

Title

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1 Van Neumann Stability Analysis: Explicit Scheme

The explicit scheme for the two dimensional heat equation is

$$\frac{T_{j,k}^{n+1} - T_{j,k}^n}{\Delta t} = \frac{T_{j-1,k}^n - 2T_{j,k}^n + T_{j+1,k}^n}{(\Delta x)^2} + \frac{T_{j,k-1}^n - 2T_{j,k}^n + T_{j,k+1}^n}{(\Delta y)^2} \Leftrightarrow \quad (1)$$

$$\Leftrightarrow T_{j,k}^{n+1} = T_{j,k}^n + \frac{1}{\rho} (T_{j-1,k}^n - 2T_{j,k}^n + T_{j+1,k}^n + T_{j,k-1}^n - 2T_{j,k}^n + T_{j,k+1}^n) \quad (2)$$

assuming $\Delta x = \Delta y =: h$ and setting $\rho = \frac{h^2}{\Delta t}$. We begin our Van Neumann stability analysis by inserting the fourier series

$$T_{j,k}^n = \sum_{p,q=0}^{N-1} \hat{T}_{p,q}^n e^{ipx_j + iqy_k} \quad (3)$$

into the numerical scheme. We obtain

$$\begin{aligned} \sum_{p,q=0}^{N-1} \hat{T}_{p,q}^{n+1} e^{i(px_j + qy_k)} &= \sum_{p,q=0}^{N-1} (\hat{T}_{p,q}^n e^{i(px_j + qy_k)} + \frac{1}{\rho} (\hat{T}_{p,q}^n e^{i(px_{j-1} + qy_k)} + \hat{T}_{p,q}^n e^{i(px_{j+1} + qy_k)} + \\ &\quad + \hat{T}_{p,q}^n e^{i(px_j + qy_{k-1})} + \hat{T}_{p,q}^n e^{i(px_j + qy_{k+1})} - 4\hat{T}_{p,q}^n e^{i(px_j + qy_k)})). \end{aligned} \quad (4)$$

Using that $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$ and that the discretization in x and y are the same we get

$$\sum_{p,q=0}^{N-1} (\hat{T}_{p,q}^{n+1} - \hat{T}_{p,q}^n - \frac{1}{\rho} (\hat{T}_{p,q}^n e^{-iph} + \hat{T}_{p,q}^n e^{iph} + \hat{T}_{p,q}^n e^{-iqh} + \hat{T}_{p,q}^n e^{iqh} - 4\hat{T}_{p,q}^n)) e^{i(px_j + qy_k)} = 0 \quad (5)$$

and because $(e^{i(px_j+qy_k)})_{j,k \in \{0, \dots, N-1\}}$ is a basis of the trigonometrical polynomials of degree N in two variables we have for all $p, q \in \{0, \dots, N-1\}$ that

$$\begin{aligned}
& \widehat{T}_{p,q}^{n+1} - \widehat{T}_{p,q}^n - \frac{1}{\rho}(\widehat{T}_{p,q}^n e^{-iph} + \widehat{T}_{p,q}^n e^{iph} + \widehat{T}_{p,q}^n e^{-iqh} + \widehat{T}_{p,q}^n e^{iqh} - 4\widehat{T}_{p,q}^n) = 0 & \Leftrightarrow \\
& \Leftrightarrow \widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^n + \frac{1}{\rho}(\widehat{T}_{p,q}^n e^{-iph} + \widehat{T}_{p,q}^n e^{iph} + \widehat{T}_{p,q}^n e^{-iqh} + \widehat{T}_{p,q}^n e^{iqh} - 4\widehat{T}_{p,q}^n) & \Leftrightarrow \\
& \Leftrightarrow \widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^n \left(1 + \frac{1}{\rho}(e^{-iph} + e^{iph} + e^{-iqh} + e^{iqh} - 4)\right) & \Leftrightarrow \\
& \Leftrightarrow \widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^n \left(1 + \frac{1}{\rho}(2\cos(ph) + 2\cos(qh) - 4)\right) & \Leftrightarrow \\
& \Leftrightarrow \widehat{T}_{p,q}^{n+1} = \widehat{T}_{p,q}^n \underbrace{\left(1 - \frac{4}{\rho} \left(\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right)\right)\right)}_{=: G_{p,q}},
\end{aligned}$$

where in the last step the identity $\sin^2(\alpha) = \frac{1}{2}(1 - \cos(\alpha))$ was used. For van Neumann stability we need that $|G_{p,q}| < 1$. We therefore want to find conditions on Δt and h such that

$$-1 \leq 1 - \frac{4}{\rho} \left(\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right) \right) \leq 1. \quad (6)$$

The right hand side inequality is trivial and for the left hand side inequality we need that $\rho \geq 4$.