

Meaningful Generalization of the Exponent

Formal Outline: Course in Introductory Calculus Required

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1. Abstract

The exponential function: enter a^n into a calculator, and the output will be $a \cdot a \cdot \dots n$ times. But, what if the expression 2^π were entered into a calculator? Why does an actual output result? How is a meaningful value to be drawn out from an expression like 2^π ?

2. Definition

natural logarithm: $\ln x = \int_1^x \frac{1}{s} ds$ $0 < x$

*...Strictly increasing invertible function
(derivative always positive for $0 < x$):*

natural exponential function: $\exp = \text{inverse ln}$

$$\Rightarrow x = \ln(\exp x) \quad 0 < \exp x$$

$$x = \exp(\ln x) \quad 0 < x$$

partial differentiation: differentiation with respect to one variable,
with all other variables treated as constant

factorial: $n! = 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \dots \cdot n$

integer conversion: $\text{int}(x) = \begin{cases} \text{real number } x \text{ converted to integer} \\ \text{by discarding all digits after decimal} \end{cases}$

absolute value: $|x| = \begin{cases} x & 0 \leq x \\ -x & x < 0 \end{cases}$

3. Definition of the Simple Exponential Function

The purpose of the exponential function is to reiterate multiplication a given number of times.

A meaningful output results from any real base $a \neq 0$ raised to the exponent of any integer input:

- Multiplication a positive number of times:

$$a^n = 1 \cdot \overbrace{a \cdot a \cdot \dots}^{n \text{ times}}$$

- Multiplication zero number of times:

$$a^0 = 1$$

- Multiplication a negative number of times translates to the inverse operation of multiplication:

$$a^{-n} = 1 \div \overbrace{a \div a \div \dots}^{n \text{ times}} \quad a \neq 0$$

4. Definition of the Generalized Exponential Function

The exponential function can be generalized to output a real value from any real input as long as base a is restricted to positive real numbers:

$$\text{output} = a^{\text{input}} \quad 0 < a$$

A defining property is drawn out directly from the definition of the simple exponential function:

$$a^{n+m} = \overbrace{a \cdot a \cdot \dots}^{n \text{ times}} \cdot \overbrace{a \cdot a \cdot \dots}^{m \text{ times}} = a^n a^m$$

The three defining properties allow the output for the exponential function to be populated from any integer input without restricting the input to integers:

- a. $a^1 = a \quad 0 < a$
- b. $a^{-1} = 1 \div a \quad 0 < a$
- c. $a^{x+y} = a^x a^y$ for all real numbers $x, y \quad 0 < a$

The inclusion of three generalization properties produces one unique function of real input for the generalized exponential function with base a restricted to positive real numbers:

- d. The function outputs a positive real value and is differentiable with respect to input.
- e. If $a \neq 1$, given any positive real output, a unique inverse exists as input and has a positive derivative.
- f. Given any real input, $1 = 1^{\text{input}}$.

5. Meaningful Generalization of the Exponent

- a. Case $a \neq 1$:

...Function defined as generalized exponential function:

$$f_a(\text{input}) = a^{\text{input}} \quad 0 < a$$

$$\text{output} = f_a(\text{input}) \quad 0 < a$$

...Application of property (4. d.):

$$0 < \text{output} \quad 0 < a$$

...Application of property (4. e.):

$$f_a^{-1}(\text{output}) = \text{input} \quad 0 < a$$

$$\dots \text{output} = f_a(\text{input}) \quad 0 < a$$

$$= f_a(f_a^{-1}(\text{output})) \quad 0 < a$$

Positive real numbers x, y selected as output variables:

$$0 < x$$

$$0 < y$$

$$x = f_a(f_a^{-1}(x)) \quad 0 < a$$

$$y = f_a(f_a^{-1}(y)) \quad 0 < a$$

...Application of property (4. c.):

$$\begin{aligned} f_a(f_a^{-1}(x) + f_a^{-1}(y)) &= f_a(f_a^{-1}(x)) f_a(f_a^{-1}(y)) & 0 < a \\ &= xy & 0 < a \end{aligned}$$

...Partial differentiation (4. d. and 4. e.)

with respect to output variable x:

$$\begin{aligned} f'_a(f_a^{-1}(x) + f_a^{-1}(y))(f_a^{-1})'(x) &= y & 0 < a \\ f'_a(f_a^{-1}(x) + f_a^{-1}(y)) &= \frac{y}{(f_a^{-1})'(x)} & 0 < a \end{aligned}$$

...Partial differentiation (4. d. and 4. e.)

with respect to output variable y:

$$\begin{aligned} f'_a(f_a^{-1}(x) + f_a^{-1}(y))(f_a^{-1})'(y) &= x & 0 < a \\ f'_a(f_a^{-1}(x) + f_a^{-1}(y)) &= \frac{x}{(f_a^{-1})'(y)} & 0 < a \end{aligned}$$

...Both results equated:

$$\begin{aligned} \frac{y}{(f_a^{-1})'(x)} &= \frac{x}{(f_a^{-1})'(y)} & 0 < a \\ (f_a^{-1})'(y)y &= (f_a^{-1})'(x)x & 0 < a \end{aligned}$$

...No dependence implied on either output variable:

$$\begin{aligned} (f_a^{-1})'(y)y &= \text{constant}_a & 0 < a \\ (f_a^{-1})'(x)x &= \text{constant}_a & 0 < a \\ (f_a^{-1})'(\text{output})\text{ output} &= \text{constant}_a & 0 < a \\ (f_a^{-1})'(\text{output}) &= \frac{\text{constant}_a}{\text{output}} & 0 < a \end{aligned}$$

...Antidifferentiation with respect to output,

utilizing dummy variable of integration s:

$$\begin{aligned} f_a^{-1}(\text{output}) &= \int_1^{\text{output}} \frac{\text{constant}_a}{s} ds + c & 0 < a \\ f_a^{-1}(1) &= \int_1^1 \frac{\text{constant}_a}{s} ds + c & 0 < a \\ &= c & 0 < a \end{aligned}$$

...Unique input (4. e.) that results in output = 1:

$$\begin{aligned} a^0 &= a^{1-1} \stackrel{4.c.}{=} a^1 a^{-1} \stackrel{4.b.}{=} a^1 (1 \div a) \stackrel{4.a.}{=} a(1 \div a) = 1 & 0 < a \\ f_a(0) &= 1 & 0 < a \\ 0 &= f_a^{-1}(1) & 0 < a \\ &= c \end{aligned}$$

...Substitution for constant c and \ln definition
back into previous equation:

$$\dots f_a^{-1}(\text{output}) = \text{constant}_a \ln(\text{output})$$

$$\text{input} = \text{constant}_a \ln(\text{output})$$

$$0 < a$$

$$0 < a$$

...Generalized exponential function well-defined
(inverse not everywhere equal to 0):

$$\text{constant}_a \neq 0$$

$$0 < a$$

...Equation solved for output:

$$\text{output} = \exp\left(\frac{\text{input}}{\text{constant}_a}\right)$$

$$0 < a$$

$$f_a(\text{input}) = \exp\left(\frac{\text{input}}{\text{constant}_a}\right)$$

$$0 < a$$

$$f_a(1) = \exp\left(\frac{1}{\text{constant}_a}\right)$$

$$0 < a$$

...Application of property (4. a.):

$$a = \exp\left(\frac{1}{\text{constant}_a}\right)$$

$$0 < a$$

$$\ln a = \frac{1}{\text{constant}_a}$$

$$0 < a$$

...Substitution back into previous equation:

$$\dots f_a(\text{input}) = \exp(\text{input} \ln a)$$

- $a^{\text{input}} = \exp(\text{input} \ln a)$

$$0 < a$$

$$0 < a$$

b. Case $a = 1$:

...Application of property (4. f.):

$$1^{\text{input}} = 1$$

...Composition of inverse functions:

$$= \exp(\ln 1)$$

...Substitution for \ln definition:

$$= \exp\left(\int_1^1 \frac{1}{s} ds\right)$$

$$= \exp(0)$$

$$= \exp(\text{input} \cdot 0)$$

$$= \exp(\text{input} \ln 1)$$

- $a^{\text{input}} = \exp(\text{input} \ln a)$

$$0 < a$$

6. Expression for \ln

a. Infinite Series

...Only simple exponents found in each term:

$$y = 1 + x^1 + x^2 + x^3 + \cdots + x^n \quad -1 < x < 1$$

$$xy = x^1 + x^2 + x^3 + x^4 + \cdots + x^{n+1} \quad -1 < x < 1$$

$$y - xy = 1 - x^{n+1} \quad -1 < x < 1$$

$$y = \frac{1 - x^{n+1}}{1 - x} \quad -1 < x < 1$$

$$\frac{1 - x^{n+1}}{1 - x} = 1 + x^1 + x^2 + x^3 + \cdots + x^n \quad -1 < x < 1$$

...Terms vanish as n approaches infinity:

- $\frac{1}{1-x} = 1 + x^1 + x^2 + x^3 + \cdots \quad -1 < x < 1$

- $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots \quad -1 < x < 1$

b. Proposed Infinite Series

...Only simple exponents found in each term:

- $-\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \cdots \quad -1 < x < 1$

c. Condition: Proposed Infinite Series Results in a Single Finite Value

...The even infinite series factored out:

$$-\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \cdots = -2x \left(1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \cdots \right) \quad -1 < x < 1$$

...Term-by-term comparisons:

$$\frac{x^2}{3} \leq x^2$$

$$\frac{x^4}{5} \leq x^4$$

$$\frac{x^6}{7} \leq x^6$$

$$\dots \leq \dots$$

...Implied inequality:

$$1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots \leq 1 + x^2 + x^4 + x^6 + \dots \quad -1 < x < 1$$

...Substitution for infinite series (6. a.):

$$1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots \leq \frac{1}{1-x^2} \quad -1 < x < 1$$

...Nondecreasing with each successive term
and existence of a finite upper bound:

$$1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots = (\text{finite value}) \quad -1 < x < 1$$

...Substitution back into previous equation:

$$\dots -\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \dots = -2x \quad (\text{finite value}) \quad -1 < x < 1$$

Proposed infinite series results in a single finite value:

$$\bullet \quad \rho(x) = -\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \dots \quad -1 < x < 1$$

d. Expression for \ln

...Differentiation with respect to x :

$$\frac{d}{dx}\rho(x) = -2 - 2x^2 - 2x^4 - 2x^6 - \dots \quad -1 < x < 1$$

...Substitution for infinite series (6. a.):

$$= \frac{-2}{1-x^2} \quad -1 < x < 1$$

...Insertion of an input transformation
before differentiation with respect to x :

$$\begin{aligned} \frac{d}{dx}\rho\left(\frac{1-x}{1+x}\right) &= \left(\frac{-2}{1-(1-x)^2/(1+x)^2}\right) \frac{d}{dx}\left(\frac{1-x}{1+x}\right) & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \left(\frac{-2}{1-(1-x)^2/(1+x)^2}\right) \left(\frac{-(1+x)-(1-x)}{(1+x)^2}\right) & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \frac{4}{(1+x)^2 - (1-x)^2} & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \frac{4}{(1+2x+x^2) - (1-2x+x^2)} & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \frac{4}{4x} & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \frac{1}{x} & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \end{aligned}$$

...Equivalent inequalities:

$$\begin{aligned}x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ \Leftrightarrow \left(\begin{array}{l} 0 < 1+x \\ -1-x < 1-x < 1+x \end{array} \right) \text{ or } \left(\begin{array}{l} 1+x < 0 \\ -1-x > 1-x > 1+x \end{array} \right) \\ \Leftrightarrow \left(\begin{array}{l} -1 < x \\ -2 < 0 < 2x \end{array} \right) \text{ or } \left(\begin{array}{l} x < -1 \\ -2 > 0 > 2x \end{array} \right) \\ \Leftrightarrow 0 < x\end{aligned}$$

...Substitution for equivalent inequality
back into previous equation:

$$\dots \frac{d}{dx} \rho \left(\frac{1-x}{1+x} \right) = \frac{1}{x} \quad 0 < x$$

...Antidifferentiation with respect to x ,
utilizing dummy variable of integration s :

$$\rho \left(\frac{1-x}{1+x} \right) = \int_1^x \frac{1}{s} ds + c \quad 0 < x$$

...Substitution for ln definition:

$$\rho \left(\frac{1-x}{1+x} \right) = \ln x + c \quad 0 < x$$

$$\ln x = -c + \rho \left(\frac{1-x}{1+x} \right) \quad 0 < x$$

$$= -c - \frac{2}{1} \left(\frac{1-x}{1+x} \right)^1 - \frac{2}{3} \left(\frac{1-x}{1+x} \right)^3 - \frac{2}{5} \left(\frac{1-x}{1+x} \right)^5 - \frac{2}{7} \left(\frac{1-x}{1+x} \right)^7 - \dots \quad 0 < x$$

$$\ln 1 = -c$$

...Substitution for ln definition:

$$\int_1^1 \frac{1}{s} ds = -c$$

$$0 = c$$

...Substitution for constant c
back into previous equation:

$$\bullet \quad \ln x = -\frac{2}{1} \left(\frac{1-x}{1+x} \right)^1 - \frac{2}{3} \left(\frac{1-x}{1+x} \right)^3 - \frac{2}{5} \left(\frac{1-x}{1+x} \right)^5 - \frac{2}{7} \left(\frac{1-x}{1+x} \right)^7 - \dots \quad 0 < x$$

7. Expression for exp

a. Key Infinite Series

...Only simple exponents found in each term:

- $1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

b. Condition: Key Infinite Series Results in a Single Finite Value

Key infinite series split at n^{th} term, where $n = |\text{int}(x)|$:

...Initial portion as finite series (when $0 < n$)

and remainder portion as infinite series:

$$\begin{aligned} 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots &= 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{x^{n-1}}{(n-1)!} \\ &\quad + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \frac{x^{n+3}}{(n+3)!} + \frac{x^{n+4}}{(n+4)!} + \frac{x^{n+5}}{(n+5)!} + \dots \end{aligned}$$

*...Remainder portion separated
into even/odd infinite series:*

$$\begin{aligned} &= 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{x^{n-1}}{(n-1)!} \\ &\quad + \frac{x^n}{n!} + \frac{x^{n+2}}{(n+2)!} + \frac{x^{n+4}}{(n+4)!} + \dots \\ &\quad + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+3}}{(n+3)!} + \frac{x^{n+5}}{(n+5)!} + \dots \end{aligned}$$

...Even/odd infinite series factored:

$$\begin{aligned} &= 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{x^{n-1}}{(n-1)!} \\ &\quad + \frac{x^n}{n!} \left(1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \dots \right) \\ &\quad + \frac{x^{n+1}}{(n+1)!} \left(1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \dots \right) \end{aligned}$$

...Key inequality:

$$-1 < \frac{x}{n+1} < 1$$

...Term-by-term comparisons:

$$\begin{aligned} \frac{x^2}{(n+1)(n+2)} &\leq \left(\frac{x}{n+1}\right)^2 \\ \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} &\leq \left(\frac{x}{n+1}\right)^4 \\ &\dots \leq \dots \\ \frac{x^2}{(n+2)(n+3)} &\leq \left(\frac{x}{n+1}\right)^2 \\ \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} &\leq \left(\frac{x}{n+1}\right)^4 \\ &\dots \leq \dots \end{aligned}$$

...Implied inequalities:

$$\begin{aligned} 1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \dots &\leq 1 + \left(\frac{x}{n+1}\right)^2 + \left(\frac{x}{n+1}\right)^4 + \dots \\ 1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \dots &\leq 1 + \left(\frac{x}{n+1}\right)^2 + \left(\frac{x}{n+1}\right)^4 + \dots \end{aligned}$$

...Substitution for infinite series (6. a.):

$$\begin{aligned} 1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \dots &\leq \frac{1}{1 - (x/(n+1))^2} \\ 1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \dots &\leq \frac{1}{1 - (x/(n+1))^2} \end{aligned}$$

...Nondecreasing with each successive term

and existence of a finite upper bound:

$$\begin{aligned} 1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \dots &= (\text{finite value } \#1) \\ 1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \dots &= (\text{finite value } \#2) \end{aligned}$$

...Substitution back into previous equation:

$$\begin{aligned} \dots 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots &= 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{x^{n-1}}{(n-1)!} \\ &\quad + \frac{x^n}{n!} (\text{finite value } \#1) + \frac{x^{n+1}}{(n+1)!} (\text{finite value } \#2) \end{aligned}$$

Key infinite series results in a single finite value:

- $k(x) = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

c. Key Properties

$$k(x) = 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

i. $k(0) = 1$

$$k'(x) = 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

ii. $k'(x) = k(x)$

...At least one of $k(x)$ or $k(-x)$

must be positive:

$$0 \leq x \Rightarrow 0 < 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

$$\Rightarrow 0 < k(x)$$

$$x < 0 \Rightarrow 0 < 1 - \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

$$\Rightarrow 0 < k(-x)$$

$$x \text{ is a real number} \Rightarrow 0 < k(x) \text{ or } 0 < k(-x)$$

$$0 = k(-x)k(x) - k(x)k(-x)$$

...Application of key property (ii):

$$= k(-x)k'(x) - k(x)k'(-x)$$

$$= k(-x)\frac{d}{dx}k(x) + k(x)\frac{d}{dx}k(-x)$$

...Application of product rule of differentiation:

$$= \frac{d}{dx}(k(-x)k(x))$$

$$c = k(-x)k(x)$$

$$c = k(-0)k(0)$$

...Application of key property (i):

$$c = 1$$

*...Substitution for constant c
back into previous equation:*

$$\dots 1 = k(-x)k(x)$$

*...Positive product with at least one factor positive
implies that both factors must be positive:*

iii. $0 < k(x)$

d. Solution for All Functions Preserving the Key Properties

...Application of key property (iii):

$$0 < k(x)$$

...Application of key property (ii):

$$\frac{k'(x)}{k(x)} = 1$$

...Antidifferentiation with respect to x :

$$\ln k(x) = x + c$$

$$\ln k(0) = c$$

...Application of key property (i):

$$\ln 1 = c$$

...Substitution for \ln definition:

$$\int_1^1 \frac{1}{s} ds = c$$

$$0 = c$$

...Substitution for constant c
back into previous equation:

$$\dots \ln k(x) = x$$

- $k(x) = \exp x$

- $\exp x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

8. Conclusion

$$2^\pi = \exp(\pi \ln 2)$$

$$= 1$$

$$\begin{aligned}
 &+ \left((2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \dots \right)^1 / 1 \\
 &+ \left((2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \dots \right)^2 / (1 \cdot 2) \\
 &+ \left((2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \dots \right)^3 / (1 \cdot 2 \cdot 3) \\
 &+ \left((2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \dots \right)^4 / (1 \cdot 2 \cdot 3 \cdot 4) \\
 &+ \dots \\
 &\approx 8.824977827
 \end{aligned}$$

The expression 2^π is meaningful because the generalization preserves the three defining properties that populate the simple exponential function.

9. Appendix: Proof by Excel Spreadsheet

- a. Formula Input: Copy With Formatting (Ctrl + h to replace all “ \approx ” with “=” and produce formula output.)

output	a	input	ln(a)	a^input	exp(input ln(a))	
calculator:	≈ 2	$\approx \text{PI}()$	$\approx \text{LN}(B2)$	$\approx B2^C2$	$\approx \text{EXP}(C2 * \text{LN}(B2))$	
	a	input	i	ln(a)	i!	exp(input ln(a))
manually:	≈ 2	$\approx \text{PI}()$	0	$\approx -2/(D4*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D4*2+1)$	≈ 1	≈ 1
			1	$\approx E4-2/(D5*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D5*2+1)$	$\approx D5^F4$	$\approx G4+(\$C\$4*\$E\$23)^D5/F5$
			2	$\approx E5-2/(D6*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D6*2+1)$	$\approx D6^F5$	$\approx G5+(\$C\$4*\$E\$23)^D6/F6$
			3	$\approx E6-2/(D7*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D7*2+1)$	$\approx D7^F6$	$\approx G6+(\$C\$4*\$E\$23)^D7/F7$
			4	$\approx E7-2/(D8*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D8*2+1)$	$\approx D8^F7$	$\approx G7+(\$C\$4*\$E\$23)^D8/F8$
			5	$\approx E8-2/(D9*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D9*2+1)$	$\approx D9^F8$	$\approx G8+(\$C\$4*\$E\$23)^D9/F9$
			6	$\approx E9-2/(D10*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D10*2+1)$	$\approx D10^F9$	$\approx G9+(\$C\$4*\$E\$23)^D10/F10$
			7	$\approx E10-2/(D11*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D11*2+1)$	$\approx D11^F10$	$\approx G10+(\$C\$4*\$E\$23)^D11/F11$
			8	$\approx E11-2/(D12*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D12*2+1)$	$\approx D12^F11$	$\approx G11+(\$C\$4*\$E\$23)^D12/F12$
			9	$\approx E12-2/(D13*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D13*2+1)$	$\approx D13^F12$	$\approx G12+(\$C\$4*\$E\$23)^D13/F13$
			10	$\approx E13-2/(D14*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D14*2+1)$	$\approx D14^F13$	$\approx G13+(\$C\$4*\$E\$23)^D14/F14$
			11	$\approx E14-2/(D15*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D15*2+1)$	$\approx D15^F14$	$\approx G14+(\$C\$4*\$E\$23)^D15/F15$
			12	$\approx E15-2/(D16*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D16*2+1)$	$\approx D16^F15$	$\approx G15+(\$C\$4*\$E\$23)^D16/F16$
			13	$\approx E16-2/(D17*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D17*2+1)$	$\approx D17^F16$	$\approx G16+(\$C\$4*\$E\$23)^D17/F17$
			14	$\approx E17-2/(D18*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D18*2+1)$	$\approx D18^F17$	$\approx G17+(\$C\$4*\$E\$23)^D18/F18$
			15	$\approx E18-2/(D19*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D19*2+1)$	$\approx D19^F18$	$\approx G18+(\$C\$4*\$E\$23)^D19/F19$
			16	$\approx E19-2/(D20*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D20*2+1)$	$\approx D20^F19$	$\approx G19+(\$C\$4*\$E\$23)^D20/F20$
			17	$\approx E20-2/(D21*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D21*2+1)$	$\approx D21^F20$	$\approx G20+(\$C\$4*\$E\$23)^D21/F21$
			18	$\approx E21-2/(D22*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D22*2+1)$	$\approx D22^F21$	$\approx G21+(\$C\$4*\$E\$23)^D22/F22$
			19	$\approx E22-2/(D23*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D23*2+1)$	$\approx D23^F22$	$\approx G22+(\$C\$4*\$E\$23)^D23/F23$

- b. Formula Output

output	a	input	ln(a)	a^input	exp(input ln(a))	
calculator:	2	3.14159265	0.693147181	8.824977827	8.8249778271	
	a	input	i	ln(a)	i!	exp(input ln(a))
manually:	2	3.14159265	0	0.666666667	1	1.000000000000
			1	0.691358025	1	3.1775860903
			2	0.693004115	2	5.5485266806
			3	0.693134757	6	7.2695024308
			4	0.693146047	24	8.2063956446
			5	0.693147074	120	8.6144287707
			6	0.693147170	720	8.7625166474
			7	0.693147180	5040	8.8085843760
			8	0.693147180	40320	8.8211239316
			9	0.693147181	362880	8.8241579274
			10	0.693147181	3628800	8.8248186061
			11	0.693147181	39916800	8.8249493956
			12	0.693147181	479001600	8.8249731294
			13	0.693147181	6227020800	8.8249771049
			14	0.693147181	87178291200	8.8249777233
			15	0.693147181	1.30767E+12	8.8249778131
			16	0.693147181	2.09228E+13	8.8249778253
			17	0.693147181	3.55687E+14	8.8249778269
			18	0.693147181	6.40237E+15	8.8249778271
			19	0.693147181	1.21645E+17	8.8249778271