1. Suppose that M is a free left R-module, over a ring R, having a basis with n elements, as well as a basis with $n+1$ elements with $n \in \mathbb{Z}^+$. Show that M has a basis with m elements, for each $m \in \mathbb{Z}^+$ satisfying $m \geq n$.

   Solution.
   Let M, R, and n be as given above. We will proceed by induction. By supposition, M has a basis with n elements. Now, assume that M has a basis with $m \geq n$ elements. So, we need a basis for M with $m+1$ elements. By our other supposition, M has a basis with $n+1$ elements. Hence, M has a basis with $m+1$ elements. Therefore, by mathematical induction, M has a basis with m elements, for each $m \in \mathbb{Z}^+$ satisfying $m \geq n$.

2. 2. Give an example of a ring R and left R-modules M, N, L such that $M \oplus L \cong N \oplus L$, but $M \ncong N$.

Solution.
Consider the ring $\mathbb{R}$ and the left $\mathbb{R}$-modules $\{0\}, \bigoplus_{i\in\mathbb{N}} \mathbb{R}$, and $\bigoplus_{i\in\mathbb{N}} \mathbb{R}$. Then, we have that $\{0\} \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{R} \cong \bigoplus_{i\in\mathbb{N}} \mathbb{R}$. Also, $\bigoplus_{i\in\mathbb{N}} \mathbb{R} \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{R} \cong \bigoplus_{i\in\mathbb{N}} \mathbb{R}$ via the map $\pi : \bigoplus_{i\in\mathbb{N}} \mathbb{R} \to \bigoplus_{i\in\mathbb{N}} \mathbb{R} \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{R}$ defined by $\pi((a_i)_{i\in\mathbb{N}}) = ((a_{2i})_{i\in\mathbb{N}}, (a_{2i+1})_{i\in\mathbb{N}})$. Clearly, π is a module homomorphism. Also, we see that $\ker(\pi) = \{0\}$ as 0 is the only element in $\bigoplus_{i\in\mathbb{N}} \mathbb{R}$ such that $\pi(0) = 0 = (0, 0)$. Thus, π is $1-1$. Also, π is onto since for any element in $\bigoplus_{i\in\mathbb{N}} \mathbb{R} \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{R}$, there are finitely many nonzero terms in each tuple. Since we have that $\bigoplus_{i\in\mathbb{N}} \mathbb{R} \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{R} \cong \bigoplus_{i\in\mathbb{N}} \mathbb{R}$ and $\{0\} \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{R} \cong \bigoplus_{i\in\mathbb{N}} \mathbb{R}$, $\bigoplus_{i\in\mathbb{N}} \mathbb{R} \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{R} \cong \{0\} \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{R}$. However, $\{0\} \ncong \bigoplus_{i\in\mathbb{N}} \mathbb{R}$.

3. 3. Let R be a ring, and consider the following commuting diagram of left R-modules and R-module homomorphisms, where each row is a short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "\phi_1"] \arrow[d, "\rho_1"'] & M_2 \arrow[r, "\phi_2"] \arrow[d, "\rho_2"'] & M_3 \arrow[r] \arrow[d, "\rho_3"'] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "\psi_1"] & N_2 \arrow[r, "\psi_2"] & N_3 \arrow[r] & 0
\end{tikzcd}$$

(a) Show that if ρ_1 and ρ_3 are $1-1$, then so is ρ_2.
(b) Show that if ρ_1 and ρ_3 are onto, then so is ρ_2.

Solution.
Let the diagram be as given above.

(a) Assume that ρ_1 and ρ_3 are $1-1$. Now, let $m \in \ker(\rho_2)$. Hence, $\rho_2(m) = 0$. Since the above diagram commutes, we have that $(\rho_3(\phi_2(m)) = (\psi_2(\rho_2(m))$. Since $\rho_2(m) = 0$ and $\psi_2(0) = 0$ by ψ_2 be an R-module homomorphism, $(\rho_3(\phi_2(m)) = 0$. By assumption ρ_3 is $1-1$, and therefore, $\ker(\rho_3) = \{0\}$. That is, $\phi_2(m) = 0$. Since each row is a short exact sequence, we have that $\phi_1(M_1) = \ker(\phi_2)$. So, $m \in \phi(M_1)$ means that there exists $m' \in M_1$ such that $\phi(m') = m$. Again, since the given diagram commutes, we have that $\psi_1(\rho_1(m')) = \rho_2(\phi_1(m'))$. Since $\phi(m') = m$ and $\rho_2(m) = 0$, $\psi_1(\rho_1(m')) = 0$. We also know that ψ_1 is $1-1$ since the each row is a short exact sequence by Observation 8. Hence, $\rho_1(m') = 0$ and since ρ_1 is $1-1$ by assumption, $m' = 0$. Thus, $\phi_1(m') = \phi_1(0) = 0 = m$. Hence, $\ker(\rho_2) = \{0\}$ and so ρ_2 is $1-1$ as desired.

(b) Assume that ρ_1 and ρ_3 are onto. Let $z \in N_3$. Since we assume that ρ_3 is onto, there exists $c \in M_3$ such that $\rho_3(c) = z$. Since each row is a short exact sequence, we have that ϕ_2 is also onto. Hence there exists $b \in M_2$ such that $\phi_2(b) = c$. So, $\rho_3(\phi_2(b)) = z$. Since the diagram commutes, we have that $\rho_3(\phi_2(b)) = \psi_2(\rho_2(b)) = z$. Since each row is a short exact sequence, we also know that there exists $y \in N_2$ such that $\psi_2(y) = z$. Now, let $y' \in \ker(\psi_2)$. Then, we have that $y' \in \psi_1(N_1)$ since the bottom row is a short exact sequence. So, there exists $x \in N_1$ such that $\psi_1(x) = y'$. Since we assume that ρ_1 is onto, we have that there exists $a \in M_1$ such that $\rho_1(a) = x$. So, $\psi_1(\rho_1(a)) = y'$. Again, since the diagram commutes, we have that $\rho_2(\phi_1(a)) = \psi_1(\rho_1(a)) = y'$. Thus, we have that ρ_2 is also onto as desired.

4. 4. Prove that $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module.

Solution.
Assume by way of contradiction that $\mathbb{Q}$ is projective as a $\mathbb{Z}$-module. By a fact from class, we know that $\mathbb{Q}$ is projective if and only if $\mathbb{Q}$ is free. So, let $S \subseteq \mathbb{Q}$ be a basis for $\mathbb{Q}$. Let $q, r \in S$. So, $q = \frac{x_1}{y_1}$ and $r = \frac{x_2}{y_2}$ for some $x_1, x_2, y_1, y_2 \in \mathbb{Z}^*$. Now consider $aq + br$. If $a = x_2 y_1$ and $b = -x_1 y_2$, $aq + br = x_2 y_1 \frac{x_1}{y_1} - x_1 y_2 \frac{x_2}{y_2} = x_1 x_2 - x_1 x_2 = 0$. However, we have that $x_1, x_2, y_1, y_2 \in \mathbb{Z}^*$, so $a, b \neq 0$. Thus S is not linearly independent, and is hence not a basis for $\mathbb{Q}$. Since S was arbitrary, we have that there is not basis for $\mathbb{Q}$. Thus, $\mathbb{Q}$ is not free and is thus not projective as a $\mathbb{Z}$-module.

5. 5. Let R be a ring, and M_i $(i \in I)$ left R-modules. Show that if $\bigoplus_{i\in I} M_i$ is projective, then each M_i is projective.

Solution.
Let R and M_i for $i \in I$ be as given. Since $\bigoplus_{i\in I} M_i$ is projective, then for any left R-modules N_1, N_2, any onto R-module homomorphism $\phi : N_1 \to N_2$, and any R-module homomorphism $\psi : \bigoplus_{i\in I} M_i \to N_2$, there exists an R-module homomorphism $\rho : \bigoplus_{i\in I} M_i \to N_1$ such that $\phi \circ \rho = \psi$ by part 3 of Theorem 10. Now consider $\iota_i : M_i \to \bigoplus_{i\in I} M_i$ the natural inclusion map. Then, for an arbitrary $a \in M_i$, $\iota_i(a) = b \in \bigoplus_{i\in I} M_i$. Now, $\psi(\iota_i(a)) = \psi(b)$. By our assumption, $\psi(b) = \phi(\rho(b))$. That is, $\psi(\iota_i(a)) = \phi(\rho(\iota_i(a)))$. So, we have each M_i is projective by considering $\rho \circ \iota_i$ and $\psi \circ \iota_i$.

6. 6. Let R be a ring, and M_i ($i \in I$) left R-modules. Show that if each M_i is projective, then $\bigoplus_{i \in I} M_i$ is projective.

Solution.
Let R and M_i for $i \in I$ be as stated. Since M_i is projective, then for any left R-modules N_1, N_2, any onto R-module homomorphism $\phi : N_1 \to N_2$, and any R-module homomorphism $\psi : M_i \to N_2$, there exists an R-module homomorphism $\rho : M_i \to N_1$ such that $\phi \circ \rho = \psi$ by part 3 of Theorem 10. Now consider $\pi_i : \bigoplus_{i \in I} M_i \to M_i$ the natural projection map. Then, for an arbitrary $a \in \bigoplus_{i \in I} M_i$, $\pi_i(a) = b \in M_i$. Now, $\psi(\pi_i(a)) = \psi(b)$. By our assumption, $\psi(b) = \phi(\rho(b))$. That is, $\psi(\pi_i(a)) = \phi(\rho(\pi_i(a)))$. So, we have $\bigoplus_{i \in I} M_i$ is projective by considering $\rho \circ \pi_i$ and $\psi \circ \pi_i$.

7. 7. Show that if M is an injective $\mathbb{Z}$-module, then for all $m \in M$ and $n \in \mathbb{Z} \setminus \{0\}$, there exists $m' \in M$ such that $nm' = m$.

Solution.
Let M be an injective $\mathbb{Z}$-module and let $m \in M$ and $n \in \mathbb{Z} \setminus \{0\}$. Consider the map $\psi : n\mathbb{Z} \to M$ induced by $n \mapsto m$. Also consider the map $\iota : n\mathbb{Z} \to \mathbb{Z}$ the natural inclusion map. Then ι is injective. Hence, there exists a $\mathbb{Z}$-module homomorphism ρ such that $\rho \circ \iota = \psi$ by part 3 of Theorem 11. Now, define $m' = \rho(1)$. Without loss of generality, assume that $n > 0$. Then, $n = 1 + \cdots + 1$. Now, $\psi(n) = \rho(\iota(n))$. That is $m = \rho(n)$ by definition of ψ and ι. Whence, $m = \rho(1 + \cdots + 1) = \rho(1) + \cdots + \rho(1)$ since ρ is a $\mathbb{Z}$-module homomorphism. Therefore, $m = m' + \cdots + m' = nm'$. Thus, if M is an injective $\mathbb{Z}$-module, then for all $m \in M$ and $n \in \mathbb{Z} \setminus \{0\}$, there exists $m' \in M$ such that $nm' = m$ as desired.

8. 8. Prove that no finite nonzero $\mathbb{Z}$-module can be injective.

Solution.
Assume by way of contradiction that there exists a finite nonzero $\mathbb{Z}$-module that is injective. Call said $\mathbb{Z}$-module M. Since M is finite and nonzero, we have that $|M| = m$ for some $m \in \mathbb{N}$ such that $m \geq 2$. Now, let $a \in M$ such that $a \neq 0$. Then, $|\langle a \rangle| = n$ where n divides m and $n > 1$. So, $na = 0$ by definition of the order of an element. Since we assume that M is injective, we have that every short exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} M_3 \longrightarrow 0$$

is split by Theorem 11. Since the sequence is short exact, we know that ϕ_1 is $1-1$ by observation 8. Therefore, $\ker(\phi_1) = \{0\}$. Now, $\phi_1(0) = \phi_1(na) = \phi_1(a + \cdots + a) = \phi_1(a) + \cdots + \phi_1(a) \neq 0$ since $a \neq 0$ and ϕ_1 is a $\mathbb{Z}$-module homomorphism, a contradiction. Thus, there are no finite nonzero $\mathbb{Z}$-modules that are injective as desired.

9. 9. Prove that for any ring R, every left R-module is projective if and only if every left R-module is injective.

Solution.
Let R be a ring. Assume first that every left R-module is projective. Then, we have that every short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} M_3 \longrightarrow 0$$

is split for all left R-modules M_1, M_2, M_3. So, if we fix the left R-module M_1, then the short exact sequence is still split for all left R-modules M_2, M_3. Since M_1 is an arbitrary left R-module, we have that every left R-module is injective. Conversely, assume that every left R-module is injective. Hence we have that every short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} M_3 \longrightarrow 0$$

is split for all left R-modules M_1, M_2, M_3. Now, fix the left R-module M_3. Whence every short exact sequence is still split for all left R-modules M_1, M_2. Since M_3 is arbitrary as a left R-modules, we have that every left R-module is projective. Thus, every left R-module is projective if and only if every left R-module is injective.