Title

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9. Januar 2021

1 Van Neumann Stability Analysis: Explicit Scheme

The explicit scheme for the two dimensional heat equation is

$$\frac{T_{j,k}^{n+1} - T_{j,k}^{n}}{\Delta t} = \frac{T_{j-1,k}^{n} - 2T_{j,k}^{n} + T_{j+1,k}^{n}}{(\Delta x)^2} + \frac{T_{j,k-1}^{n} - 2T_{j,k}^{n} + T_{j,k+1}^{n}}{(\Delta y)^2} \Leftrightarrow \tag{1}$$

$$\Leftrightarrow T_{j,k}^{n+1} = T_{j,k}^{n} + \frac{1}{\rho}\left(T_{j-1,k}^{n} - 2T_{j,k}^{n} + T_{j+1,k}^{n} + T_{j,k-1}^{n} - 2T_{j,k}^{n} + T_{j,k+1}^{n}\right) \tag{2}$$

assuming $\Delta x = \Delta y =: h$ and setting $\rho = \frac{h^2}{\Delta t}$. We begin our Van Neumann stability analysis by inserting the fourier series

$$T_{j,k}^{n} = \sum_{p,q=0}^{N-1} \widehat{T}_{p,q}^{n} e^{ipx_j + iqy_k} \tag{3}$$

into the numerical scheme. We obtain

$$\sum_{p,q=0}^{N-1} \widehat{T}_{p,q}^{n+1} e^{i(px_j+qy_k)} = \sum_{p,q=0}^{N-1} (\widehat{T}_{p,q}^{n} e^{i(px_j+qy_k)} + \frac{1}{\rho}(\widehat{T}_{p,q}^{n} e^{i(px_{j-1}qy_k)} + \widehat{T}_{p,q}^{n} e^{i(px_{j+1}+qy_k)} + \\ + \widehat{T}_{p,q}^{n} e^{i(px_j+qy_{k-1})} + \widehat{T}_{p,q}^{n} e^{i(px_j+qy_{k+1})} - 4\widehat{T}_{p,q}^{n} e^{i(px_j+qy_k)})). \tag{4}$$

Using that $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$ and that the discretization in x and y are the same we get

$$\sum_{p,q=0}^{N-1} (\widehat{T}_{p,q}^{n+1} - \widehat{T}_{p,q}^{n} - \frac{1}{\rho}(\widehat{T}_{p,q}^{n} e^{-iph} + \widehat{T}_{p,q}^{n} e^{iph} + \widehat{T}_{p,q}^{n} e^{-iqh} + \widehat{T}_{p,q}^{n} e^{iqh} - 4\widehat{T}_{p,q}^{n}))\, e^{i(px_j+qy_k)} = 0 \tag{5}$$

and because $(e^{i(px_j+qy_k)})_{j,k\in\{0,\dots,N-1\}}$ is a basis of the triginometrical polynomials of degree N in two variables we have for all $p,q \in \{0,\dots,N-1\}$ that

$$\begin{aligned}
&\widehat{T}^{n+1}_{p,q} - \widehat{T}^{n}_{p,q} - \frac{1}{\rho}\left(\widehat{T}^{n}_{p,q}e^{-iph} + \widehat{T}^{n}_{p,q}e^{iph} + \widehat{T}^{n}_{p,q}e^{-iqh} + \widehat{T}^{n}_{p,q}e^{iqh} - 4\widehat{T}^{n}_{p,q}\right) = 0 && \Leftrightarrow\\
\Leftrightarrow &\widehat{T}^{n+1}_{p,q} = \widehat{T}^{n}_{p,q} + \frac{1}{\rho}\left(\widehat{T}^{n}_{p,q}e^{-iph} + \widehat{T}^{n}_{p,q}e^{iph} + \widehat{T}^{n}_{p,q}e^{-iqh} + \widehat{T}^{n}_{p,q}e^{iqh} - 4\widehat{T}^{n}_{p,q}\right) && \Leftrightarrow\\
\Leftrightarrow &\widehat{T}^{n+1}_{p,q} = \widehat{T}^{n}_{p,q}\left(1 + \frac{1}{\rho}\left(e^{-iph} + e^{iph} + e^{-qh} + e^{qh} - 4\right)\right) && \Leftrightarrow\\
\Leftrightarrow &\widehat{T}^{n+1}_{p,q} = \widehat{T}^{n}_{p,q}\left(1 + \frac{1}{\rho}\left(2\cos(ph) + 2\cos(qh) - 4\right)\right) && \Leftrightarrow\\
\Leftrightarrow &\widehat{T}^{n+1}_{p,q} = \widehat{T}^{n}_{p,q}\underbrace{\left(1 - \frac{4}{\rho}\left(\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right)\right)\right)}_{=:G_{p,q}},
\end{aligned}$$

where in the last step the identity $\sin^2(\alpha/2) = \frac{1}{2}(1-\cos(\alpha))$ was used. For van Neumann stability we need that $|G_{p,q}| < 1$. We therefor want to find conditions on Δt and h such that

$$-1 \leq 1 - \frac{4}{\rho}\left(\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right)\right) \leq 1. \tag{6}$$

The rhight hand side inequality is trivial and for the left hand side inequality we need that $\rho \geq 4$.