1. Let R be a ring, M a left R-module, and $N_1 \subseteq N_2 \subseteq \cdots$ an ascending chain of R-submodules of M. Prove that $\bigcup_{i=1}^{\infty} N_i$ is a left R-submodule of M.

   Solution.
   Let R, M, and N_i for all $i \in \mathbb{N}$ be as given above. Let $a \in R$ and $m, n \in \bigcup_{i=1}^{\infty} N_i$. We see that $\bigcup_{i=1}^{\infty} N_i \neq \emptyset$ since for each $i \in \mathbb{N}$, N_i is an R-submodule of M and so $N_i \neq \emptyset$. Now, since $m, n \in \bigcup_{i=1}^{\infty} N_i$, we have that $m \in N_j$ $n \in N_k$ for some $j, k \in \mathbb{N}$. Let $\ell = maxj, k$. Since N_i for $i \in \mathbb{N}$ is an ascending chain of R-submodules of M, $m, n \in N_{ell}$. Since N_ℓ is an R-submodule of M, $m + n \in N_\ell$ and so $m + n \in \bigcup_{i=1}^{\infty} N_i$. Also since $n \in N_k$ for some $k \in \mathbb{N}$ and N_k is an R-submodule of M, $an \in N_k$. Hence, $an \in \bigcup_{i=1}^{\infty} N_i$. Therefore, by the submodule criterion, $\bigcup_{i=1}^{\infty} N_i$ is a left R-submodule of M as desired.

2. Let R be a ring and $\phi : M \to N$ be a homomorphism of left R-modules. Prove that ϕ is $1-1$ if and only if $\ker(\phi) = \{0\}$.

   Solution.
   Let R and ϕ be as given, and let M, N be left R-modules. First, we will show that $\phi(0_M) = 0_N$. Since ϕ is a homomorphism and $0_M = 0_M + 0_M$, we have that $\phi(0_M) = \phi(0_M + 0_M) = \phi(0_M) + \phi(0_M)$. Now, $\phi(0_M) = 0_N + \phi(0_M)$. Therefore, $0_N + \phi(0_M) = \phi(0_M) + \phi(0_M)$ and by cancellation, $0_N = \phi(0_M)$. Thus we have that $0 \in \ker(\phi)$. Now, assume first that ϕ is injective. Let $m \in M$ such that $\phi(m) = 0$. We will show that $m = 0$. We have that $\phi(m) = \phi(m+0) = \phi(m) + \phi(0)$ since ϕ is a homomorphism. By our assumption and by what was shown above, $\phi(m) + \phi(0) = 0 + \phi(0) = \phi(0) + \phi(0)$. So by cancellation, $\phi(m) = \phi(0)$. Since ϕ is $1-1$, we have that $m = 0$ and since $m \in M$ was arbitrary and we showed that $0 \in \ker(\phi)$, we have that $\ker(\phi) = \{0\}$. On the other hand, asssume that $\ker(\phi) = \{0\}$. Let $m, n \in M$ and assume that $\phi(m) = \phi(n)$. We can now see that $\phi(m) - \phi(n) = 0$ and since ϕ is a homomorphism, $\phi(m - n) = 0$. Since $\ker(\phi) = \{0\}$, we have that $m - n = 0$. Hence, $m = n$. So by definition, ϕ is $1 - 1$. Thus ϕ is $1 - 1$ if and only if $\ker(\phi) = \{0\}$ as desired.

3. Let F be a field, $n > 1$, $R = \mathbb{M}_n(F)$, and $M \subseteq R$ the set of all matricies that have arbitrary entries in the first column but zeros elsewhere. Show that M is an R-submodule of ${}_RR$, but not an R-submodule of R_R.

Solution.
Let $F, n, R,$ and M be as given. Let $z \in R$ and $x, y \in M$. Clearly $M \neq \emptyset$ since the zero $n \times n$ matrix

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M$$

Since $x, y \in M$ we have that

$$x = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}$$

and

$$y = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ b_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & \cdots & 0 \end{pmatrix}$$

where $a_i, b_i \in F$ for $1 \leq i \leq n$. Then

$$x + y = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ b_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} (a_1 + b_1) & 0 & \cdots & 0 \\ (a_2 + b_2) & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ (a_n + b_n) & 0 & \cdots & 0 \end{pmatrix} \in M$$

So, M is closed under addition. Since

$$z \in R, z = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

where $c_{ij} \in F$ for $1 \leq i, j \leq n$. Hence,

$$zx = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (a_1c_{11} + a_2c_{12} + \cdots + a_nc_{1n}) & 0 & \cdots & 0 \\ (a_1c_{21} + a_2c_{22} + \cdots + a_nc_{2n}) & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ (a_1c_{n1} + a_2c_{n2} + \cdots + a_nc_{nn}) & 0 & \cdots & 0 \end{pmatrix} \in M$$

However,

$$xz = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_1c_{11} & a_1c_{12} & \cdots & a_1c_{1n} \\ a_2c_{11} & a_2c_{12} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_nc_{11} & a_nc_{12} & \cdots & a_nc_{1n} \end{pmatrix} \notin M$$

Thus, we have that M is an R-submodule of ${}_RR$ but is not an R-submodule of R_R since z, x, y were arbitrary.

4. Give an example of a ring R, left R-modules M and N and a map $\phi : M \to N$ such that ϕ is a group homomorphism but not an R-module homomorphism.

Solution.
Consider the ring $\mathbb{C}$ and the left $\mathbb{C}$-modules $M = N = \mathbb{C}$. That is consider $\mathbb{C}$ as a left R-module of itself. Further consider the function $\phi : M \to N$ defined by $\phi(x) = \bar{x}$ where $\bar{x}$ is the complex conjugate of x. Let $x, y \in M$. Then $x = a + bi$ and $y = c + di$ where $a, b, c, d \in \mathbb{R}$. Clearly ϕ is well defined. Also,

$$\phi(x+y) = \phi(a+bi+c+di) = \phi((a+c)+(b+d)i) = (a+c)-(b+d)i = a+c-bi-di$$

$$= a - bi + c - di = \phi(x) + \phi(y)$$

Hence, ϕ is a group homomorphism by definition. However, consider $1 + i \in \mathbb{C}$ and $5 \in M$. Then $(1 + i)\phi(5) = (1 + i)5 = 5 + 5i \neq 5 - 5i = \phi(5 + 5i) = \phi(5(1 + i))$. Therefore, ϕ is indeed not a $\mathbb{C}$-module homomorphism.

5. Let R be a commutative ring, and S be a ring. Prove that S is an R-algebra if and only if there is a ring homomorphism $\phi : R \to S$ such that for all $a \in \phi(R)$ and $b \in S$, $ab = ba$.

   Solution.
   Let R and S be as given. First, assume first that there exists a ring homomorphism $\phi : R \to S$ such that for all $a \in \phi(R)$ and $b \in S$, $ab = ba$. Define the action on S by $\phi(a)b = ab$ for all $a \in R$ and $b \in S$. Now, let $a \in R$ and $b, c \in S$. Then we have that $a(bc) = \phi(a)(bc) = (\phi(a)b)c = (ab)c$ by the definition of ϕ and by associativity on S. Then, $(ab)c = (\phi(a)b)c = (b\phi(a))c = b(\phi(a)c) = b(ac)$ by our assumption about the homomorphism ϕ, the action we defined for ϕ, and by the associativity on S. Thus, we have that S is an R-algebra. Now assume that S is an R-algebra. Define the function $\phi : R \to S$ by $\phi(a) = a$. So clearly, $\phi(1_R) = 1_S$. Now, let $x, y \in R$. Then, $\phi(x + y) = (x + y) = x + y = \phi(x) + \phi(y)$ and $\phi(xy) = (xy) = xy = \phi(x)\phi(y)$ by the definition of ϕ. Thus, ϕ is a ring homomorphism. Let $x \in R$ and $y \in S$. Then, by definition of ϕ, $x = \phi(x) \in \phi(R)$. Since S is an R-algebra, we have that $xy = x(y1_S) = y(x1_S) = yx$. Thus, since x and y were arbitrary, we have that there exists a ring homomorphism such that for all $a \in \phi(R)$ and $b \in S$, $ab = ba$.

6. Let R be a ring, M a left R-module, and $\phi : M \to M$ an R-module homomorphism such that $\phi \circ \phi = \phi$. Show that $M = \phi(M) + \ker(\phi)$ and $\phi(M) \cap \ker(\phi) = \{0\}$.

   Solution.
   Let R, M, and ϕ be as given. We will first show that $\phi(M) \cap \ker(\phi) = \{0\}$. Since we showed $\phi(0) = 0$ for an arbitrary left R-module homomorphism, we clearly have that $\{0\} \subseteq \phi(M) \cap \ker(\phi)$. Now, let $m \in \phi(M) \cap \ker(\phi)$. Then $\phi(m) = 0$ and $m \in M$ such that $m = \phi(n)$ for some $n \in M$. Since $\phi \circ \phi = \phi$, we have that $\phi(\phi(n)) = \phi(m) = 0$ and $\phi(\phi(n)) = \phi(n)$. So, $\phi(n) = m = 0$. Thus we have that $m \in \{0\}$. Therefore $\phi(M) \cap \ker(\phi) = \{0\}$. Now we will show that $M = \phi(M) + \ker(\phi)$. Clearly, since $\phi(M), \ker(\phi) \subseteq M$ and M is closed under addition, $\phi(M) + \ker(\phi) \subseteq M$. So, let $x \in M$. Now, $\phi(M) + \ker(\phi) = \{m + n | m \in \phi(M), n \in \ker(\phi)\}$. We need to show that $x = \phi(m) + n$ for some $\phi(m) + n \in \phi(M) + ker(\phi)$. Take $\phi(x) = \phi(\phi(m) + n)$. Since ϕ is a homomorphism, we have that $\phi(x) = \phi(\phi(m)) + \phi(n)$. By our assumption of ϕ, $\phi(x) = \phi(m) + \phi(n)$ and since $n \in \ker(\phi)$, $\phi(x) = \phi(m)$. Thus, we have that $x \in \phi(M) + ker(\phi)$. Therefore $M = \phi(M) + ker(\phi)$ as desired.

7. 7. Let R be a ring, M a left R-module, and $A \subseteq B$ R-submodules of M. Prove that $(M/A)/(B/A) \cong (M/B)$.

   Solution.
   Let R, M, A and B be as given above. By theorem 8, we have that M/A is a left R-module. Similarly, we have that M/B is a left R-module. Now consider $\phi : (M/A) \to (M/B)$ defined by $\phi(m + A) = m + B$. Let $a \in R$ and $m + A, n + A \in M/A$. Then we have that $\phi((m+A)+(n+A)) = \phi((m+n)+A) = (m+n)+B = (m+B)+(n+B) = \phi(m + A) + \phi(n + A)$ by the definition of ϕ and M/A, M/B as a left R-modules. Also, $a\phi(m + A) = a(m + B) = (am) + B = \phi((am) + A)$ by the definition of the action on quotient modules and ϕ. So, we have that ϕ is an R-module homomorphism. Now, $\phi(M/A) = \{m + B \in M/B | \text{there exists} m + A \in M/A such that \phi(m + A) = m + B\} = \{m + B|m \in M\}$. So, $\phi(M/A) = M/B$ by our definition of ϕ. Further, $\ker(\phi) = \{m + A \in M/A|\phi(m + A) = 0\} = \{m + A \in M/A|m + B = 0\} = \{m + A|m \in B\}$. That is $\ker(\phi) = M/B$. Thus, by the Fundamental Homomorphism Theorem, we have that $(M/A)/(B/A) \cong (M/B)$ as desired.

8. 8. Let R be a ring, M a left R-module, and N an R-submodule of M. Prove that if M/N and N are both finitely generated as left R-modules, then so is M.

Solution.
Let R, M, and N be as given, and suppose that M/N and N are both finitely generated as left R-modules. We then have that $N = RS$ and $M/N = RT$ for some finite sets $S \subseteq N$ and $T \subseteq M/N$. Now, define $\phi : M \to M/N$ by $\phi(m) = m + N$. By theorem 8, we have that ϕ is an R-module homomorphism. Now, take $X = \{x_1, \ldots, x_n\} \subseteq M$ for some $n \in \mathbb{N}$ such that $\phi(X) = T$. Let $m \in M$. if $m \in \ker(\phi)$, then we have that $\phi(m) = 0$. Thus, $m \in N$ and $m = a_1 s_1 + \cdots + a_\alpha s_\alpha$ where $\alpha \in \mathbb{N}$, $a_i \in R$ and $s_i \in S$ for $1 \leq i \leq \alpha$. If, however, $m \notin \ker(\phi)$, then $\phi(m) \in M/N$. Hence, since ϕ is a homomorphism,

$$\phi(m) = a_1\phi(x_1) + \cdots + a_n\phi(x_n) = \phi(a_1x_1 + \cdots + a_nx_n)$$

where $a_i \in R$ for $1 \leq i \leq n$. Then,

$$\phi(m) - \phi(a_1x_1 + \cdots + a_nx_n) = \phi(m - (a_1x_1 + \cdots + a_nx_n)) = 0$$

since ϕ is a homomorphism. Therefore $m - (a_1x_1 + \cdots + a_nx_n) \in N$. So,

$$m - (a_1x_1 + \cdots + a_nx_n) = b_1s_1 + \cdots + b_\alpha s_\alpha$$

where $b_i \in R$ for $1 \leq i \leq \alpha$. Thus,

$$m = (b_1s_1 + \cdots + b_\alpha s_\alpha) + (a_1x_1 + \cdots + a_nx_n)$$

and so $m \in R(S \cup X)$. Since $m \in m$ was arbitrary and S and X were both finite, we have that M is finitely generated as $S \cup X$ is also finite.

9. 9. Let R be a ring, M a left R-module, $N_1, \ldots, N_n$ submodules of M, and $\pi : N_1 \times \cdots \times N_n \to N_1 + \cdots + N_n$ defined by $\pi(m_1, \ldots, m_n) = m_1 + \cdots + m_n$. Prove that the statement π is an isomorphism is equivalent to $N_i \cap (N_1 + \cdots + N_{i-1} + N_{i+1} + \cdots + N_n) = \{0\}$ for all $i \in \{1, \ldots, n\}$.

Solution.
Let R, M, N_i for $1 \le i \le n$, and π be as given. Assume first that $N_i \cap (N_1 + \cdots + N_{i-1} + N_{i+1} + \cdots + N_n) = \{0\}$ for all $i \in \{1, \ldots, n\}$.Now, let $a \in R$ and $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^n N_i$. Then,

$$\pi((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) = \pi((x_1 + y_1, \ldots, x_n + y_n))$$
$$= x_1 + y_1 + \cdots + x_n + y_n = x_1 + \cdots + x_n + y_1 + \cdots + y_n = \pi((x_1 \ldots, x_n)) + \pi((y_1, \ldots, y_n))$$

since $\sum_{i=1}^n N_i$ is an abelian group by definition of being an R-module. Also,

$$\begin{aligned} a\pi((x_1, \ldots, x_n)) &= a((x_1 + \cdots + x_n)) = ax_1 + \cdots + ax_n \\ &= \pi((ax_1, \ldots, ax_n)) = \pi(a(x_1, \ldots, x_n)) \end{aligned}$$

by the definition of the action on $\sum_{i=1}^n N_i$. So, we have that π is an R-module homomorphism. Now, let $z \in \sum_{i=1}^n N_i$. Then $z = z_1 + \cdots + z_n$ where $z_i \in N_i$ for $1 \le i \le n$. Hence, we have that $\pi((z_1, \ldots, z_n)) = z_1 + \cdots + z_n$ and so π is onto. Now, with $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^n N_i$, assume that $\pi((x_1, \ldots, x_n)) = \pi((y_1, \ldots, y_n))$. So,

$$\begin{aligned} \pi((x_1, \ldots, x_n)) - \pi((y_1, \ldots, y_n)) &= \pi((x_1, \ldots, x_n) - (y_1, \ldots, y_n)) \\ &= \pi((x_1 - y_1, \ldots, x_n - y_n)) = 0 \end{aligned}$$

since π is a homomorphism. Therefore, $x_1 - y_1 + \cdots + x_n - y_n = 0 = \underbrace{0 + \cdots + 0}_{n \text{ times}}$. Since $N_i \cap (N_1 + \cdots + N_{i-1} + N_{i+1} + \cdots + N_n) = \{0\}$ for all $i \in \{1, \ldots, n\}$, we have that $x_i \neq y_j$ for all $i \neq j$. Thus, $x_1 - y_1 + \cdots + x_n - y_n = x_1 + \cdots + x_n - y_1 - \cdots - y_n$ and so $x_1 + \cdots + x_n = y_1 + \cdots + y_n$. Therefore, $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$. Thus, we have that π is an isomorphism. Now, assume that π is an R-module isomorphism. Take $(0, \ldots, 0) \in \prod_{i=1}^n N_i$. Then, $\pi(0, \ldots, 0) = \underbrace{0 + \cdots + 0}_{n \text{ times}}$. Since π is an isomorphism, then π is $1 - 1$. Hence, $\ker(\pi) = \{(0, \ldots, 0)\}$. That is, there exists no $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^n N_i$ such that if $\pi((x_1, \ldots, x_n) - (y_1, \ldots, y_n)) = 0$, then $x_i = y_j$ for all $i \neq j$ with $1 \le i, j \le n$. Hence, we have that $N_i \cap (N_1 + \cdots + N_{i-1} + N_{i+1} + \cdots + N_n) = \{0\}$ for all $i \in \{1, \ldots, n\}$. Thus, the two statements are equivalent as desired.