

# Meaningful Generalization of the Exponent

## Formal Outline: Course in Introductory Calculus Required

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### 1. Abstract

The exponential function: enter  $a^n$  into a calculator, and the output will be  $a \cdot a \cdot \dots$   $n$  times. But, what if the expression  $2^\pi$  were entered into a calculator? Why does an actual output result? How is a meaningful value to be drawn out from an expression like  $2^\pi$ ?

### 2. Definition

natural logarithm:  $\ln x = \int_1^x \frac{1}{s} ds$   $0 < x$

*...Strictly increasing invertible function  
(derivative always positive for  $0 < x$ ):*

natural exponential function:  $\exp = \text{inverse } \ln$

$$\Rightarrow x = \ln(\exp x) \quad 0 < \exp x$$

$$x = \exp(\ln x) \quad 0 < x$$

partial differentiation: differentiation with respect to one variable,  
with all other variables treated as constant

factorial:  $n! = 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \dots \cdot n$

integer conversion:  $\text{int}(x) = \left( \begin{array}{l} \text{real number } x \text{ converted to integer} \\ \text{by discarding all digits after decimal} \end{array} \right)$

absolute value:  $|x| = \begin{cases} x & 0 \leq x \\ -x & x < 0 \end{cases}$

### 3. Definition of the Simple Exponential Function

The purpose of the exponential function is to reiterate multiplication a given number of times.

A meaningful output results from any real base  $a \neq 0$  raised to the exponent of any integer input:

- Multiplication a positive number of times:

$$a^n = 1 \cdot \overbrace{a \cdot a \cdot \dots}^{n \text{ times}}$$

- Multiplication zero number of times:

$$a^0 = 1$$

- Multiplication a negative number of times translates to the inverse operation of multiplication:

$$a^{-n} = 1 \div \overbrace{a \div a \div \dots}^{n \text{ times}} \quad a \neq 0$$

#### 4. Definition of the Generalized Exponential Function

The exponential function can be generalized to output a real value from any real input as long as base  $a$  is restricted to positive real numbers:

$$\text{output} = a^{\text{input}} \quad 0 < a$$

A defining property is drawn out directly from the definition of the simple exponential function:

$$a^{n+m} = \underbrace{a \cdot a \cdot \dots}_{n \text{ times}} \cdot \underbrace{a \cdot a \cdot \dots}_{m \text{ times}} = a^n a^m$$

The three defining properties allow the output for the exponential function to be populated from any integer input without restricting the input to integers:

- a.  $a^1 = a$   $0 < a$
- b.  $a^{-1} = 1 \div a$   $0 < a$
- c.  $a^{x+y} = a^x a^y$  for all real numbers  $x, y$   $0 < a$

The inclusion of three generalization properties produces one unique function of real input for the generalized exponential function with base  $a$  restricted to positive real numbers:

- d. The function outputs a positive real value and is differentiable with respect to input.
- e. If  $a \neq 1$ , given any positive real output, a unique inverse exists as input and has a positive derivative.
- f. Given any real input,  $1 = 1^{\text{input}}$ .

#### 5. Meaningful Generalization of the Exponent

- a. Case  $a \neq 1$ :

*...Function defined as generalized exponential function:*

$$f_a(\text{input}) = a^{\text{input}} \quad 0 < a$$

$$\text{output} = f_a(\text{input}) \quad 0 < a$$

*...Application of property (4. d.):*

$$0 < \text{output}$$

*...Application of property (4. e.):*

$$f_a^{-1}(\text{output}) = \text{input} \quad 0 < a$$

$$\dots \text{output} = f_a(\text{input}) \quad 0 < a$$

$$= f_a(f_a^{-1}(\text{output})) \quad 0 < a$$

Positive real numbers  $x, y$  selected as output variables:

$$0 < x$$

$$0 < y$$

$$x = f_a(f_a^{-1}(x)) \quad 0 < a$$

$$y = f_a(f_a^{-1}(y)) \quad 0 < a$$

...Application of property (4. c.):

$$\begin{aligned} f_a(f_a^{-1}(x) + f_a^{-1}(y)) &= f_a(f_a^{-1}(x)) f_a(f_a^{-1}(y)) \\ &= xy \end{aligned}$$

$$0 < a$$

$$0 < a$$

...Partial differentiation (4. d. and 4. e.)

with respect to output variable x:

$$f'_a(f_a^{-1}(x) + f_a^{-1}(y)) (f_a^{-1})'(x) = y$$

$$0 < a$$

$$f'_a(f_a^{-1}(x) + f_a^{-1}(y)) = \frac{y}{(f_a^{-1})'(x)}$$

$$0 < a$$

...Partial differentiation (4. d. and 4. e.)

with respect to output variable y:

$$f'_a(f_a^{-1}(x) + f_a^{-1}(y)) (f_a^{-1})'(y) = x$$

$$0 < a$$

$$f'_a(f_a^{-1}(x) + f_a^{-1}(y)) = \frac{x}{(f_a^{-1})'(y)}$$

$$0 < a$$

...Both results equated:

$$\frac{y}{(f_a^{-1})'(x)} = \frac{x}{(f_a^{-1})'(y)}$$

$$0 < a$$

$$(f_a^{-1})'(y) y = (f_a^{-1})'(x) x$$

$$0 < a$$

...No dependence implied on either output variable:

$$(f_a^{-1})'(y) y = \text{constant}_a$$

$$0 < a$$

$$(f_a^{-1})'(x) x = \text{constant}_a$$

$$0 < a$$

$$(f_a^{-1})'(\text{output}) \text{output} = \text{constant}_a$$

$$0 < a$$

$$(f_a^{-1})'(\text{output}) = \frac{\text{constant}_a}{\text{output}}$$

$$0 < a$$

...Antidifferentiation with respect to output,

utilizing dummy variable of integration s:

$$f_a^{-1}(\text{output}) = \int_1^{\text{output}} \frac{\text{constant}_a}{s} ds + c$$

$$0 < a$$

$$f_a^{-1}(1) = \int_1^1 \frac{\text{constant}_a}{s} ds + c$$

$$0 < a$$

$$= c$$

$$0 < a$$

...Unique input (4. e.) that results in output = 1:

$$a^0 = a^{1-1} \stackrel{4.c}{=} a^1 a^{-1} \stackrel{4.b}{=} a^1 (1 \div a) \stackrel{4.a}{=} a(1 \div a) = 1$$

$$0 < a$$

$$f_a(0) = 1$$

$$0 < a$$

$$0 = f_a^{-1}(1)$$

$$0 < a$$

$$= c$$

...Substitution for constant  $c$  and  $\ln$  definition  
back into previous equation:

$$\begin{aligned} \dots f_a^{-1}(\text{output}) &= \text{constant}_a \ln(\text{output}) & 0 < a \\ \text{input} &= \text{constant}_a \ln(\text{output}) & 0 < a \end{aligned}$$

...Generalized exponential function well-defined  
(inverse not everywhere equal to 0):

$$\text{constant}_a \neq 0 \quad 0 < a$$

...Equation solved for output:

$$\text{output} = \exp\left(\frac{\text{input}}{\text{constant}_a}\right) \quad 0 < a$$

$$f_a(\text{input}) = \exp\left(\frac{\text{input}}{\text{constant}_a}\right) \quad 0 < a$$

$$f_a(1) = \exp\left(\frac{1}{\text{constant}_a}\right) \quad 0 < a$$

...Application of property (4. a.):

$$a = \exp\left(\frac{1}{\text{constant}_a}\right) \quad 0 < a$$

$$\ln a = \frac{1}{\text{constant}_a} \quad 0 < a$$

...Substitution back into previous equation:

$$\dots f_a(\text{input}) = \exp(\text{input} \ln a) \quad 0 < a$$

$$\bullet \quad a^{\text{input}} = \exp(\text{input} \ln a) \quad 0 < a$$

**b. Case  $a = 1$ :**

...Application of property (4. f.):

$$1^{\text{input}} = 1$$

...Composition of inverse functions:

$$= \exp(\ln 1)$$

...Substitution for  $\ln$  definition:

$$= \exp\left(\int_1^1 \frac{1}{s} ds\right)$$

$$= \exp(0)$$

$$= \exp(\text{input} \cdot 0)$$

$$= \exp(\text{input} \ln 1)$$

$$\bullet \quad a^{\text{input}} = \exp(\text{input} \ln a) \quad 0 < a$$

## 6. Expression for ln

### a. Infinite Series

...Only simple exponents found in each term:

$$y = 1 + x^1 + x^2 + x^3 + \dots + x^n \quad -1 < x < 1$$

$$xy = x^1 + x^2 + x^3 + x^4 + \dots + x^{n+1} \quad -1 < x < 1$$

$$y - xy = 1 - x^{n+1} \quad -1 < x < 1$$

$$y = \frac{1 - x^{n+1}}{1 - x} \quad -1 < x < 1$$

$$\frac{1 - x^{n+1}}{1 - x} = 1 + x^1 + x^2 + x^3 + \dots + x^n \quad -1 < x < 1$$

...Terms vanish as  $n$  approaches infinity:

$$\bullet \quad \frac{1}{1 - x} = 1 + x^1 + x^2 + x^3 + \dots \quad -1 < x < 1$$

$$\bullet \quad \frac{1}{1 - x^2} = 1 + x^2 + x^4 + x^6 + \dots \quad -1 < x < 1$$

### b. Proposed Infinite Series

...Only simple exponents found in each term:

$$\bullet \quad -\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \dots \quad -1 < x < 1$$

### c. Condition: Proposed Infinite Series Results in a Single Finite Value

...The even infinite series factored out:

$$-\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \dots = -2x \left( 1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots \right) \quad -1 < x < 1$$

...Term-by-term comparisons:

$$\frac{x^2}{3} \leq x^2$$

$$\frac{x^4}{5} \leq x^4$$

$$\frac{x^6}{7} \leq x^6$$

$$\dots \leq \dots$$

...Implied inequality:

$$1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots \leq 1 + x^2 + x^4 + x^6 + \dots \quad -1 < x < 1$$

...Substitution for infinite series (6. a.):

$$1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots \leq \frac{1}{1-x^2} \quad -1 < x < 1$$

...Nondecreasing with each successive term  
and existence of a finite upper bound:

$$1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots = (\text{finite value}) \quad -1 < x < 1$$

...Substitution back into previous equation:

$$\dots -\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \dots = -2x (\text{finite value}) \quad -1 < x < 1$$

Proposed infinite series results in a single finite value:

$$\bullet \quad \rho(x) = -\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \dots \quad -1 < x < 1$$

#### d. Expression for ln

...Differentiation with respect to x:

$$\frac{d}{dx}\rho(x) = -2 - 2x^2 - 2x^4 - 2x^6 - \dots \quad -1 < x < 1$$

...Substitution for infinite series (6. a.):

$$= \frac{-2}{1-x^2} \quad -1 < x < 1$$

...Insertion of an input transformation  
before differentiation with respect to x:

$$\begin{aligned} \frac{d}{dx}\rho\left(\frac{1-x}{1+x}\right) &= \left(\frac{-2}{1-(1-x)^2/(1+x)^2}\right) \frac{d}{dx}\left(\frac{1-x}{1+x}\right) & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \left(\frac{-2}{1-(1-x)^2/(1+x)^2}\right) \left(\frac{-(1+x) - (1-x)}{(1+x)^2}\right) & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \frac{4}{(1+x)^2 - (1-x)^2} & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \frac{4}{(1+2x+x^2) - (1-2x+x^2)} & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \frac{4}{4x} & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \\ &= \frac{1}{x} & x \neq -1, -1 < \frac{1-x}{1+x} < 1 \end{aligned}$$

...Equivalent inequalities:

$$x \neq -1, -1 < \frac{1-x}{1+x} < 1$$

$$\Leftrightarrow \left( \begin{array}{c} 0 < 1+x \\ -1-x < 1-x < 1+x \end{array} \right) \text{ or } \left( \begin{array}{c} 1+x < 0 \\ -1-x > 1-x > 1+x \end{array} \right)$$

$$\Leftrightarrow \left( \begin{array}{c} -1 < x \\ -2 < 0 < 2x \end{array} \right) \text{ or } \left( \begin{array}{c} x < -1 \\ -2 > 0 > 2x \end{array} \right)$$

$$\Leftrightarrow 0 < x$$

...Substitution for equivalent inequality

back into previous equation:

$$\dots \frac{d}{dx} \rho \left( \frac{1-x}{1+x} \right) = \frac{1}{x}$$

$$0 < x$$

...Antidifferentiation with respect to  $x$ ,

utilizing dummy variable of integration  $s$ :

$$\rho \left( \frac{1-x}{1+x} \right) = \int_1^x \frac{1}{s} ds + c$$

$$0 < x$$

...Substitution for  $\ln$  definition:

$$\rho \left( \frac{1-x}{1+x} \right) = \ln x + c$$

$$0 < x$$

$$\ln x = -c + \rho \left( \frac{1-x}{1+x} \right)$$

$$0 < x$$

$$= -c - \frac{2}{1} \left( \frac{1-x}{1+x} \right)^1 - \frac{2}{3} \left( \frac{1-x}{1+x} \right)^3 - \frac{2}{5} \left( \frac{1-x}{1+x} \right)^5 - \frac{2}{7} \left( \frac{1-x}{1+x} \right)^7 - \dots$$

$$0 < x$$

$$\ln 1 = -c$$

...Substitution for  $\ln$  definition:

$$\int_1^1 \frac{1}{s} ds = -c$$

$$0 = c$$

...Substitution for constant  $c$

back into previous equation:

$$\bullet \quad \ln x = -\frac{2}{1} \left( \frac{1-x}{1+x} \right)^1 - \frac{2}{3} \left( \frac{1-x}{1+x} \right)^3 - \frac{2}{5} \left( \frac{1-x}{1+x} \right)^5 - \frac{2}{7} \left( \frac{1-x}{1+x} \right)^7 - \dots$$

$$0 < x$$

## 7. Expression for exp

### a. Key Infinite Series

...Only simple exponents found in each term:

$$\bullet \quad 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

### b. Condition: Key Infinite Series Results in a Single Finite Value

Key infinite series split at  $n^{\text{th}}$  term, where  $n = |\text{int}(x)|$ :

...Initial portion as finite series (when  $0 < n$ )

and remainder portion as infinite series:

$$\begin{aligned} 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots &= 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{x^{n-1}}{(n-1)!} \\ &+ \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \frac{x^{n+3}}{(n+3)!} + \frac{x^{n+4}}{(n+4)!} + \frac{x^{n+5}}{(n+5)!} + \dots \end{aligned}$$

...Remainder portion separated  
into even/odd infinite series:

$$\begin{aligned} &= 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{x^{n-1}}{(n-1)!} \\ &+ \frac{x^n}{n!} + \frac{x^{n+2}}{(n+2)!} + \frac{x^{n+4}}{(n+4)!} + \dots \\ &+ \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+3}}{(n+3)!} + \frac{x^{n+5}}{(n+5)!} + \dots \end{aligned}$$

...Even/odd infinite series factored:

$$\begin{aligned} &= 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{x^{n-1}}{(n-1)!} \\ &+ \frac{x^n}{n!} \left( 1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \dots \right) \\ &+ \frac{x^{n+1}}{(n+1)!} \left( 1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \dots \right) \end{aligned}$$

...Key inequality:

$$-1 < \frac{x}{n+1} < 1$$



...Term-by-term comparisons:

$$\begin{aligned}\frac{x^2}{(n+1)(n+2)} &\leq \left(\frac{x}{n+1}\right)^2 \\ \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} &\leq \left(\frac{x}{n+1}\right)^4 \\ &\dots \leq \dots \\ \frac{x^2}{(n+2)(n+3)} &\leq \left(\frac{x}{n+1}\right)^2 \\ \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} &\leq \left(\frac{x}{n+1}\right)^4 \\ &\dots \leq \dots\end{aligned}$$

...Implied inequalities:

$$\begin{aligned}1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \dots &\leq 1 + \left(\frac{x}{n+1}\right)^2 + \left(\frac{x}{n+1}\right)^4 + \dots \\ 1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \dots &\leq 1 + \left(\frac{x}{n+1}\right)^2 + \left(\frac{x}{n+1}\right)^4 + \dots\end{aligned}$$

...Substitution for infinite series (6. a.):

$$\begin{aligned}1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \dots &\leq \frac{1}{1 - (x/(n+1))^2} \\ 1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \dots &\leq \frac{1}{1 - (x/(n+1))^2}\end{aligned}$$

...Nondecreasing with each successive term

and existence of a finite upper bound:

$$\begin{aligned}1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \dots &= (\text{finite value \#1}) \\ 1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \dots &= (\text{finite value \#2})\end{aligned}$$

...Substitution back into previous equation:

$$\begin{aligned}\dots \quad 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots &= 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{x^{n-1}}{(n-1)!} \\ &\quad + \frac{x^n}{n!} (\text{finite value \#1}) + \frac{x^{n+1}}{(n+1)!} (\text{finite value \#2})\end{aligned}$$

Key infinite series results in a single finite value:

- $k(x) = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

**c. Key Properties**

$$k(x) = 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

i.  $k(0) = 1$

$$k'(x) = 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

ii.  $k'(x) = k(x)$

...At least one of  $k(x)$  or  $k(-x)$

must be positive:

$$0 \leq x \quad \Rightarrow \quad 0 < 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

$$\Rightarrow 0 < k(x)$$

$$x < 0 \quad \Rightarrow \quad 0 < 1 - \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

$$\Rightarrow 0 < k(-x)$$

$$x \text{ is a real number} \quad \Rightarrow \quad 0 < k(x) \text{ or } 0 < k(-x)$$

$$0 = k(-x) k(x) - k(x) k(-x)$$

...Application of key property (ii):

$$= k(-x) k'(x) - k(x) k'(-x)$$

$$= k(-x) \frac{d}{dx} k(x) + k(x) \frac{d}{dx} k(-x)$$

...Application of product rule of differentiation:

$$= \frac{d}{dx} (k(-x) k(x))$$

$$c = k(-x) k(x)$$

$$c = k(-0) k(0)$$

...Application of key property (i):

$$c = 1$$

...Substitution for constant  $c$

back into previous equation:

$$\dots \quad 1 = k(-x) k(x)$$

...Positive product with at least one factor positive

implies that both factors must be positive:

iii.  $0 < k(x)$

**d. Solution for All Functions Preserving the Key Properties***...Application of key property (iii.):*

$$0 < k(x)$$

*...Application of key property (ii.):*

$$\frac{k'(x)}{k(x)} = 1$$

*...Antidifferentiation with respect to x:*

$$\ln k(x) = x + c$$

$$\ln k(0) = c$$

*...Application of key property (i.):*

$$\ln 1 = c$$

*...Substitution for ln definition:*

$$\int_1^x \frac{1}{s} ds = c$$

$$0 = c$$

*...Substitution for constant c  
back into previous equation:*

$$\dots \ln k(x) = x$$

$$\bullet \quad k(x) = \exp x$$

$$\bullet \quad \exp x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

**8. Conclusion**

$$2^\pi = \exp(\pi \ln 2)$$

$$= 1$$

$$\begin{aligned} &+ \left( (2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \dots \right)^1 / 1 \\ &+ \left( (2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \dots \right)^2 / (1 \cdot 2) \\ &+ \left( (2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \dots \right)^3 / (1 \cdot 2 \cdot 3) \\ &+ \left( (2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \dots \right)^4 / (1 \cdot 2 \cdot 3 \cdot 4) \\ &+ \dots \\ &\approx 8.824977827 \end{aligned}$$

The expression  $2^\pi$  is meaningful because the generalization preserves the three defining properties that populate the simple exponential function.

## 9. Appendix: Proof by Excel Spreadsheet

a. Formula Input: Copy With Formatting (Ctrl + h to replace all “≈” with “=” and produce formula output.)

output	a	input		$\ln(a)$	$a^{\text{input}}$	$\exp(\text{input } \ln(a))$
calculator:	≈2	≈PI()		≈LN(B2)	≈B2^C2	≈EXP(C2*LN(B2))
	a	input	i	$\ln(a)$	i!	$\exp(\text{input } \ln(a))$
manually:	≈2	≈PI()	0	≈-2/(D4*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D4*2+1)	≈1	≈1
			1	≈E4-2/(D5*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D5*2+1)	≈D5*F4	≈G4+(\$C\$4*\$E\$23)^D5/F5
			2	≈E5-2/(D6*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D6*2+1)	≈D6*F5	≈G5+(\$C\$4*\$E\$23)^D6/F6
			3	≈E6-2/(D7*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D7*2+1)	≈D7*F6	≈G6+(\$C\$4*\$E\$23)^D7/F7
			4	≈E7-2/(D8*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D8*2+1)	≈D8*F7	≈G7+(\$C\$4*\$E\$23)^D8/F8
			5	≈E8-2/(D9*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D9*2+1)	≈D9*F8	≈G8+(\$C\$4*\$E\$23)^D9/F9
			6	≈E9-2/(D10*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D10*2+1)	≈D10*F9	≈G9+(\$C\$4*\$E\$23)^D10/F10
			7	≈E10-2/(D11*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D11*2+1)	≈D11*F10	≈G10+(\$C\$4*\$E\$23)^D11/F11
			8	≈E11-2/(D12*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D12*2+1)	≈D12*F11	≈G11+(\$C\$4*\$E\$23)^D12/F12
			9	≈E12-2/(D13*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D13*2+1)	≈D13*F12	≈G12+(\$C\$4*\$E\$23)^D13/F13
			10	≈E13-2/(D14*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D14*2+1)	≈D14*F13	≈G13+(\$C\$4*\$E\$23)^D14/F14
			11	≈E14-2/(D15*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D15*2+1)	≈D15*F14	≈G14+(\$C\$4*\$E\$23)^D15/F15
			12	≈E15-2/(D16*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D16*2+1)	≈D16*F15	≈G15+(\$C\$4*\$E\$23)^D16/F16
			13	≈E16-2/(D17*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D17*2+1)	≈D17*F16	≈G16+(\$C\$4*\$E\$23)^D17/F17
			14	≈E17-2/(D18*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D18*2+1)	≈D18*F17	≈G17+(\$C\$4*\$E\$23)^D18/F18
			15	≈E18-2/(D19*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D19*2+1)	≈D19*F18	≈G18+(\$C\$4*\$E\$23)^D19/F19
			16	≈E19-2/(D20*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D20*2+1)	≈D20*F19	≈G19+(\$C\$4*\$E\$23)^D20/F20
			17	≈E20-2/(D21*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D21*2+1)	≈D21*F20	≈G20+(\$C\$4*\$E\$23)^D21/F21
			18	≈E21-2/(D22*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D22*2+1)	≈D22*F21	≈G21+(\$C\$4*\$E\$23)^D22/F22
			19	≈E22-2/(D23*2+1)*((1-\$B\$4)/(1+\$B\$4))^(D23*2+1)	≈D23*F22	≈G22+(\$C\$4*\$E\$23)^D23/F23

b. Formula Output

output	a	input		$\ln(a)$	$a^{\text{input}}$	$\exp(\text{input } \ln(a))$
calculator:	2	3.14159265		0.693147181	8.824977827	8.8249778271
	a	input	i	$\ln(a)$	i!	$\exp(\text{input } \ln(a))$
manually:	2	3.14159265	0	0.666666667	1	1.0000000000
			1	0.691358025	1	3.1775860903
			2	0.693004115	2	5.5485266806
			3	0.693134757	6	7.2695024308
			4	0.693146047	24	8.2063956446
			5	0.693147074	120	8.6144287707
			6	0.693147170	720	8.7625166474
			7	0.693147180	5040	8.8085843760
			8	0.693147180	40320	8.8211239316
			9	0.693147181	362880	8.8241579274
			10	0.693147181	3628800	8.8248186061
			11	0.693147181	39916800	8.8249493956
			12	0.693147181	479001600	8.8249731294
			13	0.693147181	6227020800	8.8249771049
			14	0.693147181	87178291200	8.8249777233
			15	0.693147181	1.30767E+12	8.8249778131
			16	0.693147181	2.09228E+13	8.8249778253
			17	0.693147181	3.55687E+14	8.8249778269
			18	0.693147181	6.40237E+15	8.8249778271
			19	0.693147181	1.21645E+17	8.8249778271