~~problem are typically in the form of non-separable extremely anisotropic state. This results from the fact that a Pareto front of optimal non-dominated solutions is sought. If the Φ assignment technique is based on Pareto dominance all members of the current front of non-dominated solutions have $\Phi = 0$ (or very small values compared to the dominated members of the population).~~

4.3 Improved Parameterization

The aim here is to find the optimal linear orthogonal transformation of the original set of design variables which will lead to a well-posed optimization problem. This is equivalent to finding the orthogonal matrix U ($UU^T = I$) so as the transformation $\vec{x}_{opt} = U^T\vec{x}$ will result in the "optimal" parameterization. This comprises a simple coordinates system rotation so as that the new basis is optimal in the sense that it transforms the ill-posed optimization problem in hand into a well-posed one. In this thesis, it is proposed that U can be extracted through the topological characteristics of the Pareto front. To extract the topological characteristics of the Pareto front in a way which best explains the data variance, principal component analysis (PCA),(86, 103), is used. In that sense, the optimal linear orthogonal transformation of the original set of design variables would be the matrix U which contains the Principal Directions (PDs) as they were computed by PCA applied on the members of the Pareto front.

As it was proven earlier in this chapter, knowledge of the optimal parametrization can lead to a significantly faster exploration of the design space, this though requires knowledge of the Pareto front, the finding of which is the final goal of the optimization process itself. However, one can state that, as the evolution proceeds generating better and better designs, the current elite-set will gradually become better approximation to the Pareto front. Furthermore, in the same way that the elite-set represents an approximation to the Pareto front, PDs computed based on the elite-set will be an approximation to the optimal parametrization, becoming increasingly reliable as the elite-set approaches the true Pareto front.

Applying PCA on the current elite first demands the transformation of its members into a standardized data-set (X) with $\mu = 0$ and $\sigma = 1$ in all directions.

The empirical covariance matrix P, (55, 103) can be calculated as,

$$P_{N\times N} = \frac{1}{e}XX^T \tag{4.5}$$

where e is the size of the elite set and X the matrix which contains the members of the elite-set as columns, we can use the spectral decomposition theorem, (16, 55), to write P as

$$P_{N\times N} = U\Lambda U^T \tag{4.6}$$

where $\Lambda = diag(\lambda_1, ..., \lambda_n)$ is the diagonal matrix with the eigenvalues of P and U is a $N\times N$ matrix containing the eigenvectors as columns also known as principal components or directions.

The most efficient method to perform PCA is by means of the so-called singular value decomposition (SVD), (156), which allows the matrix X to be expressed as

$$X = U\Lambda V^T \tag{4.7}$$

where, V^T a matrix containing the eigenvectors of X^TX.

In industrial scale optimization problems, such as the ones concerning this thesis, where a design evaluation requires time of the order of minutes or even hours, the portion of the total computational cost attributable to PCA computation is negligible.

~~To better demonstrate the variable correlation estimation in MOO problems the welded beam test case is used (fig.~~ 4.7~~). The minimization of both the cost~~ (K~~) and the deflection (Δ) of a welded beam subject to a force~~ P ~~(fig.~~ 4.7) ~~is desired. There are two design variables: the welding length (X_1) and the side length of the square cross section (~~X_2) (fig. 4.7~~). The design is subject to constraints of shear stress (τ), bending stress (σ) and buckling load (~~P_c).

~~The problem is formulated formulated as,~~

~~Minimize Cost,~~

$$\begin{aligned} \cancel{K} &= 1.\cancel{10471}h^2l + 0.\cancel{04811}t^2(14.0 + l) \\ &= 1.\cancel{10471}h^2X_1 + 0.\cancel{04811}X_2^2(14.0 + X_1) \end{aligned} \tag{4.8}$$