10.3 10.3. Suppose $a > 0$ and consider the following skewed leapfrog method for solving the advection equation $u_t + au_x = 0$:

$$U_j^{n+1} = U_{j-2}^{n-1} - (\frac{ak}{h} - 1)(U_j^n - U_{j-2}^n)$$

a. What is the order of accuracy of this method?
b. For what range of Courant number $\frac{ak}{h}$ does this method satisfy the CFL condition?
c. Show that the method is in fact stable for this range of Courant numbers by doing von Neumann analysis. Hint: Let $\gamma(\xi) = e^{i\xi h}g(\xi)$ and show that γ satisfies a quadratic equation closely related to the equation (10.34) that arises from a von Neumann analysis of the leapfrog method

Solution.
a. It can be noted that we need to look at the local truncation error of this method. This gives us

$$\tau(x,t) = \frac{u(x,t+k) - u(x,t) + u(x-2h,t) - u(x-2h,t-k)}{2k} + a\frac{u(x,t) - u(x-2h,t)}{2h}$$

Now, expanding terms about $u(x,t)$ using multivariate Taylor series yeilds

$$u(x,t+k) = u(x,t) + ku_t + \frac{1}{2}k^2u_{tt} + \mathcal{O}(k^3)$$
$$u(x-2h,t) = u(x,t) - 2hu_x + 2h^2u_{xx} + \mathcal{O}(h^3)$$

and

$$u(x-2h,t-k) = u(x,t) - 2hu_x - ku_t + 2h^2u_{xx} + 2hku_{xt} + \frac{1}{2}k^2u_{tt} + \mathcal{O}(k^3) + \mathcal{O}(h^3)$$

So, the local trunction error becomes

$$\tau = \frac{ku_t + \frac{1}{2}k^2u_{tt} - 2hu_x + 2h^2u_{xx} + 2hu_x + ku_t - 2h^2u_{xx} - 2hku_{xt} - \frac{1}{2}k^2u_{tt} + \mathcal{O}(k^3) + \mathcal{O}(h^3)}{2k} + a\frac{2hu_x - 2h^2u_{xx} + \mathcal{O}(h^3)}{2h}$$
$$= u_t - hu_{xt} + au_x - 2hau_{xx} + \mathcal{O}(k^2) + \mathcal{O}(h^2)$$

Since $u_t = -au_x$, we have that is second order accurate in both space and time.

b. We know that the advection equation has characteristic $\frac{1}{a}$, so we need the overall Courant number to be less than 1. From the method as well as the stencil of the method, we know that we need $\frac{k}{2h} < \frac{1}{a}$. That is $\frac{ak}{h} < 2$. So, the range of Courant numbers is $(0, 2)$ to satisfy the CFL condition.

c. If we let $\gamma(\xi) = e^{i\xi h}\lambda$, then we get the quadratic equation $\lambda^2 + (\nu - 1)(1 - e^{2i\xi h})\lambda - 1 = 0$. Letting $z = e^{2i\xi h}$ and applying the quadratic formula, we see that $\lambda_{1,2} = \frac{-(\nu-1)(1-z)\pm\sqrt{(\nu-1)^2(1-z)^2+4z}}{2}$. So, having λ_1 the positive and λ_2 the negative, we have that $|\lambda_1||\lambda_2| = |z|$ and $\lambda_1 + \lambda_2 = -(\nu - 1)(1 - z)$ which yields $|\lambda_1| = |\lambda_2| = 1$. In the complex plain, this tells us that the method is stable for all Courant numbers.

10.4 10.4. Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1})$$

for the advection equation $u_t + au_x = 0$ on $0 \leq x \leq 1$ with periodic boundary conditions.

a. This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix A? Don't forget the boundary conditions.

b. Suppose we want to fix the Courant number $\frac{ak}{h}$ as $k, h \to 0$. For what range of Courant numbers will the method be stable if $a > 0$? If $a < 0$? Justify your answers in terms of eigenvalues of the matrix A from part (a) and the stability regions of the trapezoidal method.

c. Apply von Neumann stability analysis to the method (E10.4a). What is the amplification factor $g(\xi)$?

d. For what range of $\frac{ak}{h}$ will the CFL condition be satisfied for this method (with periodic boundary conditions)?

e. Suppose we use the same method (E10.4a) for the initial-boundary value problem with $u(0,t) = g_0(t)$ specified. Since the method has a one-sided stencil, no numerical boundary condition is needed at the right boundary (the formula (E10.4a) can be applied at x_{m+1}). For what range of $\frac{ak}{h}$ will the CFL condition be satisfied in this case? What are the eigenvalues of the A matrix for this case and when will the method be stable?

Solution.

a. The matrix A is given by

$$A = \begin{pmatrix} -a & 0 & \cdots & \cdots & a \\ a & -a & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & -a \end{pmatrix}$$

with the consideration of the periodic boundry conditions.

b.

c.

d.

e.

10.8 10.8. The m-file advection_LW_pbc.m implements the Lax-Wendroff method for the advection equation on $0 \leq x \leq 1$ with periodic boundary conditions.
a. Observe how this behaves with $m + 1 = 50, 100, 200$ grid points. Change the final time to $tfinal = 0.1$ and use the m-files error_table.m and error_loglog.m to verify second order accuracy.
b. Modify the m-file to create a version advection_up_pbc.m implementing the upwind method and verify that this is first order accurate.
c. Keep m fixed and observe what happens with advection_up_pbc.m if the time step k is reduced, e.g. try $k = 0.4h, k = 0.2h, k = 0.1h$. When a convergent method is applied to an ODE we expect better accuracy as the time step is reduced and we can view the upwind method as an ODE solver applied to an MOL system. However, you should observe decreased accuracy as $k \to 0$ with h fixed. Explain this apparent paradox. Hint: What ODE system are we solving more accuracy? You might also consider the modified equation (10.44).

Solution.
a.

b.

c.

10.9 10.9.a. Modify the m-file to create a version advection_lf_pbc.m implementing the leapfrog method and verify that this is second order accurate. Note that you will have to specify two levels of initial data. For the convergence test set $U_j^1 = u(x_j, k)$, the true solution at time k.

b. Modify advection_lf_pbc.m so that the initial data consists of a wave packet $\eta(x) = exp(-\beta(x-0.5)^2)\sin(\xi x)$. Work out the true solution $u(x,t)$ for this data. Using $\beta = 100$, $\xi = 80$ and $U_j^1 = u(x_j, k)$, test that your code still exhibits second order accuracy for k and h sufficiently small.

c. Using $\beta = 100$, $\xi = 150$ and $U_j^1 = u(x_j, k)$, estimate the group velocity of the wave packet computed with leapfrog using $m = 199$ and $k = 0.4h$. How well does this compare with the value (10.52) predicted by the modified equation?

Solution.

a.

b.

c.