4.9 Let R be a J-semisimple domain and a be a nonzero central element of R. Show that the intersection of all maximal left ideals not containing a is zero.

Solution.
Let R and a be as above. If a is not in any maximal left ideal of R, then the intersection of all maximal left ideals of R not containing a is $radR = \{0\}$ by assumption. Similarly, if a is in all nonzero maximal ideals of R, then the intersection of all maximal left ideals of R not containing a is simply the set $\{0\}$. So, let X be the collection of all maximal left ideals of R not containg a. Assume by way of contradiction that $\bigcap X \neq \{0\}$. So, there exists $b \in \bigcap X$ with $b \neq 0$ Since X is also an ideal, $ab \in X$. Cosider $\mathcal{R}$ the intersection of all maximal left ideals of R. So, $\mathcal{R} \setminus X = Y$ the intersection of all maximal ideals of R containing a. So, $ab \in Y$. That is, $ab = ba \in X \cap Y = radR$. Since $a, b \neq 0$ and R is a domain, $ab \neq 0$. This contradicts R being J-semisimple. Thus, $\bigcap X = \{0\}$ as desired.

In case a is to be considered an idempotent: Let R be a J-semisimple domain and a a nonzero idempotent. Let X be the intersection of all maximal ideals of R not containg a. If X is all of the maximal ideals of R, then $X = radR = \{0\}$. If a is in every nonzero maximal ideal, then $X = \{0\}$. Now consider $b \in X$. Now, $ab = a^2b$ as a is an idempotent. So, $ab - a^2b = a(b - ab) = a(1 - a)b = 0$. Since R is a domain, either $a = 0, 1 - a = 0$, or $b = 0$. By assumption, $a \neq 0$, and if $1 - a = 0$, then $1 = a$ and no maximal ideal of R can contain 1 which would result in the trivial case above. So, we have that $b = 0$ and since b was arbitrary, $X = \{0\}$ as desired.

4.11 If an ideal $I \subseteq R$ is such that R/I is J-semisimple, show that $I \supseteq radR$. (Therefore, $radR$ is the smallest ideal $I \subseteq R$ such that R/I is J-semisimple.)

Solution.
Let R be a ring and I an ideal of R such that R/I is J-semisimple. Denote $radR = J$. If $I = J$, we are done. Now, assume by way of contradiction that $I \subset J$. So, $radR/I = (radR)/I = J/I$. Since $I \subset J$, we have that I is contained in a maximal ideal of R. Since $I \neq J$, there exists $a \in J$ such that $a + I \in J/I$ with $a + I \neq 0$. This gives us that R/I is not J-semisimple, a contradiction. So, $I \supseteq J$ as desired.

4.12A Let $\mathfrak{A}_i$ $(i \in I)$ be ideals in a ring R, and let $\mathfrak{A} = \bigcap_i \mathfrak{A}_i$. True or False: "If each $R/\mathfrak{A}_i$ is J-semisimple, then so is $R/\mathfrak{A}$"?

Solution.
Let $R, \mathfrak{A}_i$ $(i \in I)$, and $\mathfrak{A}_i$ be as above. This statement is true. Since each $\mathfrak{A}_i$ is an ideal of R, we have that each $\mathfrak{A}_i$ is contained in a maximal ideal of R. Whence, $\mathfrak{A}$ is also contained in a maximal ideal of R. So, each $\mathfrak{A}_i \subseteq radR = J$ and $\mathfrak{A} \subseteq J$. Hence, $\{0\} = rad(R/\mathfrak{A}_i) = (rad(R))/\mathfrak{A}_i$ since each $R/\mathfrak{A}_i$ is J-semisimple. So, $\{0\} = \bigcap_i \{0\} = \bigcap_i ((radR)/\mathfrak{A}_i) = (radR)/\mathfrak{A} = rad(R/\mathfrak{A})$. Thus, $R/\mathfrak{A}$ is also J-semisimple.

4.14 Show that a ring R is von Neumann regular iff $IJ = I \cap J$ for every right ideal I and every left ideal J in R.

Solution.
Let R be a ring. Assume first that $IJ = I \cap J$ for every right ideal I and every left ideal J in R. Let $a \in R$. Consider the ideals aR and Ra. By our assumption, $(aR)(Ra) = aRa = aR \cap Ra$. Now, observe that $1a = a1 = a$. Hence, $a \in aR$ and $a \in Ra$. That is $a \in aR \cap Ra = aRa$. So, there exists some $b \in R$ such that $a = aba$. That is, R is von Neumann regular as desired. Conversely assume that R is von Nuemann regular. Let I, J be left and right ideals of R respectively. Let $a \in IJ$. So, there exists $i \in I$ and $j \in J$ such that $a = ij$. Since I is a left ideal and J is a right ideal, we have that $a = ij \in I$ and $a = ij \in J$. That is $a \in I \cap J$. now, let $a \in I \cap J$. Since R is von Nuemann regular, there exists $b \in R$ such that $a = aba$. Since I and J are left and right ideals respectively and $b \in R$, we have that $a = aba \in IJ$. So, $IJ = I \cap J$ as desired.

4.14B For any ring R, show that the following are equivalent:

1. For any $a \in R$, there exists a unit $u \in U(R)$ such that $a = aua$.
2. Every $a \in R$ can be written as a unit times an idempotent.

2'. Every $a \in R$ can be written as an idempotent times a unit.

If R satisfies 1, it is said to be unit-regular.

3. Show that any unit-regular ring R is Dedekind-finite.

Solution.

Let R be a ring.

$(1 \Leftrightarrow 2)$. Let $a \in R$ and let $u \in U(R)$ such that $a = aua$. Since u is a unit in R, there exists $v \in R$ such that $uv = vu = 1$. Since $a = aua$, $ua = uaua$. Hence, ua is an idempotent in R. Whence, setting $ua = e$, we have that $vua = a = ve$. Since v is also a unit in R, a is a unit times an idempotent. Conversely assume that $a \in R$ is a unit v times an idempotent e. That is $a = ve$. Since v is a unit, there exists $u \in U(R)$ such that $vu = uv = 1$. Therefore, $uve = e = ua$. That is ua is an idempotent. So, $ua = uaua$. Whence, $ua - uaua = u(a - aua) = 0$. Thus, $vu(a - aua) = a - aua = 0$. Hence, $a = aua$ for some unit u in R.

$(1 \Leftrightarrow 2')$. Let $a \in R$ and let $u \in U(R)$ such that $a = aua$. Since u is a unit in R, there exists $v \in R$ such that $uv = vu = 1$. Since $a = aua$, $au = auau$. Hence, au is an idempotent in R. Whence, setting $au = e$, we have that $auv = a = ev$. Since v is also a unit in R, a is an idempotent times a unit. Conversely assume that $a \in R$ is an idempotent e times a unit v. That is $a = ev$. Since v is a unit, there exists $u \in U(R)$ such that $vu = uv = 1$. Therefore, $evu = e = au$. That is au is an idempotent. So, $au = auau$. Whence, $au - auau = (a - aua)u = 0$. Thus, $(a - aua)uv = a - aua = 0$. Hence, $a = aua$ for some unit u in R.

$(1 \Rightarrow 3)$. Let $a \in R$ and let $u \in U(R)$ such that $a = aua$. Let $b \in R$ such that $ab = 1$. We will show that $ba = 1$. Since $ab = 1$, we have that $aba = a$. By assumption, that means b is a unit in R. Since b is a unit in R, we have that $ab = ba = 1$ and thus R is Dedekind finite as desired.

4.16 A left R-module M is said to be cohopfian if any injective R-endomorphism of M is an automorphism.
1. Show that any artinian module M is cohopfian.
2. Show that the left regular module ${}_RR$ is cohopfian iff every non right-0-divisor in R is a unit. In this case, show that ${}_RR$ is also hopfian

Solution.
1. Let R be a ring and M an artinian R-module. Let f be an injective R-endomorphism of M. Consider the decending chain of submodules of M by $M \supseteq f(M) \supseteq f^2(M) \supseteq f^3(M) \supseteq \ldots$. Since M is artinian, this chain must stabilize for some $n \in \mathbb{Z}^+$. That is $f^n(M) = f^{n+1}(M) = \ldots$. Hence, $f(f^{n-1}(M)) = f(f^n(M))$. Since f is $1-1$, we have that $f^{n-1}(M) = f^n(M)$. Repeating this process for $n-1$ more iterations, using that f is injective, yields $M = f(M)$. Thus, f is surjective and therefore an automorphism. That is M is cohopfian as desired.

2. Let R be a ring. First assume that ${}_RR$ is cohopfian. Let $u \in R$ be a non right-0-divisor. Consider that map $f : {}_RR \to {}_RR$ given by $f(r) = ru$ for all $r \in {}_RR$. Now, let $a, b \in {}_RR$ and $m \in R$. Then, $f(a+b) = (a+b)u = au + bu = f(a) + f(u)$ and $f(ma) = (ma)u = mau = m(au) = f(a)$. So, f is an endomorphism of ${}_RR$. Now assume that $f(a) = f(b)$. Whence, $au = bu$ and therefore, $au - bu = (a-b)u = 0$. Since u is not a right-0-divisor, we have that $a - b = 0$. So, $a = b$. That means f is also $1-1$ and so by assumption, f is an automorphism. So, there exists $v \in R$ such that $f(v) = vu = 1$. Hence, $uvu = u$ and so $uvu - u = (uv-1)u = 0$. Since u is not a right-0-divisor, $uv - 1 = 0$. That is $uv = 1$ and thus u is a unit in R. Conversley assume that every non right-0-divisor in R is a unit. Let g be an injective endomorphism of ${}_RR$. So, $g(r) = ru$ for all $r \in {}_RR$ and some $u \in {}_RR$. Since g is $1-1$, if $g(r) = 0$, then $r = 0$. So, if $g(r) = ru = 0$, $r = 0$. This means that u is not a right-0-divisor. So by assumption, u is a unit in R. Hence, there exists $v \in R$ such that $uv = vu = 1$. Now, let $s \in {}_RR$. Since ${}_RR$ is an R-module, $sv \in {}_RR$. Whence, $g(sv) = (sv)u = svu = s$. Thus, g is sujective making ${}_RR$ cohopfian since g was arbitrary. Now, assume that ${}_RR$ is cohopfian. Let h be a surjective endomorphism of ${}_RR$. Since h is surjective, there exists $x \in {}_RR$ such that $h(x) = 0$. We know that every endomorphism has the form $h(x) = xa$ for all $x \in {}_RR$ and some $a \in R$. So, $h(x) = xa = 0$. Let $x, y \in {}_RR$ and assume that $h(x) = h(y)$. Then, $h(x) - h(y) = h(x-y) = 0$ since h is an endomorphism. Hence, $(x-y)a = 0$. Clearly, $a \neq 0$ since h is onto. So, since a is not a right-0-divisor, a is a unit in R. Therefore, there exists $b \in R$ such that $ab = ba = 1$. Whence, $(x-y)ab = x - y = 0$ and so $x = y$. Thus, h is $1-1$ and an automorphism. Therfore, ${}_RR$ is hopfian.

6.3 Let G be a finite group whose order is a unit in a ring k, and let $W \subseteq V$ be left kG-modules.
1. If W is a direct summand of V as k-modules, show that W is a direct summand of V as kG-modules.
2. If V is projective as a k-module, show that V is projective as a kG-module.

Solution.
Let G, k, W, and V be as given.
1. Let W be a direct summand of V as k-modules. That is, $V = W \oplus X$ where $W \cap X = 0$ for some left k-module X. Let $\gamma : kG \to k$ be the augmentation map given in class, and let Y and Z be the preimages of W and X respectively. Since γ is a surjective map, we have that $W \subseteq Y$. Hence, $Y = W \oplus (Y \setminus W)$. Therefore, we have that $V = W \oplus (Y \setminus W) \oplus Z = Y \oplus Z$ and thus W is a direct summand of V as kG-modules.

2. Let W be a direct summand of V as k-modules. Consider the kG modules Y and Z. Let $\phi : Y \to Z$ be a surjective kG-module homomorphism, and let $\psi : V \to Z$ be a kG-module homomrphism. Let $\gamma : kG \to k$ be the augmentation map as given in class. Then, we have that $\gamma(Y)$ and $\gamma(Z)$ are k-modules. Now, ϕ restricted to a k-module homomorphism between $\gamma(Y)$ and $\gamma(Z)$ is still surjective. Thus, since V is projective as a k-module, there exists $\rho : V \to \gamma(Y)$ and we can restirict ρ to be between V and Y such that $\rho \circ \phi = \psi$. Thus, V is projective as a kG-module.

6.6 Let H be a normal subgroup of G. Show that $I = kG \cdot rad\ kH$ is an ideal of kG. If $rad\ kH$ is nilpotent, show that I is also nilpotent. (In particular, if H is finite, I is always nilpotent.)

Solution.
Let H, G, and I be as stated above. Define $f : kH \to kH$ by $f(h) = ghg^{-1}$. Then, we see that f is onto. Now, let $x, y \in kH$. Then, $f(x+y) = g(x+y)g^{-1} = gxg^{-1}+gyg^{-1} = f(x) + f(y)$ and $f(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = f(x)f(y)$. So, f is a surjective homomorphism. Now, let $x \in \ker(f)$. Then, $f(x) = gxg^-1 = 0$. Hence, $x = 0$. Thus, f is an automorphism. We can extend this to be a mapping of kG and f will preserve kH as well as $rad(kH)$. We can conclude that $g \cdot rad(kH) = rad(kH) \cdot g$ for all $g \in kG$. So, now let $r \in kG$ and $i \in I$. Then, $ri \in I$ due to the closure of kG. Also, $ir = ri \in I$ by our conclusion and the closure of kG. Thus, I is an ideal of kG as desired.

6.12 Assume $char(k) = 3$, and let $G = S_3$ (symmetric group on three letters)

1. Compute the Jacobson radical $J = rad(kG)$, and the factor ring kG/J.
2. Determine the index of nilpotency for J, and find a k-basis for J^i for each i.

Solution.

1. Consider the augmentation map $\phi : kG \to k$ as defined in class. Then, we have that the kernel $\ker(\phi) = J$ and thus, $kG/J = \{b + \ker(\phi) | b \in kG\}$ is the quotient ring.

2. The index of nilpotency for J is 18 which is the product of the characteristic of k and the order of G.