1.7 Let $B_1, \ldots, B_n$ be left ideals (resp. ideals) in a ring R. Show that $R = B_1 \oplus \ldots \oplus B_n$ if and only if there exist idempotents (resp. central idempotents) $e_1, \ldots, e_n$ with sum 1 such that $e_ie_j = 0$ whenever $i \neq j$, and $B_i = Re_i$ for all i. In the case where the B_i's are ideals, if $R = B_1 \oplus \ldots \oplus B_n$, then each B_i is a ring with identity e_i, and we have an isomorphism between R and the direct product of rings $B_1 \times \cdots \times B_n$. Show that any isomorphism of R with a finite direct product of rings arises in this way.

Solution.
Let R and B_i for $1 \leq i \leq n$ be as given above. Assume first that $R = B_1 \oplus \ldots \oplus B_n$. Consider $e_i \in B_i$ for $1 \leq i \leq n$ such that $1 = e_1 + \ldots + e_n$. Now, multiply this equation on the left by e_i. Then we have $e_i = e_ie_1 + \ldots + e_ie_n$. Since we have that B_j for $1 \leq j \leq n$ are left ideals of R, $e_ie_j \in B_j$. Since $e_i \in B_i$, we have that $e_ie_j = 0$ for all $i \neq j$. Hence, $e_i = e_ie_i = e_i^2$. That is e_i is an idempotent for B_i for all $1 \leq i \leq n$. Also, we see that $B_i = Re_i$ for all i. So, now assume the opposite direction. Now, let $x \in R$. Then, $x = x1 = x(e_1 + \ldots + e_n)$. Since $B_i = Re_i$, we have that $(e_1 + \ldots + e_n) \in B_1 + \ldots + B_n$. So, $R = B_1 + \ldots + B_n$. Now, consider $x \in B_i \cap B_j$ with $i \neq j$. Then, again since $B_i = Re_i$, $x = r_1e_i = r_2e_j$. Now, multiplying by e_i on the right, we get $r_1e_ie_i = r_1e_i = r_2e_je_i = 0$ since $e_ie_j = 0$ whenever $i \neq j$. So, $x = 0$. Thus, $R = B_1 \oplus \ldots \oplus B_n$ as desired.

1.12 A left R-module M is hopfian if any surjective R-endomorphism of M is an automorphism.

a. Show that any noetherian module M is hopfian.

b. Show that the left regular module ${}_RR$ is hopfian if and only if R is Dedekind-finite.

c. Deduce from a and b above that any left noetherian ring R is Dedekind-finite.

Solution.
Let R be a ring.

a. Let M be an arbitrary noetherian R-module, and let $f : M \to M$ be a surjective R-homomorphism. Since M is noetherian, we have that the submodules of M satisfy $N_1 \subset N_2 \subset \ldots \subset N_n = N_{n+1} = \ldots$ for some $n \in \mathbb{Z}^+$ where N_i are R-submodules of M. Now, let $x \in N_1$ such that $f(x) = 0$. Hence, $x \in N_i$ for all $i \geq 1$. So, $f^n(x) = 0$. That is, $x \in \ker(f^n)$. Now, we know from a previous problem that $f^n(M) \cap \ker(f^n) = \{0\}$. Therefore, we have that $x = 0$. So, f is $1 - 1$ and is thus an R-automorphism. Since f was arbitrary, any surjective R-endomorphism is an R-automorphism. Whence, M is hopfian as desired.

b. Assume first that ${}_RR$ is hopfian. Let $a, b \in R$ such that $ab = 1$. Let $\phi : M \to M$ be a surjective R-endomorphism. Since ${}_RR$ is hopfian, we have that ϕ is an automorphism. Since $ab = 1$, $b\phi(ab) = \phi(b(ab)) = \phi((ba)b) = ba\phi(b) = b\phi(1) = \phi(b)$ since ϕ is an automorphism. Hence, $ba = 1$ and so R is Dedkind finite. Now, assume that R is Dedekind finite and let $\phi : M \to M$ be a surjective R-endomorphism. Let $a, b \in R$ such that $ab = 1$, and assume that $\phi(a) = \phi(b)$. Since R is Dedekind finite, we have that $ba = 1$. So, $\phi(1) = \phi(ab) = \phi(ba)$. Since ϕ is a homomorphism, we get that $a\phi(b) = b\phi(a)$. By supposition, $\phi(a) = \phi(b)$, whence $a = b$ by cancellation on the right. Thus, ϕ is $1 - 1$ and since ϕ was an arbitrary surjective endomorphism, every such endomorphism is an automorphism.

c. Let R be a left noetherian ring. So we have that every left R-module of R is noetherian. In particular, ${}_RR$ is noetherian. By part a, we have that ${}_RR$ is hopfian and by part b, we have that R is Dedekind finite as desired.

1.19 Let R be a domain. If R has a minimal left ideal, show that R is a division ring.

Solution.
Let R be a domain, and let I be a minimal left ideal of R. Let $x \in R \setminus \{0\}$ and let $a \in I \setminus \{0\}$. Consider the left ideal Ra. We have that $Ra \subseteq I$ and by the minimality of I, $Ra = I$. Hence, $R(xa) = Ra$. So, there exists $y \in R$ such that $a = yxa$. Whence $a - yxa = (1 - yx)a = 0$. Since $a \neq 0$ and R is a domain, $1 - yx = 0$. That is $yx = 1$. Since $x \neq 0$ was arbitrary, we have that R is a division ring as desired.

1.20 Let $E = End_R(M)$ be the ring of endomorphisms of an R-module M, and let nM denote the direct sum of n copies of M. Show that $End_R(nM)$ is isomorphic to $\mathbb{M}_n(E)$.

Solution.
Let E, R, M, and nM be as given above. Consider $\pi_j : nM \to M$ and $\iota_i : M \to nM$ the natural projection and inclusion maps respectively with respect to the j^{th} component. Now, let $\phi \in End_R(nM)$. Then, we see that $\pi_j(\phi(\iota_i)) \in E$. Now, define $\psi : End_R(nM) \to \mathbb{M}_n(E)$ by $\psi(\phi) = [\pi_j(\phi(\iota_i))]$ where $\pi_j(\phi(\iota_i))$ is the ij^{th} element of the $n \times n$ matrix. Clearly, we see that $\psi(\phi) = (0)_{n\times n}$ if and only if $\phi = 0$. So, ψ is $1-1$. We can also see that ψ is onto. Now, let $\phi, \tau \in End_R(nM)$. Then,

$$\begin{gathered}\psi(\phi + \tau) = [(\pi_j(\phi + \tau(\iota_i)))] \\ = [(\pi_j(\phi(\iota_i)))] + [(\pi_j(\tau(\iota_i)))] = \psi(\phi) + \psi(\tau)\end{gathered}$$

and

$$\begin{gathered}\psi(\phi\tau) = [(\pi_j(\phi\tau(\iota_i)))] \\ = [(\pi_j(\phi(\iota_i)))][(\pi_j(\tau(\iota_i)))] = \psi(\phi)\psi(\tau)\end{gathered}$$

since $\iota_i\pi_j = e$ the identity map when $i = j$ but $\iota_i\pi_j = 0$ when $i \neq j$. Therefore, ψ is a ring isomorphism and thus $End_R(nM) \cong \mathbb{M}_n(E)$ as desired.

1.21 Let R be a finite ring. Show that there exists an infinite sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers such that, for any $x \in R$, we have $x^{n_1} = x^{n_2} = x^{n_3} = \cdots$.

Solution.
Let R be a ring with $|R| = m$ and $x_i \in R$ for $1 \leq i \leq m$. Since R is finite, we have that there exists $k_i \in \mathbb{N}$ such that $x_i^{k_i} = 1$ and k_i divides the order of the ring for all $1 \leq i \leq m$. So, take $n_1 = \prod_{i=1}^{m} k_i$. Then, $x^{n_1} = 1$ for all $x \in R$. Now, take $n_j = jn_1$ for $j \geq 2$. Then, $x^{n_j} = 1$ for all $x \in R$ and all $j \geq 2$. Therefore, $x^{n_1} = x^{n_2} = \cdots$ for the infinite sequence $n_1 < n_2 < \cdot$ as desired.

2.1 Is any subring of a left semisimple ring left semisimple? Can any ring be embedded as a subring of a left semisimple ring?

Solution.
No. Consider the ring $\mathbb{R}$. We know that $\mathbb{R}$ is left semisimple since it is a field and also since ${}_{\mathbb{R}}\mathbb{R}$ is simple. Now consider the subring $\mathbb{Z}$ of $\mathbb{R}$. Clearly $\mathbb{Z}$ is not left semisimple even though it is a subring of a left semisimple ring since ${}_{\mathbb{Z}}\mathbb{Z}$ is not simple. However, we can embed any ring as a subring of a left semisimple ring.

2.3 What are semisimple $\mathbb{Z}$-modules?

Solution.
We know that simple modules of $\mathbb{Z}$ are exactly the ones of the form $\mathbb{Z}_p$ where p is prime. Since semisimple modules are the direct sum of simple modules, we have that the semisimple modules of $\mathbb{Z}$ are of the form $\bigoplus_p \mathbb{Z}_p$ where p is prime. That is, the semisimple $\mathbb{Z}$-modules are the direct sums of $\mathbb{Z}_p$ over the primes.

2.7 Show that for a semisimple module M over any ring R , the following conditions are equivalet:
(1) M is finitely generated
(2) M is neotherian
(3) M is artinian
(4) M is a finite direct sum of simple modules

Solution.
Let R and M be as given above. By a theorem, we have that 1 is equivalent to 4. Also, by theorem 3 from the noetherian unit, we have that M being noetherian imples that M is finitely generated. So, 2 imples 1. Now, assume that M is a finite direct sum of simple modules.