A. Bogobowicz[1]
Visiting Asst. Professor.

L. Rothenburg
Assoc. Professor.

M. B. Dusseault
Professor.

Department of Civil Engineering,
University of Waterloo,
Waterloo, Ontario N2L 3G1
Canada

# Solutions for Non-Newtonian Flow Into Elliptical Openings

*A semi-analytical solution for plane velocity fields describing steady-state incompressible flow of nonlinearly viscous fluid into an elliptical opening is presented. The flow is driven by hydrostatic pressure applied at infinity. The solution is obtained by minimizing the rate of energy dissipation on a sufficiently flexible incompressible velocity field in elliptical coordinates. The medium is described by a power creep law and solutions are obtained for a range of exponents and ellipse eccentricites. The obtained solutions compare favorably with results of finite element analysis.*

## Introduction

Determination of slow closure rates of elliptical openings in a nonlinearly viscous media is a problem of considerable interest to salt rock and potash mining. Although openings in salt rock are rarely excavated in an elliptical shape, development of fractures around deep openings and subsequent detachment of roof and floor slabs and pillar spalling result in openings of nearly elliptical shape (Mraz, 1973). Formation of an elliptical opening around a rectangular room is schematically illustrated in Fig. 1.

In deep salt and potash mines, where ground stresses are high enough to cause fracture of rock around openings, the modification of the initial opening geometry occurs within several months after excavation. Eventually, a couple of years later, the elliptical opening will experience steady-state closure.

In this paper, a semi-analytical solution for steady-state closure rates of infinitely long elliptical openings in an infinite medium is presented. Material behavior is described by steady-state power creep law. The solution is developed for a wide range of ellipse eccentricities. Stress field at infinity is assumed to be hydrostatic.

Closed-form solutions for nonlinear flow problems can be only obtained in effectively one-dimensional cases (flow between two plates, closure of a circular opening). FEM analysis is rather standard in this area, but computations are numerically intensive, and parametric studies that are frequently required in engineering applications are costly in terms of computer time involved.

There are a number of practically important problems that have been addressed in an analytical form by utilizing variational principles for nonlinearly viscous media and using kinematically admissible trial functions with adjustable parameters (Lioboutry, 1987 gives several examples). Gilormini and Montheillet (1986) successfully used trial velocity fields of a corresponding linear problem to study deformations of an incompressible elliptical inclusion in a nonlinearly viscous matrix. The problem of flow into an elliptical opening is addressed here in a similar manner, except it is found that velocity fields of linear and nonlinear problems are substantially different for high exponents of power-law viscosity.

## Material Model

It is assumed that the material is described by a power creep law of the form:

$$\dot{\epsilon}'_{ij} = \frac{3}{2}\,\dot{\epsilon}_o\,\frac{\sigma'_{ij}}{\sigma_{ef}}\left(\frac{\sigma_{ef}}{\sigma_o}\right)^M \tag{1}$$

where $\dot{\epsilon}_{ij}$ is the strain rate, $\sigma'_{ij}$ is the stress deviator, $\sigma_{ef} = (3/2\sigma'_{ij}\sigma'_{ij})^{1/2}$ is the so-called "effective stress" assumed to govern creep rates in generalized stress conditions, $M$ is the power law exponent, and $\sigma_o$, $\dot{\epsilon}_o$ are material constants (only one of which is independent). In the subsequent text the inverted form of the above creep law will be extensively used:

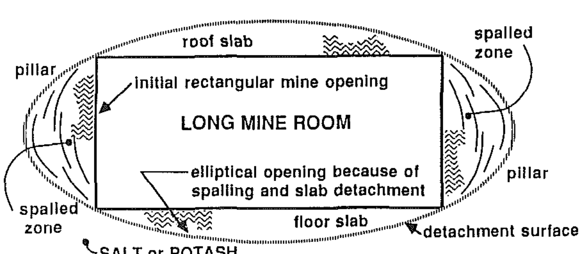


Fig. 1 Formation of an elliptical opening around a rectangular room

[1]Permanent address: Institute of Geophysics, Polish Academy of Sciences, Warsaw, Poland.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the JOURNAL OF APPLIED MECHANICS.

Discussion on this paper should be addressed to the Technical Editor, Prof. Leon M. Keer, The Technological Institute, Northwestern University, Evanston, IL 60208, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Nov. 9, 1989; final revision, June 20, 1990.

$$\sigma'_{ij} = \frac{2}{3}\,\sigma_o\,\frac{\dot\epsilon'_{ij}}{\dot\epsilon_{ef}}\left(\frac{\dot\epsilon_{ef}}{\dot\epsilon_o}\right)^m \quad (\dot\epsilon_{kk}=0) \qquad (2)$$

where $m = 1/M$ and $\dot\epsilon_{ef} = (2/3\,\dot\epsilon'_{ij}\,\dot\epsilon'_{ij})^{1/2}$ is the "effective strain rate." The definitions of effective stress and strain are such that the relationship (1) takes the form $\dot\epsilon_1 = \dot\epsilon_0\left(\frac{\sigma_1-\sigma_3}{\sigma_0}\right)^M$ in conditions of uniaxial compression.

It should be noted that the material law (1) is unlikely to be applicable for the entire range of stresses and certainly not for high stress deviators approaching uniaxial compressive strength. It seems that for salt rocks, the expression (1) is reasonable for $\sigma_{ef} < \sigma_o \approx 10$ MPa with $M = 3$. For higher stress deviators the creep mechanism changes and can be also approximated by a power law, but with a much higher exponent (Dusseault et al., 1987). The present study is limited to a single mechanism power law, although the technique described below is sufficiently general to be applicable for more complex creep models.

## Variational Approach to Steady-State Solutions

In general, exact analytical solution for nonlinearly viscous flow can be obtained in effectively one-dimensional cases like flow through a pipe or closure of a circular opening. More complex problems can be addressed by noting that a velocity field in a material obeying the constitutive relationship (1) and resulting in an equilibrium stress field minimizes the following functional (Hill, 1956) on a set of all incompressible velocity fields ($\partial v_i/\partial x_i = 0$):

$$D = \frac{\sigma_o\epsilon_o}{m+1}\int_V \left(\frac{\dot\epsilon_{ef}}{\dot\epsilon_o}\right)^{m+1} dV - \int_B \sigma_{ij} n_j v_i\, dB. \qquad (3)$$

The first term in equation (3) represents the rate of energy dissipation within the volume $V$ of the material, while the second term is the power of boundary tractions $\sigma_{ij} n_j$, where $n$ is the outward normal to the boundary $B$ surrounding $V$. In order to apply this principle to a particular problem without undue analytical complications, it is necessary to select a sufficiently wide set of physically realistic incompressible velocity fields specific to the problem.

## Incompressible Velocity Fields in Elliptical Coordinates

In further analysis, it is convenient to describe the shape of the opening and to seek the solution in elliptical coordinates that transform the exterior of a circle in ($\rho$, $\theta$) polar coordinates into the exterior of the ellipse on an $x$, $y$-plane as follows:

$$x = \left(\rho + \frac{\bar r^2 \epsilon}{\rho}\right)\cos\theta;\ y = \left(\rho - \frac{\bar r_e^2}{\rho}\right)\sin\theta \qquad (4)$$

where $\bar r$ is the mean of the ellipse semi-axes $a = \bar r\,(1+e)$, $b = \bar r\,(1-e)$ and $e$ is the ellipse eccentricity. In further analysis, $\bar r$ will be taken as unity and final results for any size opening will be obtained by simple scaling. Figure 2 illustrates families of $\rho$ = const and $\theta$ = const lines that form a mesh similar to that used for finite element computations. Elements of length $dl_\theta$, $dl_\rho$ in $x$, $y$ coordinates along $\theta$, $\rho$-lines can be calculated by differentiating (3) as follows:

$$dl_\theta = \rho g d\theta;\ dl_\rho = g d\rho$$
$$g = \left(1 - \frac{2e}{\rho^2}\cos 2\theta + \frac{e^2}{\rho^4}\right)^{1/2}. \qquad (5)$$

The simplest incompressible velocity field in elliptical coordinates physically describes D'Arcy flow through porous media. In this case fluid velocities in $\theta$-direction are zero and each flow channel between two $\theta$ = const lines carries the same quantity of flow $dq$, i.e., $v_\rho dl_\theta$ = const = $dq$. Considering (4), the velocity field of subsequent interest is as follows:

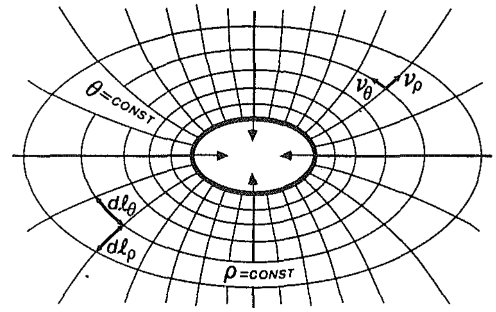


Fig. 2 Elliptical coordinates

$$v_\rho(\rho,\theta) = \frac{\text{const}}{\rho g};\ v_\theta(\rho,\theta) = 0. \qquad (6)$$

A somewhat more complex incompressible velocity field corresponds to a linearly viscous (Newtonian) flow into an elliptical opening. This solution can be obtained by noting a direct analogy between displacements of a linearly elastic incompressible solid (Poisson's ratio 0.5) and velocities of an incompressible viscous fluid in the same boundary value problem. The velocity field obtained using this analogy can be derived by extending complex variables elasticity solution (e.g., Love, 1927) and is as follows:

$$v_\rho(\rho,\theta) = v_o\,\frac{1-\xi\cos 2\theta}{\rho g}\left(\xi = \frac{2e}{1+e^2}\right);\ v_\theta(\rho,\theta) = 0 \qquad (7)$$

where $v_o$ is the mean closure rate of the opening (averaging over the entire opening surface).

In the subsequent analysis it will be convenient to characterize the response of the opening in terms of the average of vertical and horizontal closure rates $\bar v = 1/2(v_{\text{vert}} + v_{\text{horiz}})$ and in terms of the relative difference between closure rates in vertical and horizontal directions, i.e., $\chi = (v_{\text{vert}} - v_{\text{horiz}})/(v_{\text{vert}} + v_{\text{horiz}})$. For the solution (7), $\bar v = v_o/(1+e^2)$, $\chi = e$.

In connection with (7), an important question is the extent to which the velocity field for non-Newtonian flow is different from the velocity field (7) of a corresponding linear problem. While one cannot generally expect that both are identical, except for a circular opening, the incompressibility of flow and identical boundary conditions for linear and nonlinear problems do suggest that the velocity fields in both cases should be not far different.

The velocity field (7) has the property that

$$v_\rho(\rho,0) + v_\rho(\rho,\pi/2) = 2v_\rho(\rho,\pi/4). \qquad (7)$$

Several numerical solutions of nonlinear problems with integer exponents ($M$ = 2, 3, 4, 5, 6) all suggest that this property of the velocity field of the linear problem is preserved in nonlinear cases. Also, detailed analysis of numerical solutions show that velocity fields in nonlinear cases generally contain terms of the order $1/\rho^2$, $1/\rho^3$ and possibly of higher orders of $1/\rho$. Nevertheless, the variation of the velocity field along the coordinate line $\theta = \pi/4$ is exactly according to (7). These observations on numerical solutions suggest that the velocity fields for non-Newtonian flow can be taken in the form:

$$v_\rho(\rho,\theta) = v_o\left(\frac{1}{g\rho} + \frac{f_1(\cos 2\theta)}{g\rho} + \frac{f_2(\cos 2\theta)}{g\rho^2} + \frac{f_3(\cos 2\theta)}{g\rho^3} + \ldots\right). \qquad (8)$$

The fact that $f_1, f_2, f_3$ are functions of $\cos 2\theta$ follows from the symmetry of the opening geometry. Also, since for $\theta = \pi/4$ the solution degenerates into (6), $f_1(0) = f_2(0) = f_3(0) = 0$. In the subsequent analysis, both functions will be in the form $f_i(\cos 2\theta) = c_i \cos 2\theta$, where $c_i$ are constants to be determined by minimizing (3) numerically. Minimization, with respect to $v_o$ in (8), will be performed analytically.

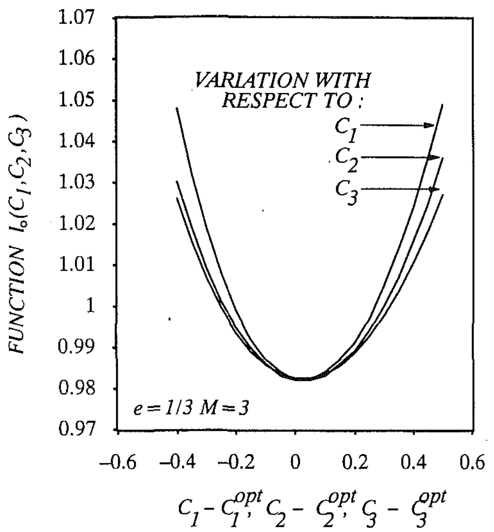


**Fig. 3 Function $I_o$ in the vicinity of minimum ($e$ = 1/3, $M$ = 3)**

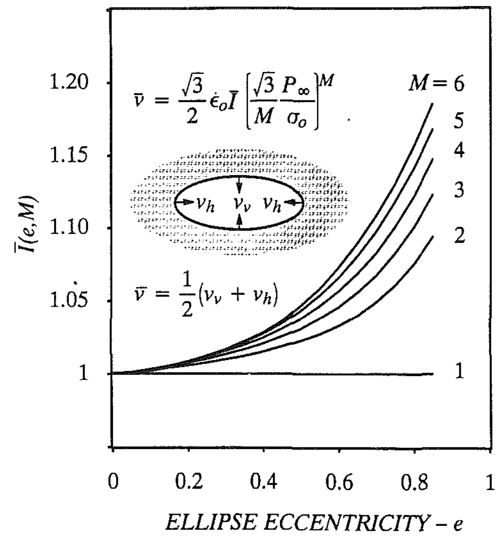


**Fig. 4 Calculation of average closure rates of elliptical openings of different eccentricities and power-law exponents**

Before proceeding, it is necessary to determine the form of the component $v_\theta$ associated with the radial component of the velocity field (8). The incompressibility condition is sufficient to solve this problem. The required expressions for strain rates are as follows:

$$\dot{\epsilon}_{\rho\rho}=\frac{1}{g}\frac{\partial v_\rho}{\partial \rho}+\frac{v_\theta}{\rho g^2}\frac{\partial g}{\partial \theta}$$
$$\dot{\epsilon}_{\theta\theta}=\frac{v_\rho}{\rho g^2}\frac{\partial(\rho g)}{\partial \rho}+\frac{1}{\rho g}\frac{\partial v_\rho}{\partial \rho} \tag{9}$$
$$2\dot{\epsilon}_{\rho\theta}=\rho\frac{\partial}{\partial\rho}\left(\frac{v_\theta}{\rho g}\right)+\frac{\partial}{\partial\theta}\left(\frac{v_\rho}{\rho g}\right).$$

These can be obtained from general expressions for strains in curvilinear coordinates given in (Love, 1927). Calculation of strains for a single term in (8) of the form $v_\rho = f(\theta)/g\rho^n$ and substitution into the incompressibility condition $\dot{\epsilon}_{\rho\rho} + \dot{\epsilon}_{\theta\theta} = 0$ gives the following differential equation for $v_\theta$:

$$\frac{\partial v_\theta}{\partial \theta}+v_\theta=(n-1)v_r.$$

The solution of this equation is as follows:

$$v_\theta(\rho,\theta)=(n-1)\frac{1}{g\rho^n}\int f(\theta)d\theta.$$

The above solution must be a single-valued function of $\theta$. In summary, the following incompressible velocity field will be used to describe flow into an elliptical opening:

$$v_\rho=v_o\left[\frac{1}{g\rho}+\cos 2\theta\left(\frac{c_1}{g\rho}+\frac{c_2}{g\rho^2}+\frac{c_3}{g\rho^3}\right)\right]$$
$$v_\theta=v_o\left[-\frac{1}{2}\sin 2\theta\left(\frac{c_2}{g\rho^2}+\frac{2c_3}{g\rho^3}\right)\right]. \tag{10}$$

## Minimization Procedure

Minimization of the functional (3) will be carried out in conditions of a hydrostatic stress field at infinity ($\sigma_{ij} = p_\infty\delta_{ij}$). In this case the boundary tractions term in (3) becomes $p_\infty\int_B v_\rho dB = 2\pi p_\infty v_o$. This can be demonstrated by noting that the element of length in elliptical coordinates is $dB = g\rho d\theta$ and $\theta$-related terms in (10) result in zero contributions, since integration with respect to $\theta$ is between 0 and $2\pi$. Further, considering that $\dot{\epsilon}_{ef}$ is linear with respect to $v_o$, the functional (3) can be written in the form:

$$D=2\pi\epsilon_o\sigma_o\left[\frac{I}{m+1}\left(\frac{v_o}{\dot{\epsilon}_o}\right)^{m+1}-\frac{p_\infty}{\sigma_o}\left(\frac{v_o}{\dot{\epsilon}_o}\right)\right]$$
$$I=\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}(\bar{\epsilon}_{ef})^{m+1}\rho g d\rho d\theta, \tag{11}$$

where $\bar{\epsilon}_{ef}$ is the effective strain rate computed with strain rates (9) for the velocity field (10) with $v_o$ taken as unity.

The minimum of (11), with respect to $v_o$, can be obtained by equating to zero the derivative of $D$ with respect to $v_o$ to obtain:

$$v_o=\epsilon_o\left(I^{-1}\frac{p_\infty}{\sigma_o}\right)^M. \tag{12}$$

At the point of minimum with respect to $v_o$, (11) becomes $-2\pi\frac{M}{M+1}v_o p_\infty$ with $v_o$ given by the above expression. Further minimization of the functional $D$ with respect to constants $c_1$, $c_2$, $c_3$ is equivalent to minimizing $I$ in the form (11). This can only be done numerically.

It is worth noting that for a circular opening ($e = 0$, $c_1 = c_2 = c_3 = 0$), $I$ in the form (11) can be calculated analytically to obtain $M(2/\sqrt{3})^{1/M}/\sqrt{3}$. If $I$ is normalized by this constant to introduce new $I_o$ that is unity for a circular, the solution (12) can be rewritten as follows:

$$v_o=\frac{\sqrt{3}}{2}\epsilon_o\left(\frac{\sqrt{3}}{MI_o}\frac{p_\infty}{\sigma_o}\right)^M. \tag{13}$$

Expression (14) becomes an exact solution for a closure rate of a circular opening of a unit radius when $e = 0$ and $I_o = 1$. The function $\bar{I}_o(M, e, c_1, c_2, c_3)$ remains close to unity for nonzero ellipse eccentricities and for fixed $M$, $e$ is very insensitive to variation of the remaining arguments $c_1$, $c_2$, $c_3$, typically changing within one percent within a physically reasonable range of values. Because of this feature, minimization of $I_o$ poses considerable numerical difficulties and has been carried out using the conjugate gradient method with linear search based on a parabolic approximation in the search direction and combined with logarithmic gold section and skew testing. The integral in (11) has been evaluated numerically using Gauss-Legendre quadratures after the transformation $z = 1/\rho$ that give finite integration limits for $z$. Figure 3 presents a typical variation of $I_o$ in the vicinity of its minimum (for $e = 1/3$ and $M = 3$).

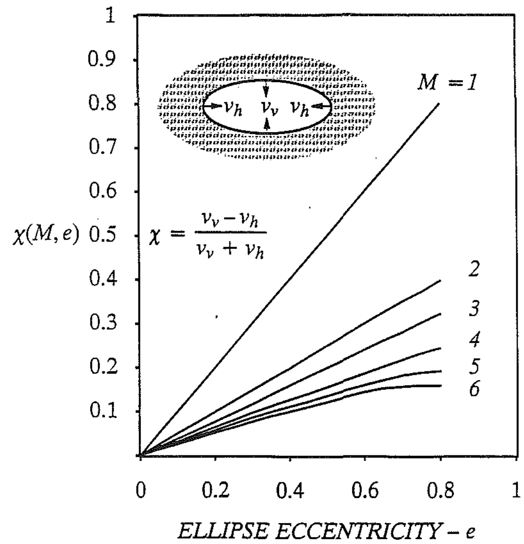


**Fig. 5 Relative difference between vertical and horizontal closure rates for different ellipse eccentricities and power-law exponents**

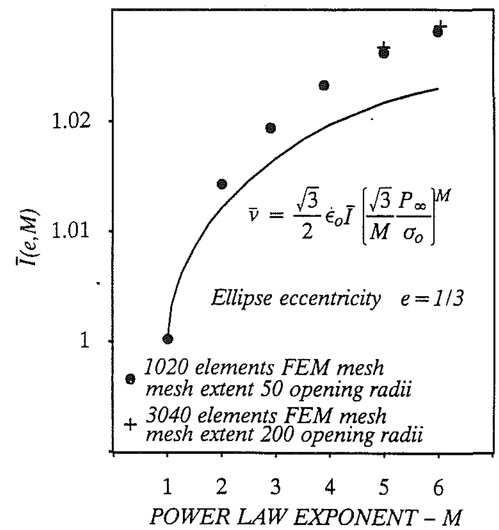


**Fig. 7 Comparison of semi-analytical and FEM solutions for the average closure rate ($e$ = 1/3)**

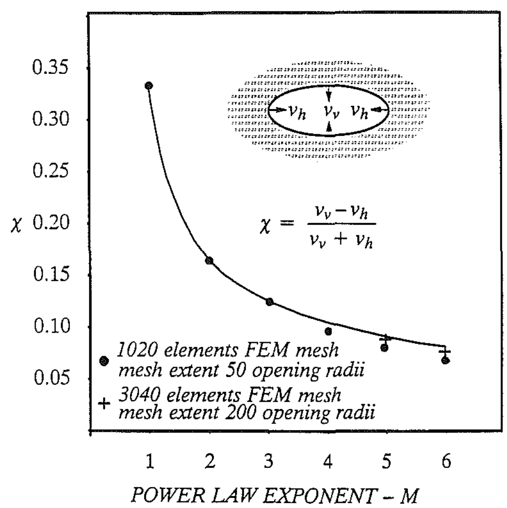


**Fig. 6 Comparison of semi-analytical and FEM solutions for the relative difference between vertical and horizontal closure rates ($e$ = 1/3)**

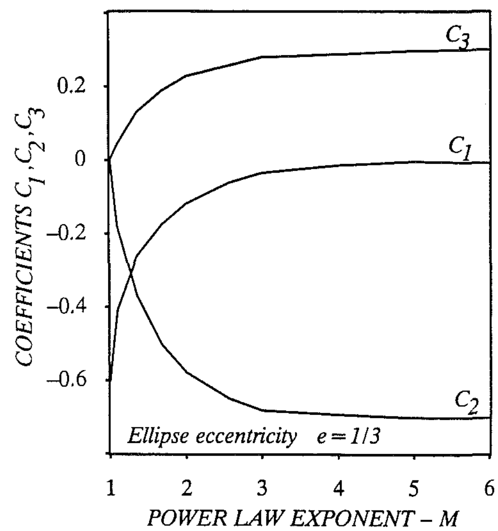


**Fig. 8 Variation of the minimization coefficients with the power-law exponent ($e$ = 1/3)**

## Results

The final result is convenient to present in the form similar to (13), but written for the average of vertical and horizontal closure rates $\bar{v}$, since this characteristic can be easily determined in practical situations of closure measurements. The solution for $\bar{v}$ can be presented in a simple form by redefining $I_o$ in (13) as follows:

$$\bar{v} = \bar{r}\,\frac{\sqrt{3}}{2}\,\dot{\epsilon}_o\bar{I}\left(\frac{\sqrt{3}}{M}\frac{p_\infty}{\sigma_o}\right)^M \tag{14}$$

where the form of $\bar{I}\,(e, M)$ is shown in Fig. 4 for a range of ellipse eccentricities and power-law exponents. Note that (14) is presented for an ellipse of an arbitrary mean radius $\bar{r}$.

In practical situations of in-situ closure measurements, horizontal (wall-to-wall) and vertical (roof-to-floor) closure rates frequently become available. The relative difference between vertical and horizontal closure rates $\chi = (v_v - v_h)/(v_v + v_h)$ is independent on creep parameters $\dot{\epsilon}_o$, $\sigma_o$ and is presented for different ellipse eccentricities and power-law exponents in Fig. 5. The parameter $\chi$ was calculated on the basis of coefficients $c_1$, $c_2$, $c_3$ obtained by minimization. In-situ measurements of $\chi$ can be used for determination of the power-law exponent while $\bar{v}$, calculated according to (14) using $\bar{I}\,(e, M)$ from Fig. 5, can be used for determination of $\dot{\epsilon}_o$.

Figures 6 and 7 illustrate comparison of steady-state FEM solutions with the solution obtained by the described method. It should be mentioned that for high values of $M$, very large FEM meshes are required to achieve an accurate solution. This is because of a slow decay of shear stresses away from the opening (of the order of $1/\rho^{2/M}$. To account for the effect of finite boundaries in FEM calculations, the comparison with FEM runs has been made with the solution of the form:

$$\bar{v} = \bar{r}\,\frac{\sqrt{3}}{2}\,\dot{\epsilon}_o\bar{I}\left(\frac{\sqrt{3}}{M}\frac{p_r}{\sigma_o}\frac{1}{[1-(\bar{r}/r)^{2/M}]}\right)^M \tag{15}$$

where $r$ denotes the position of the boundary where pressure $p_r$ is applied. The above formula is equivalent to (14) for $r = \infty$. For $e = 0$ ($I = 1$), it is identical to the solution for the closure rate of a thick cylinder with inner radius $\bar{r}$ and outer radius $r$. It is not difficult to show that the solution technique presented in the paper leads to the solution of the above form for a problem of a thick elliptical cylinder.

To appreciate the effects of finite boundaries, it is worth giving a numerical example: Presence of a boundary 200 mean ellipse radii away from the opening increases the closure rate by a factor 7.2 compared to the infinite media case for the power-law exponent of 6. The discrepancy between infinite media and finite media solutions is still a factor of 2 when the boundary is 500 mean radii away from the opening.

The values of coefficients $c_1$, $c_2$, $c_3$, controlling the shape of the steady-state velocity field around elliptical openings, are illustrated in Fig. 8. It should be noted that for the power-law exponents greater than 3, the dominant terms in the velocity field are controlled by coefficients $c_2$, $c_3$, while for exponents close to unity $c_1$ is dominant.

## Closing Remarks

The described technique for obtaining steady-state solutions for closure rates of an elliptical opening in nonlinearly viscous materials, like salt rock, is rather general, and can be used for more complex constitutive models for steady-state creep. The accuracy of the method for the power creep law was assessed by comparing solutions obtained by minimizing functional (3) with results of FEM computations.

An interesting qualitative feature of the obtained solution is that mean closure rate of an elliptical opening is very close to a closure of an equivalent circular opening of a mean radius. Also, it is interesting to note that vertical and horizontal closure rates differ little for a practical range of power-law exponents ($M$ = 3 and higher). The higher the exponent, the less is the difference between horizontal and vertical closure rates (Fig. 5).

## Acknowledgments

The research described in this paper was supported by the National Science and Engineering Research Council of Canada.

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