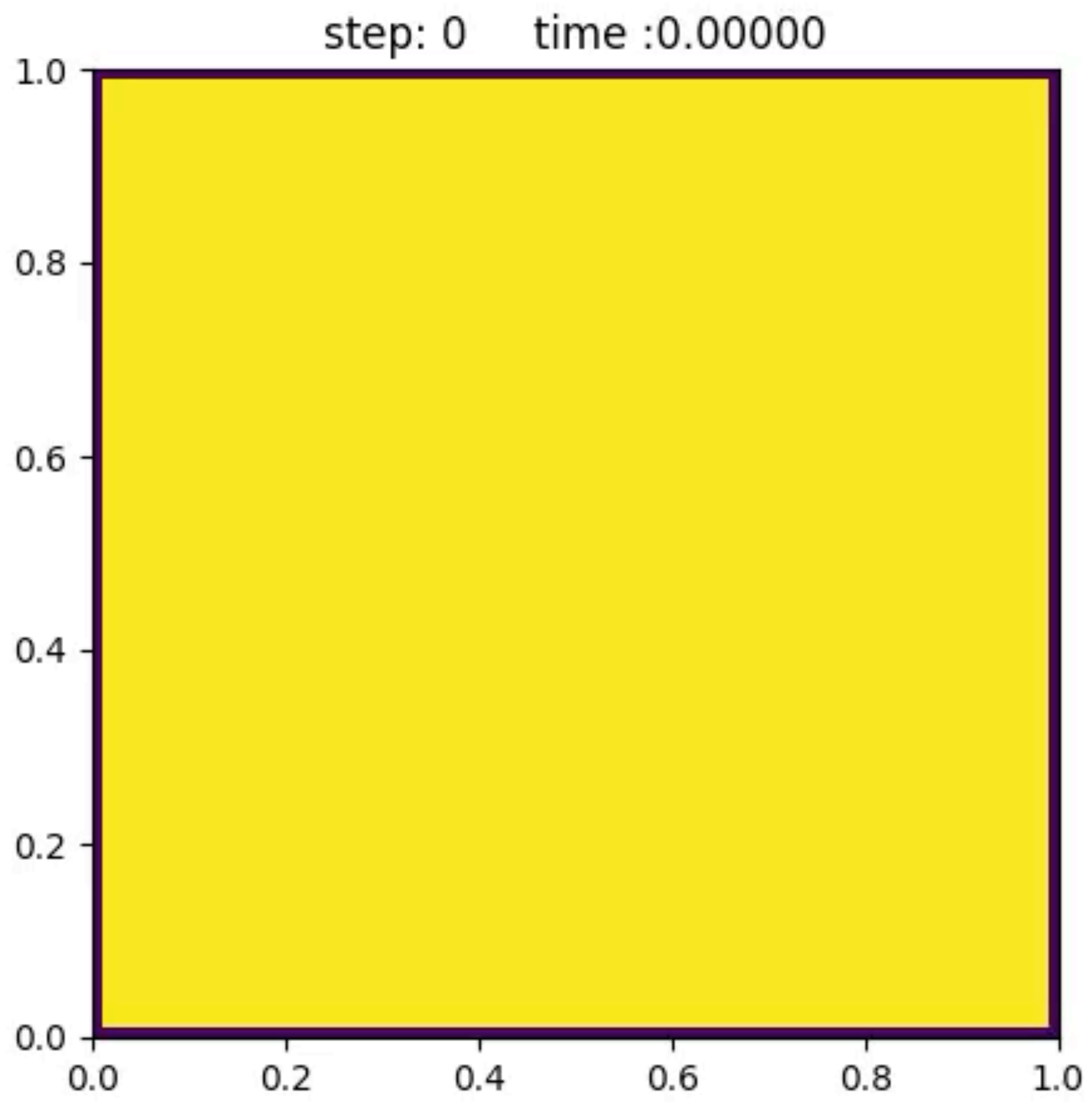
$$\hat{T}_{p,q}^{2n+2} = \hat{T}_{p,q}^{2n} \underbrace{\left(\frac{\rho - 4\sin^2\left(\frac{qh}{2}\right)}{\rho + 4\sin^2\left(\frac{ph}{2}\right)} \right) \left(\frac{\rho - 4\sin^2\left(\frac{ph}{2}\right)}{\rho + 4\sin^2\left(\frac{qh}{2}\right)} \right)}_{=:G_{p,q}}$$

Stable for all time steps

Alternate Direction Implicit Method



Alternate Direction Implicit Method



Analytic Solution

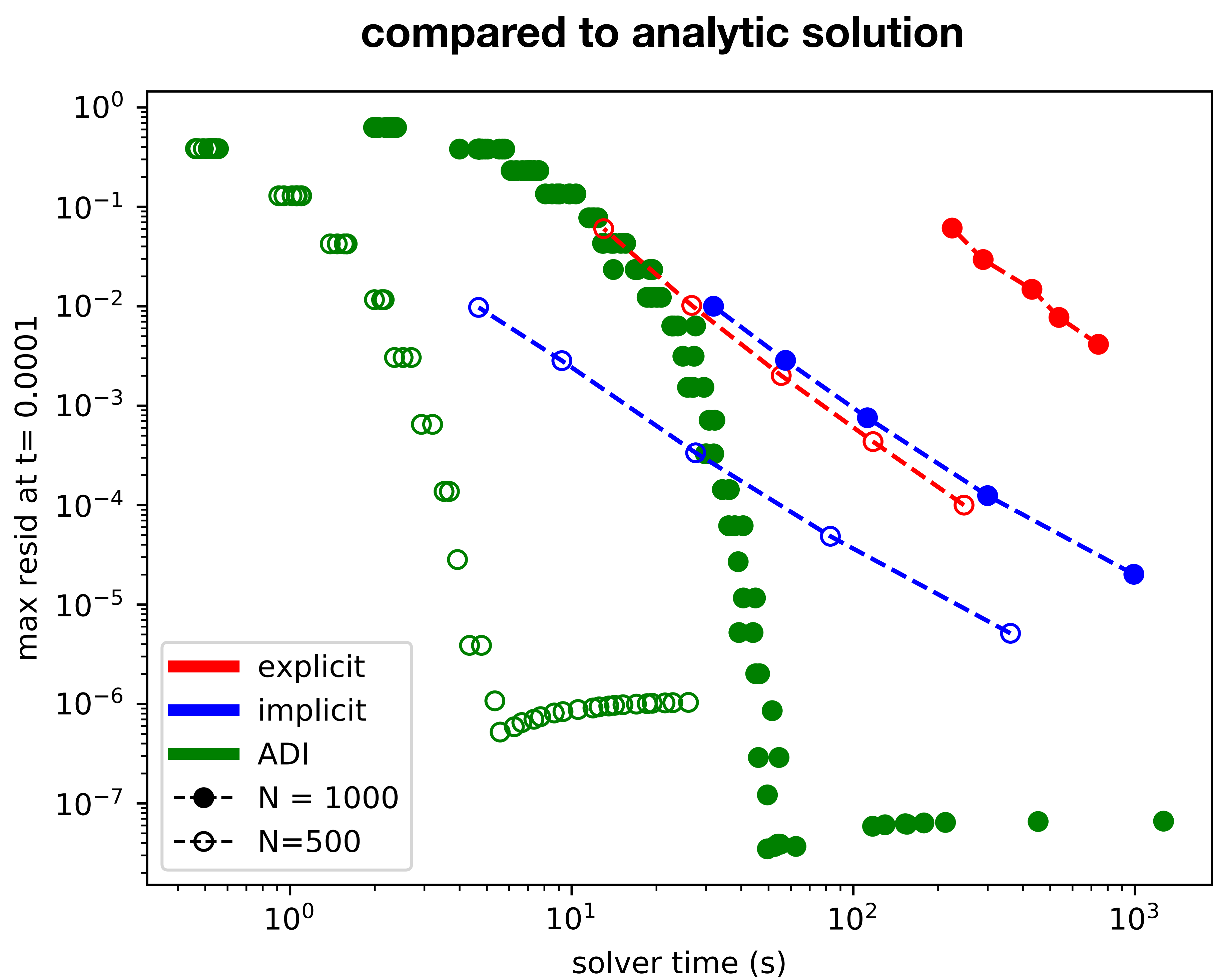
$$\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial^2 T(x,y,t)}{(\partial x)^2} + \frac{\partial^2 T(x,y,t)}{(\partial y)^2}$$

boundary condition: $T(0,y,t) = T(1,y,t) = T(x,0,t) = T(x,1,t) = 0$

Initial condition: $T(x,y,0) = 1$

$$\Longrightarrow T(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16}{(2m-1)(2n-1)\pi^2} \sin(n\pi x)\sin(m\pi y)\exp(-(m^2+n^2)\pi^2 t)$$

Comparison



compared to analytic solution
max resid at t= 0.0001
solver time (s)
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^0
10^1
10^2
10^3
explicit
implicit
ADI
N = 1000
N=500