

Title

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1 Van Neumann Stability Analysis: Explicit Scheme

The explicit scheme for the two dimensional heat equation is

$$\frac{T_{j,k}^{n+1} - T_{j,k}^n}{\Delta t} = \frac{T_{j-1,k}^n - 2T_{j,k}^n + T_{j+1,k}^n}{(\Delta x)^2} + \frac{T_{j,k-1}^n - 2T_{j,k}^n + T_{j,k+1}^n}{(\Delta y)^2} \Leftrightarrow \quad (1)$$

$$\Leftrightarrow T_{j,k}^{n+1} = T_{j,k}^n + \frac{1}{\rho} (T_{j-1,k}^n + T_{j+1,k}^n + T_{j,k-1}^n + T_{j,k+1}^n - 4T_{j,k}^n) \quad (2)$$

assuming $\Delta x = \Delta y =: h$ and setting $\rho = \frac{h^2}{\Delta t}$. We begin our Van Neumann stability analysis by inserting the fourier series

$$T_{j,k}^n = \sum_{p,q=0}^{N-1} \hat{T}_{p,q}^n e^{ipx_j + iqy_k} \quad (3)$$

into the numerical scheme. We obtain

$$\begin{aligned} \sum_{p,q=0}^{N-1} \hat{T}_{p,q}^{n+1} e^{i(px_j + qy_k)} &= \sum_{p,q=0}^{N-1} (\hat{T}_{p,q}^n e^{i(px_j + qy_k)} + \frac{1}{\rho} (\hat{T}_{p,q}^n e^{i(px_{j-1} + qy_k)} + \hat{T}_{p,q}^n e^{i(px_{j+1} + qy_k)} + \\ &\quad + \hat{T}_{p,q}^n e^{i(px_j + qy_{k-1})} + \hat{T}_{p,q}^n e^{i(px_j + qy_{k+1})} - 4\hat{T}_{p,q}^n e^{i(px_j + qy_k)})). \end{aligned} \quad (4)$$

Using that $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$ and that the discretization in x and y are the same we get

$$\sum_{p,q=0}^{N-1} (\hat{T}_{p,q}^{n+1} - \hat{T}_{p,q}^n - \frac{1}{\rho} (\hat{T}_{p,q}^n e^{-iph} + \hat{T}_{p,q}^n e^{iph} + \hat{T}_{p,q}^n e^{-iqh} + \hat{T}_{p,q}^n e^{iqh} - 4\hat{T}_{p,q}^n)) e^{i(px_j + qy_k)} = 0 \quad (5)$$

and because $(e^{i(px_j+qy_k)})_{j,k \in \{0, \dots, N-1\}}$ is a basis of the trigonometrical polynomials of degree N in two variables we have for all $p, q \in \{0, \dots, N-1\}$ that

$$\begin{aligned}
& \hat{T}_{p,q}^{n+1} - \hat{T}_{p,q}^n - \frac{1}{\rho} \left(\hat{T}_{p,q}^n e^{-iph} + \hat{T}_{p,q}^n e^{iph} + \hat{T}_{p,q}^n e^{-iqh} + \hat{T}_{p,q}^n e^{iqh} - 4\hat{T}_{p,q}^n \right) = 0 \quad \Leftrightarrow \\
& \Leftrightarrow \hat{T}_{p,q}^{n+1} = \hat{T}_{p,q}^n + \frac{1}{\rho} \left(\hat{T}_{p,q}^n e^{-iph} + \hat{T}_{p,q}^n e^{iph} + \hat{T}_{p,q}^n e^{-iqh} + \hat{T}_{p,q}^n e^{iqh} - 4\hat{T}_{p,q}^n \right) \quad \Leftrightarrow \\
& \Leftrightarrow \hat{T}_{p,q}^{n+1} = \hat{T}_{p,q}^n \left(1 + \frac{1}{\rho} \left(e^{-iph} + e^{iph} + e^{-iqh} + e^{iqh} - 4 \right) \right) \quad \Leftrightarrow \\
& \Leftrightarrow \hat{T}_{p,q}^{n+1} = \hat{T}_{p,q}^n \left(1 + \frac{1}{\rho} (2 \cos(ph) + 2 \cos(qh) - 4) \right) \quad \Leftrightarrow \\
& \Leftrightarrow \hat{T}_{p,q}^{n+1} = \hat{T}_{p,q}^n \underbrace{\left(1 - \frac{4}{\rho} \left(\sin^2 \left(\frac{ph}{2} \right) + \sin^2 \left(\frac{qh}{2} \right) \right) \right)}_{=: G_{p,q}},
\end{aligned}$$

where in the last step the identity $\sin^2(\alpha/2) = \frac{1}{2}(1 - \cos(\alpha))$ was used. For van Neumann stability we need that $|G_{p,q}| < 1$. We therefor want to find conditions on Δt and h such that

$$-1 \leq 1 - \frac{4}{\rho} \left(\sin^2 \left(\frac{ph}{2} \right) + \sin^2 \left(\frac{qh}{2} \right) \right) \leq 1. \quad (6)$$

The right hand side inequality is trivial and for the left hand side inequality we need that $\rho \geq 4$. This gives us the CFL condition

$$\frac{h^2}{\Delta t} = \rho \geq 4 \Leftrightarrow \Delta t \leq \frac{h^2}{4} = \frac{1}{4N^2}. \quad (7)$$

We conclude that $4N^2 t$ time steps are needed to integrate up to a time t .

2 Van Neumann Stability Analysis: Implicit Scheme

The implicit scheme for the two dimensional heat equation is

$$\frac{T_{j,k}^{n+1} - T_{j,k}^n}{\Delta t} = \frac{T_{j-1,k}^{n+1} - 2T_{j,k}^{n+1} + T_{j+1,k}^{n+1}}{(\Delta x)^2} + \frac{T_{j,k-1}^{n+1} - 2T_{j,k}^{n+1} + T_{j,k+1}^{n+1}}{(\Delta y)^2} \Leftrightarrow \quad (8)$$

$$\Leftrightarrow T_{j,k}^{n+1} = T_{j,k}^n + \frac{1}{\rho} \left(T_{j-1,k}^{n+1} + T_{j+1,k}^{n+1} + T_{j,k-1}^{n+1} + T_{j,k+1}^{n+1} - 4T_{j,k}^{n+1} \right) \quad (9)$$

assuming $\Delta x = \Delta y =: h$ and setting $\rho = \frac{h^2}{\Delta t}$. Again, we begin our Van Neumann stability analysis by inserting the fourier series

$$T_{j,k}^n = \sum_{p,q=0}^{N-1} \hat{T}_{p,q}^n e^{ipx_j + iqy_k} \quad (10)$$

into the numerical stencil. We bring all terms to one side of the equation and again by using $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$ and that the discretization in x and y are the same and also by placing the exponential outside of the brackets we get

$$\sum_{p,q=0}^{N-1} \left(\widehat{T}_{p,q}^{n+1} - \widehat{T}_{p,q}^n - \frac{1}{\rho} \left(\widehat{T}_{p,q}^{n+1} e^{-iph} + \widehat{T}_{p,q}^{n+1} e^{iph} + \widehat{T}_{p,q}^{n+1} e^{-iqh} + \right. \right. \quad (11)$$

$$\left. \left. + \widehat{T}_{p,q}^{n+1} e^{iph} + \widehat{T}_{p,q}^{n+1} \right) \right) e^{i(px_j + qy_k)} = 0.$$