IGIMF Theory supplementary
Theoretical background and a modification of Optimal sampling to generate custom composite IMFs

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Abstract

This supplementary document has the aim to document and explain the theoretical background of the published code. It allows to sample composite IMFs (cIMFs) and IGIMFs within the IGIMF Theory, but also gives the possibility to sample custom cIMFs for the study of sub-galactic regions.
First, the general context is laid out, explaining the initial mass function (IMF) in Sec. 1, the embedded cluster mass function (ECMF) in Sec. 2 and the IGIMF theory in Sec. 3.
Sec. 3.1.3 describes how the Optimal sampling algorithm used in the IGIMF Theory can be modified to create custom star cluster samples that not only follow a given total mass, but also a given most massive cluster mass. This can be useful e.g. to create cIMFs that fit empirical results.
Sec. 3.1.4 then goes into how this could be used to study composite IMFs in star-forming sites. It also points out how one has to be careful with interpreting results using this approach with regards to the overarching IGIMF Theory.
The modification of the Optimal sampling algorithm should not be seen as an extension of the IGIMF theory, but rather as a tool to help study star-formation on sub-galactic scales.
However, the reformulation of the Optimal Sampling algorithm could potentially be of great use for the eventual development of a sub-galactic extension of the IGIMF Theory.
Also, the modified algorithm is not constricted to only describe star clusters and could also be interesting for the ICIMF Theory to sample e.g. clump masses in molecular clouds.

1 The initial mass function

The initial mass function (IMF), $\xi(m)$, describes the number distribution of initial stellar masses in a stellar population. It is defined as

$$\xi(m) = \frac{\mathrm{d}N}{\mathrm{d}m}\,, \tag{1}$$

with $\mathrm{d}N$ the number of stars in the mass interval $[m, m + \mathrm{d}m]$.
The IMF of stars in the Solar neighborhood, the canonical IMF, has first been described by Salpeter (1955) as a single power law with an index $\alpha_{\text{Salpeter}} = 2.35$ for stars with masses of m between $\approx 0.4 - 10.0\,M_\odot$. The general form of the IMF is

$$\xi(m) = \frac{\mathrm{d}N}{\mathrm{d}m} = km^{-\alpha}\,, \tag{2}$$

with k as a normalization constant.
Different formulations of the IMF have been published following Salpeter covering larger mass ranges, e.g. the Kroupa IMF (Kroupa 2001) which is a three-part powerlaw with the exponents:

$$\alpha = \begin{cases} 1.3, & 0.08M_\odot \leq m < 0.5M_\odot \\ 2.3, & 0.5M_\odot \leq m < 1.0M_\odot \\ 2.3, & 1.0M_\odot \leq m < 150M_\odot \end{cases}$$

1.1 The stellar IMF

Observations of very young stars suggest that these form in embedded clusters (Kroupa 1995a; Kroupa 1995b; Lada & Lada 2003). An embedded cluster being "a cluster that is fully or partially embedded in interstellar gas and dust" (Lada & Lada 2003).
The *stellar* IMF describes the population of a single embedded cluster, born from one molecular cloud clump. This is opposed to a *composite* IMF (cIMF) which describes a stellar population that consists of stars from various embedded clusters. An example for a composite IMF is the Salpeter IMF. The composite IMF of a whole galaxy is called *galaxy-wide* IMF (gwIMF).
Integrating over the stellar IMF, gives the number of stars N inside the initial cluster

$$N = \int_{m_{\min}}^{m_{\max}} \xi(m)\,\mathrm{d}m = \int_{m_{\min}}^{m_{\max}} k_{\text{str}} m^{-\alpha}\,\mathrm{d}m\,, \tag{3}$$

with $m_{\max}$ being the mass of the most massive star of the cluster, $m_{\min}$ the mass of the least massive star in the cluster and k_{str} the normalization constant of the stellar IMF.
The total mass of an embedded cluster is similarly given as

$$\frac{M_{\text{ecl}}}{M_\odot} = \int_{m_{\min}}^{m_{\max}} \xi(m)\; m\,\mathrm{d}m\,. \tag{4}$$

Stellar IMFs follow the $m_{\rm max}-M_{\rm ecl}$ relation, a correlation between the maximum stellar mass that can be found inside an embedded cluster, $m_{\rm max}$, and the stellar mass of this embedded cluster $M_{\rm ecl}$. This has been shown e.g. in Weidner & Kroupa (2006), Weidner et al. (2010), Weidner et al. (2013a) and Weidner et al. (2014). The relation indicates that embedded clusters need to have a certain mass in order to form certain stellar masses.

1.2 The varying stellar IMF

More recent research has shown, that the stellar IMF seems to be sensitive to metallicity and density in extreme environments such as star burst clusters (Marks et al. 2012) and ultra-compact dwarf galaxies (Dabringhausen et al. 2012).
The variation of the high-mass part of the stellar IMF, α_3, with density and metallicity has originally been determined by Marks et al. (2012), while varying slopes of the low-mass parts of the stellar IMF, α_1 and α_2, were formulated by Jeřábková et al. (2018). In this work, the formulation of the varying IMF from Yan et al. (2021) is adapted. The dependence of the IMF on metallicity and density is such that

$$\begin{aligned} \alpha_1 &= 1.3 + \Delta\alpha(Z - Z_\odot), \qquad 0.08 \leq m/M_\odot < 0.5\,, \\ \alpha_2 &= 2.3 + \Delta\alpha(Z - Z_\odot), \qquad 0.5 \leq m/M_\odot < 1.0\,, \end{aligned}$$

with Z the metallicity of the targeted system and and $Z_\odot = 0.0142$ the solar metallicity. For this solar metallicity Yan et al. (2021) empirically determined a value of $\Delta\alpha = 63$.

$$\begin{aligned} \alpha_3 &= \begin{cases} 2.3, x < -0.87, \\ -0.41x + 1.94, x > -0.87 \end{cases} \qquad 1.0 \leq m/M_\odot < m_{\rm max}\,, \\ x &= -0.14[Z] + 0.99\log_{10}(\rho_{\rm cl}/(10^6))\,, \end{aligned}$$

with $\log_{10}(\rho_{\rm cl}) = 0.61\log_{10}(M_{\rm ecl}) + 2.85$, where $\rho_{\rm cl}$ represents the density of the cluster.
For the metallicity $[Z/X] = \log_{10}(Z/X) - \log_{10}(Z_\odot/X_\odot)$, one can assume the initial hydrogen mass fraction X of any star to be the same as in the sun ($X_\odot \approx X$) leading to $[Z/X] \approx \log_{10}(Z/Z_\odot) = [Z]$.
The higher the density in a region or the lower the metallicity, the smaller α_3 becomes. This is equivalent to an excess of massive stars compared to a Kroupa IMF. The stellar IMF becomes *top-heavy*.
Note, that since the cluster density scales with cluster mass, this means that more massive clusters have top-heavier stellar IMFs.

2 The embedded cluster mass function

A similar distribution function as the IMF for stellar masses also exists for star clusters, the embedded cluster mass function (ECMF). *Embedded clusters* are young clusters (< 1 Myr) that are still surrounded by gas and dust from the molecular cloud in which they formed (Lada & Lada 2003; Kroupa et al. 2024). The canonical embedded cluster mass function follows a powerlaw of the form.

$$\frac{\mathrm{d}N_{\mathrm{ecl}}}{\mathrm{d}M_{\mathrm{ecl}}} = \xi_{\mathrm{ecl}} \propto M_{\mathrm{ecl}}^{-\beta} \,,$$

with N_{ecl} the number of clusters in the mass range $[M_{\mathrm{ecl}}, M_{\mathrm{ecl}} + \mathrm{d}M_{\mathrm{ecl}}]$. The power-law index of the canonical ECMF is $\beta \approx 2$ (Lada & Lada 2003).

The ECMF also has been suggested to vary with star formation rate (SFR) of a galaxy by Weidner et al. (2013b) in order to match the variation of galaxy-wide IMFs with SFR observed by Gunawardhana et al. (2011). In order to fix the variation of the IMF above $1\,M_\odot$, Weidner et al. (2013b) proposed the following variation of the ECMF, if the SFR $> 1\,\frac{M_\odot}{\mathrm{yr}}$:

$$\beta = \begin{cases} 2.00, & \mathrm{SFR} < 1\,M_\odot/\mathrm{yr} \\ -0.106\log_{10}(\mathrm{SFR}) + 2.00, & \mathrm{SFR} \geq 1\,M_\odot/\mathrm{yr} \end{cases} \tag{5}$$

An empirical relation between the present galaxy-wide SFR and the most massive young ($<$10 Myr) cluster mass has been found as well, the $M_{\mathrm{ecl,max}} - \mathrm{SFR}$ relation (Randriamanakoto et al. (2013)).

3 The IGIMF Theory

Kroupa & Weidner (2003) were the first to point out that the fact that stars primarily form in star clusters predicts that the galactic-field IMF is the sum of all the stellar IMFs.

The integrated galactic IMF (IGIMF) as the integral over all stellar IMFs is given as

$$\xi_{\mathrm{IGIMF}}(m) = \int_{M_{\mathrm{ecl,min}}}^{M_{\mathrm{ecl,max}}} \xi(m \leq m_{\mathrm{max}})\xi_{\mathrm{ecl}}(M_{\mathrm{ecl}})\;\mathrm{d}M_{\mathrm{ecl}} \,,$$

with ξ_{ecl} the ECMF and $\xi(m \leq m_{\mathrm{max}})$ the stellar IMF of a particular cluster with m_{max} the maximum stellar mass within this cluster.

The mass of the most massive star was defined in their paper such that

$$1 = \int_{m_{\mathrm{max}}}^{m_{\mathrm{max}*}} \xi(m)\,\mathrm{d}m \,, \tag{6}$$

$$M_{\mathrm{ecl}} = \int_{m_{\mathrm{l}}}^{m_{\mathrm{max}}} \xi(m)\,\mathrm{d}m \,, \tag{7}$$

where m_{l} and m_{max_*} are the physical lower and upper limit for stellar masses, respectively. This formulation already suggested a functional relation between m_{max} and M_{ecl} which has later been verified as already mentioned.

The formulation of the theory changed slightly over time.[1] But in general, the IGIMF theory is built on the following axioms, that are described and justified in more detail in Yan et al. (2017):

- All stars form in embedded clusters.
- The galaxy-wide IMF is the sum of stellar IMFs of all the embedded clusters. This also implies that even if the stellar IMF is universal, the galaxy-wide IMF can vary.
- The stellar IMF generally follows a Kroupa IMF, but can vary with density and metallicity as described in Sec. 1.2. A stellar upper mass limit of $150\,M_\odot$ is adopted[2].
- The ECMF is a single slope power-law that varies with the galaxy-wide SFR, following the form in Weidner et al. (2013b) (see, Eq. 5). Although the true shape of the ECMF might be more complicated, e.g. a Schechter-type form as shown in Lieberz & Kroupa (2017), a power law is assumed as a good approximation.
- An ensemble of embedded clusters that optimally populates[3] the ECMF is formed within a period of $\delta t = 10\,\mathrm{Myr}$ with a constant SFR. The period, or star formation epoch, δt is related to the total mass of embedded clusters M_{tot} formed in the time δt such that $M_{\mathrm{tot}} = \mathrm{SFR} \cdot \delta t$.

It it important to note, that the form of the IGIMF as a sum of stellar IMFs can significantly deviate from a pure power law. Also, one IGIMF only describes the galaxy-wide stellar population formed within one star-formation epoch $\delta t = 10\,\mathrm{Myr}$.
Successes of the IGIMF theory are e.g. that it can explain the Hα cutoff in disk galaxies (Pflamm-Altenburg & Kroupa 2008) and the gwIMFs variation with SFR at the high-mass end (Yan et al. 2017).
It also predicts the existence of a $m_{\mathrm{max}} - M_{\mathrm{ecl}}$ and $M_{\mathrm{ecl,max}} - \mathrm{SFR}$ relation. Both arise from the fact that the IGIMF is assumed to be an optimally sampled function. Optimal sampling will be explained in the following Sec. 3.1.

3.1 Optimal Sampling

Optimal sampling is in contrast to random sampling a sampling method without Poisson noise and is designed to perfectly distribute stars along a defined distribution (Kroupa et al. 2013). Due to the drawn samples following the

[1] The theory mainly changed regarding the optimal sampling procedure (see, Kroupa et al. 2013; Schulz et al. 2015). The most up-to-date description of the sampling is given in the sections below.

[2] The origin of stars more massive than $150\,M_\odot$, as currently observed in R136 (e.g. Shenar et al. 2023), is not directly adressed by the IGIMF Theory.

[3] "optimally populates" here means that the ECMF is an optimally sampled function(see, Sec. 3.1)

desired distribution function tightly, optimal sampling corresponds to highly self-regulated systems (Yan et al. 2017).
Observations of the Orion Nebula Cluster reveal a deficit of massive stars, which is in tension with a randomly sampled stellar IMF (Hsu et al. 2012). If stellar masses formed according to a mainly random process inside the embedded clusters, an ensemble of low-mass clusters should be able to produce the same amount of massive stars as a single cluster of equal total mass. A stochastic IMF was ruled out in their study with 3-4 sigma significance.
An advantage of the optimal sampling method over random sampling is also that a $m_{\rm max} - M_{\rm ecl}$ and $M_{\rm ecl,max} - {\rm SFR}$ relation emerge directly from it. Since stellar populations seem to follow the $m_{\rm max} - M_{\rm ecl}$ relation rather tightly (see, e.g. Weidner et al. 2013a; Yan et al. 2023) this indicates a high self-regulation of the star formation process and therefore that the stellar IMF may be well described by optimal sampling.
In order to explain the general optimal sampling algorithm used to draw from the IMF and ECMF, the notation presented in Tab. 1 is introduced which is only valid in this section.
The documentation of the `galIMF` code available on Github[4] goes deeper into how to actually realize an optimal sampling code and was used as a guideline for the following sections.

Symbol	Meaning	Additional Notes
m	Integration variable for stellar mass	
$m_{\rm max}$	Integration upper limit for stellar mass	
m_U	Upper limit for stellar mass	$150.0\,M_\odot$ from Weidner & Kroupa (2004)
m_L	Lower limit for stellar mass	hydrogen burning mass limit
$m_{\rm str}$	Stellar mass	
$m_{\rm str,max}$	Most massive stellar mass	
M	Integration variable for cluster mass	
$M_{\rm max}$	Integration upper limit for cluster mass	
M_U	Upper limit for cluster mass	$10^9\,M_\odot$
$M_{\rm L}$	Lower limit for cluster mass	$5\,M_\odot$
$M_{\rm ecl}$	Embedded cluster mass	
$M_{\rm ecl,max}$	Most massive embedded cluster mass	
$M_{\rm tot}$	Total mass formed within δt	Sum of all $M_{\rm ecl}$ formed within δt

Table 1: Notation for the optimal sampling algorithm, which is only valid within Sec. 3.1

3.1.1 Optimal sampling from the IMF

For $\xi(m_{\rm str}) = k_{\rm str} m_{\rm str}^{-\alpha}$, the stellar IMF of a specific embedded cluster, one defines exactly one star to be contained within the mass interval $[m_{i+1}, m_i]$

$$1 = \int_{m_{i+1}}^{m_i} \xi(m)\,{\rm d}m\,, \tag{8}$$

[4] https://github.com/Azeret/galIMF

with the mass of the star in this interval given by

$$m_{\rm str,i} = \int_{m_{i+1}}^{m_i} \xi(m) m \, \mathrm{d}m \, . \tag{9}$$

The maximum stellar mass inside this cluster is accordingly

$$m_{\rm str,max} = m_{\rm str,1} = \int_{m_2}^{m_1 = m_{\rm max}} \xi(m) m \, \mathrm{d}m \, .$$

The mass of the cluster $M_{\rm ecl}$ is defined as

$$M_{\rm ecl} = \int_{m_L}^{m_{\rm max}} \xi(m) m \, \mathrm{d}m \, , \tag{10}$$

with $m_L = 0.08 M_\odot$ the lower stellar mass limit.
Lastly, the optimal sampling normalization condition is defined as

$$I_{\rm str} = 1 = \int_{m_{\rm max}}^{m_U} \xi(m) \, \mathrm{d}m \, . \tag{11}$$

Eq.11 uses a virtual star as a mathematical tool in order to help draw the rest of the stars. The use of this condition "is only justified by its ability to describe observations" (Yan et al. 2017).
The set of two equations, Eq. 10 and Eq. 11, makes it possible to determine the normalization constant of the stellar IMF, $k_{\rm str}$, and the upper integration limit $m_{\rm max}$.
From these and knowing that each integral follows the rule that between two integration limits, the integral has to be equal to one (Eq. 8), the lower integration bound m_2 can be determined. From this all other integral bounds follow in an iterative way. With the determined integration limits the stellar masses can be computed from top down via Eq. 9.
Note that Eq. 10 and Eq. 11 are similar to Eq. 6 and Eq. 7, leading to the emergence of an $m_{\rm str,max} - M_{\rm ecl}$ relation.

3.1.2 Optimal sampling from the ECMF

A similar reasoning as for the IMF can be applied in order to draw from the ECMF $\xi_{\rm ecl} = \frac{\mathrm{d}N}{\mathrm{d}M_{\rm ecl}}$. Here, the cluster mass is replaced with the total stellar mass, that has been formed in a time-frame of 10 Myr via the corresponding mean SFR ($\overline{\rm SFR}$) during that period such that $M_{\rm tot} = \overline{\rm SFR} \times \delta t$.
Similarly to Eq. 8 and Eq. 9 the corresponding integrals for the ECMF are

$$1 = \int_{M_{i+1}}^{M_i} \xi_{\rm ecl}(M) \, \mathrm{d}M \, ,$$

and

$$M_{{\rm ecl},i} = \int_{M_{i+1}}^{M_i} \xi_{\rm ecl}(M) M \, \mathrm{d}M \, ,$$

where the mass of the most massive embedded cluster is given as

$$M_{\mathrm{ecl,max}} = M_{\mathrm{ecl},1} = \int_{M_2}^{M_{\max}} \xi_{\mathrm{ecl}}(M) M \,\mathrm{d}M \,.$$

If the mean SFR over a 10 Myr timespan δt is known, one can employ the condition that

$$\overline{\mathrm{SFR}} \times \delta t = M_{\mathrm{tot}} = \int_{M_L}^{M_{\max}} \xi_{\mathrm{ecl}}(M) M \,\mathrm{d}M \,, \tag{12}$$

where $M_L = 5M_\odot$ is a defined lower limit for cluster masses and $M_{\max}$ is the upper integration limit of the ECMF.
The optimal sampling normalization condition for the ECMF is defined as

$$I_{\mathrm{ecl}} = 1 = \int_{M_{\max}}^{M_U} \xi_{\mathrm{ecl}}(M) \,\mathrm{d}M \,, \tag{13}$$

with the same justification as for the IMF.
This again yields a system of two equations, Eq. 12 and Eq. 13, which can be used to solve for the normalization constant of the ECMF function k_{ecl} and the upper integration limit $M_{\max}$ from which all the other integration limits can be calculated from top down as well as all the cluster masses.
In a code that samples an IGIMF, first the cluster masses are drawn and the determined cluster masses are used as an input to draw the corresponding IMFs, yielding an optimally sampled galaxy-wide IMF.

3.1.3 The ambiguity of the optimal sampling conditions

As mentioned before, the fact that one chooses $I_{\mathrm{ecl}} = 1$ is only justified through the fact, that it seems to fit observations well. I_{ecl} can be interpreted as one of the input values of the optimal sampling algorithm, along with the mean SFR. It is needed to solve the set of two equations (Eq. 12 and Eq. 13) in order to optimally sample the population of star clusters. This also means, that one might reformulate this algorithm such that an input can be substituted by a different input that is backed by observations.
For this sake, the optimal sampling normalization condition for clusters is replaced with the following equation:

$$M_{\mathrm{ecl},1} = \int_{M_2}^{M_{\max}} \xi_{\mathrm{ecl}}(M_{\mathrm{ecl}}) M_{\mathrm{ecl}} \,\mathrm{d}M_{\mathrm{ecl}} \,.$$

This condition demands, that the mass of the most massive cluster is given by the integral over the ECMF times M_{ecl} for the mass range $[M_2,\, M_{\max}]$, which is simply how one would draw the most massive cluster mass under usual circumstances.

In simple terms, instead of M_{tot} and I_{ecl} as input to solve the equation system, one uses M_{tot} and $M_{\mathrm{ecl},1}$. M_{tot} can then be the total stellar mass formed within a region, instead of the mass formed within the whole galaxy.
This formulation is essentially now like usual quantile sampling, but instead of giving the number of data points as input, one uses the sum of data points.

For completeness, it is also possible to use $M_{\mathrm{ecl},1}$ and I_{ecl} as inputs, if e.g. M_{tot} is unknown, though this formulation may be less useful in most contexts.
As an example, it could be used to investigate how the physical upper limit M_U, which is closely connected to I_{ecl}, could be modified for sub-galactic regions to fit local relations between total mass and most massive cluster mass.

3.1.4 A word of warning and possible uses

The varying ECMF suggested by Weidner et al. (2013b) uses the usual optimal sampling normalization condition ($I_{\mathrm{ecl}} = 1$) to perform an empirical fit to observed galaxy-wide IMFs. The variation of β is therefore gauged to fit observations. If one uses $M_{\mathrm{ecl},1}$ instead of $I_{\mathrm{ecl}} = 1$ as input β cannot vary in the same way. Due to this, value of β will be unknown. The use of the modified sampling algorithm is a clear deviation from the standard IGIMF Theory.
Therefore, results derived from the use of the modified sampling algorithm cannot be directly interpreted within the IGIMF framework. It is possible to draw IGIMFs using the modified algorithm, but be aware that then you are no longer using the established IGIMF theory. Instead, the modification should be mainly used for the sampling of custom composite IMFs.
Another important thing to note is that one gives up the emergence of the *maximum value - total value* relations that arise from the existence of physical limits. The reformulations may therefore be seen as using optimal sampling as a tool, rather than part or extension of an overarching theory. It is a tool one can use to draw samples without randomness, that follow a pre-defined distribution function very well, and takes the total sum of data points as input rather than a number of data points.

The modified algorithm can be used to optimally sample stellar populations on sub-galactic scales. If the mass of the most massive cluster within a star-forming region and the local SFR are known, one can investigate e.g. the behavior of local composite IMF for different assumptions on the properties of the local cluster population and compare this with observations, like IMFs and spectra. From the study of many star-forming sites one could then investigate relations between environmental conditions (local SFR, initial cloud mass, metallicity, ...) and properties of the emerging cluster populations (M_1, β, the local M_{tot}, ...). Alternatively, it could be used to try to find a new empirical ECMF function, where fits to galaxy-wide IMFs are made using empirical fits to $M_{\mathrm{ecl,max}} - \mathrm{SFR}$ relations as starting point.
Also, it may be useful for the study of the ICIMF Theory, that describes the stellar population of molecular clouds (Zhou et al. 2025).

4 Derivations

In this section the derivations to determine the upper integration limit $M_{\max}$ for the two other possible variations of the Optimal sampling algorithm are documented. The derivations of the standard way can be found in the supplementary documentation of the galIMF module on GitHub.
The notation used is shown in Tab. 2

Symbol	Meaning	Additional Notes
m_i	Integration bounds for cluster masses	
$M_{\max} \equiv m_1$	Upper integration limit for cluster masses	
$M_{\mathrm{U,ecl}}$	Upper limit for cluster masses	$10^9\, M_\odot$
$M_{\mathrm{L,ecl}}$	Lower limit for cluster masses	$5\, M_\odot$
M_{ecl}	Embedded cluster mass	
M_1	Most massive embedded cluster mass	
M_{tot}	Total mass formed within δt	Sum of all M_{ecl} formed within δt

Table 2: Notation for the optimal sampling algorithm used within Sec. 4

4.1 Optimally sampling from a power law ECMF using the mass of the most massive cluster and the total mass formed

In this section the algorithm to optimally sample a number of clusters is presented, that can be used when the initial mass of the most massive cluster M_1 as well as the total stellar mass formed in a galaxy within δt, M_{tot}, is known. In this case, the variable I_{ecl}, coming in from the optimal sampling normalization condition is not needed. This has the advantage that no upper physical limit has to be assumed for the cluster mass, but the drawback that instead M_1 needs to be known reasonably well.
The goal is to determine the upper integration limit $M_{\max}$ in order to draw all other integration limits and from this all cluster masses. First, by definition we know that only one cluster can exist in the interval $[m_2,\, m_1 = M_{\max}]$ such that the integral over the power-law ECMF $\xi_{\mathrm{ecl}} = k_{\mathrm{ecl}} M_{\mathrm{ecl}}^{-\beta}$ is

$$\begin{aligned} 1 &= \int_{m_2}^{M_{\max}} k_{\mathrm{ecl}} M_{\mathrm{ecl}}^{-\beta} dM_{\mathrm{ecl}} = \frac{k_{\mathrm{ecl}}}{1-\beta} \cdot (M_{\max}^{1-\beta} - m_2^{1-\beta}) \\ \Leftrightarrow \frac{1-\beta}{k_{\mathrm{ecl}}} &= M_{\max}^{1-\beta} - m_2^{1-\beta} \\ \Leftrightarrow m_2^{1-\beta} &= M_{\max}^{1-\beta} - \frac{1-\beta}{k_{\mathrm{ecl}}} \end{aligned}$$

$$\Rightarrow m_2 = (M_{\max}^{1-\beta} - \frac{1-\beta}{k_{\mathrm{ecl}}})^{\frac{1}{1-\beta}} \quad \text{and also} \quad m_{i+1} = (m_i^{1-\beta} - \frac{1-\beta}{k_{\mathrm{ecl}}})^{\frac{1}{1-\beta}} \tag{14}$$

It is assumed that $\beta \neq 1$. The same is assumed for all following calculations. One has to make a case distinction between $\beta \neq 2$ and $\beta = 2$.

4.1.1 For $\beta \neq 2$

Consider first the case that $\beta \neq 2$. Assume that we have given the mass of the most massive cluster M_1 and the mass of the total system M_{tot}. The following two equations have to hold true. First, since M_1 is assumed to be known one can follow from this that

$$\begin{aligned} M_1 &= \int_{m_2}^{M_{\max}} k_{\text{ecl}} M_{\text{ecl}}^{1-\beta} dM_{\text{ecl}} \overset{\beta\neq 2}{=} \frac{k_{\text{ecl}}}{2-\beta}(M_{\max}^{2-\beta} - m_2^{2-\beta}) \\ &\Leftrightarrow \frac{2-\beta}{k_{\text{ecl}}} M_1 = M_{\max}^{2-\beta} - m_2^{2-\beta} \\ &\Leftrightarrow m_2^{2-\beta} = M_{\max}^{2-\beta} - \frac{2-\beta}{k_{\text{ecl}}} M_1 \end{aligned}$$

$$\Rightarrow m_2 = (M_{\max}^{2-\beta} - \frac{2-\beta}{k_{\text{ecl}}} M_1)^{\frac{1}{2-\beta}} \tag{15}$$

Also, we know that the total mass of the system M_{tot} has to be

$$M_{\text{tot}} = \int_{M_{L,\text{ecl}}}^{M_{\max}} k_{\text{ecl}} M_{\text{ecl}}^{1-\beta} dM_{\text{ecl}} \overset{\beta\neq 2}{=} \frac{k_{\text{ecl}}}{2-\beta} \cdot (M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta})$$

$$\Rightarrow k_{\text{ecl}} = \frac{(2-\beta) M_{\text{tot}}}{M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta}} \tag{16}$$

$M_{L,\text{ecl}}$ here refers to the lower physical limit of cluster masses and is typically set to $M_{L,\text{ecl}} = 5 M_\odot$.
Plugging in the derived k_{ecl} from Eq. 16 in Eq. 14 and Eq. 15 yields

$$m_2 \overset{\text{Eq.14}}{=} (M_{\max}^{1-\beta} - \frac{1-\beta}{k_{\text{ecl}}})^{\frac{1}{1-\beta}} \overset{\text{Eq.16}}{=} (M_{\max}^{1-\beta} - \frac{1-\beta}{[\frac{(2-\beta)M_{\text{tot}}}{M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta}}]})^{\frac{1}{1-\beta}} = (M_{\max}^{1-\beta} - \frac{1-\beta}{2-\beta} \frac{M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta}}{M_{\text{tot}}})^{\frac{1}{1-\beta}} \tag{17}$$

and

$$m_2 \overset{\text{Eq.15}}{=} (M_{\max}^{2-\beta} - \frac{2-\beta}{k_{\text{ecl}}} M_1)^{\frac{1}{2-\beta}} \overset{\text{Eq.16}}{=} (M_{\max}^{2-\beta} - \frac{2-\beta}{[\frac{(2-\beta)M_{\text{tot}}}{M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta}}]} M_1)^{\frac{1}{2-\beta}} = (M_{\max}^{2-\beta} - \frac{M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta}}{M_{\text{tot}}} M_1)^{\frac{1}{2-\beta}} \tag{18}$$

Setting Eq. 17 and Eq. 18 equal yields the final equality.

$$\begin{aligned} (M_{\max}^{2-\beta} - \frac{M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta}}{M_{\text{tot}}} M_1)^{\frac{1}{2-\beta}} &= (M_{\max}^{1-\beta} - \frac{1-\beta}{2-\beta} \frac{M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta}}{M_{\text{tot}}})^{\frac{1}{1-\beta}} \\ \Leftrightarrow M_{\max}^{2-\beta} - \frac{M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta}}{M_{\text{tot}}} M_1 &= (M_{\max}^{1-\beta} - \frac{1-\beta}{2-\beta} \frac{M_{\max}^{2-\beta} - M_{L,\text{ecl}}^{2-\beta}}{M_{\text{tot}}})^{\frac{2-\beta}{1-\beta}} \end{aligned}$$

This equation has to be solved numerically for $M_{\max}$ and since it is the upper integration limit all remaining integration limits can be drawn using the generalization of Eq.14.

4.1.2 For $\beta = 2$

Similar to before, knowing M_1 and M_{tot} yields the following two equations:

$$\begin{aligned}
M_1 &= \int_{m_2}^{M_{\max}} k_{\text{ecl}} M_{\text{ecl}}^{1-\beta} dM_{\text{ecl}} \overset{\beta=2}{=} \int_{m_2}^{M_{\max}} k_{\text{ecl}} M_{\text{ecl}}^{-1} dM_{\text{ecl}} = k_{\text{ecl}} \cdot (\ln(M_{\max}) - \ln(m_2)) \\
\Leftrightarrow & \quad \frac{M_1}{k_{\text{ecl}}} = \ln(M_{\max}) - \ln(m_2) = \ln(\frac{M_{\max}}{m_2}) \\
\Leftrightarrow & \quad \mathrm{e}^{\frac{M_1}{k_{\text{ecl}}}} = \frac{M_{\max}}{m_2}
\end{aligned}$$

$$\Rightarrow m_2 = \frac{M_{\max}}{\mathrm{e}^{\frac{M_1}{k_{\text{ecl}}}}} = M_{\max} \cdot \mathrm{e}^{-\frac{M_1}{k_{\text{ecl}}}} \tag{19}$$

$$M_{\text{tot}} = \int_{M_{L,\text{ecl}}}^{M_{\max}} k_{\text{ecl}} M_{\text{ecl}}^{1-\beta} dM_{\text{ecl}} \overset{\beta=2}{=} \int_{M_{L,\text{ecl}}}^{M_{\max}} k_{\text{ecl}} M_{\text{ecl}}^{-1} dM_{\text{ecl}} = k_{\text{ecl}}(\ln(M_{\max}) - \ln(M_{L,\text{ecl}}))$$

$$\Rightarrow k_{\text{ecl}} = \frac{M_{\text{tot}}}{\ln(M_{\max}) - \ln(M_{L,\text{ecl}})} \tag{20}$$

Then one can plug in Eq. 20 in Eq. 14 and Eq. 19. This yields

$$\begin{aligned}
m_2 &\overset{\text{Eq.14}}{=} (M_{\max}^{1-\beta} - \frac{1-\beta}{k_{\text{ecl}}})^{\frac{1}{1-\beta}} \\
&\overset{\text{Eq.20}}{=} (M_{\max}^{1-\beta} - \frac{1-\beta}{[\frac{M_{\text{tot}}}{\ln(M_{\max})-\ln(M_{L,\text{ecl}})}]})^{\frac{1}{1-\beta}} \\
&= (M_{\max}^{1-\beta} - \frac{1-\beta}{M_{\text{tot}}} \cdot [\ln(M_{\max}) - \ln(M_{L,\text{ecl}})])^{\frac{1}{1-\beta}} \\
&= (M_{\max}^{1-\beta} - \frac{1-\beta}{M_{\text{tot}}} \cdot \ln(\frac{M_{\max}}{M_{L,\text{ecl}}}))^{\frac{1}{1-\beta}} \\
&= (M_{\max}^{1-\beta} + \frac{\beta-1}{M_{\text{tot}}} \cdot \ln(\frac{M_{\max}}{M_{L,\text{ecl}}}))^{\frac{1}{1-\beta}}
\end{aligned} \tag{21}$$

and

$$m_2 \overset{\text{Eq.19}}{=} \frac{M_{\max}}{\mathrm{e}^{\frac{M_1}{k_{\mathrm{ecl}}}}} = M_{\max} \cdot \mathrm{e}^{-\frac{M_1}{k_{\mathrm{ecl}}}} \overset{\text{Eq.20}}{=} M_{\max} \cdot \mathrm{e}^{-\frac{M_1}{\left[\frac{M_{\mathrm{tot}}}{\ln(M_{\max})-\ln(M_{L,\mathrm{ecl}})}\right]}} = M_{\max} \cdot \mathrm{e}^{-\frac{M_1 \cdot (\ln(M_{\max})-\ln(M_{L,\mathrm{ecl}}))}{M_{\mathrm{tot}}}}$$

$$\Leftrightarrow \quad m_2 \cdot \mathrm{e}^{\frac{M_1 \cdot (\ln(M_{\max})-\ln(M_{L,\mathrm{ecl}}))}{M_{\mathrm{tot}}}} = M_{\max}$$

$$\Leftrightarrow \quad m_2 \cdot \mathrm{e}^{\frac{M_1 \cdot \ln(\frac{M_{\max}}{M_{L,\mathrm{ecl}}})}{M_{\mathrm{tot}}}} = M_{\max}$$

$$\Leftrightarrow \quad m_2 \cdot \mathrm{e}^{\ln(\frac{M_{\max}}{M_{L,\mathrm{ecl}}}) \frac{M_1}{M_{\mathrm{tot}}}} = M_{\max}$$

$$\Leftrightarrow \quad m_2 \cdot (\frac{M_{\max}}{M_{L,\mathrm{ecl}}})^{\frac{M_1}{M_{\mathrm{tot}}}} = M_{\max}$$

$$\Rightarrow m_2 = M_{\max} \cdot (\frac{M_{\max}}{M_{L,\mathrm{ecl}}})^{-\frac{M_1}{M_{\mathrm{tot}}}} \tag{22}$$

Setting Eq. 21 and Eq. 22 equal, again yields the final equality of

$$M_{\max} \cdot (\frac{M_{\max}}{M_{L,\mathrm{ecl}}})^{-\frac{M_1}{M_{\mathrm{tot}}}} = (M_{\max}^{1-\beta} + \frac{\beta-1}{M_{\mathrm{tot}}} \cdot \ln(\frac{M_{\max}}{M_{L,\mathrm{ecl}}}))^{\frac{1}{1-\beta}}$$

From this again, $M_{\max}$ can be determined numerically and all other integration limits by using the generalization of Eq. 14.

4.2 Optimally sampling from a power law ECMF using only the mass of the most massive cluster

In this section the algorithm to optimally sample a number of clusters is presented, that can be used to draw from a region, when the mass of the most massive cluster M_1 is known, but not the total mass of the region M_{tot}.
It relies on the variable I_{ecl} that is canonically set to 1, which might not always be correct. Again, the goal is to determine the upper integration limit $M_{\max}$.
Please note, that your assumption on the ECMF exponent β then influences the value of M_{tot}. For the following calculations, it is always assumed that $\beta \neq 1$.

4.2.1 For $\beta \neq 2$

Use that the mass of the most massive known cluster is

$$M_1 = \int_{m_2}^{M_{\max}} k_{\mathrm{ecl}} M_{\mathrm{ecl}}^{1-\beta} dM_{\mathrm{ecl}} = \frac{k_{\mathrm{ecl}}}{2-\beta}(M_{\max}^{2-\beta} - m_2^{2-\beta})$$

$$\Leftrightarrow \quad \frac{2-\beta}{k_{\mathrm{ecl}}} M_1 = M_{\max}^{2-\beta} - m_2^{2-\beta}$$

$$\Leftrightarrow \quad m_2^{2-\beta} = M_{\max}^{2-\beta} - \frac{2-\beta}{k_{\mathrm{ecl}}} M_1$$

$$\Rightarrow m_2 = (M_{\max}^{2-\beta} - \frac{2-\beta}{k_{\mathrm{ecl}}} M_1)^{\frac{1}{2-\beta}} \tag{23}$$

Also, we know that

$$I_{\text{ecl}} = I = \int_{M_{\text{max}}}^{M_{\text{U,ecl}}} k_{\text{ecl}} M_{\text{ecl}}^{-\beta} dM_{\text{ecl}} = \frac{k_{\text{ecl}}}{1-\beta} \cdot (M_{\text{U,ecl}}^{1-\beta} - M_{\text{max}}^{1-\beta})$$

$$\Rightarrow k_{\text{ecl}} = \frac{I(1-\beta)}{M_{\text{U,ecl}}^{1-\beta} - M_{\text{max}}^{1-\beta}} \tag{24}$$

And lastly, we know that only one star can exist in the interval $m_1 = M_{max}$ to m_2

$$\begin{aligned}
1 &= \int_{m_2}^{M_{\text{max}}} k_{\text{ecl}} M_{\text{ecl}}^{-\beta} dM_{\text{ecl}} = \frac{k_{\text{ecl}}}{1-\beta} \cdot (M_{\text{max}}^{1-\beta} - m_2^{1-\beta}) \\
\Leftrightarrow & \frac{1-\beta}{k_{\text{ecl}}} = M_{\text{max}}^{1-\beta} - m_2^{1-\beta} \\
\Leftrightarrow & m_2^{1-\beta} = M_{\text{max}}^{1-\beta} - \frac{1-\beta}{k_{\text{ecl}}}
\end{aligned}$$

$$\Rightarrow m_2 = (M_{\text{max}}^{1-\beta} - \frac{1-\beta}{k_{\text{ecl}}})^{\frac{1}{1-\beta}} \tag{25}$$

We therefore have three equations for three unknowns m_2, M_{max} and k_{ecl} which means we can solve for them.
Plugging in Eq. 24 in the other two equation yields

$$\begin{aligned}
m_2 &= \left(M_{\text{max}}^{2-\beta} - \frac{2-\beta}{k_{\text{ecl}}} M_1\right)^{\frac{1}{2-\beta}} \\
&\overset{Eq.\ 24}{=} \left(M_{\text{max}}^{2-\beta} - \frac{2-\beta}{[\frac{I(1-\beta)}{M_{\text{U,ecl}}^{1-\beta} - M_{\text{max}}^{1-\beta}}]} M_1\right)^{\frac{1}{2-\beta}}
\end{aligned}$$

$$\Rightarrow m_2 = \left(M_{\text{max}}^{2-\beta} - \frac{2-\beta}{I(1-\beta)} (M_{\text{U,ecl}}^{1-\beta} - M_{\text{max}}^{1-\beta}) \cdot M_1\right)^{\frac{1}{2-\beta}} \tag{26}$$

and

$$\begin{aligned}
m_2 &= \left(M_{\text{max}}^{1-\beta} - \frac{1-\beta}{k_{\text{ecl}}}\right)^{\frac{1}{1-\beta}} \\
&\overset{Eq.\ 24}{=} \left(M_{\text{max}}^{1-\beta} - \frac{1-\beta}{[\frac{I(1-\beta)}{M_{\text{U,ecl}}^{1-\beta} - M_{\text{max}}^{1-\beta}}]}\right)^{\frac{1}{1-\beta}} \\
&= \left(M_{\text{max}}^{1-\beta} - \frac{1-\beta}{I(1-\beta)} \cdot (M_{\text{U,ecl}}^{1-\beta} - M_{\text{max}}^{1-\beta})\right)^{\frac{1}{1-\beta}}
\end{aligned}$$

$$\Rightarrow m_2 = \left(M_{\text{max}}^{1-\beta} - \frac{(M_{\text{U,ecl}}^{1-\beta} - M_{\text{max}}^{1-\beta})}{I}\right)^{\frac{1}{1-\beta}} \tag{27}$$

Giving the final equality of Eq. 26=Eq. 27:

$$\left(M_{\max}^{2-\beta} - \frac{2-\beta}{I(1-\beta)}(M_{\mathrm{U,ecl}}^{1-\beta} - M_{\max}^{1-\beta}) \cdot M_1\right)^{\frac{1}{2-\beta}} = \left(M_{\max}^{1-\beta} - \frac{(M_{\mathrm{U,ecl}}^{1-\beta} - M_{\max}^{1-\beta})}{I}\right)^{\frac{1}{1-\beta}}$$

$$\Leftrightarrow M_{\max}^{2-\beta} - \frac{2-\beta}{I(1-\beta)}(M_{\mathrm{U,ecl}}^{1-\beta} - M_{\max}^{1-\beta}) \cdot M_1 = \left(M_{\max}^{1-\beta} - \frac{(M_{\mathrm{U,ecl}}^{1-\beta} - M_{\max}^{1-\beta})}{I}\right)^{\frac{2-\beta}{1-\beta}} \tag{28}$$

From Eq. 28 $M_{\max}$ can be determined and from that m_2 and k_{ecl} which are needed the calculate the rest of the cluster masses using the optimal sampling method.

4.2.2 For $\beta = 2$

If $\beta = 2$, one has to recalculate Eq. 23

$$\begin{aligned} M_1 &= \int_{m_2}^{M_{\max}} k_{\mathrm{ecl}} M_{\mathrm{ecl}}^{1-\beta} dM_{\mathrm{ecl}} \overset{\beta=2}{=} \int_{m_2}^{M_{\max}} k_{\mathrm{ecl}} M_{\mathrm{ecl}}^{-1} dM_{\mathrm{ecl}} = k_{\mathrm{ecl}} \cdot (\ln(M_{\max}) - \ln(m_2)) \\ &\Leftrightarrow \frac{M_1}{k_{\mathrm{ecl}}} = \ln(M_{\max}) - \ln(m_2) = \ln(\frac{M_{\max}}{m_2}) \\ &\Leftrightarrow e^{\frac{M_1}{k_{\mathrm{ecl}}}} = \frac{M_{\max}}{m_2} \end{aligned}$$

$$\Rightarrow m_2 = \frac{M_{\max}}{e^{\frac{M_1}{k_{\mathrm{ecl}}}}} = M_{\max} \cdot e^{-\frac{M_1}{k_{\mathrm{ecl}}}} \tag{29}$$

When directly plugging in $\beta = 2$, Eq. 24 becomes

$$k_{\mathrm{ecl}} = \frac{I(1-\beta)}{M_{\mathrm{U,ecl}}^{1-\beta} - M_{\max}^{1-\beta}} \overset{\beta=2}{=} \frac{I(1-2)}{M_{\mathrm{U,ecl}}^{1-2} - M_{\max}^{1-2}} = \frac{I(-1)}{M_{\mathrm{U,ecl}}^{-1} - M_{\max}^{-1}} = \frac{I}{M_{\max}^{-1} - M_{\mathrm{U,ecl}}^{-1}} \tag{30}$$

In the limit of $M_{\mathrm{U,ecl}} \to \infty$, Eq. 30 simplifies to $k_{\mathrm{ecl}} = I \cdot M_{\max}$.
Finally, Eq. 25 becomes

$$m_2 = \left(M_{\max}^{1-\beta} - \frac{1-\beta}{k_{\mathrm{ecl}}}\right)^{\frac{1}{1-\beta}} \overset{\beta=2}{=} \left(M_{\max}^{1-2} - \frac{1-2}{k_{\mathrm{ecl}}}\right)^{\frac{1}{1-2}} = \left(M_{\max}^{-1} + \frac{1}{k_{\mathrm{ecl}}}\right)^{-1} \tag{31}$$

Setting Eq. 29=Eq. 31, this gives then the final equality of

$$M_{\max} \cdot e^{-\frac{M_1}{k_{\mathrm{ecl}}}} = \left(M_{\max}^{1-\beta} - \frac{1-\beta}{k_{\mathrm{ecl}}}\right)^{\frac{1}{1-\beta}} \tag{32}$$

from which $M_{\max}$ can be determined numerically.

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