A complete Proof of the Yoneda Lemma

$\mathcal{C} : \mathbf{Cat}$

$\mathcal{S} : \mathbf{Cat} \coloneqq \mathrm{Hom}(\mathcal{C} \times \mathrm{Hom}(\mathcal{C}, \mathbf{Set}), \mathbf{Set})$

$L : \mathcal{S} \coloneqq ((X, F) \mapsto \mathrm{Hom}(\mathrm{Hom}(X, -), F), (f, \eta) \mapsto \theta \mapsto \eta \circ \theta \circ (- \circ f))$

$R : \mathcal{S} \coloneqq ((X, F) \mapsto FX, (f, \eta) \mapsto \eta_Y \circ Ff)$

$\alpha : \mathrm{Hom}(L, R) \coloneqq (X, F) \mapsto (-)_X \mathsf{id}_X$

$\beta : \mathrm{Hom}(R, L) \coloneqq (X, F) \mapsto x \mapsto Z \mapsto F(-)x$

$$\begin{array}{l|l}
\alpha_{(Y,G)} \circ L(f, \eta) & \text{definitions of } \alpha, L \\
= (-)_Y \mathsf{id}_Y \circ \eta \circ \theta \circ (- \circ f) & \text{definition of } \circ \\
= \theta \mapsto (\eta_Y \circ \theta_Y \circ (- \circ f)_Y) \mathsf{id}_Y & \text{application} \\
= \theta \mapsto \eta_Y(\theta_Y(\mathsf{id}_Y \circ f)) & \text{identities} \\
= \theta \mapsto \eta_Y(\theta_Y(f \circ \mathsf{id}_X)) & \text{naturality of } \theta \\
= \theta \mapsto \eta_Y((Ff \circ \theta_X)\mathsf{id}_X) & \text{definition of } \circ \\
= \eta_Y \circ Ff \circ (-)_X \mathsf{id}_X & \text{definitions of } \alpha, R \\
= R(f, \eta) \circ \alpha_{(X,F)} &
\end{array}$$

$\Rightarrow \alpha$ is natural.

$$\begin{array}{l|l}
\beta_{(Y,G)} \circ R(f, \eta) & \text{definitions of } \beta, R \\
= (x \mapsto Z \mapsto g \mapsto Ggx) \circ \eta_Y \circ Ff & \text{definition of } \circ \\
= x \mapsto Z \mapsto g \mapsto (Gg \circ \eta_Y)(Ffx) & \text{naturality of } \eta \\
= x \mapsto Z \mapsto g \mapsto (\eta_X \circ Fg)(Ffx) & \text{definition of } \circ \\
= x \mapsto Z \mapsto g \mapsto (\eta_X(Fg \circ Ff)x) & \text{functoriality of } F \\
= x \mapsto Z \mapsto g \mapsto (\eta_X(F(g \circ f)x)) & \text{definition of } \circ \\
= x \mapsto \eta \circ (Z \mapsto g \mapsto Fgx) \circ (- \circ f) & \text{definition of } \circ \\
= (\theta \mapsto \eta \circ \theta \circ (- \circ f)) \circ (x \mapsto Z \mapsto F(-)x) & \text{definitions of } \beta, L \\
= L(f, \eta) \circ \beta_{(X,F)} &
\end{array}$$

$\Rightarrow \beta$ is natural.

$$\begin{array}{l|l}
(\alpha \circ \beta)_{(X,F)} x & \text{definitions of } \alpha, \beta \\
= F\mathsf{id}_X x & \text{functoriality of } F \\
= \mathsf{id}_{FX} x & \text{identity} \\
= x & \text{definition of } \mathsf{id}_R
\end{array}$$

$\Rightarrow \alpha \circ \beta = \mathsf{id}_R$

$$\begin{array}{l|l}
(\beta \circ \alpha)_{(X,F)} \eta_Y f & \text{definitions of } \alpha, \beta \\
= (Ff \circ \eta_X)\mathsf{id}_X & \text{naturality of } \eta \\
= (\eta_Y \circ \mathrm{Hom}\ (X, -)f)\mathsf{id}_X & \text{definition of } \circ \\
= \eta_Y(f \circ \mathsf{id}_X) & \text{identity} \\
= \eta_Y f & \text{definition of } \mathsf{id}_L
\end{array}$$

$\Rightarrow \beta \circ \alpha = \mathsf{id}_L$

$\Rightarrow (\alpha, \beta) : L \cong R$