From $1 + 1 = 2$ to ∞-Categories

and a few other structures along the way

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0. Introduction

While category theory has already made a name for itself as the study of **abstract nonsense** in mathematics, ∞-categories in particular are heralded as the poster eldrich child of modern pure mathematics gone too far. With this article I hope to show that despite their dramatic name, at least the basic definitions of these ∞-categories are in themselves not so scary, and actually just a few small hops of generalization away from familiar structures like monoids.

This is a journey through a wide variety of mathematical structures, and while in theory knowledge of the theory of these structures is not necessary to have, the information may be quite dense to take in all at once, perhaps an alternative title might be "speedrunning ∞-categories from scratch". As such, this article consists mostly of definitions, which is great news for lovers of definitions, but if you are curious about the theories of any of the structures introduced here in more depth there is plenty of material out there to dig into. If nothing else, I hope that if you henceforth encounter an ∞ -category in the wild, you will approach it with powerful curiosity rather than dread.

We will begin with monoids and categories, and while their definitions are a bit cumbersome to write down in their entirity one of the rewards of this journey will be seeing how all of their properties can be essentially reduced to just one operation of horn fillers.

Most of this article is based on the infinity confusing but very helpful website known as the nLab and the first subchapter of Lurie's Higher Topos Theory.

1. Monoids and Groups

As promised by the title, we will begin with $1 + 1$. These are two natural numbers, and we can add them to get another natural number, 2. More generally, we have a set $\mathbb{N}$ together with an operation $+$ which takes two natural numbers and gives us another one, this is an example of an **instance** of a mathematical structure.

A **structure** itself consists of *data, operations and properties.* An instance of such a structure is then an instance of the data together with operations that satisfy the properties. To motivate monoids, we will therefore choose some of the data, operations and properties of $\mathbb{N}$ and $+$ we like and write them down in terms of variables. In this case, we choose to keep the operations of the **identity element** 0 and **composition** $+$, and the properties of **identity** and **associativity**. The data is just our one set, $\mathbb{N}$, this might seem unnecessary to point out in this way but will make more sense later.

Identity is the fact that adding 0 to anything does nothing and associativity is the theorem that it does not matter in which order you evaluate chains of addition, $a + (b + c) = (a + b) + c$. This means that any chain of additions and an arbitrary number of zeroes inserted has one unique sum.

Abstractly, a **monoid** is a tuple $(X, \mathsf{id} : X, \circ : X \times X \to X)$ where the identiy element is id and $\circ$ is associative. This notation means that id is an element of X and $\circ$ is a function from pairs of X to X, this lines up with 0 and $+$.

Another example of monoids is $(\mathbb{N}^+, 1, \cdot)$, where $\mathbb{N}^+$ is the set of natural numbers without 0.

Whenever we create a structure, it will be useful to consider *nice maps* (or **homo-morphisms** in greek) between instances of the structure. A homomorphism is a function between the data that **commutes** with the operations, for example a monoid homomorphism from $(X, \mathsf{id}, \circ)$ to $(X', \mathsf{id}', \circ')$ is a function $f : X \to X'$ such that $f(\mathsf{id}) = \mathsf{id}'$ and $f(x \circ y) = f(x) \circ' f(y)$.

These homomorphisms tell us about the ways different monoids interact with each other, for example the function $x \mapsto 2^x$ is a homomorphism between the above examples $(\mathbb{N}, 0, +)$ and $(\mathbb{N}^+, 1, \cdot)$.

A variation of monoids are **groups**, there we additionally have an **inverse** operation $\mathsf{inv} : X \to X$ which, when multiplied with, cancels to the identity, specifically $x \circ \mathsf{inv}(x) = \mathsf{inv}(x) \circ x = \mathsf{id}$ (**left** and **right inverse**). The above examples of monoids are not groups, however there is a group of integers and addition $(\mathbb{Z}, 0, +, x \mapsto -x)$.

Group homomorphisms are analogous to monoid homomorphisms, though actually commuting with inverses is not necessary to spell out as it can be proven from the other properties, we have $f(x) \circ' f(\mathsf{inv}(x)) = f(x \circ \mathsf{inv}(x)) = f(\mathsf{id}) = \mathsf{id}'$, so $f(\mathsf{inv}(x))$ is really the a left inverse of $f(x)$ and by symmetry also a right inverse.

The study of monoids, groups and their homomorphism is a large field in its own right with many different theorems and applications, for now we will shelve them for a moment as we consider another kind of structure:

2. Graphs

While monoids and groups are about composing elements to get more elements, graphs add some shape into the mix: they have edges *between* elements. For example, you might have two vertices and two edges going from one to the other, or one vertex with many edges from and to itself. In general, a **graph** is a tuple $(X_0, X_1, \sigma : X_1 \to X_0, \tau : X_1 \to X_0)$ where X_0 is the set of **vertices**, X_1 the set of **edges** and σ, τ assign to each edge its **source** and **target**. The first above example would therefore formally be $(\{0, 1\}, \{0, 1\}, f \mapsto 0, f \mapsto 1)$, or visually

$$\begin{tikzcd} 0 \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & 1 \end{tikzcd}.$$

The data in this case are *two* sets, and we have no properties for now.

For $x, y : X_0$, we will also write $x \to_{\sigma\tau} y$ for the set of edges between them $\{f : X_1 \mid \sigma(f) = x \land \tau(f) = y\}$, and leave out the subscript if σ and τ can be inferred.

A note on terminolgy is that these are often instead referred to as **directed graphs**, as opposed to "undirected graphs" where instead of source and target each edge has an unordered two-element boundary. For each graph, swapping the source and target of every edge gives the **opposite graph**, written as C^{op} for some graph C.

Of course, a graph with no edges is the same as just a set X_0, but as will be important later a graph with exactly one vertex is *also* the same as a set X_1, as σ and τ just send everything to the one vertex and therefore contain no additional information.

We also get a notion of **graph homomorphism**, which is a pair of functions $(F_0 : X_0 \to {X_0}', F_1 : X_1 \to {X_1}')$ such that $\sigma'(F_1(f)) = F_0(\sigma(f))$ and $\tau'(F_1(f)) = F_0(\tau(f))$.

This means that for any two vertices $x, y : X_0$, F_1 is a map from $x \to y$ to $F_0(x) \to F_0(y)$.

Besides the various finite graphs you can construct, one can also consider the graph whose vertices are *all sets* and edges *all functions*, with source and target being the domain (i. e. input set) and codomain (i. e. output set) of a given function. Indeed, for any structure, there is a graph of its instances and homomorphisms. Attentive set theorists might here complain about *size issues*, after all there is no set of all sets, this can be resolved by saying **class** instead of set, using universes (there is a 1-set of all 0-sets, and so on) or just ignoring it.

Graphs also have a rich theory and plenty of applictions, but for our needs we will use them as the building blocks for our next structure:

3. Categories

Categories combine the structures of monoids and graphs, a **category** is a graph with a monoid-like structure on its **composable edges**. This might seem odd at first, one way to motivate composable edges is such that the above examples of graphs of instances and homomorphisms actually all turn out to be categories, with composition being actual *function composition* and identities being *identity functions*. Specifically, for any functions $f : Y \to Z$ and $g : X \to Y$ you can form the composite function $f \circ g : X \to Z$ defined as $x \mapsto f(g(x))$, and for any set X there is an identity function $\mathsf{id}_X : X \to X$ defined as $x \mapsto x$.

This tells us that in general two edges are composable if the target of the second is the source of the first, motivating the proper domain for $\circ$ in a category: for a graph (X_0, X_1, σ, τ) we can form the set $X_1 \times_{\sigma=\tau} X_1 = \{(f, g) : X_1 \mid \sigma(f) = \tau(g)\}$, we then have $\circ : X_1 \times_{\sigma=\tau} X_1 \to X_1$ with the properties that $\sigma(f \circ g) = \sigma(g), \tau(f \circ g) = \tau(f)$. Moreover, instead of having just one identiy element, we have an id_x for each $x : X_0$, with the property $\sigma(\mathsf{id}_x) = \tau(\mathsf{id}_x) = x$. These properties ensure that we can simply copy-paste the properties of associativity and identity from monoids to categories, the domains and codomains all check out. Just like for monoids, another way to view associativity is that for any *chain of composable edges* there is one unique composite edge, remember this fact.

Everything in the above example remains true if you replace sets with instances of some structure and functions with homomorphisms, as long as the composition of homomorphisms is itself a homomorphism and the identity function is also one, thus in addition to the *category of sets* we also have categories of monoids, groups, graphs, and also of (for the sake of attentive set theorists *small*) categories. These categories are then shorthandedly referred to by names such as $\mathbf{Set}, \mathbf{Grp}, \mathbf{Graph}, \mathbf{Cat}$.

For this reason, category theorists will call their vertices **objects** and edges **morphisms**. While these large categories packaging entire structures and homomorphism sets are very useful and often the way category theory is introduced, just like graphs or monoids there is plenty of fun to be had with small categories as well:

For example we can augment our two-vertices-two-edges graph from above into a category by adding identity edges for both objects, giving a category that looks like

$$\begin{tikzcd} 0 \arrow[loop left, "\mathsf{id}_0"] \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & 1 \arrow[loop right, "\mathsf{id}_1"] \end{tikzcd}.$$

the only composable pairs are with one identity each so composition is trivial. In an act of subtle foreshadowing, we will call this category $\boldsymbol{G}$.

If we also require that our edges have inverses like they do for groups, we move from categories to **groupoids**. The relevant properties for composability are that $\sigma(\mathsf{inv}(e)) = \tau(e)$ and $\tau(\mathsf{inv}(e)) = \sigma(e)$.

The above examples of sets and functions are not groupoids, as not every function is invertible, but there is a groupoid of sets and *bijections*, which are exactly the invertible functions.

This process of graph-ifying a structure (from sets to graphs, monoids to categories and groups to groupoids) can itself be abstracted, the technical term is **oidification** and according to this an alternative name for categories would be *monoidoids* and for graphs *set-oids*, though *setoid* unfortunately and confusingly already means something else.

A property of this oidification is that the set of edges from a vertex to *itself* is for graphs a set, for categories a monoid and for groupoids a group. This also means that just as a one-vertex graph is exactly a set, a one-vertex category is exactly a monoid and a one-vertex groupoid is a group. A synonym of one-vertex is **reduced**.

It's easy to see that just by reversing the direction of composition, for each category the opposite graph can also be given a category structure, appropriately called the **opposite category**. The opposite category of a groupoid is also a groupoid.

Category homomorphisms are exactly the graph homomorphisms which are also monoid homomorphisms, the graph homomorphism structure ensures that the monoid homomorphism properties work properly, for example $F_1(\mathsf{id}_x) = \mathsf{id}'_{F_0(x)}$. These category homomorphisms are so important that they've been given their own special name, they are called **functors**. An easy example of functors are the so-called **forgetful functors** between categories of instances that "forget" certain operations or properties, for example there are such forgetful functors from the category of groups to monoids, or monoids to sets, as it's easy to see that a group homomorphism is also necessarily a monoid homomorphism, and of course is always a function.

A note about notation is that it's somewhat common to conflate a functor with its two maps, for example for a functor $F = (F_0, F_1)$ it's common to write $F(x)$ or $F(f)$ with x an object and f a morphism, the correct function is inferred from the context.

One reason functors are special is that some structures can actually be expressed as functors between specific categories: for example, if you recall our friend $\boldsymbol{G}$ from earlier, graphs are exactly the same thing as functors from $\boldsymbol{G}^{\mathsf{op}}$ to $\mathbf{Set}$! Given such a functor (F_0, F_1), the set of vertices is $F_0(0)$, the set of edges is $F_0(1)$, and source and target respectively are exactly what F_1 maps the two edges to. This construction of functors from the opposite category to $\mathbf{Set}$ comes up quite often, so it's also been given the mysterious name **presheaf**. In this case we've seen that we can encode the edge structure of graphs in this small source category $\boldsymbol{G}$.

Another important example of functors are the **hom-functors**: for each object x of a category C there is a functor $h^x : C \to \mathbf{Set}$ mapping each object y to the set of morphisms $x \to y$, and then each edge $f : y \to z$ to the function from $h^x(y)$ to $h^x(z)$ given by $g \mapsto f \circ g$. There is also the functor h_x from C^{op} to $\mathbf{Set}$ given analogously by mapping y to $y \to x$, this is therefore a presheaf.

An interesting aspect of functors is that there is also a notion of homomorphisms between them: if F and F' are two functors between categories $(X_0, X_1, \circ), ({X_0}', {X_1}', \circ')$ then a transformation η between them is in a sense a family of "corrections", for each vertex $x : X_0$ an edge η_x from $F(x)$ to $F'(x)$.

The "niceness" property of functor homomorphisms is called *naturality* and is defined by $\eta_y \circ' F(f) = F'(f) \circ \eta_x$ for any morphism $f : x \to y$, functor homomorphisms are therefore also called "natural transformations".

This means that for any two categories C, D, we actually get a *category of functors* $C \to D$. his also means that with the above redefinition of graphs as $\boldsymbol{G}^{\mathsf{op}} \to \mathbf{Set}$, we automatically obtain the category structure of $\mathbf{Graph}$ as well.

This redefinition of graphs might then naturally raise the question of what kinds of categories we can get by taking presheaves from small categories other than $\boldsymbol{G}$.

4. Simplicial Sets

To map out the road ahead, the goal for a definition of ∞-categories is to find a suitable generalization of graphs and then an analogous "higher" category structure. The ∞ here has the following meaning: *sets* have one kind of object (and are also trivially presheaves from the category with one object and one morphism), graphs have two kinds of objects, vertices and edges between them (this is encoded in the source category $\boldsymbol{G}$). Thinking geometrically, vertices are 0-dimensional and edges 1-dimensional, and thus our "∞-graphs" would have a (countably) infinite number of kinds of objects, each consisting of objects of lower dimensions.

One such notion is that of **simplicial sets**, which are built by generalizing vertices and edges to *triangles*, *tetrahedra*, and so on. These higher dimensional triangles are the simplest n-dimensional polyhedra respectively and are called *simplices*. What defines an n-simplex is that it consists of $n-1$-simplices, $n-2$-simplices and so on, laid out in a compatible shape. This generalizes the source and target functions, where instead of two of them for edges we have many of them mapping each simplex to its lower dimensional components. A perhaps unintuitive aspect is that just like graphs can have edges from a vertex to itself, there is also nothing preventing triangle from containing a vertex more than once.

To describe exactly what this *compatible shape* means, consider the example of triangles (2-simplices). They go between three vertices $\{0, 1, 2\}$ and then three edges, specifically from 0 to 1, 0 to 2 and 1 to 2. This means that there are three maps from the set of triangles to the set of vertices, and three maps to the set of edges. Note that edge maps correspond to exactly the ways that an edge can fit into a triangle while keeping the *order* of the vertices: the target always gets mapped to a vertex with a number greater or equal to the source. With some thinking you can convince yourself that this pattern in general describes exactly the ways n-simplices fit into m-simplices: they correspond exactly to the *order-preserving maps* from $\{0...n\}$ to $\{0...m\}$.

An alternative description of this is that for each n we can define a *category* $[n]$ whose vertices are $\{0...n\}$ and where for each $k \leq l$ there is exactly and only one edge from *k* to *l*, then these maps are simply the *functors* from $[n]$ to $[m]$.

This motivates the **simplex category** Δ: It has countably infinite vertices, thus $X_0 = \mathbb{N}$, and as edges the functors $[n] \to [m]$.

Then, a **simplicial set** is a presheaf : $\Delta^{\text{op}} \to \mathbf{Set}$: for each such order-preserving map we get a map in the simplicial set sending an m-simplex to an n-simplex component (note the reversal of m and n, this is a presheaf after all). This definition is quite abstract but these simplicial sets really are each just a bunch of edges, triangles etc. stuck together.

One indicator that this really is a generalization of graphs is that the subcategory of Δ consisting of only the objects 0 and 1 is exactly $\boldsymbol{G}$, meaning we can redefine a graph to be a simplicial set mapping n to $\{\}$ for $n \geq 2$.

Since they are just functors, together with natural transformations we also obtain a category $\mathbf{Set}\boldsymbol{\Delta}$ of simplicial sets.

5. ∞-Categories

As promised in the introduction, we will now show that all of the different operations and properties of categories and groupoids respectively can be seen as just one concept for simplicial sets, that of *horn fillers*. Specifically, for each category C we obtain a simplicial set $N(C)$ called the **nerve** of C whose n-simplices are the functors $[n] \to C$, in other words *composable chains* of n morphisms in C, we will now examine the properties of these nerves.

First, note that for any $n : \mathbb{N}$ the $\boldsymbol{n}$**-simplex** Δ^n is itself a simplicial set, it can be concisely described as hom-functor h_n in Δ, i. e. its sets of m-simplices are exactly the order-preserving maps from m to n, pictured is Δ^2.

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

Then, for each $k : \{0...n\}$ the **(n, k)-horn** $\Lambda^n_k \subseteq \Delta^n$ is the result of deleting the face opposite k as well as the inner volume from Δ^n. More formally, it is the sub-presheaf of Δ^n obtained by setting its set of n-simplices to $\{\}$ (removing the inner volume), and from the set of $(n-1)$-simplices removing the one *not* mapping any index to k.

The horns Λ^n_0 and Λ^n_n are the **outer horns**, the others are the **inner horns**, pictured is the inner horn Λ^2_1.

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$

Because Λ^n_k is a sub-presheaf Δ^n, there is a unique inclusion morphism $\iota_{nk} : \Lambda^n_k \to \Delta^k$.

With all this setup, we say that for a simplicial set S and a *horn in* S $\lambda : \Lambda^n_k \to S$, a **horn filler** of λ is a simplex $\varphi : \Delta^n \to S$ such that $\varphi \circ \iota_{nk} = \lambda$, this is to say the horn λ *extends* to a simplex.

An example may illuminate things, consider an inner horn $\lambda : \Lambda^2_1 \to N(C)$.

$$\begin{tikzcd} & y \arrow[dr, "f"] & \\ x \arrow[ur, "g"] & & z \end{tikzcd}$$

To this we can construct a horn filler by adding the edge $f \circ g : x \to z$ and the 2-simplex (g, f).

$$\begin{tikzcd} & y \arrow[dr, "f"] \arrow[d, phantom, "{(g, f)}"] & \\ x \arrow[ur, "g"] \arrow[rr, "f \circ g"'] & {} & z \end{tikzcd}$$

Moreover, any other horn filler, since it extends a horn consisting of g and f, must therefore also have the same inner 2-simplex (g, f) and therefore also, by the definition of a composition, the same composite edge $f \circ g$, this means that all such horn fillers are equal.

More generally, let $\lambda : \Lambda^n_k \to N(C)$ be an inner horn having vertices x_i with $i : \{0...n\}$. For $i > 0$, let $f_i : x_{i-1} \to x(i)$ be the morphism obtained from λ by restricting it to $\{i - 1, i\} \subseteq \Delta^n$. The chain of composable morphisms given by $\varphi = (f_1, ..., f_n)$ will then be our desired horn filler, unique for the same reason as above that this chain is uniquely determined by the horn and each such chain has a unique composite.

Therefore, *the nerve of a category has unique inner horn fillers*!

Additionally, the nerve of a *groupoid* has fillers for the outer horns as well: consider an outer horn Λ^2_0

$$
\begin{tikzcd}
& y & \\
x \arrow[ur, "f"] \arrow[rr, "g"'] & & z
\end{tikzcd}
$$

The unique horn filler of this has the 2-simplex $(f, g \circ \mathsf{inv}(f))$, symmetrically for the other outer horn it would be $\mathsf{inv}(f) \circ g$.

What is perhaps more surprising, and what we have been working towards, is that the converses are true as well: *a simplicial set with unique inner horn fillers is always the nerve of a category, and a simplicial set with unique horn fillers is always the nerve of a groupoid*! This means that categories (respectively groupoids) are the same thing as simplicial sets with unique fillers for all inner (respectively all) horns.

Let S be a simplicial set with unique inner horn fillers, we will now constuct the category this is canonically the nerve of.

The set of objects X_0 is exactly the set of vertices $S(0)$, and the set of morphisms is the set of edges $\Delta^1 \to S$, with source and target being the images of 0 and 1 respectively.

For each object $x : X_0$ we then set $\mathsf{id}_X : x \to x$ to be the morphism from Δ^1 mapping both vertices 0 and 1 to x.

Then for composition we simply use the construction from before: let $f : y \to z$ and $g : x \to y$, together they determine an inner horn $\lambda : \Lambda^2_1 \to S$, then the composition $f \circ g : x \to z$ is exactly the edge added by the unique filler of λ.

What remains is to prove the properties of identity and associativity:

For a morphism $f : x \to y$, together with $\mathsf{id}_x : x \to x$ this also gives an inner horn $\lambda : \Lambda^2_1 \to S$

$$
\begin{tikzcd}
& x \arrow[dr, "f"] & \\
x \arrow[ur, "\mathsf{id}_x"] & & y
\end{tikzcd}
$$

which can be extended by f itself to a filler

$$
\begin{tikzcd}
& x \arrow[dr, "f"] & \\
x \arrow[ur, "\mathsf{id}_x"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=4, "(\mathsf{id}_x, f)"] & & y
\end{tikzcd}
.
$$

Because this is exactly the construction for the composite $f \circ \mathsf{id}_x$, by uniqueness of fillers we have $f \circ \mathsf{id}_x = x$. Symmetrically we also get $f \circ \mathsf{id}_y = f$.

Then, for a composable chain of morphisms (h, g, f) we can construct an inner horn $\lambda : \Lambda^3_2 \to S$, pictured are only the vertices and edges.

$$
\begin{tikzcd}
 & z & \\
 & x \arrow[u, "f \circ g"] \arrow[dr, "g"] & \\
w \arrow[ur, "h"] \arrow[uur, "f \circ (g \circ h)"] \arrow[rr, "g \circ h"'] & & y \arrow[uul, "f"']
\end{tikzcd}
$$

If φ is now the unique horn filler of this with inner volume (h, g, f), then the added *bottom face* of φ is the diagram for the composition $f \circ (g \circ h)$ with composite edge $f \circ (g \circ h)$, thus this 3-simplex witnesses exactly the associativity of this double composition.

As for groupoids, we can similarly use the outer horn fillers to obtain inverse morphisms: let $f : x \to y$, from f and id_x we construct an outer horn $\lambda : \Lambda^2_0 \to S$

$$
\begin{tikzcd}
 & y & \\
x \arrow[ur, "f"] \arrow[rr, "\mathsf{id}_x"'] & & x
\end{tikzcd}
.
$$

Then the unique horn filler adds the edge $\mathsf{inv}(f) : y \to x$ and witnesses $f \circ \mathsf{inv}(f) = \mathsf{id}_x$, symmetrically the other outer horn proves that this is a right inverse as well.

Returning to the concept of data, operations, properties, we summarize that a category (respectively groupoid) has the data and properties of a simplicial set as a functor in $\Delta^{\mathsf{op}} \to \mathbf{Set}$, the operation of inner (respectively all) horn fillers and the property of these horn fillers being unique. Despite having infinitely many dimensions, the only actual data is then in the dimensions up to 2, everything else is determined completely by this uniqueness.

One might then wonder what happens if we disregard this uniqueness property entirely, would we be able to have “data in all dimensions”? As it turns out the answer is yes, and doing so gives us precisely the definition of ∞-categories! Specifically, an **∞-category** is a simplicial set with fillers for inner horns, an **∞-groupoid** is a simplicial set with fillers for all horns. The nonuniqueness of fillers has the following consequence: composition of morphisms is no longer a function that outputs another morphism, rather it gives a *space* of morphisms that correspond to all the different horn fillers. Associativity then no longer holds as an identity, but rather we once again there are spaces of *associators* which fill the respective higher horns.

To motivate an example of an ∞-category which is not a nerve and thus justifies its existence, we can look at one of the main applications (yes, applications!) of ∞-category theory, which is studying the *shapes of spaces*.

Consider, for a concrete example, *paths on a sphere*. These naturally form a groupoid whose objects are points on the sphere (we can consider the unit sphere, so $S^2 = \{(x, y, z) : \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$) and for two such points p, q the set of morphisms $p \to q$ is the set of continuous paths on the sphere from p to q, formally continuous functions $f : [0, 1] \to S^2$ with $f(0) = p$ and $f(1) = q$. The nerve of this groupoid then consists of chains of paths, which are really just the same as paths, reinforcing the idea that nerves have no “higher-dimensional information”.

However, we can construct a different natural ∞-groupoid whose sets of n-simplices are exactly n-simplices on the sphere, for example there are 3-simplices with one vertex at the north pole and a triangle of vertices below. More formally the n-simplices are continuous maps from Δ^n seen as a space to the sphere, generalizing the above definition of paths. This serves an example of horn fillers being non-unique: consider a 2-horn i. e. three points p, q, r with arcs $f : p \to q, g : q \to r$ on the sphere, then for any path $h : p \to r$ the spherical triangle bounded by f, g, h is a horn filler, thus this is not the nerve of any groupoid. Note how here we have a literal space of horn fillers.

This is called the **fundamental ∞-groupoid** Π_∞ of the sphere, to give an example of what we might mean by higher-dimensional information consider the fact that you can continuously move any point to any other point and the same is true of paths (even long paths which wrap around the sphere can be "unravelled") and triangles, but *not* of 3-simplces: they can either be "printed" flat on the sphere or envelop it entirely. In fact, while this may be hard to visualize, they can wrap around the sphere any number of times. For higher n-simplices things become more complicated, in fact classifying these for any n is an open problem! The general keyword to look up here is **homotopy theory** of spheres and a proper introduction to it would go beyond the goals of this article, the main lesson for our purposes is that $\Pi_\infty(S^2)$ is an example of an ∞-groupoid with a highly nontrivial higher-dimensional structure.

As alluded to earlier, simplicial sets are not the only way to build infinity categories, you can replace Δ with other "categories of n-dimensional spaces" and obtain other models for higher dimensional graphs. The challenge for each of these models is to then come up with an appropriate generalisation of composition, associativity, inverses, and this is where simplices are especially nice because using simplicial sets those all turn into special cases of of horn fillers.

6. Full Triangle

Coming back to the beginning, given the fact that monoids are reduced categories and categories are ∞-categories where the horn fillers are unique, this means that *a monoid is a reduced simplicial set where every inner horn has a unique filler*, analogously we obtain a definition of groups. If you read that sentence and it actually made sense to you then you, then I must congratulate you, only madness lies ahead.

One can organize some of the structures and properties we can add to simplicial sets in a table:

	reduced simplicial set	simplicial set
unique inner horn fillers	monoid	category
unique horn fillers	group	groupoid
inner horn fillers	∞-monoid	∞-category
horn fillers	∞-group	∞-groupoid

Consider the monoid $(\mathbb{N}, +)$ from the very beginning: as an ∞-category this has one object 0, countably many edges from that object to itself whose composition is given by addition, and then n-simplices which are lists of n natural numbers.

$1+1$ is therefore the inner horn

$$
\begin{tikzcd}
 & 0 \arrow[dr, "1"] & \\
0 \arrow[ur, "1"] & & 0
\end{tikzcd}
$$

which has a unique filler extending it to the simplex

$$
\begin{tikzcd}
 & 0 \arrow[dr, "1"] & \\
0 \arrow[ur, "1"] \arrow[rr, "2"'] & (1,1) & 0
\end{tikzcd}
$$

witnessing $1+1=2$.