Answers to questions in

Lab 1: Filtering operations

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Instructions: Complete the lab according to the instructions in the notes and respond to the questions stated below. Keep the answers short and focus on what is essential. Illustrate with figures only when explicitly requested.

Question 1: Repeat this exercise with the coordinates p and q set to (5, 9), (9, 5), (17, 9),

(17, 121), (5, 1) and (125, 1) respectively. What do you observe?

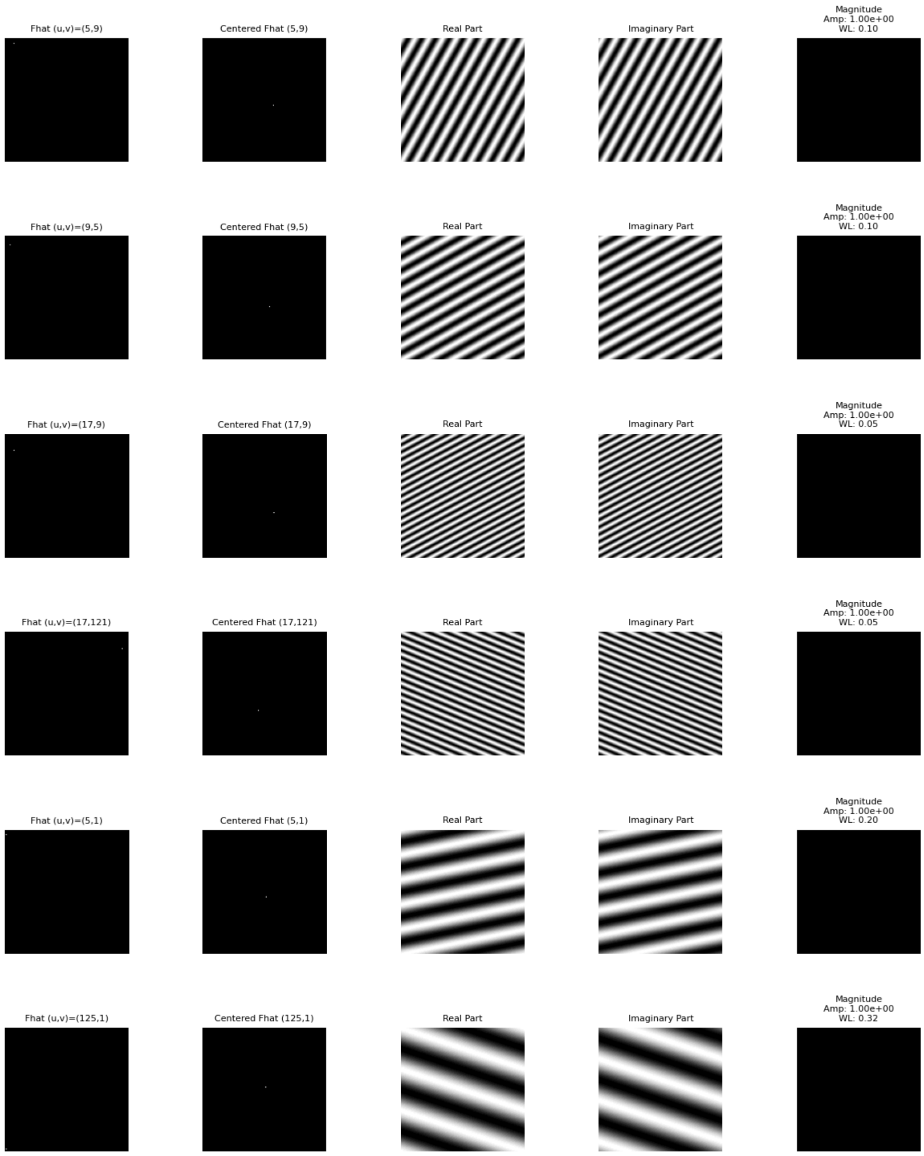
Answers:

Each pair of (p, q) positions produces a sine wave in the spatial domain with variations in orientation, wavelength, and frequency depending on the distance of (p, q) from the origin and their values. Points closer to the center (low frequencies) result in longer wavelengths (more gradual oscillations), while points farther from the center (high frequencies) result in shorter wavelengths (more rapid oscillations).

Symmetry properties of the Fourier transform may cause mirroring effects for points with coordinates exceeding half the image size. For example, (125, 1), it is the frequency equivalent of (125 - 128, 1) = (-3, 1).

Question 2: Explain how a position (p, q) in the Fourier domain will be projected as a sine wave in the spatial domain. Illustrate with a Matlab figure.

Answers:



A position (p, q) in the Fourier domain represents a specific frequency in the spatial domain. When the inverse Fourier transform is applied, the spatial domain will show a sinusoidal pattern corresponding to the frequency specified by (p, q).

The Fourier basis function is of complex exponential form $e^{2\pi i \frac{px+qy}{N}}$, which is essentially equivalent to a combination of sine and cosine.

The frequency of this sine wave is proportional to the distance from the center of the Fourier domain to the point (p, q). The direction of the sine wave is perpendicular to the line from the origin to (p, q).

Question 3: How large is the amplitude? Write down the expression derived from Equation (4) in the notes. Complement the code (variable amplitude) accordingly.

Answers:

$$\mathrm{f}(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \hat{\mathrm{f}}(u,v) e^{+2\pi i \left(\frac{xu}{M} + \frac{yv}{N}\right)}$$

The amplitude of each basis function in the discrete Fourier transform is typically scaled by 1/MN. Given that the magnitude here is 1, the amplitude is 1/MN.

```
amplitude = 1.0 / sz * sz   # Fabsmax == 1
```

Question 4: How does the direction and length of the sine wave depend on p and q? Write down the explicit expression that can be found in the lecture notes. Complement the code (variable wavelength) accordingly.

Answers:

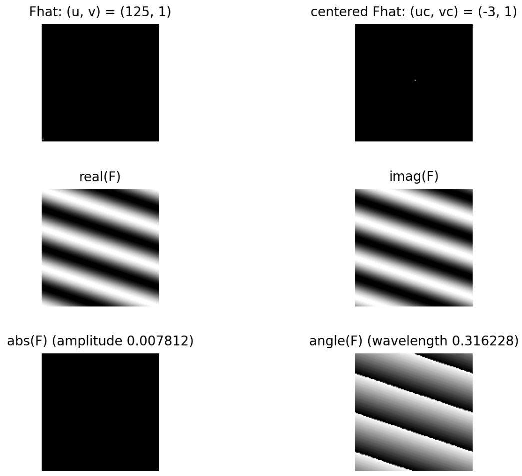
The wavelength of the sinusoid is: $\lambda = \frac{1}{\sqrt{u^2+v^2}}$, where (u, v) are the frequencies along (x, y) and the periods are $1/u$ and $1/v$.

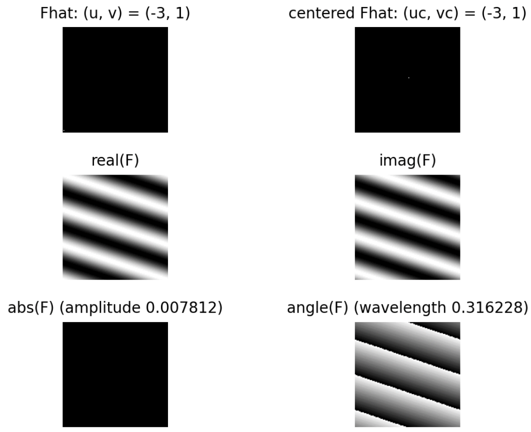
The direction of the sine wave is perpendicular to the vector (p, q), and the wavelength of the sine wave is inversely proportional to $\sqrt{p^2 + q^2}$.

```
wavelength = 1 / np.sqrt(uc**2 + vc**2)
```

Question 5: What happens when we pass the point in the center and either p or q exceeds half the image size? Explain and illustrate graphically with Matlab!

Answers:





Due to mirrored effects in the spatial representation, when p or q exceeds half the image size, the frequency values "fold back" to the lower frequency region. For example, with an image size N = 128, the frequency of the coordinate (125, 1) is equivalent to (125 - 128, 1) = (-3, 1), which places it near the center in the wrapped domain, making it appear as a low frequency.

This "opposite direction" is due to how coordinates are interpreted in the wrapped space rather than a true directional change in the spatial domain. This periodic feature is inherent to the Fourier transform. This periodicity creates a symmetry where high frequencies near the edges (in the unwrapped domain) wrap around and appear as low frequencies near the center (in the wrapped domain).

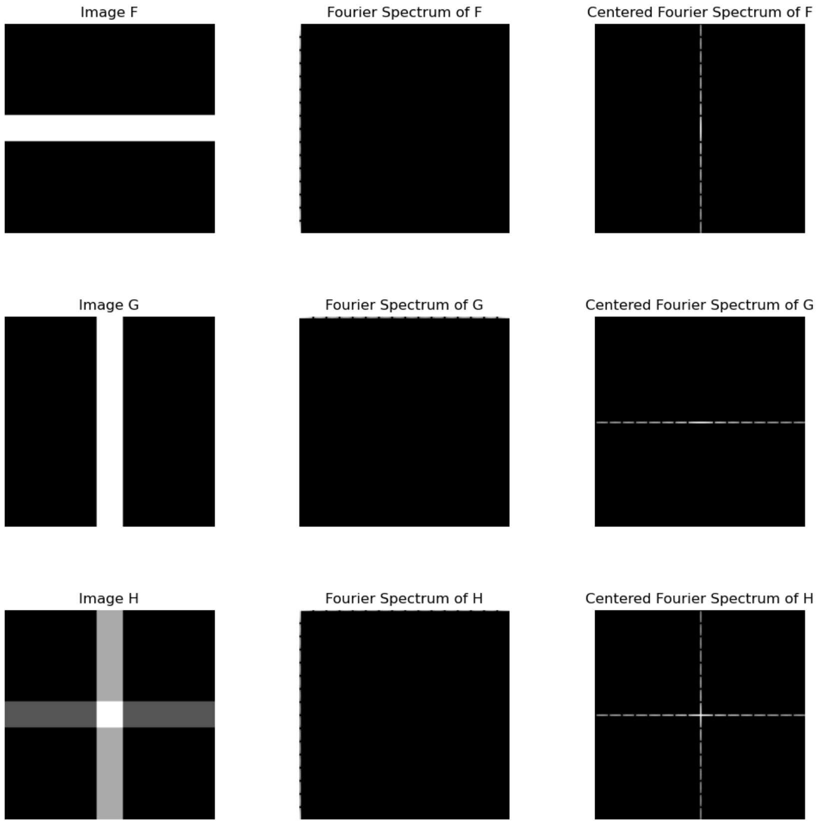
Question 6: What is the purpose of the instructions following the question *What is done by these instructions?* in the code?

Answers:

Used to convert frequency coordinates (u, v) to centered coordinates (uc, vc). This moves the low-frequency components in the frequency domain to the center of the image, making the distribution of high-frequency components and low-frequency components more intuitive and helpful for spectral analysis.

Question 7: Why are these Fourier spectra concentrated to the borders of the images? Can you give a mathematical interpretation? Hint: think of the frequencies in the source image and consider the resulting image as a Fourier transform applied to a 2D function. It might be easier to analyze each dimension separately!

Answers:



In the default FFT output, the origin is located in the upper left corner of the image, while the high-frequency components are concentrated at the edges.

For image F, which is a horizontal strip, there is only variation in the vertical direction. This results in zero horizontal frequency components, so the Fourier spectrum is concentrated along the left edge of the frequency domain. The periodic vertical transitions create high-frequency components, forming a dotted line along this edge. These dots represent the harmonics of the vertical edges, resulting from constructive and destructive interference in the Fourier domain.

Image G, on the other hand, is a vertical strip with variation only in the horizontal direction. Consequently, the Fourier spectrum of G is concentrated along the top edge, where vertical frequency components are zero. The periodic horizontal transitions produce discrete high-frequency components that appear as a dotted line along the top edge, similar to the pattern in F but rotated.

Image H combines both the horizontal strip from F and the vertical strip from G, leading to a Fourier spectrum that is a combination of the two. In the frequency domain, H's spectrum displays both a vertical dotted line along the left edge (from the horizontal strip) and a horizontal dotted line along the top edge (from the vertical strip).

Question 8: Why is the logarithm function applied?

Answers:

The range of values in a Fourier spectrum is usually highly variable, and the amplitude of the high-frequency portion may be very high while the low-frequency portion is relatively low.

Displaying amplitude values directly may result in high amplitude values appearing particularly bright, while low amplitude values are barely visible. By logarithmically processing the amplitude, we can 'compress' the amplitude range, making the lower amplitude values easier to see, enhancing the overall contrast of the image, and contributing to a more complete view of the spectral features.

Question 9: What conclusions can be drawn regarding linearity? From your observations can you derive a mathematical expression in the general case?

Answers:

For the image H = F + 2G, the Fourier transform Hhat = Fhat + 2 · Ghat, which is consistent with the form of the operation in the spatial domain, illustrates the linear nature of the Fourier transform.

Mathematically, if the Fourier transforms of f(x) and g(x) are fhat(ω) and ghat(ω), respectively, then the Fourier transform of the image hhat(x) = afhat(ω) + bghat(ω) for any constants a and b.

Question 10: Are there any other ways to compute the last image? Remember what multiplication in Fourier domain equals to in the spatial domain! Perform these alternative computations in practice.

Answers:

Yes, the last image can be obtained by applying convolution on Fhat and Ghat.

Convolution in the spatial domain is the same as multiplication in the Fourier (frequency) domain, while multiplication in the spatial domain is the same as convolution in the Fourier (frequency) domain.

```
showgrey(np.log(1 + np.abs(convolve2d(Fhat, Ghat, mode='same', boundary='wrap') / (128 ** 2))), False)
```

Question 11: What conclusions can be drawn from comparing the results with those in the previous exercise? See how the source images have changed and analyze the effects of scaling.

Answers:

When an image is compressed in the spatial domain (made smaller), its Fourier spectrum expands, covering a wider range of frequencies. Conversely, when an image is expanded in the spatial domain (made larger), its Fourier spectrum compresses, concentrating around lower frequencies.

This effect is due to compression in the spatial domain introduces rapid transitions, which generate higher frequencies and expand the Fourier spectrum. Expansion in the spatial domain, on the other hand, reduces the frequency content, causing the Fourier spectrum to concentrate around the center.

Question 12: What can be said about possible similarities and differences? Hint: think of the frequencies and how they are affected by the rotation.

Answers:

When an image is rotated in the spatial domain, its Fourier spectrum rotates by the same angle in the frequency domain.

The magnitude distribution remains unchanged in shape, only rotating with the image, while the phase pattern adjusts to reflect the new orientation. Each rotation maintains the overall structure of the frequency components, showing that the Fourier transform preserves the image's frequency content under rotation, only altering the orientation.

Question 13: What information is contained in the phase and in the magnitude of the Fourier transform?

Answers:

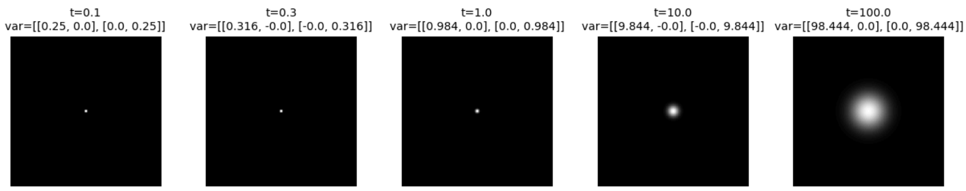
The magnitude represents the intensity of each frequency, contributing to the overall texture and contrast, and the brightness intensity.

The phase determines the precise arrangement and structure of the features in the spatial domain and is essential for accurate spatial reconstruction of the image.

While both are important, phase is often more critical for maintaining the recognizable shape and layout in an image reconstruction.

Question 14: Show the impulse response and variance for the above-mentioned t-values. What are the variances of your discretized Gaussian kernel for t = 0.1, 0.3, 1.0, 10.0 and 100.0?

Answers:




```
Variance for t=0.1: [[0.25, 0.0], [0.0, 0.25]]
Variance for t=0.3: [[0.316, -0.0], [-0.0, 0.316]]
Variance for t=1.0: [[0.984, 0.0], [0.0, 0.984]]
Variance for t=10.0: [[9.844, -0.0], [-0.0, 9.844]]
Variance for t=100.0: [[98.444, 0.0], [0.0, 98.444]]
```

Question 15: Are the results different from or similar to the estimated variance? How does the result correspond to the ideal continuous case? Lead: think of the relation between spatial and Fourier domains for different values of t.

Answers:

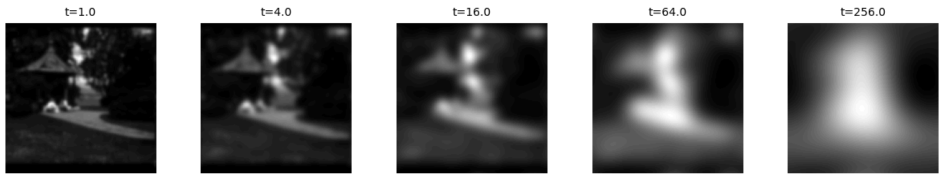
The discrete Gaussian kernel approximates the continuous Gaussian distribution well within a finite image size.

As the t value increases, the variance of the Gaussian expands in the spatial domain. A smaller variance keeps the Gaussian kernel more concentrated, while a larger variance results in a broader spread.

Higher variance approximates the continuous Gaussian in the Fourier domain, but kernels with high variance may require larger support to avoid boundary issues, as they need to cover a broader area.

Question 16: Convolve a couple of images with Gaussian functions of different variances (like t = 1.0, 4.0, 16.0, 64.0 and 256.0) and present your results. What effects can you observe?

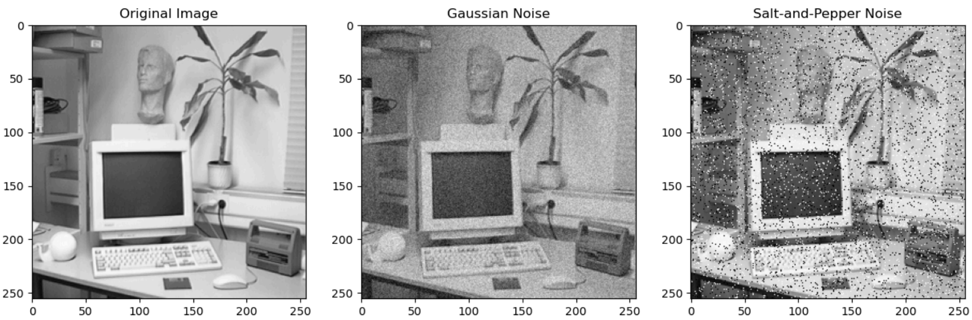
Answers:

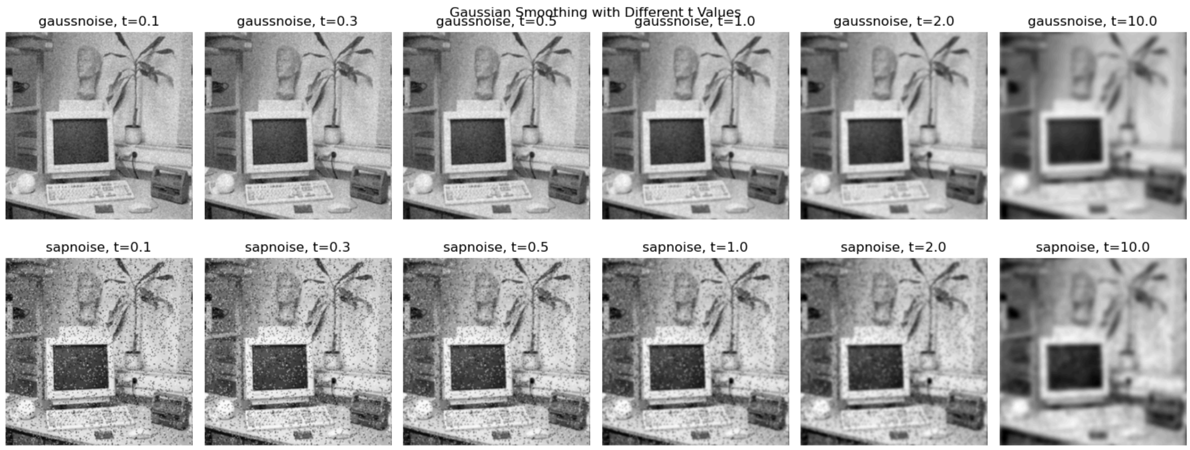


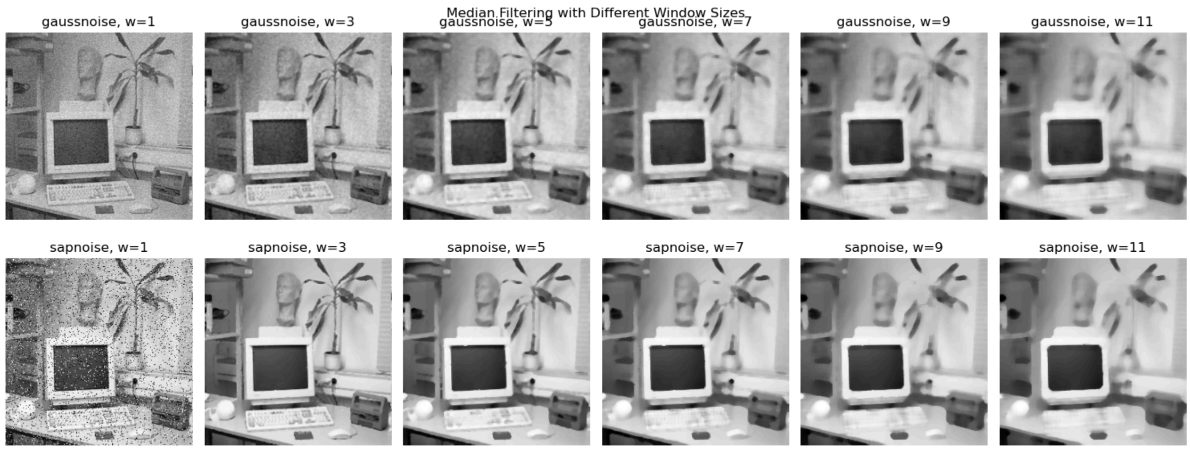
As the t value increases, the effect of blurring is enhanced. Smaller t-values preserve details, while larger t-values result in loss of details, leaving only the overall outline.

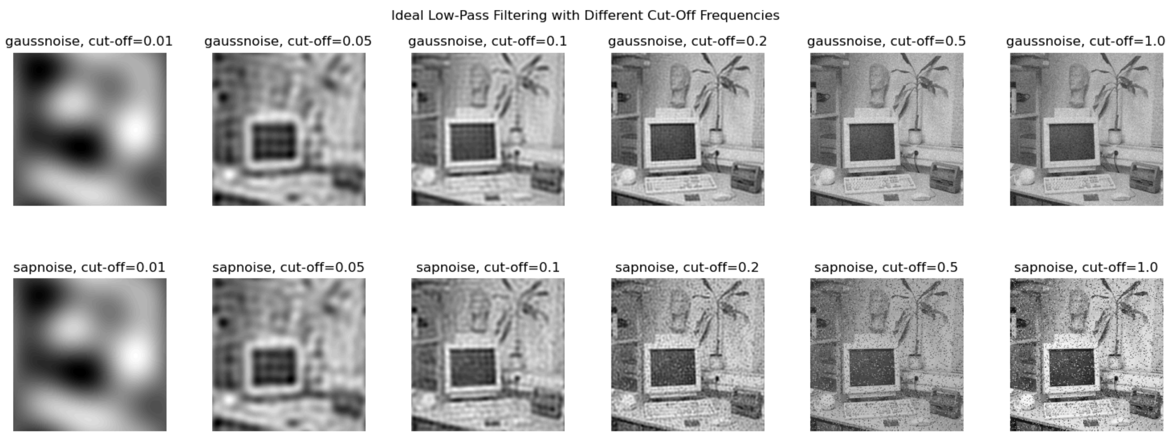
Question 17: What are the positive and negative effects for each type of filter? Describe what you observe and name the effects that you recognize. How do the results depend on the filter parameters? Illustrate with Matlab figure(s).

Answers:









Gaussian Filter

- **Positive Effects**: Effectively removes Gaussian noise, smoothing the image and reducing high-frequency noise.
- **Negative Effects**: Less effective for salt-and-pepper noise and it blurs edges, especially with larger t values, which increase blurring.

- **Parameter Effects**: Higher t values increase smoothing but lead to more blurring.

Median Filter

- **Positive Effects**: Effectively removes "salt-and-pepper" noise while preserving edges without blurring.
- **Negative Effects**: Can remove small details, such as dots or fine lines, leading to loss in detailed images.
- **Parameter Effects**: Larger window sizes improve noise removal but may also reduce image detail.

Ideal Low-Pass Filter

- **Positive Effects**: Excellent for high-frequency noise suppression.
- **Negative Effects**: Causes "ringing effect" in real images by removing high-frequency details that contribute to image clarity.
- **Parameter Effects**: Lower cutoff frequency increases smoothing but reduces detail, making it effective for noise reduction but less so for preserving fine detail.

Question 18: What conclusions can you draw from comparing the results of the respective methods?

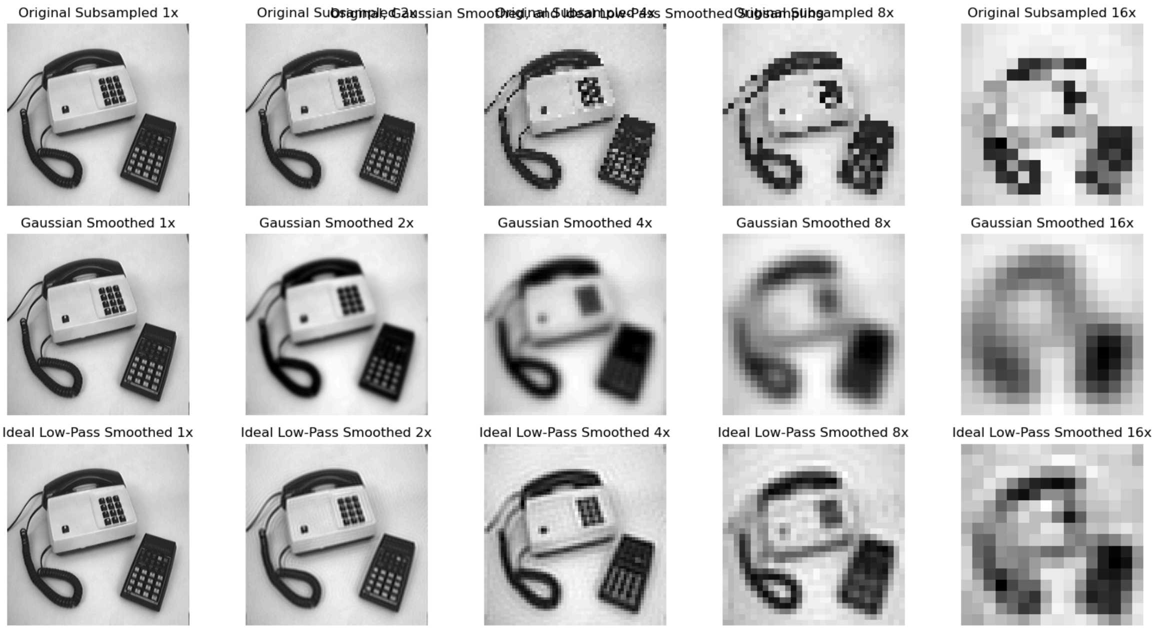
Answers:

The Gaussian filter and the Ideal low-pass filter are both low-pass filters, but the variance of the Gaussian filter and the cut-off frequency of the Ideal low-pass filter both have a similar effect: increasing the variance or decreasing the cut-off frequency results in a stronger smoothing effect.

The Median filter is the best choice for removing salt-and-pepper noises (measurement noise) and is excellent at preserving edges. It can be very good at ignoring occasional really wrong pixels.

Question 19: What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration $i = 4$.

Answers:



At the fourth downsampling (i = 4), the edges of the images with Gaussian filtering and ideal low-pass filtering are smoother than those with direct downsampling, but there is an obvious ‘ringing effect’ when using low-pass filtering. This suggests that ideal low-pass filtering is effective in reducing high-frequency components but has artifacts in edge preservation, whereas Gaussian smoothing smoothes the image while preserving the natural transition of the edges.

Question 20: What conclusions can you draw regarding the effects of smoothing when combined with subsampling? Hint: think in terms of frequencies and side effects.

Answers:

The sampling frequency should be greater than the Nyquist frequency, which is twice the highest frequency in the signal. If the condition is not met, the image will appear to be the aliasing effect. Gaussian and low-pass filters remove the high-frequency component of the image, thus reducing the Nyquist frequency of the signal, making the sampling rate closer to or even at the Nyquist frequency, thus reducing information loss. This smoothing process allows the downsampled image to maintain good quality at low resolution, avoiding the aliasing effect caused by high frequencies.
