1. Let $(S, +)$ be an abelian semigroup. Suppose S has the property that for each pair $a, b \in S$ there exists $c \in S$ for which $a + c = b$. Prove that S is a group.

Solution.
Let S be as given above. Let $a, b \in S$. We will show that S is has an identity and is closed under inverses. Assume first that $a = b$. Then, by our assumption of S, there exists a $c \in S$ such that $a+c = b = a$. Since S is abelian, we have that $a+c = c+a = a$. Therefore, S has an identity element. To show that the identity is unique, assume that there is more than one. That is, assume there exists e' such that $a + e' = e' + a = a$. Now, let $b = e$ be the identity element. Again, by our assumption, there exists $c' \in S$ such that $a + c' = b = e$. That is $a + c' = c' + a = e$ since S is an abelain semigroup. Thus, S is a group as desired.

2. Let K be any field. Let E be the Toeplitz graph $\circlearrowleft \bullet \longrightarrow \bullet$. Let A be the Jacobson Algebra $A = K\langle X, Y | XY = 1\rangle$. Explicitly write down a K-algebra isomorphism $A \cong L_K(E)$.

Solution.
Let K, E, and A be as given. Let $L_K(E)$ be the Leavitt algebra of E. Denote the vertices from left to right by u and v and the edges from left to right by e and f respctively. Consider the mapping $\rho : L_K(E) \to A$ given by:

$$\rho(u)$$

$$\rho(v)$$

$$\rho(e)$$

$$\rho(e^*)$$

$$\rho(f)$$

$$\rho(f^*)$$

3. Let K be any field. Let F be the graph $\circlearrowleft\bullet \longleftarrow \bullet$. Explicitly write down a K-algebra isomorphism $M_2(K[x, x^{-1}]) \cong L_K(F)$.

Solution.
Let K and F be as above, and $L_K(F)$ be the Leavitt algebra of F. Let $G = M_2(K[x, x^{-1}])$. Denote the edges from left to right as f and e and the vertices from left to right as v and u. Consider the mapping $\phi : L_K(F) \to G$ given by

$$\phi(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\phi(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\phi(e^*) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\phi(v) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\phi(f) = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$$

$$\phi(f^*) = \begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix}$$

Now, we will check that the relations are satisfied. Notice that:

$$\phi(u)\phi(u) - \phi(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$\phi(v)\phi(v) - \phi(v) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\phi(u)\phi(v) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \phi(v)\phi(u)$$

and

$$\begin{aligned}\phi(u)+\phi(v) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2\end{aligned}$$

So, we have that the mappings of the vertices are also idempotent, orthogonal, and sum to the identity. Now observe that:

$$\begin{aligned}\phi(v)\phi(f) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \phi(f)\phi(v)\end{aligned}$$

$$\begin{aligned}\phi(v)\phi(f^*) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \phi(f^*)\phi(v)\end{aligned}$$

$$\begin{aligned}\phi(f)\phi(f^*) &= \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \phi(v)\end{aligned}$$

$$\begin{aligned}\phi(f^*)\phi(f) &= \begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \phi(v)\end{aligned}$$

$$\begin{aligned}\phi(u)\phi(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \phi(e)\end{aligned}$$

$$\phi(v)\phi(e^*) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \phi(e^*)$$

$$\phi(e)\phi(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$\phi(e^*)\phi(v) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\phi(e)\phi(e^*) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \phi(u)$$

and

$$\phi(e^*)\phi(e) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \phi(v)$$

So, all of the relations described in the Leavitt algebra are satisfied by the mapping and thus, this mapping is a homomorphism.

Now, consider the mapping $\psi : G \to L_K(F)$ given by:

$$\psi(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = u$$

$$\psi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = e$$

$$\psi(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = e^*$$

$$\psi(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = v$$

$$\psi(\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}) = f$$

$$\psi(\begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix}) = f^*$$

Notice that, with almost identical but reversed logic, ψ is also a homomorphism of rings.

Finally, note that for all $x \in G$,

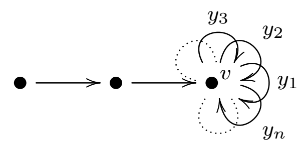
$$\phi(\psi(x)) = x$$

and for all $x \in L_K(F)$

$$\psi(\phi(x)) = x$$

Hence, each of the above derived homomorphisms has an inverse, namely the other. That is, they are both bijective. Therefore, they are isomorphisms yielding $M_2(K[x, x^{-1}]) \cong L_K(F)$ as desired.

4. Let K be any field. Let G_n be the graph



Explicitly write down a K-algebra isomorphism $\mathrm{M}_3(L_K(R_n)) \cong L_K(G_n)$ (where as usual R_n is the graph with one vertex and n loops).

Solution.
Let K, G_n be as above, and let $F = \mathrm{M}_3(L_K(R_n))$. Let v and y_i for $1 \leq i \leq n$ be the vertex and edges denoted in the graph above. Further, let t and u be the other two vertices from left to right repsctively, and let e and f be the remaining two edges from left to right respectively. Consider the mapping $\gamma : L_K(G_n) \to F$ given by:

$$\gamma(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\gamma(u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\gamma(v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\gamma(e) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\gamma(e^*) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\gamma(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\gamma(f^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\gamma(y_i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\gamma(y_i^*) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now witness, without loss of generality in terms of vertices:

$$\gamma(t)\gamma(t) - \gamma(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

That is, all vertice mappings are idempotent. Also:

$$\gamma(t)\gamma(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

So, the mappings of the vertices are also orthogonal. Next note that:

$$\gamma(t) + \gamma(u) + \gamma(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Hence, the sum of the mappings of the vertices is the identiy element.

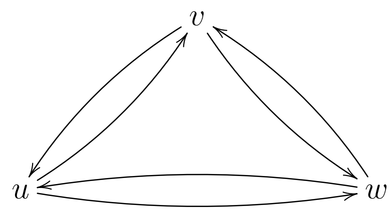
5. Modify/simplify/rewrite the argument given in The Reduction Theorem 2.2.11 of AAS, in the situation where the graph is assumed to be of the form R_n (i.e., one vertex, n loops, $n \geq 2$). More formally, prove the following.
   Let $E = R_n$ for some $n \geq 2$ (denote the unique vertex by v), and let K be an arbitrary field. For any nonzero element $\alpha \in L_K(E)$ there exists $\mu, \eta \in Path(E)$ such that $0 \neq \mu * \alpha\eta = kv$ in $L_K(E)$, for some $0 \neq k \in K$.

   Solution.
   Well,

7 Let E be a finite graph and K any field. Suppose E contains a cycle c based at a vertex v, and suppose c has an exit f at v. Prove that $L_K(E)$ is not left noetherian.

Solution.
Well,

8 Let E be the graph



(a) Compute 'directly' the monoid M_E. (First find a set of representatives of the equivalence classes in M_E, then prove that these equivalence classes are distinct.)

Well,

(b) Using the theorem of Ara / Moreno / Pardo, we know $M_E \cong \mathcal{V}(L_K(E))$. Under this identification, which element of M_E corresponds to $[1_{L_K(E)}]$?

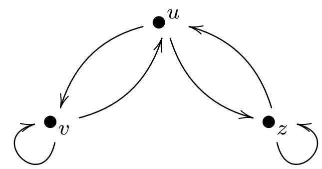
Well,

(c) Find the matrix $I - A_E^t$, and compute its Smith normal form. Show that your answer to (c) is consistent with your answer to (a).

Well,

Solution.
Well,

9 Repeat all three parts of the previous question, but this time using the graph F shown here.



Solution.
Well,

10 For $i \geq 3$, let C_n be the graph consisting of n vertices $\{v_1, v_2, \ldots, v_n\}$, and $2n$ edges $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ for which

$$s(e_i) = v_i, \quad r(e_i) = v_{i+1}, \quad s(f_i) = v_i, \quad r(f_i) = v_{i-1},$$

where indices are interpreted mod n. (So, for instance, $r(e_n) = e_1$, and $r(f_1) = v_n$.) In words, C_n is the graph with n vertices, where each vertex emits two edges, one to both of its neighboring vertices. In particular, C_3 is the graph E of the previous problem. For $n = 4$, $n = 5$, and $n = 6$, describe $\mathcal{V}(L_K(C_n))$. (Do this by Smith normal form; you need NOT do this directly as in Question 6a.) For two of these three, you will be able to use this description of $\mathcal{V}(L_K(C_n))$ to identify $L_K(C_n)$ as a "known" K-algebra. (You might need to look at the monoid $\mathcal{V}(L_K(C_n))$ directly in order to identify $[L_K(C_n)]$ within $\mathcal{V}(L_K(C_n))$.) Do that. Can you say anything about $L_K(C_n)$ for the third of these?

Solution.
Well,