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Alternate Direction Implicit Method

Solving the Heat Flow Equation in 2D by Applying Implicit Equations in One Spatial Direction at a Time

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Heat Flow Equation in 2D

$$\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial^2 T(x,y,t)}{(\partial x)^2} + \frac{\partial^2 T(x,y,t)}{(\partial y)^2}$$

Explicit Method

strong time step restriction

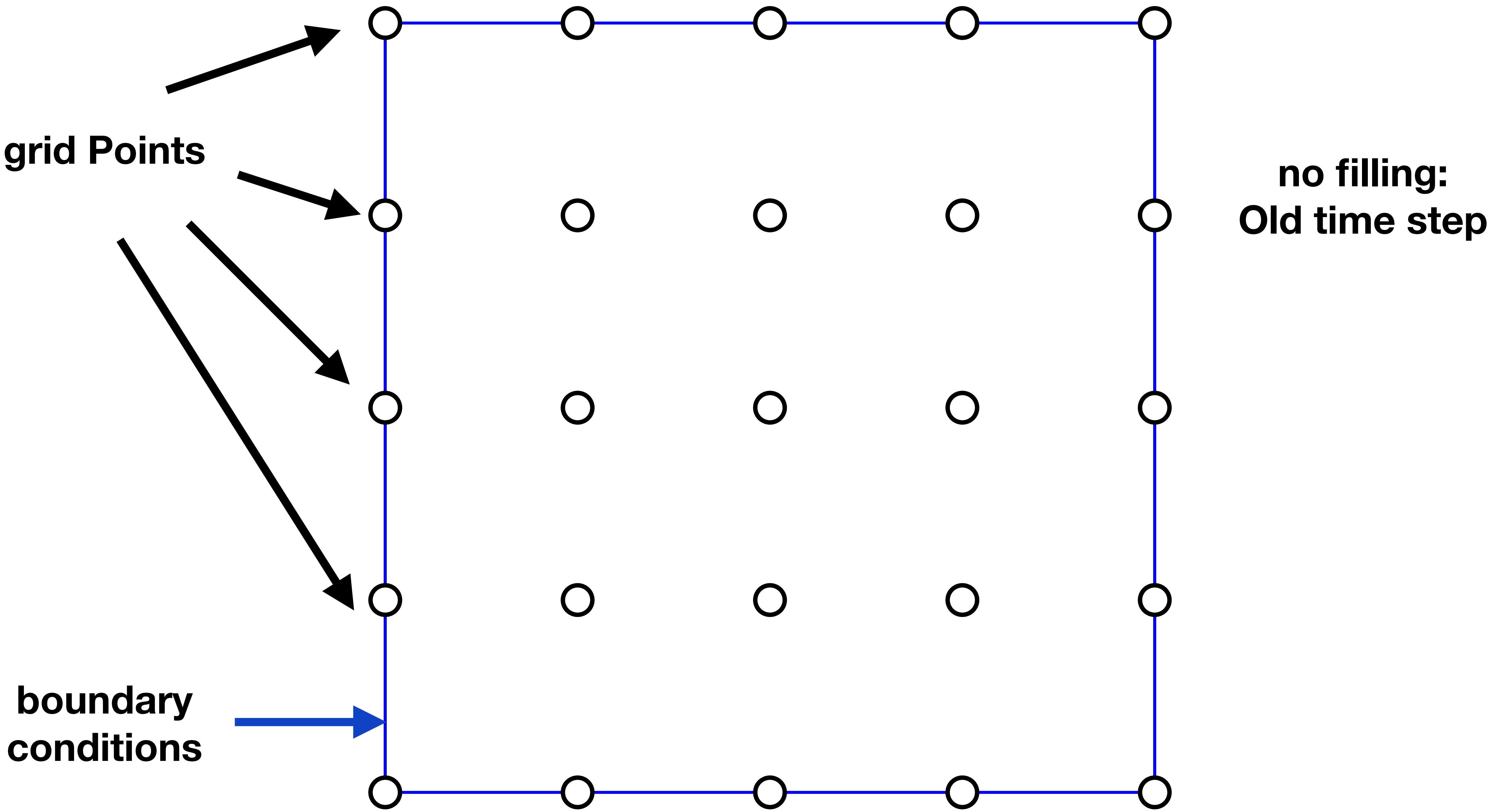
Implicit Method

no time step restriction

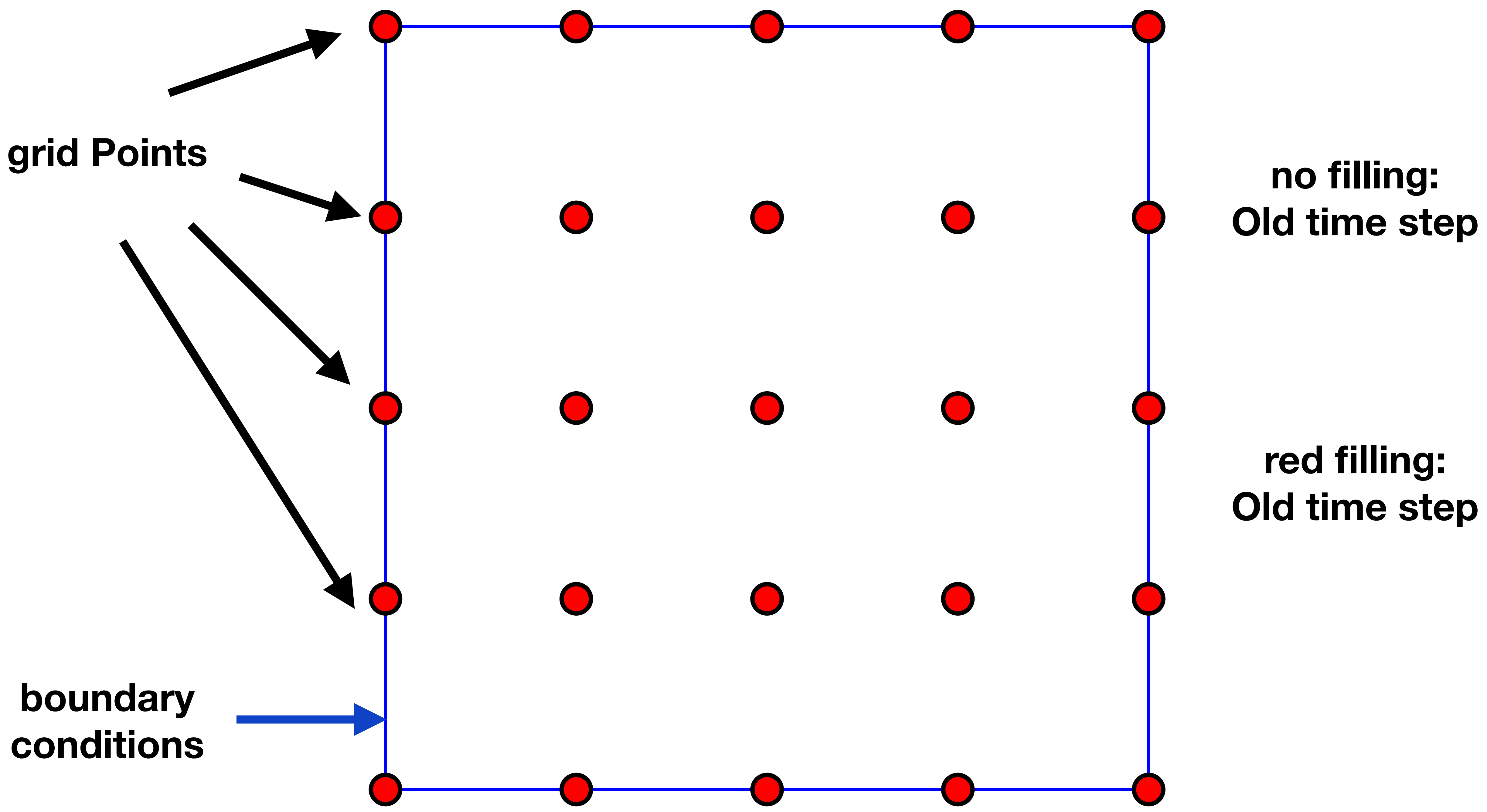
need to solve big SLE

Alternate Direction Implicit

Heat Flow Equation in 2D

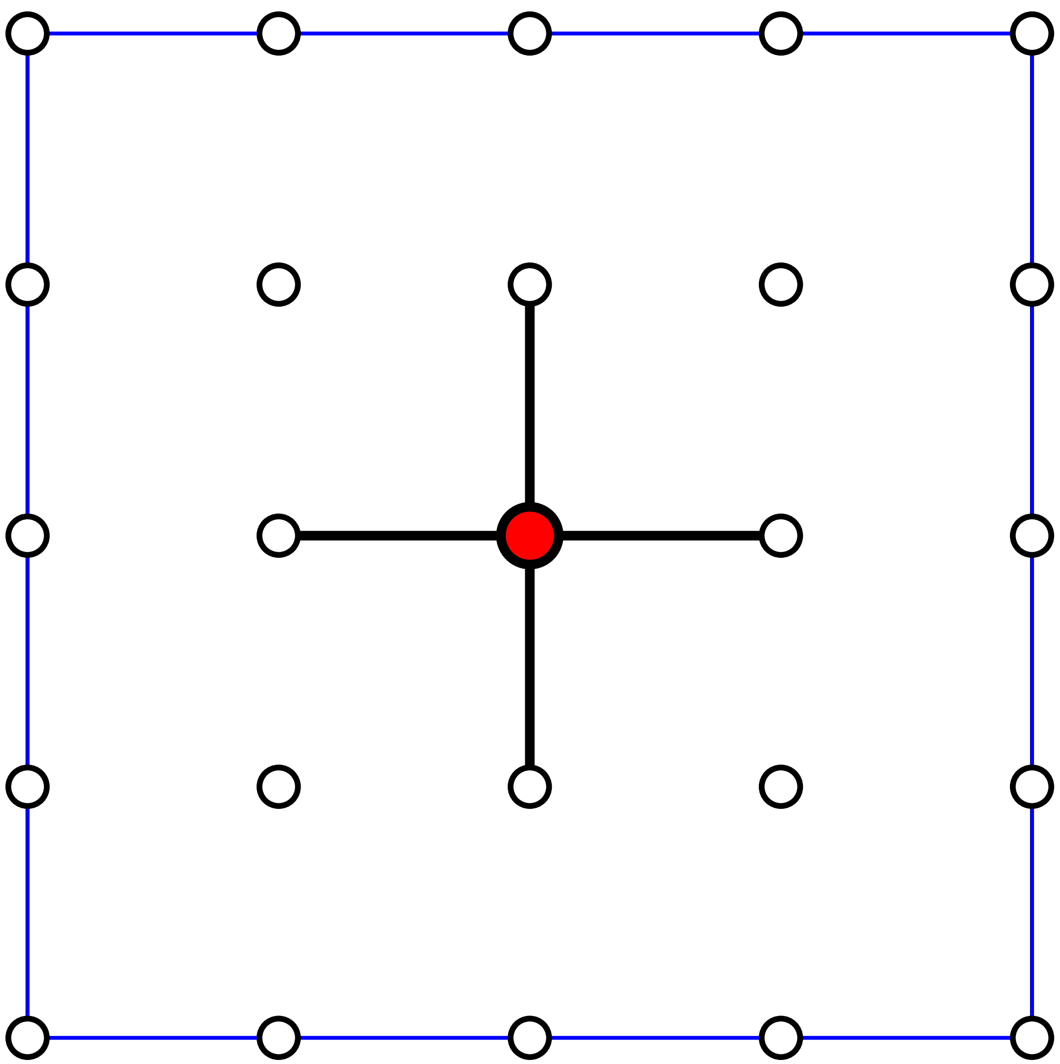


Heat Flow Equation in 2D



Explicit Method

$$\frac{T_{i,j,n+1} - T_{i,j,n}}{\Delta t} = \frac{T_{i-1,j,n} - 2T_{i,j,n} + T_{i+1,j,n}}{(\Delta x)^2} + \frac{T_{i,j-1,n} - 2T_{i,j,n} + T_{i,j+1,n}}{(\Delta y)^2}$$



only stable if

$$-1 \leq 1 - \frac{4}{\rho}\left(\sin^2 \frac{\beta_p \Delta x}{2} + \frac{\beta_q \Delta y}{2}\right) \leq 1$$

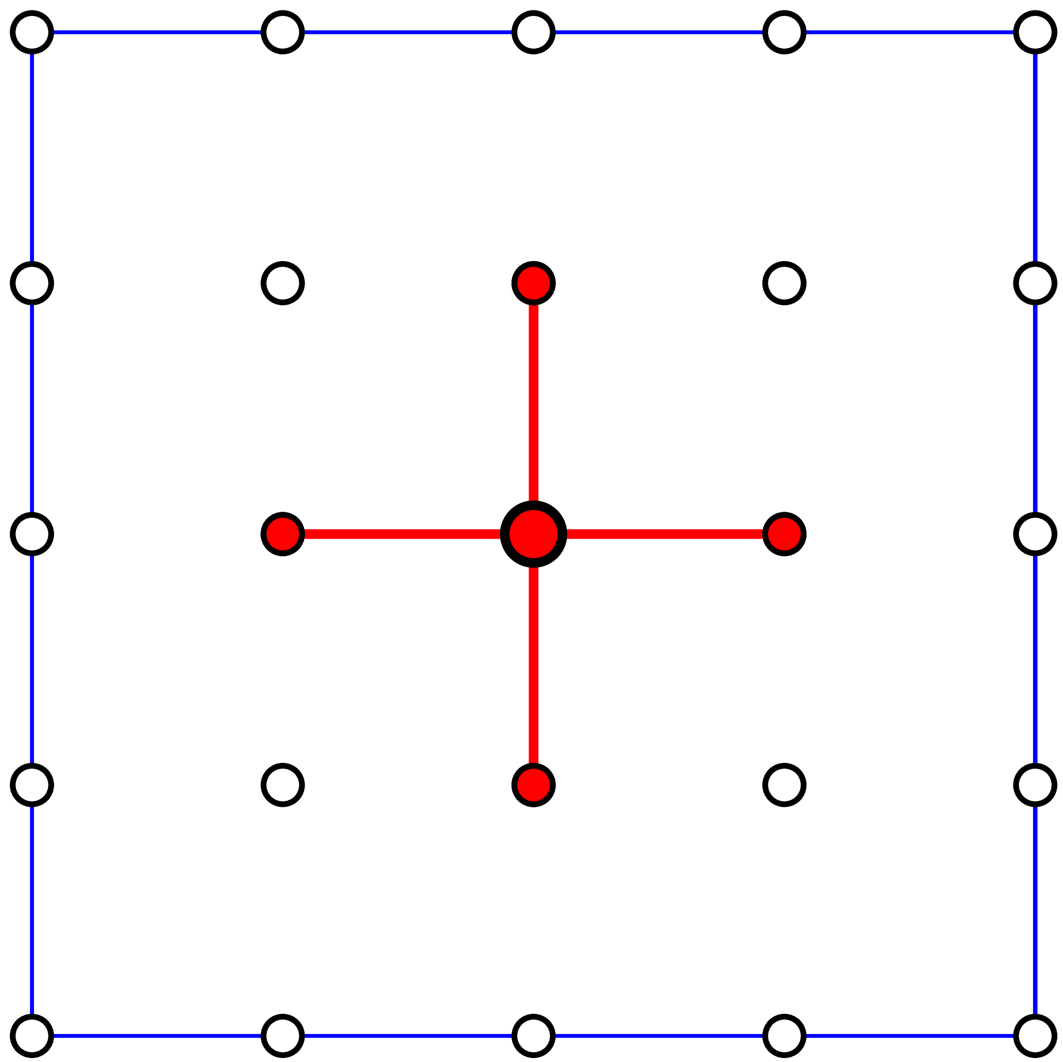
$$\Longrightarrow \rho = \frac{(\Delta x)^2}{\Delta t} = \frac{(\Delta y)^2}{\Delta t} \geq 4$$

$$\Longrightarrow \Delta t \leq \frac{(\Delta x)^2}{4} = \frac{1}{4N^2}$$

strong time step restriction

Implicit Method

$$\frac{T_{i,j,n+1} - T_{i,j,n}}{\Delta t} = \frac{T_{i-1,j,n+1} - 2T_{i,j,n+1} + T_{i+1,j,n+1}}{(\Delta x)^2} + \frac{T_{i,j-1,n+1} - 2T_{i,j,n+1} + T_{i,j+1,n+1}}{(\Delta y)^2}$$



stable for all time steps

$$T_{i-1,j,n+1} + T_{i+1,j,n+1} + T_{i,j-1,n+1} +$$
$$+ T_{i,j+1,n+1} - (4+\rho)T_{i,j,n+1} = -\rho T_{i-1,j,n}$$

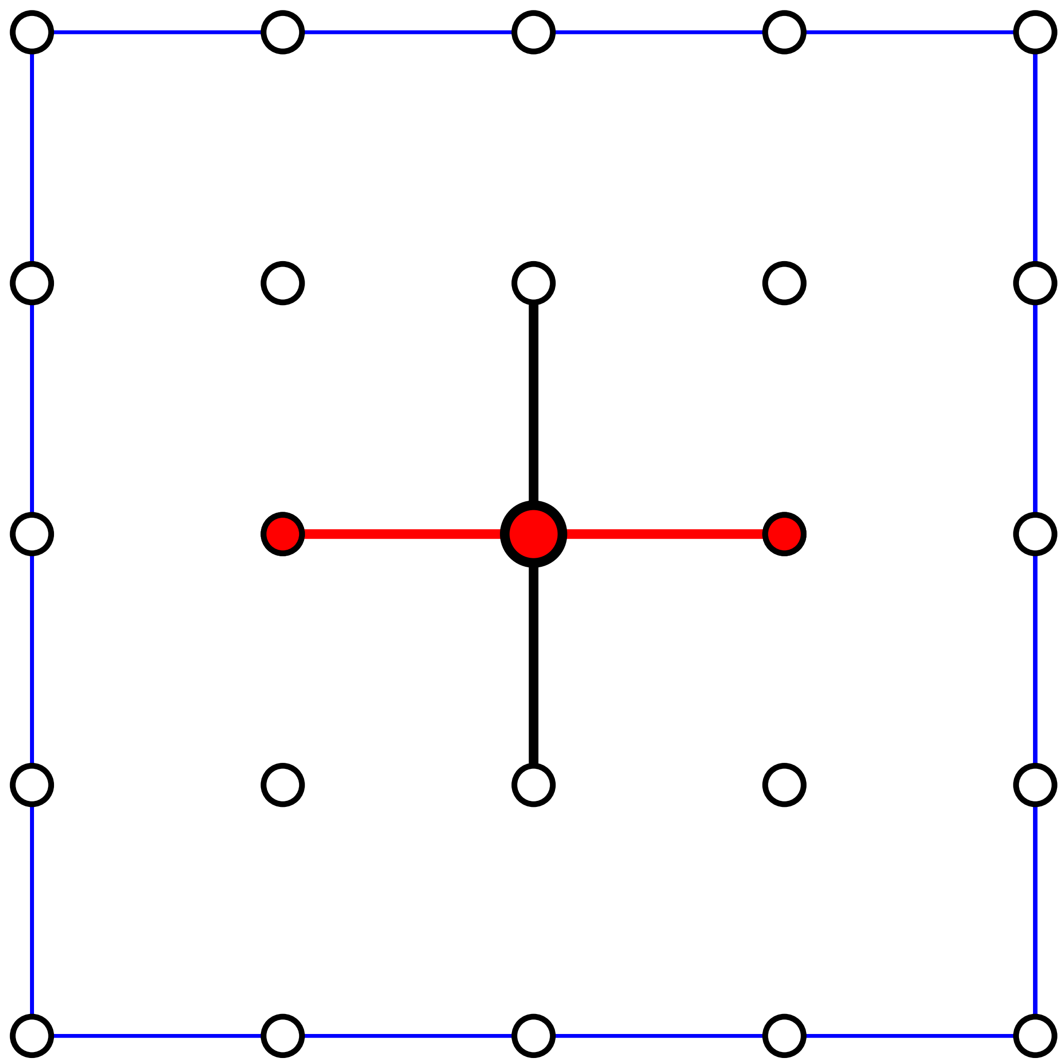
system of linear equations

$$Ax = b \qquad A \in \mathbb{R}^{N^2 \times N^2} \qquad x, b \in \mathbb{R}^{N^2}$$

hard to solve

Alternate Direction Implicit Method I

$$\frac{T_{i,j,2n+1} - T_{i,j,2n}}{\Delta t} = \frac{T_{i-1,j,2n+1} - 2T_{i,j,2n+1} + T_{i+1,j,2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1,2n} - 2T_{i,j,2n} + T_{i,j+1,2n}}{(\Delta y)^2}$$

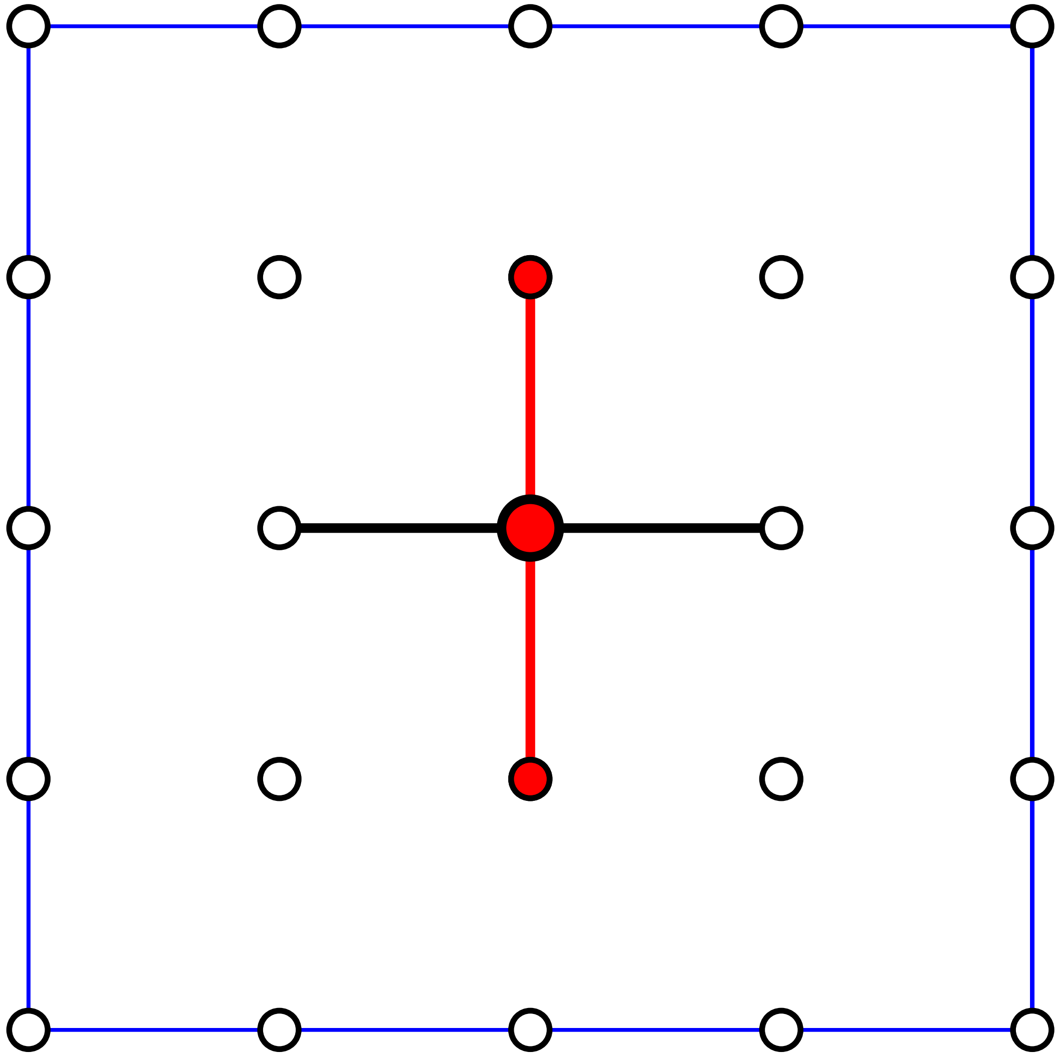


implicit in x - direction

explicit in y - direction

Alternate Direction Implicit Method II

$$\frac{T_{i,j,2n+2} - T_{i,j,2n+1}}{\Delta t} = \frac{T_{i-1,j,2n+1} - 2T_{i,j,2n+1} + T_{i+1,j,2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1,2n+2} - 2T_{i,j,2n+2} + T_{i,j+1,2n+2}}{(\Delta y)^2}$$



implicit in y - direction

explicit in x - direction

Alternate Direction Implicit Method

N sets of N simultaneous equations

$$T_{i-1,j,2n+1} - (2+\rho)T_{i,j,2n+1} + T_{i+1,j,2n+1} = -T_{i,j-1,2n} + (2-\rho)T_{i,j,2n} - T_{i,j+1,2n}$$