1.4 True or False: "If ab is a unit, then a, b are units"?
This statement is false.
Show the following for any ring R:
a. If a^n is a unit in R, then a is a unit in R.
b. If a is left-invertible and not a right 0-divisor, then a is a unit in R.
c. If R is a domain, then R is Dedekind-finite.

Let R be any ring.

Solution.
a. Let $a \in R$ and assume that a^n is a unit in R. Then, there is a $b \in R$ such that $a^n b = ba^n = 1$. We have that $a^n b = (aa^{n-1})b = a(a^{n-1})b$ and $ba^n = b(a^{n-1}a) = (ba^{n-1})a$. Whence $a(a^{n-1}b) = (ba^{n-1})a = 1$. Therefore, a is a unit in R.

b. Let $a \in R$ and assume that a is left invertible and not a right 0-divisor. So, $a \neq 0$, there exists $b \in R$ such that $ba = 1$, and for all $c \in R \setminus \{0\}$, $ca \neq 0$. Now consider $(1 - ab)a$. By distribution, we get

$$(a - ab)a = a - (ab)a = a - a(ba) = a - a = 0$$

Since a is not a right zero divisor, we have that $1 - ab = 0$. Hence, $ab = 1$, That is a is a unit in R.

c. Assume that R is a domain and let $a, b \in R \setminus \{0\}$ such that $ab = 1$. Now, consider $a(1 - ba)$. This yields $a - a(ba) = a - (ab)a = a - a = 0$. Since R is a domain and $a \neq 0$, we have that $1 - ba = 0$. That is $1 = ba$. Thus, R is Dedekind-finite as desired.

1.5 Give an example of an element x in a ring R such that $Rx \subsetneq xR$.

Solution.
Consider the ring of upper triangular 2×2 real matricies. So, $m \in R$ has the form $m = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $a, b, d \in \mathbb{R}$. Now consider $x = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Then

$$Rx = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix} | a \in \mathbb{R} \}$$

and

$$xR = \{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 0 & 0 \end{pmatrix} | a, b \in \mathbb{R} \}$$

Clearly, we have that $Rx \subsetneq xR$ as desired.

1.9 Show that for any ring R, the center of the matrix ring $\mathbb{M}_n(R)$ consists of the diagonal matrices $r \cdot I_n$,where r belongs to the center of R.

Solution.

Let R be any ring and let $r \in Z(R)$. Let $a = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \mathbb{M}_n(R)$ Hence,

$$(r \cdot I_n)a = \begin{pmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & \ddots & \vdots \\ ra_{n1} & \cdots & ra_{nn} \end{pmatrix}$$

Since $r, a_{ij} \in R$ for $1 \leq i, j \leq n$ and $r \in Z(R)$, $ra_{ij} = a_{ij}r$ for all a_{ij}. Therefore,

$$\begin{pmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & \ddots & \vdots \\ ra_{n1} & \cdots & ra_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}r & \cdots & a_{1n}r \\ \vdots & \ddots & \vdots \\ a_{n1}r & \cdots & a_{nn}r \end{pmatrix} = a(r \cdot I_n)$$

Since $a \in \mathbb{M}_n(R)$ was arbitrary, we have that $(r \cdot I_n) \in Z(\mathbb{M}_n(R))$ where $r \in Z(R)$ as desired.

1.11 Let R be a ring possibly without an identity. An element $e \in R$ is called a left (resp. right) identity for R if $ea = a$(resp.$ae = a$)for every $a \in R$.
(a) Show that a left identity for R need not be a right identity.
(b) Show that if R has a unique left identity e, then e is also a right identity.
(Hint.For(b),consider$(e + ae - a)c$ for arbitrary $a, c \in R$.)

Solution.
Let R be as above.

a. Consider the ring of 2×2 real matricies with 0's in the bottom row. That is, the ring $R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b \in \mathbb{R} \quad \text{and} \quad c, d = 0\}$. Further consider the element $el = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Now, let $m \in R$. So,

$$el \cdot m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

However,

$$m \cdot el = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Since $m \in R$ was arbitrary, we have that el is a left identity, but not a right identity.

b. Let $e \in R$ be the unique left identity for R. Let $a, c \in R$ and consider $(e + ae - a)c$. By distribution, we have that

$$(e + ae - a)c = ec + (ae)c - ac = c + a(ec) - ac = c + ac - ac = c$$

Since we have that $ec = (e + ae - a)c$ and that e is the unique left identity for R, $e = e + ae - a$. Hence, $ae - a = 0$ and thus $ae = a$. Therefore, e is also a right identity for R as desired.

1.17 Let x, y be elements in a ring R such that $Rx = Ry$. Show that there exists a right R-module isomorphism $f : xR \to yR$ such that $f(x) = y$.

Solution.
Let x, y, and R be as given. Consider the mapping $\phi : xR \to yR$ defined by $\phi(xr) = yr$ for all $r \in R$. Now, $\phi(xr + xr') = \phi(x(r + r'))$ since xR is a right R-module. Whence, $\phi(x(r + r')) = y(r + r') = yr + yr' = \phi(xr) + \phi(xr')$ since yR is also a right R module. Now, let $r' \in R$. So, $\phi(xr)r' = (yr)r' = y(rr') = \phi(x(rr')) = \phi((xr)r')$ since R is a ring and xR, yR are right R-modules. Therefore, ϕ is a right R-module homomorphism. Now, let let $xr, xr' \in xR$ and assume that $\phi(xr) = \phi(xr')$. That is $yr = yr'$. Now, let $\bar{r} \in R$ So, $\bar{r}(yr) = \bar{r}(yr')$. By associativity, $(\bar{r}y)r = (\bar{r}y)r'$. Since $Rx = Ry$ and $\bar{r} \in R$, we have that $(\bar{r}x)r = (\bar{r}x)r'$. Thus, $xr = xr'$ and ϕ is $1 - 1$. Now, let $yr \in yR$ be arbitrary. We see that $\phi(xr) = yr$, and so ϕ is also onto. It follows that ϕ is a right R-module isomorphism. Clearly, if $r = 1$ the identity of R, then we have that $\phi(x) = y$ where $\phi : xR \to yR$ is an isomorphism.

1. Let R be a ring and

$$0 \longrightarrow M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} M_3 \longrightarrow 0$$

a short exact sequence of left R-modules. Show that if M_2 is noetherian, then so are M_1 and M_3.

Solution.
Let R and the above sequence be as given. Assume that M_2 is noetherian. Let $K = \ker(\phi_2)$. We have that $K \subseteq M_2$ and since M_2 is noetheriean, so to is M_2/K. Since the sequence is short exact, we know that ϕ_2 is onto, and so $\phi_2(M_2) = M_3$. Now, by the Fundamental Homomorphism Theorem, we have that $M_2/K \cong \phi_2(M_2) = M_3$, and so M_3 is also noetherian. Now, let $N_1 \subset N_2 \subset \ldots$ be an ascending chain of left R-submodules of M_1. So, $\phi_1(N_1) \subset \phi_1(N_2) \subset \ldots$ is an ascending chain of left R-submodules of M_2. Since M_2 is noetherian, we have that $\phi_1(N_1) \subset \phi_1(N_2) \subset \ldots$ satisfies the ascending chain condition. That is for some $n \in \mathbb{Z}^+$, we have that $\phi_1(N_n) = \phi_1(N_{n+1}) = \ldots$. Since the above sequence is short exact, we have that ϕ_1 is $1-1$. Thus, $N_n = N_{n+1} = \ldots$. Whence, $N_1 \subset N_2 \subset \ldots$ also satisfis the ascending chain condition. Therefore, M_1 is also noetherian.

2. Let R be a ring and $n \in \mathbb{Z}^+$. Show that if ${}_{\mathbb{M}_n(R)}\mathbb{M}_n(R)$ is artinian, then so is ${}_RR$.

Solution.
Let R and n be as above. Assume that ${}_{\mathbb{M}_n(R)}\mathbb{M}_n(R)$ is artinian. So, let $N_1 \supseteq N_2 \supseteq \dots$ be a descending chain of submodules of ${}_RR$. Then, $\mathbb{M}_n(N_1) \supseteq \mathbb{M}_n(N_2) \supseteq \dots$ is a descending chain of submodules of ${}_{\mathbb{M}_n(R)}\mathbb{M}_n(R)$. Since ${}_{\mathbb{M}_n(R)}\mathbb{M}_n(R)$ is artinian, we have that $\mathbb{M}_n(N_m) = \mathbb{M}_n(N_{m+1}) = \dots$ for some $m \in \mathbb{Z}^+$. Clearly, I_n the $n \times n$ identity matrix is an element of $\mathbb{M}_n(N_i)$ for all $i \in \mathbb{Z}^+$. Hence, let $x_m \in N_m$. Then $x_m \cdot I_n \in \mathbb{M}_n(N_m)$ and $x_m \cdot I_n \in \mathbb{M}_n(N_{m+i})$ for all $i \in \mathbb{Z}^+$ since $\mathbb{M}_n(N_m) = \mathbb{M}_n(N_{m+1}) = \dots$. Therefore, we have that $x_m \in N_{m+i}$ for all $i \in \mathbb{Z}^+$. Since x_m was arbitrary, we have that $N_m = N_{m+1} = \dots$. Thus, the descending chain of submodules of ${}_RR$, $N_1 \supseteq N_2 \supseteq \dots$, satisfies the descending chain condition. That is, ${}_RR$ is artinian as desired.

3. Let R be any ring and M a left R-module that is both artinian and noetherian. Prove that for any R-module homomorphism $\phi : M \to M$,there exists $n \in \mathbb{Z}^+$ such that $\phi^n(M) \cap \ker(\phi^n) = \{0\}$.

Solution.
Let R and M be as given. Let $\phi : M \to M$ be an R-module homomorphism. Consider the ascending chain

$$\phi(M) \subset \phi^2(M) \subset \phi^3(M) \subset \dots$$

and the descending chain

$$\phi(M) \supset \phi^2(M) \supset \phi^3(M) \supset \dots$$

Since ϕ is an endomorphism, $\phi^i(M) \in M$ for all $i \in \mathbb{Z}^+$. Since M is noetherian, we have that

$$\phi(M) \subset \phi^2(M) \subset \phi^3(M) \subset \dots \subset \phi^l(M) = \phi^{l+1}(M) = \dots$$

for some $l \in \mathbb{Z}^+$ and since M is also artinian,

$$\phi(M) \supset \phi^2(M) \supset \phi^3(M) \supset \dots \supset \phi^m(M) = \phi^{m+1}(M) = \dots$$

for some $m \in \mathbb{Z}^+$. Now set $n = \max\{l, m\}$. Then, $\phi^n(M) = \phi^{n+1}(M) = \dots$ satisfies both the ascending and descending chain conditions on M. Clearly $0 \in \phi^n(M) \cap \ker(\phi^n)$ since ϕ is an endomorphism. Now, let $x \in \phi^n(M) \cap \ker(\phi^n)$. That is, $\phi^n(x) = 0$ and there exists $y \in M$ such that $\phi^n(y) = x$. So, $\phi^{n+1}(y) = \phi(x)$ and $\phi^{n+2}(y) = \phi^2(x)$. Continuing this process of composing with ϕ on both sides, we see that $\phi^{2n}(y) = \phi^n(x) = 0$. Whence, $x = 0$. Since $x \in \phi^n(M) \cap \ker(\phi^n)$ was arbitrary, $\phi^n(M) \cap \ker(\phi^n) \subseteq \{0\}$. Therefore, $\phi^n(M) \cap \ker(\phi^n) = \{0\}$ for some $n \in \mathbb{Z}^+$ as desired.