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№1

$$|\psi(x)\rangle = \frac{A}{x^2+a^2} \qquad \langle\psi(x)| = \frac{A}{x^2+a^2}$$

$$\langle\psi(x)|\psi(x)\rangle = \int\limits_{-\infty}^{+\infty} \frac{A^2}{(x^2+a^2)^2}dx = \left\{ \begin{array}{l} x = a\,tg\,y \\ dx = \frac{a}{\cos^2 y}dy \end{array} \right\} =$$

$$= A^2 \int\limits_{-\infty}^{+\infty} \frac{a\,dy}{\cos^2 y\,(a^2 tg^2 y + a^2)^2} = A^2 \int\limits_{-\infty}^{+\infty} \frac{dy}{a^3 \cos^2 y\,(tg^2 y + 1)^2} =$$

$$= \frac{A^2}{a^3} \int\limits_{-\infty}^{+\infty} \frac{dy}{\cos^2 y \cdot \frac{1}{\cos^4 y}} = \frac{A^2}{a^3} \int\limits_{-\infty}^{+\infty} \cos^2 y\,dy =$$

$$= \frac{A^2}{2a^3} \int\limits_{-\infty}^{+\infty} (1+\cos 2y)\,dy = \frac{A^2}{2a^3}\left(\int\limits_{-\infty}^{+\infty} dy + \frac{1}{2}\int\limits_{-\infty}^{+\infty} \cos 2y\,d2y \right) =$$

$$= \frac{A^2}{2a^3}\left(y\Big|_{-\infty}^{+\infty} + \frac{\sin 2y}{2}\Big|_{-\infty}^{+\infty} \right) =$$

$$= \frac{A^2}{2a^3}\left(arctg\frac{x}{a}\Big|_{-\infty}^{+\infty} + \underbrace{\frac{1}{2}\sin\left(2 arctg\frac{x}{a}\right)\Big|_{-\infty}^{+\infty}}_{=0} \right) = 1$$

$$\frac{A^2}{2a^3}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1 \qquad \underline{\underline{A = \pm\sqrt{\frac{2a^3}{\pi}}}}$$

№2

$$|\varphi(x)\rangle = \frac{B(x-ib)}{(x+ib)(x-ib)} = \frac{Bx}{x^2+b^2} - \frac{ibB}{x^2+b^2}$$

$$\langle\varphi(x)| = \frac{Bx}{x^2+b^2} + \frac{ibB}{x^2+b^2}$$

$$\langle\varphi(x)|\varphi(x)\rangle = \int_{-\infty}^{+\infty}\left(\frac{B^2x^2}{(x^2+b^2)^2} + \frac{b^2B^2}{(x^2+b^2)^2}\right)dx = 1$$

$$B^2\int_{-\infty}^{+\infty}\left(\frac{x^2}{(x^2+b^2)^2} + \frac{b^2}{(x^2+b^2)^2}\right)dx =$$

$$\int_{-\infty}^{+\infty}\frac{x^2dx}{(x^2+b^2)^2} = \left\{\begin{array}{l}\frac{C}{x^2+b^2} + \frac{D}{(x^2+b^2)^2} = \frac{x^2}{(x^2+b^2)^2} \\ Cx^2+Cb^2+D = x^2 \\ C=1,\ D=-b^2\end{array}\right\} =$$

$$= \int_{-\infty}^{+\infty}\frac{dx}{x^2+b^2} - \int_{-\infty}^{+\infty}\frac{b^2dx}{(x^2+b^2)^2}$$

$$= B^2\int_{-\infty}^{+\infty}\left(\frac{\cancel{dx}\,1}{x^2+b^2} - \frac{b^2}{(x^2+b^2)^2} + \frac{b^2}{(x^2+b^2)^2}\right)dx = B^2\int_{-\infty}^{+\infty}\frac{dx}{x^2+b^2} =$$

$$= \frac{B^2}{b}\,\mathrm{arctg}\,\frac{x}{b}\Big|_{-\infty}^{+\infty} = \frac{\pi B^2}{b} = 1 \qquad B = \pm\frac{\sqrt{b}}{\sqrt{\pi}}$$

№ 3

$$\langle \varphi(x) | \psi(x) \rangle = \int_{-\infty}^{+\infty} \left(\frac{ABx}{(x^2+a^2)(x^2+b^2)} + \frac{ibAB}{(x^2+a^2)(x^2+b^2)} \right) dx =$$

$$\frac{Cx+D}{x^2+b^2} + \frac{Fx+H}{x^2+a^2} = \frac{x}{(x^2+a^2)(x^2+b^2)}$$

$$Cx^3 + Dx^2 + Ca^2x + Da^2 + Fx^3 + Hx^2 + Fb^2x + Hb^2 = x$$

$$C + F = 0 \qquad C = \frac{1}{a^2-b^2}, \quad F = -\frac{1}{a^2-b^2}$$

$$D + H = 0$$

$$Ca^2 + Fb^2 = 1 \qquad D = H = 0$$

$$Da^2 + Hb^2 = 0$$

$$\frac{Cx+D}{(x^2+b^2)} + \frac{Fx+H}{(x^2+a^2)} = \frac{ib}{(x^2+a^2)(x^2+b^2)}$$

$$C + F = 0 \qquad D = \frac{ib}{a^2-b^2} \qquad H = -\frac{ib}{a^2-b^2}$$

$$D + H = 0$$

$$Ca^2 + Fb^2 = 0 \qquad C = F = 0$$

$$Da^2 + Hb^2 = ib$$

$$= AB \int_{-\infty}^{+\infty} \left(\frac{1}{a^2-b^2} \cdot \frac{x}{x^2+b^2} - \frac{1}{a^2-b^2} \frac{x}{x^2+a^2} + \frac{ib}{a^2-b^2} \cdot \frac{1}{x^2+b^2} - \frac{ib}{a^2-b^2} \cdot \frac{1}{x^2+a^2} \right) dx =$$

$$= \frac{AB}{a^2-b^2} \left(\frac{1}{2} \int \frac{d(x^2+b^2)}{x^2+b^2} - \frac{1}{2} \int \frac{d(x^2+a^2)}{x^2+a^2} + ib \int \frac{dx}{x^2+b^2} - ib \int \frac{dx}{x^2+a^2} \right) =$$

$$= \underbrace{\frac{AB}{2(a^2-b^2)} \ln \left(\frac{x^2+b^2}{x^2+a^2} \right) \Bigg|_{-\infty}^{+\infty}}_{=0} + \frac{ABib}{a^2-b^2} \left(\frac{1}{b} \operatorname{arctg} \frac{x}{b} - \frac{1}{a} \operatorname{arctg} \frac{x}{a} \right) \Bigg|_{-\infty}^{+\infty} =$$

$$= \frac{ABib}{a^2-b^2} \left(\frac{\pi}{b} - \frac{\pi}{a} \right) = \frac{\pi i AB}{a \cdot (a+b)} = \sqrt{\frac{2a^3}{\pi}} \cdot \sqrt{\frac{b}{\pi}} \cdot \frac{\pi i}{a(a+b)} = \frac{i\sqrt{2ab}}{(a+b)}$$

№ 4

$$\delta(f(x)) = \sum_{x_i} \frac{1}{|f'(x_i)|}\delta(x-x_i)$$

И-дем $\int_a^b \varphi(x)\delta[f(x)]dx$ =

Пусть $f(x)=y$, $dy=f'(x)dx$, $x=g(y)$

$$= \int_{f(a)}^{f(b)} \varphi[g(y)]\delta(y)\frac{dy}{f'[g(y)]} = \int_{f(a)}^{f(b)} \frac{\varphi[g(y)]}{f'[g(y)]}\delta(y)dy$$

Пусть $f(x)$ возр-т на $[a;b]$ и имеет ед. корень $x_0 \Rightarrow f(x_0)=0$ и $f'(x_0)>0$, тогда

$$\int_{f(a)}^{f(b)} \frac{\varphi[g(y)]}{f'(x)}\delta(y)dy = \frac{\varphi(x_0)}{f'(x_0)}$$

Пусть $f(x)$ убывает, тогда $f'(x_0)<0$ и $f(a)>f(b) \Rightarrow$

$$\Rightarrow \int_{f(a)}^{f(b)} \frac{\varphi[g(y)]}{-|f'(x)|}\delta(y)dy = \int_{f(b)}^{f(a)} \frac{\varphi[g(y)]}{|f'(x)|}\delta(y)dy = \frac{\varphi(x_0)}{|f'(x_0)|}$$

справедливо и для возр, и для убыв. ф-ии

$$\Rightarrow \quad \delta[f(x)] = \frac{\delta(x-x_0)}{|f'(x_0)|}$$

Если корней счетное кол-во, то, разбивая $[a; b]$ на элементарные участки, получим

$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \quad , x_i \text{ - корень ур-я}$$

№5

1. $|f_1\rangle = \alpha_1 e^{i\frac{\pi x}{a}} = \alpha_1\left(\cos\frac{\pi x}{a} + i\sin\frac{\pi x}{a}\right)$

$\langle f_1| = \alpha_1 e^{-i\frac{\pi x}{a}} = \alpha_1\left(\cos\frac{\pi x}{a} - i\sin\frac{\pi x}{a}\right)$

$|f_2\rangle = \alpha_2 e^{-\frac{i\pi x}{a}} = \alpha_2\left(\cos\frac{\pi x}{a} - i\sin\frac{\pi x}{a}\right)$

$$\langle f_1|f_2\rangle = \int_0^a \alpha_1\alpha_2\left(\cos^2\frac{\pi x}{a} - 2i\sin\frac{\pi x}{a}\cos\frac{\pi x}{a} - \sin^2\frac{\pi x}{a}\right)dx =$$

$$= \alpha_1\alpha_2\int_0^a\left(2\cos^2\frac{\pi x}{a} - 1 - i\sin\frac{2\pi x}{a}\right)dx = \alpha_1\alpha_2\int_0^a\left(\cos\frac{2\pi x}{a} - i\sin\frac{2\pi x}{a}\right)dx =$$

$$= \alpha_1\alpha_2\frac{a}{2\pi}\left(\sin\frac{2\pi x}{a} + i\cos\frac{2\pi x}{a}\right)\Big|_0^a = \alpha_1\alpha_2\frac{a}{2\pi}(0 + i - 0 - i) = 0$$

2. $$\langle f_1|f_1\rangle = \alpha_1^2\int_0^a\left(\cos^2\frac{\pi x}{a} + \sin^2\frac{\pi x}{a}\right)dx = 1 \qquad \alpha_1^2\int_0^a dx = 1$$

$$\alpha_1^2\cdot a = 1 \qquad \alpha_1 = \frac{1}{\sqrt{a}}$$

$$\langle f_2|f_2\rangle = \alpha_2^2\int_0^a\left(\cos^2\frac{\pi x}{a} + \sin^2\frac{\pi x}{a}\right)dx = 1$$

$$\alpha_2 = \frac{1}{\sqrt{a}}$$

3. $|\psi(x)\rangle = c_1|f_1\rangle + c_2|f_2\rangle$

$$\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a} = \frac{c_1}{\sqrt{a}}\left(\cos\frac{\pi x}{a} + i\sin\frac{\pi x}{a}\right) + \frac{c_2}{\sqrt{a}}\left(\cos\frac{\pi x}{a} - i\sin\frac{\pi x}{a}\right)$$

$$\Rightarrow \cos\frac{\pi x}{a}(c_1 + c_2) + i\sin\frac{\pi x}{a}(c_1 - c_2) = \sqrt{2}\sin\frac{\pi x}{a}$$

$c_1 + c_2 = 0 \qquad c_1 = -c_2$

$(c_1 - c_2)i = \sqrt{2} \qquad -2c_2 i = \sqrt{2}$

$$c_2 = \frac{\sqrt{2}}{-2i}\cdot\frac{i}{i} = \frac{\sqrt{2}i}{2} \qquad c_1 = -\frac{\sqrt{2}i}{2}$$