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Math 5510
Topology

Problem Set 1

1. (#1 in 2.1) Prove **Theorem 2.1.3 (ii)** (DeMorgan's law): If X is any set and $\{A_\lambda | \lambda \in \Lambda\}$ is any indexed collection of sets, then $X \backslash \bigcap_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda}(X \backslash A_\lambda)$.

   Let X be a set and $\{A_\lambda | \lambda \in \Lambda\}$ be an indexed collection of sets. First consider the case $\bigcup_{\lambda \in \Lambda}(X \backslash A_\lambda) = \emptyset$. Then $X \backslash A_\lambda = \emptyset$ for every $\lambda \in \Lambda$. So, $A_\lambda = X$ for all $\lambda \in \Lambda$. Hence $\bigcap_{\lambda \in \Lambda} A_\lambda = X$, and thus $X \backslash \bigcap_{\lambda \in \Lambda} A_\lambda = \emptyset$, preserving equality. So, assume that $X \backslash \bigcap_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Now, let $x \in X \backslash \bigcap_{\lambda \in \Lambda} A_\lambda$. Then, $x \in X$, but $x \notin \bigcap_{\lambda \in \Lambda} A_\lambda$. So, by definition of intersection, $x \notin A_\lambda$ for at least one $\lambda \in \Lambda$. So, for some $\lambda \in \Lambda$, we have that $x \in X \setminus A_\lambda$. Hence $x \in \bigcup_{\lambda \in \Lambda}(X \backslash A_\lambda)$. Conversely, assume that $x \in \bigcup_{\lambda \in \Lambda}(X \backslash A_\lambda)$. So, for some $\lambda \in \Lambda$ $x \in X \backslash A_\lambda$. So, by definition, we have that $x \in X$ and $x \notin A_\lambda$ for at least one $\lambda \in \Lambda$. Therefore $x \in X$ but $x \notin \bigcap_{\lambda \in \Lambda} A_\lambda$. Thus $x \in X \backslash \bigcap_{\lambda \in \Lambda} A_\lambda$ and therefore $X \backslash \bigcap_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda}(X \backslash A_\lambda)$ as desired.

2. (#3 in 2.1) Consider the subset D of $\mathbb{R}^2$ defined by $D = \{(x, y) | x \leq y^2\}$. Is this set a Cartesian product of two subsets of $\mathbb{R}$? Explain.

   D is not a cartesian product. Assume by way of contradiction that D is a cartesian product. Well, we can see that the points $(1, 1)$ and $(0, 0)$ are elements of D. Since we assume that D is a cartesian product, we would then have the points $(0, 1), (1, 0) \in D$. However, $1 \nleq 0^2 = 0$ which is a contradiction. Therefore, D is not a cartesian product.

3. (#3 in 2.2) Prove or disprove the following: For $B \subseteq Y$ and $f : X \to Y$, $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$.

   Prove: Let B, X, and Y be sets, $B \subseteq Y$, and $f : X \to Y$ be a function. First assume that $B = \emptyset$. Then $Y \setminus B = Y$. So, $f^{-1}(Y \setminus B) = f^{-1}(Y) = X = X \setminus f^{-1}(B)$. So, assume that $B \neq \emptyset$ Let $x \in f^{-1}(Y \backslash B)$. Then, we have that $f(x) \in Y \setminus B$. So, by definition, $f(x) \in Y$ and $f(x) \notin B$. Again by definition, $x \in X$ but $x \notin f^{-1}(B)$. Therefore $x \in X \backslash f^{-1}(B)$ and $f^{-1}(Y \backslash B) \subseteq X \backslash f^{-1}(B)$. On the other hand, assume $x \in X \backslash f^{-1}(B)$. Well, we have then that $x \in X$ and $x \notin f^{-1}(B)$. So, by definition, $f(x) \in Y$, but also $f(x) \notin B$ Thus, $f(x) \in Y \setminus B$ and so $x \in f^{-1}(Y \backslash B)$. Therefore $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$ as desired.

4. (#5 in 2.2) Prove **Theorem 2.2.5**: For $A \subseteq X$ and $f : X \to Y$ any function, we have $A \subseteq f^{-1}(f(A))$. If, in addition, f is one-to-one, then $A = f^{-1}(f(A))$.

   Let $A \subseteq X$ and $f : X \to Y$ be any function. Assume $x \in A$. Then $f(x) = y$ for some $y \in Y$. So by definition of image, we have that $f(x) \in f(A)$. Then by definition of inverse image, we have that $x \in f^{-1}(f(A))$, and so $A \subseteq f^{-1}(f(A))$ as desired. Now, assume that f is injective. Also, let $x \in f^{-1}(f(A))$. So, $f(x) \in f(A)$. Therefore, by definition, there is an $a \in A$ such that $f(x) = f(a)$. Now, since f is one-to-one, we have that $x = a$. Hence $x \in A$. So, by the preiviuos proof, we have that $A = f^{-1}(f(A))$ as was to be done.

5. (#8 a, b in 2.2) Let $f : X \to Y$ and $g : Y \to Z$ be any functions.

   (a) Prove that if f is one-to-one and g is one-to-one, then $g \circ f : X \to Z$ is one-to-one. Is the converse true?
   Let f and g be one-to-one functions as given above with X, Y, and Z as sets. Now, assume that $g(f(x_1)) = g(f(x_2))$ for some $x_1, x_2 \in X$. Since g is injective, $f(x_1) = f(x_2)$. Also, since f is injective, $x_1 = x_2$ and thus $g \circ f : X \to Z$ is injective as desired. The converse is not true however. Consider $f : \mathbb{R} \to [0, \infty)$ and $g : [0, \infty) \to [0, \infty)$ given by $f(x) = x^2$ and $g(x) = \sqrt{x}$. Clearly $g(f(x)) = x$ is one-to-one, but $f(x)$ is not.

   (b) If g is onto and f is onto, then is $g \circ f$ always onto? Is the converse true?
   Let f and g be as given with sets X, Y, and Z. Assume that f and g are onto. Now, let $z \in Z$. Since g is onto, there exists $y \in Y$ such that $g(y) = z$. Now, since f is onto, there exists $x \in X$ such that $f(x) = y$. Thus, there exists $x \in X$ such that $g(f(x)) = z$ and so $g \circ f : X \to Z$ is onto as desired. The converse is not always true. Consider $f : [0, \infty) \to \mathbb{R}$ and $g : \mathbb{R} \to [0, \infty)$ defined by $f(x) = \sqrt[3]{x}$ and $g(x) = x^2$. So clearly $g(f(x)) = x^{\frac{2}{3}}$ is onto, but f is not.

6. (#3 in 2.5) Verify that the set $\{1, 4, 7, 10, \ldots\}$ is infinite, by Definition 2.5.2.

   Let $A = \{1, 4, 7, 10, \ldots\}$. Define $B = \{4, 7, 10, 13, \ldots\} \subset A$. Also define $f : A \to B$ by $f(x) = x + 3$. Now, assume that $f(x_1) = f(x_2)$. So, $x_1 + 3 = x_2 + 3$, whence $x_1 = x_2$. Hence f is injective. Now $y \in Y$ such that $y = x + 3$ for some $x \in X$. Therefore, $x = y - 3$. So, $f(x) = f(y - 3) = (y - 3) + 3 = y$. Since y was arbitrary, we have that f is also onto. So, f is bijective. Thus, $A = B$ and we have that A is equal to a proper subset of itself. Thus by definition, A is infinite as desired.

7. Prove that if $B \subseteq A$ and B is infinite, then A is infinite. Conclude that every subset of a finite set is finite.

   Let A and B be sets, and $B \subseteq A$. Let B be infinite. Assume first that $B = A$. Then clearly A is infinite. So, assume that $B \subset A$. Now, since B is infinite, there exists an injective function $f : \mathbb{N} \to B$. So for any $x \in \mathbb{N}$, $f(x) \in B$. Now consider the function $g : B \to A$ defined by $g(x) = x$ since $B \subset A$. Now, let $x_1, x_2 \in B$ and assume that $g(x_1) = g(x_2)$. Then $x_1 = x_2$ and so $g(x)$ is injective. So, by above we have that $(g \circ f) : \mathbb{N} \to A$ is injective. Since we have an injective function from $\mathbb{N}$ to A, we have that A is infinite by definition. Still letting $B \subseteq A$, we have that if A is finite, then B is also finite by the contrapositive of the above proven statement.

8. Prove that the union of a finite collection of finite sets is finite.

   Let $X = \{A_1, \ldots, A_n\}$ where A_i is a finite set for all $1 \leq i \leq n$. Assume first that $\bigcup_{i=1}^{n} A_i = \emptyset$. So, $A_i = \emptyset$ for all $0 \leq i \leq n$, and $\bigcup_{i=1}^{n} A_i$ is finite. So, assume that not all A_i are empty. Since $B \cup \emptyset = B$ for any set B, define $Y = \{A_1, \ldots A_m\}$ where A_i is a finite nonempty set for all $0 \leq i \leq m$. Now consider $m = 1$. Then clearly $\bigcup_{i=1}^{1} A_i = A_1$ which is finite by assumption. Next consider $m = 2$, so $\bigcup_{i=1}^{2} A_i = A_1 \cup A_2$. Assume first that $A_1 \cap A_2 = \emptyset$. Now, since A_1 and A_2 are finite, there exist bijections $f : A_1 \to \{1, \ldots, k\}$ and $g : A_2 \to \{1, \ldots, l\}$ for some $k, l \in \mathbb{N}$. Define $\tilde{f} : A_1 \cup A_2 \to \{1, \ldots, k + l\}$ by

$$f(a) = \begin{cases} f(a) & \text{if } a \in A_1 \\ g(a) + k & \text{if } a \in A_2 \end{cases}$$

   Now, suppose $\tilde{f}(a_1) = \tilde{f}(a_2)$ for some $a_1, a_2 \in A_1 \cup A_2$. Then, either $f(a_1) = f(a_2)$ or $g(a_1) + k = g(a_2) + k$. In either case, we have that $a_1 = a_2$ since both f and g are bijective. Now, let $y \in \{1, \ldots, k + l\}$. again, since f and g are bijective, there exists $x \in A_1 \cup A_2$ such that $f(x) = y$. So, $\tilde{f}$ is a bijection, and $A_1 \cup A_2$ is finite. Now, assume that $A_1 \cap A_2 \neq \emptyset$. Now, $A_1 \cup A_2 = (A_1 \setminus A_2) \cup A_2$, and $(A_1 \setminus A_2) \cap A_2 = \emptyset$. Now since $(A_1 \setminus A_2) \subset A_1$, $(A_1 \setminus A_2)$ is finite, and so $(A_1 \setminus A_2) \cup A_2$ is the union of 2 disjoint finite sets. Thus, the union of 2 finite sets is finte regardless of them being disjoint or not. Then take $m = 3$. So, $\bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3$ is finite as it can be given as $(A_1 \cup A_2) \cup A_3$ which is the union of 2 finite sets. Continuing in this manner, we have that $\bigcup_{i=1}^{m} A_i = A_1 \cup \cdots \cup A_m$ is finite as desired.

9. Prove that the product of a finite collection of finite sets is finite. (Hint: First prove that the product of two finite sets is finite by writing the product $A \times B$ as a finite union.)

   Let A and B be finite nonempty sets. Then B is equivalent to $\{1, \ldots, n\}$ for some

$n \in \mathbb{N}$. That is, there is a bijection $g : \{1, \ldots, n\} \to B$ defined by $g(i) = b_i$. So, each $b_i \in B$ is the image of an element of $\{1, \ldots, n\}$ for $1 \leq i \leq n$. Now, take $A \times B$. We may write $A \times B = \bigcup_{a \in A} \{a\} \times B$. Then $(\bigcup_{a \in A} \{a\}) \times B = \bigcup_{a \in A} (\{a\} \times B) = \bigcup_{a \in A} \{(a, b_i) | b_i \in B, 1 \leq i \leq n\}$ for some $n \in \mathbb{N}$. Now, fix $a \in A$ and define $f : \{(a, b_i) | b_i \in B, 1 \leq i \leq n\} \to \{1, \ldots, n\}$ by $f((a, b_i)) = i$. Suppose $f((a, b_s)) = f((a, b_t))$ for some $b_s, b_t \in B$. Then, $s = t$ by the definition of the function, and since B is equivalent to $\{1, \ldots, n\}$ with equivalence g, $b_s = b_t$, so f is injective. Let $y \in \{1, \ldots, n\}$. Since B is equivalent to $\{1, \ldots, n\}$ with equivalence g there exists $b \in B$ such that $g(b) = y$. Therefore, there exists $b \in B$ such that $f(a, b) = y$ and so f is onto. Hence, f is bijective, so we have that $\{(a, b_i) | b_i \in B, 1 \leq i \leq n\}$ is finite. Then, by problem 8 above, we have $\bigcup_{a \in A} \{(a, b_i) | b_i \in B, 1 \leq i \leq n\}$, the finite union of finite sets, is finite. Hence, the product of two finite sets is finite. Now consider finite sets A_1, A_2, and A_3. Then $A_1 \times A_2 \times A_3 = (A_1 \times A_2) \times A_3$. Since A_1 and A_2 are finite, $A_1 \times A_2$ is finite. So, since $A_1 \times A_2$ and A_3 are finite, $(A_1 \times A_2) \times A_3 = A_1 \times A_2 \times A_3$ is also finite. Continuing in the same manner, we have that $A_1 \times A_2 \times \ldots A_m$ for some $m \in \mathbb{N}$ is finite.

10. (#11 in 2.5) Prove that if $\text{Card}(A) = n$ for any $n \in \mathbb{N}$, then $\text{Card}(\mathcal{P}(A)) = 2^n$.

First let $A = \emptyset$. Then $\text{Card}(A) = 0$. So, $\mathcal{P}(A) = \{\emptyset\}$ and $\text{Card}(\mathcal{P}(A)) = 1 = 2^0$. So, now assume that $\text{Card}(A) = n$ and that $\text{Card}(\mathcal{P}(A)) = 2^n$ for some $n \geq 0$. Define $A = \{a_1, \ldots, a_n\}$. Now take $A \cup \{a_{n+1}\}$. Then we see that $A \cap \{a_{n+1}\} = \emptyset$. So, $\text{Card}(A \cup \{a_{n+1}\}) = Card(A) + Card(\{a_{n+1}\}) - Card(A \cap \{a_{n+1}\} = \emptyset) = n+1-0 = n+1$. Now, $\mathcal{P}(A \cup \{a_{n+1}\}) = \mathcal{P}(A) \cup X$ where $X = \{B \cup \{a_{n+1}\} | B \in \mathcal{P}(A)\}$. Define the function $f : X \to \mathcal{P}(A)$ by $f(C) = B$ where $C = B \cup \{a_{n+1}\}$. Now, take $C_1, C_2 \in X$ where $C_1 = B_1 \cup \{a_{n+1}\}$ and $C_2 = B_2 \cup \{a_{N=1}\}$ and assume that $f(C_1) = f(C_2)$. Then $B_1 \cup \{a_{n+1}\} = B_2 \cup \{a_{n+1}\}$, and hence $C_1 = C_2$. So, f is injective. Now, take $B \in \mathcal{P}(A)$ and consider $C = B \cup \{a_{n+1}\}$. Then $f(C) = B$ and we have that f is also onto. Then f is bijective and so $Card(X) = Card(\mathcal{P}(A))$ Now, $\text{Card}(\mathcal{P}(A \cup \{a_{n+1}\})) = \text{Card}(\mathcal{P}(A)) + \text{Card}(X)$ and by the induction hypothesis, $\text{Card}(\mathcal{P}(A)) = 2^n$. So, clearly $\text{Card}(\mathcal{P}(A)) = \text{Card}(X) = 2^n$ since there are still 2^n elements in X. Hence $\text{Card}\mathcal{P}(A \cup \{a_{n+1}\}) = 2^n + 2^n = 2^{n+1}$. Thus, we have that if $\text{Card}(A) = n$, then $\text{Card}(\mathcal{P}(A)) = 2^n$.