10.0 Show that a nonzero central element of a prime ring R is not a zero-divisor in R. In particular, the center $Z(R)$ is a (commutative) domain, and $char R$ is either 0 or a prime number.

Solution.
Let R be a prime ring and let a be a nonzero central element. Assume by way of contradiction that a is a zero divisor. So, there exists $b \in R\setminus\{0\}$ such that $ab = ba = 0$. Now, cosider the ideal $\langle a\rangle\langle b\rangle = (RaR)(RbR) = RabR = \{0\}$ since a is a central element of R. This means that either $a = 0$ or $b = 0$ which contradicts $a, b \neq 0$. Therefore, a is not a zero divisor. Now, since $a \in Z(R)$ was arbitrary, we have that $Z(R)$ is a commutative domain since no element of the center is a zero divisor.

10.2 Let $\mathfrak{p} \subset R$ be a prime ideal, $\mathfrak{A}$ be a left ideal and $\mathfrak{B}$ be a right ideal. Does $\mathfrak{A}\mathfrak{B} \subseteq \mathfrak{p}$ imply that $\mathfrak{A} \subseteq \mathfrak{p}$ or $\mathfrak{B} \subseteq \mathfrak{p}$?

Solution.
Let $R, \mathfrak{p}, \mathfrak{A}$, and $\mathfrak{B}$ be as above. Assume that $\mathfrak{A}\mathfrak{B} \subseteq \mathfrak{p}$. Since $\mathfrak{A}$ is a left ideal, $\mathfrak{A} = aR$ for some element $a \in R$ and since $\mathfrak{B}$ is a right ideal, we have that $\mathfrak{B} = Rb$ for some $b \in R$. So, $\mathfrak{A}\mathfrak{B} = (aR)(Rb) = aRb$. Since $\mathfrak{p}$ is a prime ideal and we assume that $aRb \subseteq \mathfrak{p}$, we have that $a \in \mathfrak{A}$ or $b \in \mathfrak{B}$. So, $\mathfrak{A} \subseteq \mathfrak{p}$ or $\mathfrak{B} \subseteq \mathfrak{p}$. Thus, the above implication is true.

10.3 Show that a ring R is a domain iff R is prime and reduced.

Solution.
Let R be a ring. Assume first that R is a domain. Then, for all $a, b \in R$ if $ab = 0$, then $a = 0$ or $b = 0$. Consider the ideal $\{0\}$ and assume that $aRb \subseteq \{0\}$. That is $arb = 0$ for all $r \in R$ and inparticualr $1 \in R$. Hence, $a1b = ab = 0$. Since R is a domain, this means that $a = 0$ or $b = 0$. This is the same as saying that $a \in \{0\}$ or $b \in \{0\}$, so R is a prime ring. If $b = a$, then we have that $a = b = 0$ and that $a = 0$ is the only nilpotent element of R, so R is also reduced. Conversely, assume that R is prime and reduced. Let $a, b \in R$ be arbitrary and assume that $aRb \subseteq \{0\}$. This means that $a = 0$ or $b = 0$. If $b = a$, then we have that $a^2 = 0$ and since R is reduced, $a = 0$. If $b \neq a$, we have that $ab = 0$ implies that $a = 0$ or $b = 0$. Thus, R is a domain as desired.

10.4 Show that in a right artinian ring R, every prime ideal $\mathfrak{p}$ is maximal. (Equivalently, R is prime iff it is simple.)

Solution.
Let R be a right artinian ring and let p be a prime ideal in R. Assume there exists an ideal I of R such that $p \subseteq I \subset R$.

10.8 a. Show that a ring R is semiprime iff, for any two ideals A, B in R, $AB = 0$ implies that $A \cap B = 0$.
b. Let A, B be left (resp. right) ideals in a semiprime ring R. Show that $AB = 0$ iff $BA = 0$. If A is an ideal, show that $ann_r(A) = ann_l(A)$.

Solution.
a. Let R be a ring. Assume first that R is semiprime. Let A, B be ideal of R such that $AB = 0$. We may assume that $A, B \neq 0$. Since R is semiprime, we have that $\{0\}$ is a semiprime ideal, and there are no nonzero nilpotent ideals. Note that if $A = B$, then $AB = A^2 = 0$ and so $A = 0 = B$ which means $A \cap B = 0$. If $A \neq B$. Let $x \in A \cap B$ and observe $(BxA)^2$. Then, $(BxA)(BxA) = BxABxA = 0$ since $AB = 0$. So, $BxA = 0$ since R is semiprime. By supposition that $A, B \neq 0$, we find that $x = 0$ and since $x \in A \cap B$ was arbitrary, we have that $A \cap B = 0$ as desired. On the other hand, assume that for any two ideals $A, B \subseteq R$ if $AB = 0$, then $A \cap B = 0$. Let A be an ideal of R and consider the situation $AA = A^2 = 0$. Then $A \cap A = A = 0$ by our assumption. Then we see that R has no nonzero nilpotnet ideals since A was an arbitrary ideal. Thus, R is semiprime as desired.

b. Let R, A, and B be as given above. Our argumnet will proceed as follows. Assume first that $AB = 0$. Consider $(BA)^2$. This yields $(BA)(BA) = BABA$. Since we assume that $AB = 0$, we have that $BABA = B0A = 0$. Whence $BA = 0$ from the fact that R is semiprime. Next, assume that $BA = 0$. Similarlly observe $(AB)^2 = (AB)(AB) = ABAB$. By supposition, $BA = 0$, and so $ABAB = A0B = 0$. Thus, $AB = 0$ by R being semiprime, completing our proof.

Now, let A be an arbitrary ideal of R. We will show that $ann_r(A) = ann_l(A)$. Our proof is as follows. We have that $ann_r(A) = \{r \in R | rA = 0\}$ and $ann_l(A) = \{r \in R | Ar = 0\}$. Let $r \in ann_r(A)$ and examine $(Ar)^2$. This gives us $(Ar)(Ar) = ArAr$ and since we have that $rA = 0$ by $r \in ann_r(A)$, $ArAr = 0$. Hence, $Ar = 0$ since R is semiprime. Therefore, $r \in ann_l(A)$. Conversely assume that $r \in ann_l(A)$ and consider $(rA)^2$. Then, $(rA)(rA) = rArA$. So, $rArA = 0$ by our assumption that $Ar = 0$. Since R is semiprime, $rA = 0$ and we conclude that $r \in ann_r(A)$. Thus, $ann_r(A) = ann_l(A)$ as desired.

10.14 Show that any prime ideal p in a ring R contains a minimal prime ideal. Using this, show that the lower nilradical Nil_*R is the intersection of all the minimal prime ideals of R.

Solution.
Let R be a ring and p a prime ideal in R. Note that if p is a minimal ideal itself, we are done. Also, note that if R is prime, then we have that $\{0\} \subseteq p$ is a minimal prime ideal contained in all prime ideals p. So, assume that p is not itself minimal. Let U be an ideal such that $U \subset p$.