1.1 Determine the interpolating quadratic polynomial of u at $\bar{x}, \bar{x}-h$, and $\bar{x}-2h$, and verify $P'(\bar{x}) = D_2u(\bar{x}) = \frac{1}{2h}[3u(\bar{x}) - 4u(\bar{x}-h) + u(\bar{x}-2h)]$.

Solution.
We will use a Lagrange interpolating polynomial. So,

$$P(x) = \frac{(x-(\bar{x}-h))(x-(\bar{x}-2h))}{(\bar{x}-(\bar{x}-h))(\bar{x}-(\bar{x}-2h))}u(\bar{x}) + \frac{(x-(\bar{x}))(x-(\bar{x}-2h))}{((\bar{x}-h)-(\bar{x}))((\bar{x}-h)-(\bar{x}-2h))}u(\bar{x}-h)$$
$$+\frac{(x-(\bar{x}))(x-(\bar{x}-h))}{((\bar{x}-2h)-(\bar{x}))((\bar{x}-2h)-(\bar{x}-h))}u(\bar{x}-2h) =$$
$$\frac{(x-(\bar{x}-h))(x-(\bar{x}-2h))}{2h^2}u(\bar{x}) - \frac{(x-(\bar{x}))(x-(\bar{x}-2h))}{h^2}u(\bar{x}-h)$$
$$+\frac{(x-(\bar{x}))(x-(\bar{x}-h))}{2h^2}u(\bar{x}-2h) =$$
$$\frac{x^2-2x\bar{x}+3xh+\bar{x}^2-3\bar{x}h+2h^2}{2h^2}u(\bar{x}) - \frac{x^2-2x\bar{x}+2xh+\bar{x}^2-2\bar{x}h}{h^2}u(\bar{x}-h)+$$
$$\frac{x^2-2x\bar{x}+xh+\bar{x}^2-\bar{x}h}{2h^2}u(\bar{x}-2h)$$

So,

$$P'(x) = \frac{2x-2\bar{x}+3h}{2h^2}u(\bar{x}) - \frac{2x-2\bar{x}+2h}{h^2}u(\bar{x}-h) + \frac{2x-2\bar{x}+h}{2h^2}u(\bar{x}-2h)$$

Therefore,

$$P'(\bar{x}) = \frac{3}{2h}u(\bar{x}) - \frac{2}{h}u(\bar{x}-h) + \frac{1}{2h}u(\bar{x}-2h) =$$
$$\frac{1}{2h}[3u(\bar{x}) - 4u(\bar{x}-h) + u(\bar{x}-2h)]$$

as desired.

1.2 a Use the method of undeermined coefficients to set up the 5×5 Vandermonde system that would determine a fourth-order accurate finite difference approximation to $u''(x)$ based on 5 equally spaced points;

$$u''(x) = c_{-2}u(x-2h) + c_{-1}u(x-h) + c_0 u(x) + c_1 u(x+h) + c_2 u(x+2h) + O(h^4)$$

Solution.
We will expand each of the $u(\bar{x})$ using Taylor's Theorme to the fourth derivative. This gives us:

$$\begin{aligned}
u(\bar{x}-2h) &= u(x) - 2hu'(x) + 2h^2u''(x) - \frac{8}{6}h^3u'''(x) + \frac{2}{3}h^4u^{(4)}(x)\\
u(\bar{x}-h) &= u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(x) + \frac{1}{24}h^4u^{(4)}(x)\\
u(\bar{x}) &= u(x) + 0\\
u(\bar{x}+h) &= u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(x) + \frac{1}{24}h^4u^{(4)}(x)
\end{aligned}$$

and

$$u(\bar{x}+2h) = u(x) + 2hu'(x) + 2h^2u''(x) + \frac{8}{6}h^3u'''(x) + \frac{2}{3}h^4u^{(4)}(x)$$

So, we have the following Vandermonde system

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 2h^2 & \frac{h^2}{2} & 0 & \frac{h^2}{2} & 2h^2 \\ -\frac{8h^3}{6} & -\frac{h^3}{6} & 0 & \frac{h^3}{6} & \frac{8h^3}{6} \\ \frac{2h^4}{3} & \frac{h^4}{24} & 0 & \frac{h^4}{24} & \frac{2h^4}{3} \end{pmatrix} \begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

So taking the inverse of the Vandermonde matrix yeilds

$$\begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{12h} & -\frac{1}{12h^2} & -\frac{1}{2h^3} & \frac{1}{h^4} \\ 0 & -\frac{2}{3h} & \frac{4}{3h^2} & \frac{1}{h^3} & -\frac{4}{h^4} \\ 1 & 0 & -\frac{5}{2h^2} & 0 & \frac{6}{h^4} \\ 0 & \frac{2}{3h} & \frac{4}{3h^2} & -\frac{1}{h^3} & -\frac{4}{h^4} \\ 0 & -\frac{1}{12h} & -\frac{1}{12h^2} & \frac{1}{2h^3} & \frac{1}{h^4} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Solving this system gives us $c_{-2} = -\frac{1}{12h^2}, c_{-1} = \frac{4}{3h^2}, c_0 = -\frac{5}{2h^2}, c_1 = \frac{4}{3h^2}$, and $c_2 = -\frac{1}{12h^2}$. Since each of the coefficients c_i are of the order $\frac{1}{h^2}$ and we are using the centered finite difference formula, we have that u'' is of the order of $O(h^{6-2}) = O(h^4)$. So, we get that

$$u''(x) = -\frac{1}{12h^2}u(\bar{x}-2h) + \frac{4}{3h^2}u(\bar{x}-h) - \frac{5}{2h^2}u(\bar{x}) + \frac{4}{3h^2}u(\bar{x}+h) - \frac{1}{12h^2}u(\bar{x}+2h) + O(h^4)$$

b Compute the coefficients using MATLAB and check that they satisfy the above system.

Solution.
We have that the MATLAB code returns the same coeffiecients found above, baring the $\frac{1}{h^2}$ factor shared among the coefficients. That is, $c_{-2} = -\frac{1}{12h^2}, c_{-1} = \frac{4}{3h^2}, c_0 = -\frac{5}{2h^2}, c_1 = \frac{4}{3h^2}$, and $c_2 = -\frac{1}{12h^2}$. See the code and the output below.

c Test the finite difference formula to approximate $u''(1)$ for $u(x) = \sin(2x)$ with values of h from the MATLAB array. Make a table of the error vs. h for several values of h and compare against the predicted error from the leading term of the printed expression. Also produce a log-log plot of the absolute error vs. h.

Solution.
Since we are approximating $u''(1)$, we need $u''(x) = -4\sin(2x)$. We see that the approximation is accurate to 6 decimal places. We also have the log-log plot and table of values. See the MATLAB code and output below

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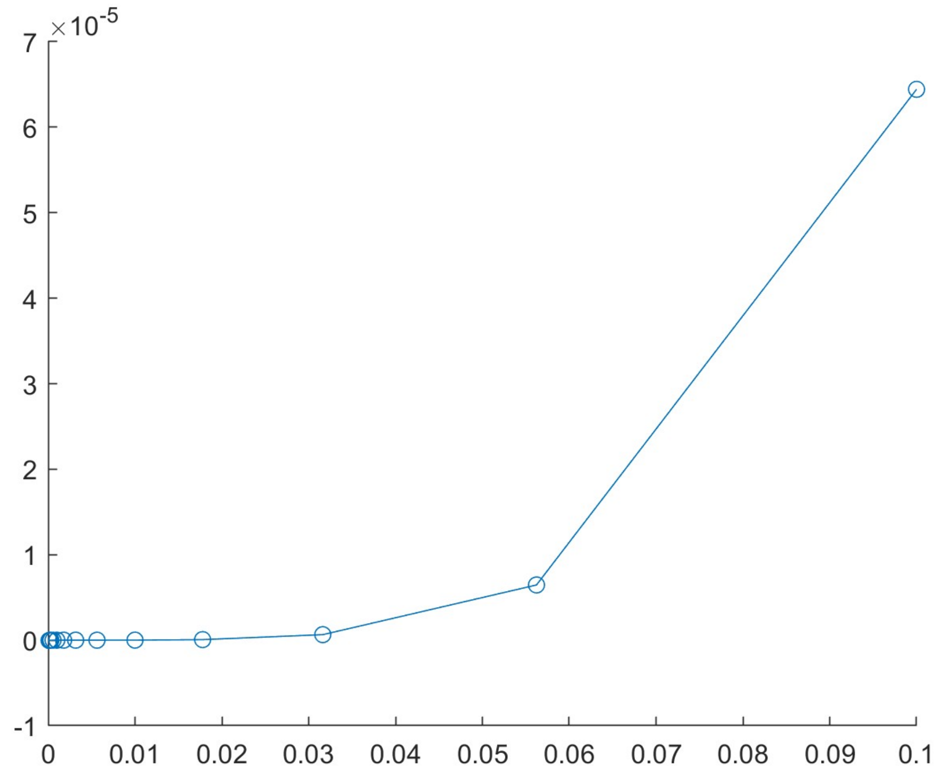
ommand Window

```
Problem 1.2
Part b.


The derivative u^(2) of u at x0 is approximated by

          1/h^2 * [
                     -8.333333333333333e-02 * u(x0-2*h) +
                      1.333333333333333e+00 * u(x0-1*h) +
                     -2.500000000000000e+00 * u(x0) +
                      1.333333333333333e+00 * u(x0+1*h) +
                     -8.333333333333333e-02 * u(x0+2*h) ]

For smooth u,
       Error = 0 * h^3*u^(5) + -0.0111111 * h^4*u^(6) + ...

   -0.0833    1.3333   -2.5000    1.3333   -0.0833
```

```
Part c.

u=sin(2*x)
u''=-4*sin(2*x)
      h                  Error
1.0000e-01           6.4431e-05

5.6234e-02           6.4588e-06

3.1623e-02           6.4638e-07

1.7783e-02           6.4653e-08

1.0000e-02           6.4608e-09

5.6234e-03           6.3753e-10

3.1623e-03           5.2040e-11

1.7783e-03           -3.9182e-11

1.0000e-03           -5.0307e-10

5.6234e-04           -4.8242e-10

3.1623e-04           -4.3333e-09

1.7783e-04           -9.9178e-09

1.0000e-04           -3.3754e-08

Approximation=-3.6371897411
Exact=-3.6371897073
```




```
%%Dallas Klumpe MATH 5670
%% Homework 1
%Part 1.2.b.
clear;
close all;
clc;
fprintf('Problem 1.2');
fprintf('\n');
fprintf('Part b.');
fprintf('\n\n');
fdstencil(2,-2:2);
c=fdcoeffF(2,0,-2:2);
disp(c);
fprintf('Part c.\n\n');
syms x;
hvals=logspace(-1,-4,13);
u=sin(2*x);
d2u=diff(diff(u));
ex=vpa(subs(d2u,x,1));
fprintf('u=%s\n',u);
fprintf('u''''=%s\n',d2u);
disp('');
disp('      h                 Error');
for i=1:length(hvals)
```

```
    h=hvals(i);
    fd=((-1/12)*sin(2*(1-2*h))+(4/3)*sin(2*(1-h))-(5/2)*sin(2*1) ...
        ...
        +(4/3)*sin(2*(1+h))-(1/12)*sin(2*(1+2*h)))/h^2;
    Err(i)=fd-ex;
    disp(sprintf('%2.4e          %2.4e\n',h,Err(i)));
end
fprintf('Approximation=%.10f\n',fd);
fprintf('Exact=%.10f\n',ex);
hold on
loglog(hvals,Err,'o-');
hold off
```