Stella Li MA677 Homework 2.

1. In the sentence "the shortest distance between two points is a taxi", the letters and number of times each appear are

T 8	S 5	d 1	n : 3	w : 2
h 2	o 3	i 4	c : 1	p : 1
e 6	r 1	a 3	b : 1	x : 1

Among the 15 letters; y is the time a letter appears

$p(y=1) = \frac{6}{15}$ $\quad p(y=2) = \frac{2}{15}$ $\quad p(y=3) = \frac{3}{15}$ $\quad p(y=4) = \frac{1}{15}$

$p(y=5) = \frac{1}{15}$ $\quad p(y=6) = \frac{1}{15}$ $\quad p(y=8) = \frac{1}{15}$

$E(y) = \frac{6}{15}\times 1 + \frac{2}{15}\times 2 + \frac{3}{15}\times 3 + \frac{1}{15}\times 4 + \frac{1}{15}\times 5 + \frac{1}{15}\times 6 + \frac{1}{15}\times 8$

$= 2.8$

2. $E(XY) = \int_0^1 \int_0^x xy f(x,y)\, dy\, dx$

$= \int_0^1 \int_0^x xy \cdot 12y^2\, dy\, dx = \int_0^1 x \int_0^x 12y^3 dy\, dx$

$= \int_0^1 3x \cdot x^4\, dx = \frac{1}{2} x^6 \Big|_0^1 = \frac{1}{2}$

3. $E(X_1) = E(X_2) = E(X_3) = \frac{1}{2}$

$E(X_1^2) = E(X_2^2) = E(X_3^2) = \int_0^1 x^2 \cdot f(x)\, dx = \int_0^1 x^2 dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3}$

$E(X_1X_2) = E(X_2X_3) = E(X_1X_3) = E(X_1)\cdot E(X_2) = \frac{1}{4}$

Therefore $E[(X_1 - 2X_2 + X_3)^2] = E(X_1^2 + 4X_2^2 + X_3^2 - 4X_1X_2 - 4X_2X_3 + 2X_1X_3)$

$= E(X_1^2) + 4E(X_2^2) + E(X_3^2) - 4E(X_1X_2) - 4E(X_2X_3) + 2E(X_1X_3)$

$= \frac{1}{2}$

4. $E(Y) = \int_0^\infty \phi(x) f(x)\, dx = \int_0^\infty e^{\frac{3}{4}x} \cdot e^{-x} dx = \int_0^\infty e^{-\frac{x}{4}} dx = -4\int_0^\infty e^{-\frac{x}{4}} d(-\frac{x}{4})$

$= -4 e^{-\frac{x}{4}} \Big|_0^\infty = 0 - (-4) = 4$

5. $P(X=1) = P(X=2) = \cdots = P(X=6) = \frac{1}{6}$ $\qquad E(X) = \frac{1}{6}\sum_{i=1}^{6} i = \frac{7}{2}$

$E(Y) = E(2X^2+1) = 2E(X^2) + 1$

$= 2\times(\frac{1}{6}\times 1^2 + \frac{1}{6}\times 2^2 + \cdots + \frac{1}{6}\times 6^2) + 1$

$= 2\times\frac{1}{6}\times 91 + 1 = 31.33$

6. $E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot 2(1-x)\,dx = -2\left(\int_0^1 x^2 dx - \int_0^1 x\,dx\right)$

$= -2 \times \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{3}$

$E(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \cdot 2(1-x)\,dx = -2\left(\int_0^1 x^3 dx - \int_0^1 x^2\,dx\right)$

$= -2 \times \left(\frac{1}{4} - \frac{1}{3}\right) = \frac{1}{6}$

$E(Y^2) = E\left((2X+1)^2\right) = E(4X^2 + 4X + 1)$

$= 4E(X^2) + 4E(X) + 1 = 4 \times \frac{1}{3} + 4 \times \frac{1}{6} + 1 = 3$

7. Proof: $E[(aX+b)^n]$

$= E\left(\sum_{i=0}^{n} \binom{n}{i} (aX)^{n-i} \cdot b^i\right]$

$= \sum_{i=0}^{n} E\left[\binom{n}{i} (aX)^{n-i} \cdot b^i\right]$ $\quad (E(X+Y) = E(X) + E(Y))$

$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \, E(X^{n-i})$ $\quad (E(aX) = aE(X))$

Therefore, $E\left((aX+b)^n\right) = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$

8. Among n parts: $n = X + Y$, $E(X) = np$

$E(X-Y) = E(X-(n-X)) = E(2X-n) = 2E(X) - n = 2np - n$

Therefore, when $n=20$, $p=5\%$, $E(X-Y) = 20 \times (-0.9) = -18$.

In other word, for a random sample of 20 parts from the shipment, it is expected that defective parts is 18 less than good parts on average.