3.4 Show that the center of a simple ring is a field, and the center of a semisimple ring is a finite direct product of fields.

Solution.
First, let R be a simple ring, and let Z be the center of R. We know that Z is a subring of R, so we need only show that multiplication is commutative in Z, there is a multiplicative identity, and that every nonzero element is invertible. It is obvious that if $1 \in R$, then $1 \in Z$. So, if we assume that $1 \notin R$, then we have that $Z = \{0\}$ which can be considered to be the trivial field. So, we will continue with the assumption that $1 \in R$. Clearly multiplication is commutative in Z since every element $a \in Z$ commutes with every element $b \in R$ and in particular every element $b \in Z$ since $Z \subseteq R$. Now, let $x \in Z \setminus \{0\}$. Then, we have that Rx is a two sided ideal of R since $x \in Z$. Since R is simple, $Rx = R$. So, there exists $y \in R$ such that $yx = 1$. Since $x \in Z \setminus \{0\}$ was arbitrary, we have that $xy = 1$ and also that $y \in Z \setminus \{0\}$. Hence, Z is closed under multiplicative inverses. Therefore, Z is a field.

Now, let R be a semisimple ring and let Z be the center of R. Since R is semisimple, $R \cong \mathbb{M}_{n_1}(D_1) \times \ldots \times \mathbb{M}_{n_m}(D_m)$ where, $m, n_1, \ldots n_m \in \mathbb{Z}^+$ and D_i are division rings. From a previous problem on homework 3, we know that the center of $\mathbb{M}_{n_i}(D_i)$ are $d_i \cdot I_{n_i}$ where I_{n_i} is the identity matrix and d_i is in the center of D_i. Since D_i is a division ring, D_i is also simple. Therefore, the center of D_i is a field by the above exercise. Now, clearly the center of $\mathbb{M}_{n_i}(D_i) = Z_i$ is a field since it contains the zero matrix, the identity matrix, additive and multiplicative inverses, and is commutative, asscociatve and distributive due to the center of D_i being a field. Thus, the center of R is $Z \cong Z_1 \times \ldots \times Z_m$ a finite direct product of fields as desired.

3.5 Let M be a finitely generated left R-module and $E = End({}_RM)$. Show that if R is semisimple (resp. simple artinian), then so is E.

Solution.
Let R, M, and E be as given. Suppose that R is semisimple. This means that M is also semisimple. Then, we have that M is the finite direct sum of simple R-modules. Group together the left simple modules that are isomorphic. Then ${}_RM \cong N_1^{n_1} \times \ldots \times N_m^{n_m}$ for some simple left R-modules N_i, $m, n_i \in \mathbb{Z}^+$, $1 \leq i \leq m$ with $N_i \ncong N_j$ for $i \neq j$. We have that $M \cong End({}_RM)$. So, $M \cong End(N_1^{n_1} \times \ldots \times N_m^{n_m})$. Since each submodule N_i is simple and $N_i \ncong N_j$ for $i \neq j$, there are no nonzero homomorphisms $N_i^{n_i} \to N_j^{n_j}$ for $i \neq j$. Therefore, we can write $M \cong End(N_1^{n_1}) \times \ldots \times End(N_m^{n_m})$. Now, $End(N_i^{n_i}) \cong \mathbb{M}_{n_i}(End(N_i))$ for all $1 \leq i \leq m$. Whence, $\mathbb{M}_{n_i}(End(N_i))$ is a matrix ring over a division ring by Schur's Lemma. Thus, E is a finite direct product of simple modules and is therefore semisimple as desired.

3.9 Let R, S be rings such that $\mathbb{M}_m(R) \cong \mathbb{M}_n(S)$. Does this imply that $m = n$ and $R \cong S$?
b. Let us call a ring A a matrix ring if $A \cong \mathbb{M}_m(R)$ for some integer $m \geq 2$ and some ring R. True or False: "A homomorphic image of a matrix ring is also a matrix ring"?

Solution.
a. This implication is not true. Consider the rings $R = \mathbb{M}_3(\mathbb{Q})$ and $\mathbb{Q}$. Then, $\mathbb{M}_3(\mathbb{M}_3(\mathbb{Q})) \cong \mathbb{M}_9(\mathbb{Q})$. However, $3 \neq 9$ and $R \ncong S$.

b. Let R be a ring and A a matrix ring. Then $A \cong \mathbb{M}_m(R)$ for some $m \geq 2$. Now, let I be an ideal of A. Then, $I = \mathbb{M}_m(J)$ where J is an ideal of R. Then, $\mathbb{M}_m(R)/I = \mathbb{M}_m(R)/\mathbb{M}_m(J) = \mathbb{M}_m(R/J)$. Let $f : \mathbb{M}_m(R) \to \mathbb{M}_m(R/J)$ be given by $f(x) = x + \mathbb{M}_m(J)$. We first see that $\ker(f) = \mathbb{M}_m(J)$. Also, note that f is surjective since any $x + \mathbb{M}_m(J)$ can be be given by $f(x)$ for $x \in \mathbb{M}_m(R)$. Now, let $x, y \in \mathbb{M}_m(R)$ and $r \in R$. Then, $f(x+y) = (x+y)+\mathbb{M}_m(J) = x+\mathbb{M}_m(J)+\mathbb{M}_m(J) = f(x)+f(y)$, and also $f(rx) = rx + \mathbb{M}_m(J) = r(x + \mathbb{M}_m(J)) = rf(x)$. Thus, f is a surjective R-module homomorphism, and therefore, by the Fundemental Homomorphism Theorem, $f(A)$ is also a matrix ring.

3.12 For a subset S in a ring R, let $ann_l(S) = \{a \in R | aS = 0\}$ and $ann_r(S) = \{a \in R | Sa = 0\}$. Let R be a semisimple ring, I be a left ideal and J be a right ideal in R. Show that $ann_l(ann_r(I)) = I$ and $ann_r(ann_l(J)) = J$.

Solution.
Let R be a semisimple ring. Then, every left and right ideal of R is generated by an idempotent. Let e be an idempotent of R. Then, we have that $e' = 1 - e$ is also an idempotent of R. Now, by problem 1.7, we have that $R = Re \oplus e'R$ since clearly $ee' = 0 = e'e$. So, let $I = Re$ and $J = e'R$. Then, $ann_r(I) = \{a \in R | aRe = 0\}$. We can see that $e' \in ann_r(I)$ as $e'Re = e're = (1-e)re = re - ere = re - re^2 = re - re = 0$ for all $r \in R$. So, $ann_r(I) = J$ since e' generates J. Then, $ann_l(J) = \{a \in R | e'Ra = 0\}$. Clearly, $e \in ann_l(J)$ since $e'Re = e're = 0$ as seen above, and since e generates I, we have that $ann_l(J) = I$. Therefore, we have that $ann_l(ann_r(I)) = I$ and $ann_r(ann_l(J)) = J$ as desired.

3.13 Let R be a simple, infinite-dimensional algebra over a field k. Show that any nonzero left R-module V is also infinite-dimensional over k.

Solution.
Let R and k be as given. Let $M \neq \{0\}$ be a left R-module over k. Assume that M is finite dimensional. Then, we have that M is an R-vector space since k is field, so M has a finite basis. Let $\{m_1, \ldots, m_n\}$ be a basis for M for $n \in \mathbb{Z}^+$. Now, observe the ideal Rm_1. Since R is simple and m_1 is an element of a basis for M, we have that $Rm_1 = R$. Since m_1 is part of a finite set, we have that R must be finite dimensional which is a contradiction. Thus, M is infinite dimensional as desired.

3.15 Let D be a division ring, $V = \bigoplus_{i=1}^{\infty} e_i D$, and $E = End(V_D)$. Show that the ring E has exactly three ideals: $0, E$, and the ideal consisting of endomorphisms of finite rank.

Solution.
Let D, V, and E be as given. Let I be the ideal consisting of endomorphisms of finite rank. It should be noted that $I \neq E$, and $I \neq 0$, so it is a proper nonzero ideal. Now assume that there exists an ideal $J \neq E$ such that $I \subseteq J$. Let $f \in J/I$. We have that V is injective since D is a division ring. Therefore, $V \cong M \oplus \ker(f)$ with M a submodule of V. By the definition of M and direct sum, M has a basis and $M \cap \ker(f) = \{0\}$. So, if we let $\{m_1, m_2, \ldots\}$ be a basis for M, we have that $\{f(m_1), f(m_2), \ldots\}$ is linearly independent. Whence, there exists $g \in E$ such that $g(f(m_i)) = v_i$ where v_i is a standard basis vector for V for $i \geq 1$. Also, there exists $h \in E$ such that $h(v_i) = m_i$ for all $i \geq 1$. Hence, $g(((f(h(v_i)))) = v_i$ and since J is an ideal, $g \circ f \circ h \in J$. Thus, $J = E$ and so I is a maximal ideal. Now assume that there exists an ideal $K \neq 0$ such that $K \subseteq I$. Again, let $f \in K/I$. Since V is injective, $V \cong N \oplus \ker(f)$ for N a submodule of V. Let $\{n_1, n_2, \ldots\}$ be a basis for N. Form the fact that $N \cap \ker(f) = \{0\}$, we have that $\{f(n_1), f(n_2), \ldots\}$ is linearly indpendent. Therefore, there exists $g, h \in E$ such that $g(f(n_i)) = v_i$ a standard basis vector for V and $h(v_i) = n_i$ for all $i \geq 1$. So, $g(f(h(v_i))) = v_i$ and since K is also an ideal of E, $g \circ f \circ h \in K$. So, $K = I$ which gives that I is also minimal. Since we have that I is both minimal and maximal, it is the only proper nonzero ideal of E. Thus, the only ideals of E are $0, I$, and E as desired.

4.10 Show that if $f : R \to S$ is a surjective ring homomorphism, then $f(radR) \subseteq radS$. Give an example to show that $f(radR)$ may be smaller than $radS$.

Solution.
Let $f : R \to S$ be as stated above. Let $a \in radR$. Then we know that $1 - ba$ is left invertible for all $b \in R$. That is there exists $c \in R$ such that $c(1 - ba) = 1$. Now, since f is a surjective homomorphism, we know that $f(1) = 1$. Hence,

$$1 = f(1) = f(c(1 - ba))$$

Since f is a ring homomorphism,

$$1 = f(c(1 - ba)) = f(c)f(1 - ba) = f(c)(f(1) - f(ba)) = f(c)(1 - f(b)f(a))$$

Since $f(a) \in S$ we have that $f(a) \in radS$. That is $f(radR) \subseteq radS$ as desired.

Consider the rings $R = \mathbb{Z}$ and $S = \mathbb{Z}/2^2\mathbb{Z}$ with $f : R \to S$ defined to be the standard map. Then, f is a surjective homomorphism and $radR = \{0\}$ and $radS = 2\mathbb{Z}/4\mathbb{Z}$. Then clearly $f(radR) \neq radS$.

4.18 The socle $soc(M)$ of a left module M over a ring R is defined to be the sum of all simple submodules of M. Show that

$$soc(M) \subseteq \{m \in M | (radR) \cdot m = 0\}$$

with equality if $R/radR$ is an artinian ring.

Solution.
Let M be a left module of a ring R. Let $x \in soc(M)$. Then, $x = m_1 + m_2 + \ldots + m_n$ where $n \in \mathbb{Z}^+$ and $m_i \in N_i$ the simple submodules of M for $i \in I$. Then, $(radR)x = (radR)m_1 + \ldots + (radR)m_n$. Since m_i is an element of a simple submodule of M for all $i \in I$, we have that $(radR)m_i = 0$ for all $i \in I$. Hence, $(radR)x = 0 + \ldots + 0 = 0$. Therefore, $x \in \{m \in M | (radR) \cdot m = 0\}$ and so $soc(M) \subseteq \{m \in M | (radR) \cdot m = 0\}$. Now, assume that $R/radR$ is artinain. Now, we have that $rad(R/radR) = \{0\}$. This means that $R/radR$ is J-semisimple. Hence, $R/radR$ is semisimple. Therefore, $R/radR$ is the sum of simple left $R/radR$-modules. Since R and $R/radR$ have the simple left mods, then $R/radR$ is the sum of simple left R-modules. So any $x \in R/radR$ is of the form $x = n_1 + \ldots + n_k$ where $k \in \mathbb{Z}^+$ and $n_i \in A_i$ with A_i a simple submodule for all $i \in J$. Since for all $x \in R/radR$, $(radR)x = 0$, we have that $x \in soc(M)$ as desired.

4.20 4.20. For any left artinian ring R with Jacobson radical J, show that

$$soc({}_RR) = \{r \in R | Jr = 0\} \text{ and } soc(R_R) = \{r \in R | rJ = 0\}$$

Using this, construct an artinian ring R in which $soc({}_RR) \neq soc(R_R)$.

Solution.
Let R and J be as stated above. By the problem above, we have that $soc({}_RR) = \{r \in R | Jr = 0\}$ if R/J is artinian. Since R is left artinian and J is an ideal of R, R/J is indeed artinian. Hence, we have that $soc({}_RR) = \{r \in R | Jr = 0\}$. Now, since R/J is artinian and $rad(R/J) = \{0\}$, then R/J is J-semisimple. This gives us that R/J is semisimple same as above.