Stella Li MA677 Homework 2.

1. In the sentense "the shortest distance between two points is a taxi", the y can take value of 1, 2, 3, 4, 6, 7, 8

$$P(Y=1)=\frac{1}{42} \quad P(Y=2)=\frac{2}{42} \quad P(Y=3)=\frac{3\times2}{42} \quad P(Y=4)=\frac{4}{42}$$

$$P(Y=6)=\frac{6}{42} \quad P(Y=7)=\frac{7}{42} \quad P(Y=8)=\frac{8\times2}{42}$$

$$E(y)=\sum_{i=1\ldots}^{7} y_i f(y_i) = 1\times\frac{1}{42}+2\times\frac{2}{42}+3\times\frac{6}{42}+4\times\frac{4}{42}+6\times\frac{6}{42}+7\times\frac{7}{42}+8\times\frac{16}{42}$$

$$=\frac{1}{42}\times(1+4+18+16+36+49+128)$$

$$=6$$

2. $$E(XY) = \int_0^1\int_0^x xy\, f(x,y)\,dy\,dx$$

$$= \int_0^1\int_0^x xy\cdot 12y^2\,dy\,dx = \int_0^1 x\int_0^x 12y^3\,dy\,dx$$

$$= \int_0^1 3x\cdot x^4\,dx = \frac{1}{2}x^6\Big|_0^1 = \frac{1}{2}$$

3. $$E(X_1)=E(X_2)=E(X_3)=\frac{1}{2}$$

$$E(X_1^2)=E(X_2^2)=E(X_3^2)=\int_0^1 x^2\cdot f(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1=\frac{1}{3}$$

$$E(X_1X_2)=E(X_2X_3)=E(X_1X_3)=E(X_1)\cdot E(X_2)=\frac{1}{4}$$

Therefore $E[(X_1-2X_2+X_3)^2] = E(X_1^2+4X_2^2+X_3^2-4X_1X_2-4X_2X_3+2X_1X_3)$

$$=E(X_1^2)+4E(X_2^2)+E(X_3^2)-4E(X_1X_2)-4E(X_2X_3)+2E(X_1X_3)$$

$$=\frac{1}{2}$$

4. $$E(Y)=\int_0^\infty \phi(x)f(x)\,dx = \int_0^\infty e^{\frac{3}{4}x}\cdot e^{-x}\,dx = \int_0^\infty e^{-\frac{x}{4}}\,dx = -4\int_0^\infty e^{-\frac{x}{4}}\,d\left(-\frac{x}{4}\right)$$

$$= -4e^{-\frac{x}{4}}\Big|_0^\infty = 0-(-4)=4$$

5. $$P(X=1)=P(X=2)=\cdots=P(X=6)=\frac{1}{6} \qquad E(X)=\frac{1}{6}\sum_{i=1}^{6} i=\frac{7}{2}$$

$$E(Y)=E(2X^2+1) = 2E(X^2)+1$$

$$=2\times\left(\frac{1}{6}\times1^2+\frac{1}{6}\times2^2+\cdots+\frac{1}{6}\times6^2\right)+1$$

$$=2\times\frac{1}{6}\times91+1=31.33$$

6. $E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot 2(1-x)\,dx = -2\left(\int_0^1 x^2 dx - \int_0^1 x\,dx\right)$

$= -2 \times \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{3}$

$E(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \cdot 2(1-x)\,dx = -2\left(\int_0^1 x^3 dx - \int_0^1 x^2\,dx\right)$

$= -2 \times \left(\frac{1}{4} - \frac{1}{3}\right) = \frac{1}{6}$

$E(Y^2) = E\left((2X+1)^2\right) = E(4X^2 + 4X + 1)$

$= 4E(X^2) + 4E(X) + 1 = 4 \times \frac{1}{3} + 4 \times \frac{1}{6} + 1 = 3$

7. Proof: $E[(aX+b)^n]$

$= E\left(\sum_{i=0}^{n} \binom{n}{i} (aX)^{n-i} \cdot b^i\right]$

$= \sum_{i=0}^{n} E\left[\binom{n}{i} (aX)^{n-i} \cdot b^i\right]$ $\quad (E(X+Y) = E(X) + E(Y))$

$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$ $\quad (E(aX) = aE(X))$

Therefore, $E((aX+b)^n) = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$

8. Among n parts: $n = X + Y$, $E(X) = np$

$E(X-Y) = E(X-(n-X)) = E(2X-n) = 2E(X) - n = 2np - n$

Therefore, when $n=20$, $p=5\%$, $E(X-Y) = 20 \times (-0.9) = -18$.

In other word, for a random sample of 20 parts from the shipment, it is expected that defective parts is 18 less than good parts on average.