Journal of Computational and Applied Mathematics 138 (2002) 219–241

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Runge–Kutta methods for numerical solution of stochastic differential equations

A. Tocino[a, *, 1], R. Ardanuy[b]

[a] Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced, 1, 37008-Salamanca, Spain

[b] Departamento de Estadística, Universidad de Salamanca, Plaza de los Caídos, 37008-Salamanca, Spain

Received 2 September 2000; received in revised form 22 February 2001

Abstract

The way to obtain deterministic Runge–Kutta methods from Taylor approximations is generalized for stochastic differential equations, now by means of stochastic truncated expansions about a point for sufficiently smooth functions of an Itô process. A class of explicit Runge–Kutta schemes of second order in the weak sense for systems of stochastic differential equations with multiplicative noise is developed. Also two Runge–Kutta schemes of third order have been obtained for scalar equations with constant diffusion coefficients. Numerical examples that compare the proposed schemes to standard ones are presented. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 65U05; 60H10; 34F05

Keywords: Stochastic differential equations; Weak approximation; Runge–Kutta methods; Weak numerical schemes; Explicit schemes

1. Introduction

Stochastic differential equations are becoming increasingly important due to its application for modelling stochastic phenomena in different fields, e.g. physics or economics (see [7,3]). Unfortunately, in many cases analytic solutions of these equations are not available and we are forced to use numerical methods to approximate them. Roughly speaking, there are two basic ways to achieve these approximations. When sample paths of the solutions need to be approximated, mean-square convergence is used and the methods so obtained are called strong. When we are only interested in the moments or other functionals of the solution, which implies a much weaker form of convergence

* Corresponding author.

E-mail addresses: bacon@gugu.usal.es (A. Tocino), raa@gugu.usal.es (R. Ardanuy).

[1] This author was partially supported by CICYT under the project BFM2000-1115.

PII: S0377-0427(01)00380-6

than that needed for pathwise approximations, the methods are called weak. In the present paper we will only refer to weak methods; for this reason words like "scheme", "order" and so on mean "weak scheme", "weak order"...

In previous works various mean-square and weak numerical methods have been derived; extended presentations on this subject are given in Milstein [12] and Kloeden and Platen [7]. Analogously with the deterministic case, stochastic Taylor schemes are obtained by truncating stochastic Taylor expansions. The practical difficulty of employing Taylor approximations is that they require to determine many derivatives. In this investigation we are interested in methods of Runge–Kutta (RK) type, i.e. one step methods which avoid the use of derivatives. RK schemes in the strong sense have been proposed, for instance, by Chang [3], Hernandez and Spigler [4,5], Klauder and Petersen [6], Mauthner [8], McShane [9] and Rümelin [14]; see also Kloeden and Platen [7], Milstein [12] and the references in Saito and Mitsui [15]. On the other hand, Milstein [10–12], Talay [16] and Kloeden and Platen [7] present RK schemes in the weak sense.

In this paper a class of explicit second order and two explicit third order RK schemes are developed. The second order class contains a scheme proposed by Milstein [10]. In analogy with the ordinary case, these RK schemes have been obtained by matching their truncated stochastic expansion about a point with the corresponding Taylor approximation. The required truncated expansions about a point for functions of the solution of a stochastic differential equation have been derived from Itô-Taylor expansions in Tocino and Ardanuy [17].

2. Weak approximations

In this paper we consider a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, an m-dimensional Wiener process $\{W_t = (W_t^1, \dots, W_t^m)\}_{t \geqslant 0}$ and a d-dimensional stochastic differential equation (sde)

$$\mathrm{d}X_t = a(t,X_t)\,\mathrm{d}t + b(t,X_t)\,\mathrm{d}W_t, \quad t_0 \leqslant t \leqslant T, \tag{1}$$

where $a=(a^1,\dots,a^d)$ denotes the d-dimensional drift vector and $b=(b^{ij})$ the $d \times m$-diffusion matrix. Let's denote $b^j = (b^{1j},\dots,b^{dj})$, $j=1,\dots,m$. The functions $a=a(t,x)$ and $b^j = b^j(t,x)$ are assumed to be defined and measurable in $[t_0,T] \times \mathbb{R}^d$ and to satisfy both Lipschitz and linear growth bound conditions in x. These assumptions ensure the existence of a unique solution of the sde (1) with the initial condition $X_{t_0} = X_0$ if X_0 is $\mathscr{F}_{t_0}$-measurable (see [1]). We shall suppose that all of the initial moments $E[|X_0|^r] < \infty$, $r=1,2,\dots$ exist; so, the moments of every X_t will exist (see [1]). Let $X_{t,x}$ denote the solution of (1) starting at time $t \in [t_0,T]$ at $x \in \mathbb{R}^d$.

Let $\mathscr{C}_{\mathrm{P}}$ denote the space of functions $f(t,x)$ defined in $[t_0,T] \times \mathbb{R}^d$ which have polynomial growth (with respect to x) and let $\mathscr{C}_{\mathrm{P}}^{\beta}$ the subspace of functions $f \in \mathscr{C}_{\mathrm{P}}$ for which all partial derivatives up to order $\beta = 1,2,\dots$ belong to $\mathscr{C}_{\mathrm{P}}$.

Next, to Eq. (1) we consider the *one-step approximation*

$$\bar{X}_{t,x}(t+h) = x + A(t,x,h,\xi), \tag{2}$$

where A is some $\mathbb{R}^d$-valued function and ξ a random vector. We shall say that the one-step approximation $\bar{X} = \bar{X}_{t,x}$ *converges weakly to* $X = X_{t,x}$ *with order* $\beta + 1$ if there exists a function $K(x) \in \mathscr{C}_{\mathrm{P}}$

such that

$$\left| E\left[\prod_{j=1}^{l}(\bar{X}^{i_j} - x^{i_j}) - \prod_{j=1}^{l}(X^{i_j} - x^{i_j}) \right] \right| \leqslant K(x)h^{\beta+1}, \quad i_j = 1,\dots,d,\ l = 1,\dots,2\beta+1, \tag{3}$$

where z^i denotes the ith component of the vector z. From (3) it's obvious that the differences between the moments (from the first up to $(2\beta+2)$th inclusively) of the vector X and the corresponding moments of its approximation $\bar{X}$ have $\beta+1$ order of smallness in h. The number $\beta+1$ will be called the *local order* of the approximation.

Let be given an equidistant discretization $\{t_0, t_1, \dots, t_N\}$ of the time interval $[t_0, T]$ with step size $\Delta = (T - t_0)/N$. From the one-step approximation (2) we construct the *discrete approximation* (also called *scheme*):

$$\begin{aligned} &\bar{X}_0 = X_0, \\ &\bar{X}_{n+1} = \bar{X}_n + A(t_n, \bar{X}_n, \Delta, \xi_n), \quad n = 0,\dots,N-1. \end{aligned} \tag{4}$$

We shall say that the discrete approximation $\bar{X} = \{\bar{X}_0, \bar{X}_1, \dots, \bar{X}_N\}$ (based on a step size Δ) *converges weakly to X with order* β if for each $g \in \mathscr{C}_{\mathrm{P}}^{2\beta+2}$ there exists a constant $K_g \geqslant 0$ (not depending on Δ) such that

$$|E[g(\bar{X}_N) - g(X_T)]| \leqslant K_g \Delta^{\beta}.$$

The number β in the above definition is the order of the scheme on an interval. The following theorem (see [11]) establishes the relation between the order of a one-step approximation and the order of the scheme generated by such approximation:

Theorem 1. *Suppose that the coefficients of Eq.* (1) *are continuous, satisfy a Lipschitz condition and a linear growth bound and belong to* $\mathscr{C}_{\mathrm{P}}^{2\beta+2}$. *Suppose that the one-step approximation* (2) *has order* $\beta+1$ *and for each* $0 < h < 1$ *verifies*

$$\begin{aligned} &|E[A(t_k, x, h, \xi_k)]| \leqslant K(1 + |x|)h, \\ &|A(t_k, x, h, \xi_k)| \leqslant M(\xi_k)(1 + |x|)h^{1/2}, \end{aligned} \tag{5}$$

where M *is a function of* $\mathscr{C}_{\mathrm{P}}$ *such that* $M(\xi_k)$ *has moments of all orders. Then the scheme* (4) *has order* β.

Based on the above result one can obtain (see [7] for the details) the *weak Taylor schemes*: next to the sde (1) consider the operators

$$\begin{aligned} &L^{(0)} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} a^i \frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d} c^{ij} \frac{\partial^2}{\partial x^i \partial x^j}, \\ &L^{(k)} = \sum_{i=1}^{d} b^{ik} \frac{\partial}{\partial x^i}, \quad k = 1,\dots,m, \end{aligned} \tag{6}$$

where $c^{ij} = \sum_{k=1}^{m} b^{ik} b^{jk}$, $i, j = 1, \ldots, d$. Given $\beta \in \mathbb{N}$, we denote by Γ_β the set of all multi-indices $\alpha = (j_1, \ldots, j_l)$, $j_k \in \{0, 1, \ldots, m\}$, of length $l \in \{1, \ldots, \beta\}$ and by v the multi-index of length zero. Given a function $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, if we drop the remainder of the Itô–Taylor expansion of $f(t, X_t)$ for the hierarchical set $\Gamma_\beta \cup \{v\}$, we obtain the order β truncated Itô–Taylor expansion

$$f(t, X_t) \simeq f(t_0, X_{t_0}) + \sum_{\alpha \in \Gamma_\beta} (L^\alpha f)(t_0, X_{t_0}) I_\alpha, \tag{7}$$

where, if $\alpha = (j_1, \ldots, j_l)$, we have denoted $L^\alpha = L^{(j_1)} \circ \cdots \circ L^{(j_l)}$, $I_\alpha = \int_t^{t+\Delta} 1 \, \mathrm{d}\alpha = \int_t^{t+\Delta} \int_t^{s_l} \cdots \int_t^{s_2} \mathrm{d}W_{s_1}^{j_1} \cdots \mathrm{d}W_{s_l}^{j_l}$ and $\mathrm{d}W^{(0)} = \mathrm{d}t$. From (7), if $f(t, x) = x$ one obtains the one-step approximation

$$\bar{X}_{t,x}(t + \Delta) = x + \sum_{\alpha \in \Gamma_\beta} (L^\alpha f)(t, x) I_\alpha. \tag{8}$$

It can be shown that if the coefficients a and b are as in Theorem 1 then the one-step approximation (8) verifies that for each $g \in \mathscr{C}_{\mathrm{P}}^{2\beta+2}$ there exist constants $K > 0$ and $r \in \mathbb{N}$ such that

$$|E[g(\bar{X}_{t,x}(t + h)) - g(X_{t,x}(t + h))]| \leqslant K(1 + |x|^{2r}) h^{\beta+1} \tag{9}$$

and hence it has order $\beta + 1$. On the other hand, if the functions $L^\alpha f$, where $f(t, x) = x$ and $\alpha \in \Gamma_\beta$, grow at most linearly with respect to x then one can see that the one-step approximation (8) verifies conditions (5). We can conclude, by Theorem 1, that under the above conditions the scheme generated by (8)

$$\bar{X}_{n+1} = \bar{X}_n + \sum_{\alpha \in \Gamma_\beta} (L^\alpha f)(t_n, \bar{X}_n) I_{\alpha,n} \tag{10}$$

converges weakly with order β. It's called the *order β weak Taylor scheme*.

The order 2 Taylor scheme was first proposed by Milstein [10]. Talay [16] proved it. Milstein [11] also proposed the third order Taylor scheme for systems with additive noises. Weak Taylor schemes of any order were constructed by Platen; see Kloeden and Platen [7].

We shall say that two one-step approximations $\bar{X}_{t,x}$ and $\bar{\bar{X}}_{t,x}$ are *β-equivalent*, $\bar{X}_{t,x}(t+h) \overset{(\beta)}{\simeq} \bar{\bar{X}}_{t,x}(t+h)$, if there exists a function $K(x) \in \mathscr{C}_{\mathrm{P}}$ such that

$$\left| E\left[\prod_{j=1}^{l} (\bar{X}^{i_j} - x^{i_j}) - \prod_{j=1}^{l} (\bar{\bar{X}}^{i_j} - x^{i_j}) \right] \right| \leqslant K(x) h^{\beta+1}, \quad i_j = 1, \ldots, d, \ l = 1, \ldots, 2\beta + 1.$$

It's obvious that if the one-step approximations $\bar{X}_{t,x}$ and $\bar{\bar{X}}_{t,x}$ are β-equivalent then either both or none of them have order $\beta + 1$. For example, by (9), an Itô process (a solution of a sde) and its truncated Itô–Taylor expansion of order β are β-equivalent approximations if the coefficients of the equation are continuous, satisfy both Lipschitz and linear growth conditions and belong to $\mathscr{C}_{\mathrm{P}}^{2\beta+2}$.

Once obtained the one-step Taylor approximation (10), which has local order $\beta + 1$, we can construct schemes of order β by means of β-equivalent approximations to it. An example (see [7]) of this technique is the *simplified order β weak Taylor scheme*

$$\bar{X}_{n+1} = \bar{X}_n + \sum_{\alpha \in \Gamma_\beta} (L^\alpha f)(t_n, \bar{X}_n) \hat{I}_{\alpha,n}, \tag{11}$$

where the variables $\hat{I}_{\alpha,n}$, $\alpha \in \Gamma_\beta$, are such that there is a constant $K > 0$ verifying

$$\left| E\left[\prod_{k=1}^{l} I_{\alpha_k,n} - \prod_{k=1}^{l} \hat{I}_{\alpha_k,n} \right] \right| \leqslant K\Delta^{\beta+1} \tag{12}$$

for all choices of multi-indices $\alpha_k \in \Gamma_\beta - \{v\}$ with $k = 1,\dots,l$ and $l = 1,\dots,2\beta+1$.

If $\beta = 2$, $d = m = 1$ (scalar case) and $\Delta\hat{W}_n$ is any variable satisfying the moment conditions

$$\begin{aligned}&|E[\Delta\hat{W}_n]| + |E[(\Delta\hat{W}_n)^2] - \Delta| + |E[(\Delta\hat{W}_n)^3]|\\ &\quad + |E[(\Delta\hat{W}_n)^4] - 3\Delta^2| + |E[(\Delta\hat{W}_n)^5]| \leqslant K\Delta^3\end{aligned} \tag{13}$$

for some constant $K > 0$, it's easy to prove that the variables $\hat{I}_{(0),n} = I_{(0),n} = \Delta$, $\hat{I}_{(0,0),n} = I_{(0,0),n} = \Delta^2/2$, $\hat{I}_{(1),n} = \Delta\hat{W}_n$, $\hat{I}_{(0,1),n} = \hat{I}_{(1,0),n} = \frac{1}{2}\Delta\Delta\hat{W}_n$, $\hat{I}_{(1,1),n} = \frac{1}{2}((\Delta\hat{W}_n)^2 - \Delta)$ satisfy (12). Then we have the *simplified order 2 Taylor scheme*

$$\begin{aligned}\bar{X}_{n+1} = {}& \bar{X}_n + b\Delta\hat{W}_n + a\Delta + \tfrac{1}{2}bb_{01}((\Delta\hat{W}_n)^2 - \Delta)\\ &+ \tfrac{1}{2}(b_{10} + ab_{01} + \tfrac{1}{2}b^2b_{02} + ba_{01})\Delta\Delta\hat{W}_n + \tfrac{1}{2}(a_{10} + aa_{01} + \tfrac{1}{2}b^2a_{02})\Delta^2,\end{aligned} \tag{14}$$

where for a function $g = g(t,x)$ with $t,x \in \mathbb{R}$ we have denoted

$$g_{ij} = \frac{\partial^{i+j}g}{\partial t^i \partial x^j}(t_n,\bar{X}_n) \tag{15}$$

and $g = g_{00} = g(t_n,\bar{X}_n)$.

Notice that a Gaussian random variable $\Delta W_n \sim \mathrm{N}(0,\Delta)$ or a three-point distributed random variable $\Delta\hat{W}_n$ with $\mathrm{P}(\Delta\hat{W}_n = \sqrt{3\Delta}) = \mathrm{P}(\Delta\hat{W}_n = -\sqrt{3\Delta}) = \frac{1}{6}$, $\mathrm{P}(\Delta\hat{W}_n = 0) = \frac{2}{3}$ satisfy the moment conditions (13).

In the multi-dimensional case, the kth component of the *simplified order 2 Taylor scheme* is given by

$$\begin{aligned}\bar{X}^k_{n+1} = {}& \bar{X}^k_n + \sum_{j=1}^{m} b^{kj}\Delta\hat{W}^j_n + a^k\Delta + \frac{1}{2}\sum_{i,j=1}^{m}\left(\sum_{l=1}^{d} b^{li}\frac{\partial b^{kj}}{\partial x^l}\right)(\Delta\hat{W}^i_n\Delta\hat{W}^j_n + V_{ij,n})\\ &+ \frac{1}{2}\sum_{j=1}^{m}\left(\sum_{i=1}^{d} b^{ij}\frac{\partial a^k}{\partial x^i} + \frac{\partial b^{kj}}{\partial t} + \sum_{i=1}^{d} a^i\frac{\partial b^{kj}}{\partial x^i} + \frac{1}{2}\sum_{i,l=1}^{d} c^{il}\frac{\partial^2 b^{kj}}{\partial x^i\partial x^l}\right)\Delta\Delta\hat{W}^j_n\\ &+ \frac{1}{2}\left(\frac{\partial a^k}{\partial t} + \sum_{i=1}^{d} a^i\frac{\partial a^k}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d} c^{ij}\frac{\partial^2 a^k}{\partial x^i\partial x^j}\right)\Delta^2,\end{aligned} \tag{16}$$

where $\Delta\hat{W}^1_n,\dots,\Delta\hat{W}^m_n$ are independent random variables satisfying the moment conditions (13) and $V_{ij,n}$, $i,j = 1,\dots,m$, are independent two-point distributed random variables with

$$\begin{aligned}&\mathrm{P}(V_{ij,n} = \Delta) = \tfrac{1}{2} = \mathrm{P}(V_{ij,n} = -\Delta) \quad \text{if } j < i,\\ &V_{ii,n} = -\Delta,\\ &V_{ij,n} = -V_{ji,n} \quad \text{if } j > i.\end{aligned} \tag{17}$$

If $\beta=3$, $d=m=1$ and the diffusion coefficient b is a constant, from the third order weak Taylor scheme one can construct the *simplified order 3 weak Taylor scheme*, given by

$$\begin{aligned}\bar{X}_{n+1} &= \bar{X}_n + b\Delta\hat{W}_n + a\Delta + ba_{01}\Delta\hat{Z}_n + \tfrac{1}{2}(a_{10} + aa_{01} + \tfrac{1}{6}b^2a_{02})\Delta^2 \\ &\quad + \tfrac{1}{6}(ba_{01}^2 + 2ba_{11} + 2aba_{02} + b^3a_{03})\Delta^2\Delta\hat{W}_n + \tfrac{1}{6}b^2a_{02}\Delta(\Delta\hat{W}_n)^2 + \tfrac{1}{6}a_{10}a_{01}\Delta^3 \\ &\quad + \tfrac{1}{6}(aa_{01}^2 + \tfrac{3}{2}b^2a_{01}a_{02} + a_{20} + 2aa_{11} + b^2a_{12} + ab^2a_{03} + a^2a_{02} + \tfrac{1}{4}b^4a_{04})\Delta^3, \end{aligned} \tag{18}$$

where $\Delta\hat{W}_n$ and $\Delta\hat{Z}_n$ are correlated Gaussian random variables with

$$\Delta\hat{W}_n \sim \mathrm{N}(0,\Delta), \quad \Delta\hat{Z}_n \sim \mathrm{N}(0,\Delta^3/3), \quad E[\Delta\hat{W}_n\Delta\hat{Z}_n] = \Delta^2/2. \tag{19}$$

In the sequel, for simplicity of notation we shall often abbreviate $I_{\alpha,n}$, $\Delta\hat{W}_n$, $\Delta\hat{Z}_n$, etc. to I_α, $\Delta\hat{W}$ and $\Delta\hat{Z}$, respectively. Obviously, we can obtain β-equivalent schemes to the order β scheme given in (11) by replacing the $\hat{I}_\alpha$'s by new variables $\tilde{I}_\alpha$'s satisfying

$$\left| E\left[\prod_{k=1}^{l}\hat{I}_{\alpha_k} - \prod_{k=1}^{l}\tilde{I}_{\alpha_k}\right]\right| \leqslant K\Delta^{\beta+1}, \quad \alpha_k \in \Gamma_\beta - \{v\},\ l = 1,\ldots,2\beta+1 \tag{20}$$

for some constant $K > 0$. For example, the new family can contain all the variables of the old one except one of them, say $\hat{I}$, replaced by $\tilde{I}$; if (20) holds (in this case it reduces to compare the products which contain $\hat{I}$ with the corresponding with $\tilde{I}$) we shall write $\hat{I}\overset{(\beta)}{\simeq}\tilde{I}$. In general, we shall say that $\hat{I}$ and $\tilde{I}$ are β-equivalent, in symbols $\hat{I}\overset{(\beta)}{\simeq}\tilde{I}$, if by replacing in an approximation the variable $\hat{I}$ by $\tilde{I}$ the new approximation is β-equivalent to the old one. For example, in the scalar case, if $\Delta\hat{W}$ is as in (13) and the function $A(t,x,\Delta,\Delta\hat{W})$ of the one-step approximation (2) is a linear combination of products of the form $\Delta^i(\Delta\hat{W})^j$, $i,j = 0,1,\ldots$ (note that the simplified Taylor approximation (14) belongs to this class), by (20) it's clear that

$$\begin{aligned} &(\Delta\hat{W})^3 \overset{(2)}{\simeq} 3\Delta\Delta\hat{W}, \\ &\Delta(\Delta\hat{W})^2 \overset{(2)}{\simeq} \Delta^2, \\ &(\Delta\hat{W})^4 \overset{(2)}{\simeq} 3\Delta^2. \end{aligned} \tag{21}$$

Each variable $\Delta^i(\Delta\hat{W})^j$ with mean-square order 5/2 (i.e. $i + j/2 = \frac{5}{2}$) is 2-equivalent to zero because it has zero mean and its product with every variable of the family has at least mean-square order 3. Obviously, the variables $\Delta^i(\Delta\hat{W})^j$ with mean-square order $3, \frac{7}{2}, 4, \ldots$ are 2-equivalent to zero:

$$\Delta^i(\Delta\hat{W})^j \overset{(2)}{\simeq} 0 \quad \text{if } i + j/2 \geqslant \tfrac{5}{2}. \tag{22}$$

In the multi-dimensional case, when $\Delta\hat{W}^1,\ldots,\Delta\hat{W}^m$ are independent random variables satisfying the moment conditions (13) and the function $A(t,x,\Delta,\Delta\hat{W})$ is a linear combination of products of the form $\Delta^i(\Delta\hat{W}^1)^{j_1}\cdots(\Delta\hat{W}^m)^{j_m}$ and variables V_{ij} satisfying (17), we have the following 2-equivalences:

$$\Delta\Delta\hat{W}^i\Delta\hat{W}^j \overset{(2)}{\simeq} \begin{cases} \Delta^2 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{23}$$

$$\Delta\hat{W}^i\Delta\hat{W}^j\Delta\hat{W}^k \overset{(2)}{\simeq} \begin{cases} 3\Delta\Delta\hat{W}^i & \text{if } i=j=k, \\ \Delta\Delta\hat{W}^i & \text{if } j=k\neq i, \\ 0 & \text{if } i\neq j,\ i\neq k,\ j\neq k, \end{cases} \tag{24}$$

$$\Delta^i(\Delta\hat{W}^1)^{j_1}\cdots(\Delta\hat{W}^m)^{j_m} \overset{(2)}{\simeq} 0 \quad \text{if } i+\frac{j_1+\cdots+j_m}{2} \geqslant \frac{5}{2}. \tag{25}$$

Similarly, in the scalar case, if $\Delta\hat{W}$ and $\Delta\hat{Z}$ verify (19) and the function $A(t,x,\Delta,\Delta\hat{W})$ which defines the approximation is a linear combination of products $\Delta^i(\Delta\hat{W})^j$, $i,j=0,1,\dots$ and $\Delta\hat{Z}$, we have the following 3-equivalences:

$$\Delta(\Delta\hat{W})^3 \overset{(3)}{\simeq} 3\Delta^2\Delta\hat{W}; \quad \Delta^2(\Delta\hat{W})^2 \overset{(3)}{\simeq} \Delta^3; \quad \Delta^i(\Delta\hat{W})^j \overset{(3)}{\simeq} 0 \quad \text{if } i+j/2 \geqslant \tfrac{7}{2}. \tag{26}$$

3. Runge–Kutta schemes

An important disadvantage of simplified Taylor schemes is that they require to determine many derivatives. Using the idea of the deterministic case we shall obtain RK schemes by replacing the derivatives in simplified Taylor schemes by new evaluations of the coefficients of the equation. An *explicit s-stage stochastic Runge–Kutta scheme* will be given by

$$\bar{X}_{n+1} = \bar{X}_n + \Delta\sum_{j=1}^{s}\alpha_j a(t_n+\mu_j\Delta,\eta_j) + \sum_{k=1}^{m}\Delta\hat{W}_n^k\sum_{j=1}^{s}\beta_j^k b^k(t_n+\mu_j\Delta,\eta_j) + R, \tag{27}$$

where $\mu_1=0$, $\eta_1=X_n$,

$$\eta_j = \bar{X}_n + \Delta\sum_{i=1}^{j-1}\lambda_{ji}a(t_n+\mu_i\Delta,\eta_i) + \sum_{k=1}^{m}\Delta\hat{W}_n^k\sum_{i=1}^{j-1}\gamma_{ji}^k b^k(t_n+\mu_i\Delta,\eta_i), \quad j=1,\dots,s,$$

and R is a fit term. The numerical constants $\alpha_j,\beta_j^k,\mu_j,\lambda_{ij},\gamma_{ij}^k$ and the term R must be chosen so that the approximation (27) is β-equivalent to the simplified order β Taylor scheme. Since the truncated expansion of order β of a process is, under appropriate conditions, β-equivalent to the process, it suffices to choose the parameters and R so that a β-equivalent approximation to the truncated expansion of order β of (27) is equal to the simplified order β Taylor scheme. Note that the fit term is free and then it's obvious that for every family of parameters it can be chosen so that the required equality is fulfilled. But our goal is to avoid the use of derivatives in the scheme. Then, for efficiency, the number of derivatives in R must be notoriously smaller than in the Taylor scheme. If for a family of parameters we have the equality with $R=0$, the scheme does not involve any derivative; it's then a Runge–Kutta scheme in strict sense. If R contains one or more derivatives we shall say that it's a Runge–Kutta type scheme. For example, the second order scheme in Milstein [12, pp. 116–117], is a RK type scheme with $s=4$ which contains one derivative.

Generalizing Butcher arrays, the coefficients occurring in (27) can be displayed

$$
\begin{array}{c|cccc|cccc|c|cccc}
\mu_2 & \lambda_{21} & & & & \gamma^1_{21} & & & & & \gamma^m_{21} & & & \\
\vdots & \vdots & \ddots & & & \vdots & \ddots & & & \vdots & \vdots & \ddots & & \\
\mu_s & \lambda_{s1} & \cdots & \lambda_{s,s-1} & & \gamma^1_{s1} & \cdots & \gamma^1_{s,s-1} & & & \gamma^m_{s1} & \cdots & \gamma^m_{s,s-1} & \\
\hline
\mathrm{R} & \alpha_1 & \cdots & \alpha_{s-1} & \alpha_s & \beta^1_1 & \cdots & \beta^1_{s-1} & \beta^1_s & \cdots & \beta^m_1 & \cdots & \beta^m_{s-1} & \beta^m_s
\end{array}
$$

where the first matrix contains the coefficients corresponding to the deterministic part and each of the remaining ones contains the coefficients corresponding to the stochastic part with respect to a Wiener component.

As in the deterministic case, in order to match the truncated expansion of (27) with the simplified Taylor scheme we need an expression of the order β truncated expansion of a process $f(t+\Delta, X_t+\Delta X)$ in terms of Δ and $\Delta X = X_{t+\Delta} - X_t$. This expression has been obtained by Tocino and Ardanuy [17] for $\beta = 2$ in the multi-dimensional case and for $\beta = 3$ in the scalar case. So, the formula

$$
\begin{aligned}
f(t+\Delta, X_t+\Delta X) \overset{(2)}{\simeq}{} & f + \frac{\partial f}{\partial t}\Delta + \sum_{i=1}^{d}\frac{\partial f}{\partial x^i}\Delta X^i + \left\{ \frac{\partial^2 f}{\partial t^2} + \sum_{i,j=1}^{d} c^{ij}\frac{\partial^3 f}{\partial t \partial x^i \partial x^j} \right. \\
& \left. + \frac{1}{2}\sum_{i,j,k=1}^{d}\left(\sum_{l=1}^{d} c^{kl}\frac{\partial c^{ij}}{\partial x^l}\right)\frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} + \frac{1}{4}\sum_{i,j,k,l=1}^{d} c^{ij}c^{kl}\frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^l} \right\}\frac{\Delta^2}{2} \\
& + \sum_{i=1}^{d}\left(\frac{\partial^2 f}{\partial t \partial x^i} + \frac{1}{2}\sum_{j,k=1}^{d} c^{jk}\frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}\right)\Delta \Delta X^i + \sum_{i,j=1}^{d}\frac{\partial^2 f}{\partial x^i \partial x^j}\frac{\Delta X^i \Delta X^j}{2},
\end{aligned}
\tag{28}
$$

where all of the functions are evaluated at (t, X_t), expresses the 2-equivalence between the process and its second order truncated expansion.

In the scalar case, $d = m = 1$, using notation (15), (28) reduces to

$$
\begin{aligned}
f(t+\Delta, X_t+\Delta X) \overset{(2)}{\simeq}{} & f_{00} + f_{10}\Delta + f_{01}\Delta X \\
& + \left(f_{20} + b^2 f_{12} + b^3 b_{01} f_{03} + \frac{b^4}{4} f_{04}\right)\frac{\Delta^2}{2} \\
& + \left(f_{11} + \frac{b^2}{2} f_{03}\right)\Delta \Delta X + f_{02}\frac{(\Delta X)^2}{2}.
\end{aligned}
\tag{29}
$$

Similarly, in the scalar case, if $b(t,x) = b$ is a constant, the formula

$$
\begin{aligned}
f(t+\Delta, X_t+\Delta X) \overset{(3)}{\simeq}{} & f_{00} + f_{10}\Delta + f_{01}\Delta X + \left(f_{20} - \frac{b^4}{4} f_{04}\right)\frac{\Delta^2}{2} + f_{11}\Delta \Delta X \\
& + f_{02}\frac{(\Delta X)^2}{2} + \left(f_{30} + \frac{3}{2}b^2 f_{22} + \frac{3}{4}b^4 f_{14} + \frac{1}{8}b^6 f_{06}\right)\frac{\Delta^3}{6}
\end{aligned}
$$

$$+\left(f_{21}+b^2 f_{13}+\frac{b^4}{4}f_{05}\right)\frac{\Delta^2\Delta X}{2}$$
$$+\left(f_{12}+\frac{b^2}{2}f_{04}\right)\frac{\Delta(\Delta X)^2}{2}+f_{03}\frac{(\Delta X)^3}{6}. \tag{30}$$

shows the 3-equivalence between the process and its third order truncated expansion.

4. Two-stage Runge–Kutta schemes

In the scalar case, for $s=2$ the RK scheme (27) takes the form

$$\bar{X}_{n+1}=\bar{X}_n+\{\alpha_1 a+\alpha_2 a(t_n+\mu\Delta,\eta)\}\Delta+\{\beta_1 b+\beta_2 b(t_n+\mu\Delta,\eta)\}\Delta\hat{W}+R, \tag{31}$$

where $\eta=\bar{X}_n+\lambda a\Delta+\gamma b\Delta\hat{W}$; for simplicity, in the above expression and from now on, when a function in a scheme is evaluated at $(t_n,\bar{X}_n)$ we shall omit such point; notice that we have also abbreviated $\Delta\hat{W}_n$ to $\Delta\hat{W}$. The coefficients a and b are supposed to belong to $\mathscr{C}_{\mathrm{P}}^6$. By (29) and the equivalences (21) and (22) we shall obtain a 2-equivalent approximation to (31) and we shall match this approximation with the simplified order 2 Taylor scheme (14).

By (29) we get

$$\begin{aligned}a(t_n+\mu\Delta,\bar{X}_n+\lambda a\Delta+\gamma b\Delta\hat{W})&\overset{(2)}{\simeq} a+a_{10}\mu\Delta+a_{01}(\lambda a\Delta+\gamma b\Delta\hat{W})\\&\quad+\frac{1}{2}\left(a_{20}+b^2a_{12}+b^3b_{01}a_{03}+\frac{b^4}{4}a_{04}\right)\mu^2\Delta^2\\&\quad+\left(a_{11}+\frac{b^2}{2}a_{03}\right)\mu\Delta(\lambda a\Delta+\gamma b\Delta\hat{W})\\&\quad+\frac{1}{2}a_{02}(\lambda a\Delta+\gamma b\Delta\hat{W})^2\end{aligned}$$

and then, using (21) and (22), we have

$$\begin{aligned}a(t_n+\mu\Delta,\bar{X}_n+\lambda a\Delta+\gamma b\Delta\hat{W})\Delta&\overset{(2)}{\simeq} a_{01}b\gamma\Delta\Delta\hat{W}+a\Delta\\&\quad+a_{10}\mu\Delta^2+aa_{01}\lambda\Delta^2+\tfrac{1}{2}a_{02}b^2\gamma^2\Delta^2.\end{aligned} \tag{32}$$

Similarly, we get

$$\begin{aligned}b(t_n+\mu\Delta,\bar{X}_n+\lambda a\Delta+\gamma b\Delta\hat{W})\Delta\hat{W}&\overset{(2)}{\simeq} b\Delta\hat{W}+bb_{01}\gamma(\Delta\hat{W})^2+b_{10}\mu\Delta\Delta\hat{W}\\&\quad+ab_{01}\lambda\Delta\Delta\hat{W}+\tfrac{3}{2}b^2b_{02}\gamma^2\Delta\Delta\hat{W}\\&\quad+b(b_{11}+\tfrac{1}{2}b^2b_{03})\mu\gamma\Delta^2+abb_{02}\lambda\gamma\Delta^2.\end{aligned} \tag{33}$$

From (32) and (33) we see that approximation (31) is 2-equivalent to

$$\begin{aligned}\bar{X}_{n+1} = \bar{X}_n &+ (\beta_1+\beta_2)b\Delta\hat{W} + (\alpha_1+\alpha_2)a\Delta + \beta_2\gamma bb_{01}(\Delta\hat{W})^2 \\ &+ (\alpha_2\gamma a_{01}b + \beta_2\mu b_{10} + \beta_2\lambda ab_{01} + \tfrac{3}{2}\beta_2\gamma^2 b^2 b_{02})\Delta\Delta\hat{W} + \alpha_2\mu a_{10}\Delta^2 \\ &+ (\alpha_2\lambda aa_{01} + \tfrac{1}{2}\alpha_2\gamma^2 a_{02}b^2 + \beta_2\mu\gamma b(b_{11} + \tfrac{1}{2}b^2 b_{03}) + \beta_2\lambda\gamma abb_{02})\Delta^2 + R. \qquad (34)\end{aligned}$$

Now, we compare the above approximation with the simplified order 2 Taylor approximation (14).

Firstly, let's suppose that $\partial b/\partial x = k$ is a constant. Since in this case $b_{11} = b_{02} = b_{03} = 0$, (34) and (14) coincide if the constants verify

$$\begin{array}{lll} \alpha_1+\alpha_2 = 1, & \beta_1+\beta_2 = 1, & \gamma\alpha_2 = \tfrac{1}{2}, \\ \mu\alpha_2 = \tfrac{1}{2}, & \mu\beta_2 = \tfrac{1}{2}, & \gamma^2\alpha_2 = \tfrac{1}{2}, \\ \lambda\alpha_2 = \tfrac{1}{2}, & \gamma\beta_2 = \tfrac{1}{2}, & \lambda\beta_2 = \tfrac{1}{2}, \end{array}$$

and $R = -\frac{1}{2}bb_{01}\Delta$. Notice that the first column contains the equations which appear in the deterministic case ($b \equiv 0$) and the second one contains the analogues for the stochastic part of the scheme. The equations in the third column contain both deterministic and stochactic parameters. The above system has the unique solution $\lambda = \mu = \gamma = 1$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1/2$, which leads to the scheme

$$\begin{aligned}\bar{X}_{n+1} = \bar{X}_n &+ \tfrac{1}{2}b\Delta\hat{W}_n + \tfrac{1}{2}b(t_n+\Delta, \bar{X}_n + a\Delta + b\Delta\hat{W}_n)\Delta\hat{W}_n \\ &+ \tfrac{1}{2}a\Delta + \tfrac{1}{2}a(t_n+\Delta, \bar{X}_n + a\Delta + b\Delta\hat{W}_n)\Delta - \frac{1}{2}b\frac{\partial b}{\partial x}\Delta. \qquad (35)\end{aligned}$$

By construction, this scheme is 2-equivalent to the simplified order 2 Taylor scheme. On the other hand, since b is continuous and we have supposed that $\partial b/\partial x$ is a constant, we have that b is a Lipschitz function which grows at most linearly and therefore it's immediate to show that if a also grows at most linearly then approximation (35) verifies conditions (5). By Theorem 1, we deduce the following.

Theorem 2. *Suppose that the coefficients of Eq.* (1) *belong to $\mathscr{C}_P^6$, that the drift a verifies both Lipschitz and linear growth conditions in x and that $\partial b/\partial x$ is a constant. Then the scheme given in* (35) *is of second order in the weak sense.*

Since $\partial b/\partial x$ shall be a known constant, at each step, in scheme (35) we have to evaluate the drift a at two points, the diffusion coefficient b at two points, as well as generating the random variable $\Delta\hat{W}_n$; remember that in the Taylor scheme we have to evaluate a, b, $\partial b/\partial t$, $\partial a/\partial x$, $\partial a/\partial t$ and $\partial^2 a/\partial x^2$ at one point and to generate $\Delta\hat{W}_n$. Notice also that this scheme coincides (in the analysed case) with the second order RK scheme proposed by Milstein [10].

The Butcher array of the scheme would be

$$\begin{array}{r|ll|ll} 1 & 1 & & 1 & \\ \hline -\frac{1}{2}bb_{01}\Delta & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

To match (34) with (14) in a more general case we can take the same parameters, but R must change because new partial derivatives could appear. For example, if $\partial^2 b/\partial x^2 = 0$ then (14) contains the same terms that in the above case but in (34) it appears the term $\beta_2\mu\gamma bb_{11}\Delta^2 = \frac{1}{2}bb_{11}\Delta^2$; so we must take $R = -\frac{1}{2}b(b_{01} + b_{11}\Delta)\Delta$. Note that now the scheme contains two derivatives. And it's clear that in more general cases we can continue in this way (i.e. with the same parameters and by increasing the number of derivatives in R) to derive second order schemes of the form (31).

Notice that we have also derived that only if $\partial b/\partial x$ is a constant there exists a second order Runge–Kutta scheme in strict sense as (27) with $s = 2$.

5. A class of second order schemes

In view of the above results we must take $s > 2$ in (27) in order to obtain for the general case a second order RK scheme that does not include most of the derivatives participating in (14). So, let us consider schemes of the form

$$\begin{aligned} \bar{X}_{n+1} = \bar{X}_n &+ \{\alpha_1 a + \alpha_2 a(t_n + \mu\Delta, \eta)\}\Delta \\ &+ \{\beta_1 b + \beta_2 b(t_n + \mu\Delta, \bar{\eta}) + \beta_3 b(t_n + \mu\Delta, \bar{\bar{\eta}})\}\Delta\hat{W} + R, \end{aligned} \tag{36}$$

where

$$\begin{aligned} \eta &= \bar{X}_n + \lambda a\Delta + \gamma b\Delta\hat{W}, \\ \bar{\eta} &= \bar{X}_n + \bar{\lambda} a\Delta + \bar{\gamma} b\Delta\hat{W}, \\ \bar{\bar{\eta}} &= \bar{X}_n + \bar{\bar{\lambda}} a\Delta + \bar{\bar{\gamma}} b\Delta\hat{W}. \end{aligned}$$

Using (29), (21) and (22) analogously as in the above section, we shall obtain that the approximation (36) is 2-equivalent to

$$\begin{aligned} \bar{X}_{n+1} = \bar{X}_n &+ (\beta_1 + \beta_2 + \beta_3) b\Delta\hat{W} + (\alpha_1 + \alpha_2) a\Delta + (\beta_2\bar{\gamma} + \beta_3\bar{\bar{\gamma}}) bb_{01}(\Delta\hat{W})^2 \\ &+ \alpha_2\gamma a_{01} b\Delta\Delta\hat{W} + (\beta_2 + \beta_3)\mu b_{10}\Delta\Delta\hat{W} + (\beta_2\bar{\lambda} + \beta_3\bar{\bar{\lambda}}) ab_{01}\Delta\Delta\hat{W} \\ &+ \tfrac{3}{2}(\beta_2\bar{\gamma}^2 + \beta_3\bar{\bar{\gamma}}^2) b^2 b_{02}\Delta\Delta\hat{W} + \alpha_2\mu a_{10}\Delta^2 + \alpha_2\lambda a a_{01}\Delta^2 + \tfrac{1}{2}\alpha_2\gamma^2 a_{02} b^2\Delta^2 \\ &+ \mu(\beta_2\bar{\gamma} + \beta_3\bar{\bar{\gamma}}) b(b_{11} + \tfrac{1}{2}b^2 b_{03})\Delta^2 + (\beta_2\bar{\lambda}\bar{\gamma} + \beta_3\bar{\bar{\lambda}}\bar{\bar{\gamma}}) abb_{02}\Delta^2 + R. \end{aligned} \tag{37}$$

This approximation will be equal to the simplified second order Taylor approximation (14) if the parameters satisfy the nonlinear system

$$\begin{array}{ll} & \beta_1 + \beta_2 + \beta_3 = 1, \\ \alpha_1 + \alpha_2 = 1, & (\beta_2 + \beta_3)\mu = \frac{1}{2}, \\ \alpha_2\mu = \frac{1}{2}, & \beta_2\bar{\lambda} + \beta_3\bar{\bar{\lambda}} = \frac{1}{2}, \\ \alpha_2\lambda = \frac{1}{2}, & \mu(\beta_2\bar{\gamma} + \beta_3\bar{\bar{\gamma}}) = 0, \\ \alpha_2\gamma = \frac{1}{2}, & \beta_2\bar{\gamma}^2 + \beta_3\bar{\bar{\gamma}}^2 = \frac{1}{6}, \\ \alpha_2\gamma^2 = \frac{1}{2}, & \beta_2\bar{\lambda}\bar{\gamma} + \beta_3\bar{\bar{\lambda}}\bar{\bar{\gamma}} = 0 \end{array} \tag{38}$$

and if $R = \frac{1}{2}bb_{01}((\Delta\hat{W})^2 - \Delta)$. The left column system has the unique solution

$$\alpha_1 = \alpha_2 = \tfrac{1}{2}, \quad \gamma = \lambda = \mu = 1. \tag{39}$$

On substituting (39) on the right column of (38), it's easy to see that the system obtained has the one-parameter solution

$$\beta_1 = \frac{1}{2}, \quad \beta_2 = \frac{1}{2+6\bar{\gamma}^2}, \quad \beta_3 = \frac{3\bar{\gamma}^2}{2+6\bar{\gamma}^2}, \quad \bar{\lambda} = \bar{\bar{\lambda}} = 1, \quad \bar{\bar{\gamma}} = \frac{-1}{3\bar{\gamma}}, \quad \bar{\gamma} \neq 0. \tag{40}$$

Then, taking $R = \frac{1}{2}bb_{01}((\Delta\hat{W})^2 - \Delta)$, for each $\bar{\gamma} \neq 0$, (39) and (40) give a scheme

$$\begin{aligned}\bar{X}_{n+1} = {} & \bar{X}_n + \frac{1}{2}b\Delta\hat{W}_n + \frac{1}{2+6\bar{\gamma}^2}b(t_n + \Delta, \bar{X}_n + a\Delta + \bar{\gamma}b\Delta\hat{W}_n)\Delta\hat{W}_n \\ & + \frac{3\bar{\gamma}^2}{2+6\bar{\gamma}^2}b\left(t_n + \Delta, \bar{X}_n + a\Delta - \frac{1}{3\bar{\gamma}}b\Delta\hat{W}_n\right)\Delta\hat{W}_n \\ & + \frac{1}{2}a\Delta + \frac{1}{2}a(t_n + \Delta, \bar{X}_n + a\Delta + b\Delta\hat{W}_n)\Delta + \frac{1}{2}b\frac{\partial b}{\partial x}((\Delta\hat{W}_n)^2 - \Delta),\end{aligned} \tag{41}$$

which, by construction, is 2-equivalent to the order 2 Taylor scheme. On the other hand, it's easy to show that it verifies (5) if a, b and $\partial b/\partial x$ have at most linear growth, and so, by Theorem 1, we have proved the following.

Theorem 3. *Suppose that the coefficients a, b of Eq.* (1) *satisfy a Lipschitz condition and belong to $\mathscr{C}_P^6$. If a, b and $\partial b/\partial x$ have at most linear growth then the RK schemes of* (41) *are of second order in the weak sense.*

At each step in a scheme of the family (41) we have to evaluate the drift a at two points, the diffusion coefficient b at three points and the function $\partial b/\partial x$ at one point, as well as generating a random variable $\Delta\hat{W}_n$ satisfying (13).

Notice that the second order RK scheme proposed by Milstein [10] belongs to the above class.

6. Multi-dimensional generalization

The construction of a second order schemes family in the multi-dimensional case can be accomplished by the procedure of the previous section. Now we shall consider schemes whose kth component can be written as

$$\begin{aligned}\bar{X}^k_{n+1} = {} & \bar{X}^k_n + \{\alpha_1 a^k + \alpha_2 a^k(t_n + \mu\Delta, \eta)\}\Delta \\ & + \sum_{j=1}^m \{\beta_1^j b^{kj} + \beta_2^j b^{kj}(t_n + \mu\Delta, \bar{\eta}) + \beta_3^j b^{kj}(t_n + \mu\Delta, \bar{\bar{\eta}})\}\Delta\hat{W}^j + R,\end{aligned} \tag{42}$$

where

$$\eta = \bar{X}_n + \lambda a\Delta + \gamma_1 b^1 \Delta\hat{W}^1 + \cdots + \gamma_m b^m \Delta\hat{W}^m,$$

$$\bar{\eta} = \bar{X}_n + \bar{\lambda} a\Delta + \bar{\gamma}_1 b^1 \Delta\hat{W}^1 + \cdots + \bar{\gamma}_m b^m \Delta\hat{W}^m,$$

$$\bar{\bar{\eta}} = \bar{X}_n + \bar{\bar{\lambda}} a\Delta + \bar{\bar{\gamma}}_1 b^1 \Delta\hat{W}^1 + \cdots + \bar{\bar{\gamma}}_m b^m \Delta\hat{W}^m.$$

Here the constants and the fit term R must be chosen so that the above scheme is 2-equivalent to the simplified order 2 Taylor scheme (16).

By (28) and the equivalences (23)–(25) we obtain that

$$\begin{aligned} & a^k \left(t_n + \mu\Delta, \bar{X}_n + \lambda a\Delta + \sum_{r=1}^{m} \gamma_r b^r \Delta\hat{W}^r \right) \Delta \\ & \overset{(2)}{\simeq} a^k \Delta + \sum_{i=1}^{d} \sum_{j=1}^{m} \frac{\partial a^k}{\partial x^i} b^{ij} \gamma_j \Delta\Delta\hat{W}^j + \frac{\partial a^k}{\partial t} \mu\Delta^2 \\ & + \sum_{i=1}^{d} a^i \frac{\partial a^k}{\partial x^i} \lambda\Delta^2 + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{r=1}^{m} \frac{\partial^2 a^k}{\partial x^i \partial x^j} b^{ir} b^{jr} \gamma_r^2 \Delta^2 \end{aligned} \tag{43}$$

and that

$$\begin{aligned} & b^{kj} \left(t_n + \mu\Delta, \bar{X}_n + \bar{\lambda} a\Delta + \sum_{r=1}^{m} \bar{\gamma}_r b^r \Delta\hat{W}^r \right) \Delta\hat{W}^j \\ & \overset{(2)}{\simeq} b^{kj} \Delta\hat{W}^j + \sum_{i=1}^{d} \sum_{r=1}^{m} \frac{\partial b^{kj}}{\partial x^i} b^{ir} \bar{\gamma}_r \Delta\hat{W}^r \Delta\hat{W}^j + \frac{\partial b^{kj}}{\partial t} \mu\Delta\Delta\hat{W}^j \\ & + \sum_{i=1}^{d} \frac{\partial b^{kj}}{\partial x^i} a^i \bar{\lambda} \Delta\Delta\hat{W}^j + \frac{1}{2} \sum_{i,l=1}^{d} \sum_{r,s=1}^{m} \frac{\partial^2 b^{kj}}{\partial x^i \partial x^l} b^{ir} b^{ls} \bar{\gamma}_r \bar{\gamma}_s \Delta\hat{W}^r \Delta\hat{W}^s \Delta\hat{W}^j \\ & + \sum_{i=1}^{d} \left(\frac{\partial^2 b^{kj}}{\partial t \partial x^i} + \frac{1}{2} \sum_{s,l=1}^{d} c^{sl} \frac{\partial^3 b^{kj}}{\partial x^i \partial x^s \partial x^l} \right) b^{ij} \mu \bar{\gamma}_j \Delta^2 + \sum_{i,l=1}^{d} \frac{\partial^2 b^{kj}}{\partial x^i \partial x^l} a^i b^{lj} \bar{\lambda} \bar{\gamma}_j \Delta^2 \end{aligned} \tag{44}$$

if $j = 1, \ldots, m$; besides, a similar equivalence to (44) can be obtained for each $b^{kj}(t_n + \mu\Delta, \bar{\bar{\eta}})\Delta\hat{W}^j$ with $\bar{\lambda}$ and $\bar{\gamma}_r$ replaced by $\bar{\bar{\lambda}}$ and $\bar{\bar{\gamma}}_r$, respectively. Using these last three equivalences we get the 2-equivalent to (42) scheme

$$\begin{aligned} \bar{X}^k_{n+1} = & \bar{X}^k_n + \sum_{j=1}^{m} (\beta^j_1 + \beta^j_2 + \beta^j_3) b^{kj} \Delta\hat{W}^j + (\alpha_1 + \alpha_2) a^k \Delta \\ & + \sum_{i,j=1}^{m} (\beta^j_2 \bar{\gamma}_i + \beta^j_3 \bar{\bar{\gamma}}_i) \sum_{l=1}^{d} \frac{\partial b^{kj}}{\partial x^l} b^{li} \Delta\hat{W}^i \Delta\hat{W}^j + \sum_{j=1}^{m} \alpha_2 \gamma_j \sum_{i=1}^{d} \frac{\partial a^k}{\partial x^i} b^{ij} \Delta\Delta\hat{W}^j \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\mu(\beta_2^j+\beta_3^j)\frac{\partial b^{kj}}{\partial t}\Delta\Delta\hat{W}^j+\sum_{j=1}^{m}\sum_{i=1}^{d}(\beta_2^j\bar{\lambda}+\beta_3^j\bar{\bar{\lambda}})\frac{\partial b^{kj}}{\partial x^i}a^i\Delta\Delta\hat{W}^j\\
&+\frac{1}{2}\sum_{j,r,s=1}^{m}(\beta_2^j\bar{\gamma}_r\bar{\gamma}_s+\beta_3^j\bar{\bar{\gamma}}_r\bar{\bar{\gamma}}_s)\sum_{i,l=1}^{d}\frac{\partial^2 b^{kj}}{\partial x^i\partial x^l}b^{ir}b^{ls}\Delta\hat{W}^r\Delta\hat{W}^s\Delta\hat{W}^j\\
&+\alpha_2\mu\frac{\partial a^k}{\partial t}\Delta^2+\alpha_2\lambda\sum_{i=1}^{d}a^i\frac{\partial a^k}{\partial x^i}\Delta^2+\frac{1}{2}\sum_{i,j=1}^{d}\sum_{r=1}^{m}\alpha_2\gamma_r^2\frac{\partial^2 a^k}{\partial x^i\partial x^j}b^{ir}b^{jr}\Delta^2\\
&+\sum_{j=1}^{m}\mu(\beta_2^j\bar{\gamma}_j+\beta_3^j\bar{\bar{\gamma}}_j)\sum_{i=1}^{d}\left(\frac{\partial^2 b^{kj}}{\partial t\partial x^i}+\frac{1}{2}\sum_{s,l=1}^{d}c^{sl}\frac{\partial^3 b^{kj}}{\partial x^i\partial x^s\partial x^l}\right)b^{ij}\Delta^2\\
&+\sum_{j=1}^{m}(\beta_2^j\bar{\lambda}\bar{\gamma}_j+\beta_3^j\bar{\bar{\lambda}}\bar{\bar{\gamma}}_j)\sum_{i,l=1}^{d}\frac{\partial^2 b^{kj}}{\partial x^i\partial x^l}a^ib^{lj}\Delta^2+R.
\end{aligned}\tag{45}
$$

By (24) we have

$$
\begin{aligned}
&\sum_{j,r,s=1}^{m}(\beta_2^j\bar{\gamma}_r\bar{\gamma}_s+\beta_3^j\bar{\bar{\gamma}}_r\bar{\bar{\gamma}}_s)\sum_{i,l=1}^{d}\frac{\partial^2 b^{kj}}{\partial x^i\partial x^l}b^{ir}b^{ls}\Delta\hat{W}^r\Delta\hat{W}^s\Delta\hat{W}^j\\
&\overset{(2)}{\simeq}3\sum_{j=1}^{m}(\beta_2^j\bar{\gamma}_j^2+\beta_3^j\bar{\bar{\gamma}}_j^2)\sum_{i,l=1}^{d}\frac{\partial^2 b^{kj}}{\partial x^i\partial x^l}b^{ij}b^{lj}\Delta\Delta\hat{W}^j\\
&\quad+\sum_{j=1}^{m}\sum_{\substack{r=1\\ r\neq j}}^{m}(\beta_2^j\bar{\gamma}_r^2+\beta_3^j\bar{\bar{\gamma}}_r^2)\sum_{i,l=1}^{d}\frac{\partial^2 b^{kj}}{\partial x^i\partial x^l}b^{ir}b^{lr}\Delta\Delta\hat{W}^j\\
&\quad+2\sum_{j=1}^{m}\sum_{\substack{r=1\\ r\neq j}}^{m}(\beta_2^r\bar{\gamma}_r\bar{\gamma}_j+\beta_3^r\bar{\bar{\gamma}}_r\bar{\bar{\gamma}}_j)\sum_{i,l=1}^{d}\frac{\partial^2 b^{kr}}{\partial x^i\partial x^l}b^{ij}b^{lr}\Delta\Delta\hat{W}^j;
\end{aligned}
$$

with this substitution, scheme (45) shall be equal to the simplified order 2 Taylor scheme (16) if the parameters verify the systems

$$
\begin{aligned}
&\alpha_1+\alpha_2=1,\\
&\alpha_2\mu=\tfrac{1}{2},\\
&\alpha_2\lambda=\tfrac{1}{2},\\
&\alpha_2\gamma_j=\tfrac{1}{2},\\
&\alpha_2\gamma_j^2=\tfrac{1}{2},\\
&\beta_1^j+\beta_2^j+\beta_3^j=1,
\end{aligned}
$$

$$(\beta_2^j + \beta_3^j)\mu = \tfrac{1}{2},$$

$$\beta_2^j \bar{\lambda} + \beta_3^j \bar{\bar{\lambda}} = \tfrac{1}{2},$$

$$(\beta_2^j \bar{\gamma}_j + \beta_3^j \bar{\bar{\gamma}}_j)\mu = 0,$$

$$\beta_2^j \bar{\gamma}_j^2 + \beta_3^j \bar{\bar{\gamma}}_j^2 = \tfrac{1}{6},$$

$$\beta_2^j \bar{\lambda}\bar{\gamma}_j + \beta_3^j \bar{\bar{\lambda}}\bar{\bar{\gamma}}_j = 0, \quad j = 1,\dots,m \tag{46}$$

and

$$\beta_2^j \bar{\gamma}_i + \beta_3^j \bar{\bar{\gamma}}_i = 0, \quad i,j = 1,\dots,m,\ i \neq j \tag{47}$$

and if

$$\begin{aligned} R = {} & \frac{1}{2}\sum_{i,j=1}^{m}\left(\sum_{l=1}^{d} b^{li}\frac{\partial b^{kj}}{\partial x^l}\right)(\Delta\hat{W}^i\Delta\hat{W}^j + V_{ij}) \\ & + \sum_{j=1}^{m}\sum_{\substack{r=1\\ r\neq j}}^{m}\left(\frac{1}{4} - \frac{1}{2}(\beta_2^j\bar{\gamma}_r^2 + \beta_3^j\bar{\bar{\gamma}}_r^2)\right)\sum_{i,l=1}^{d}\frac{\partial^2 b^{kj}}{\partial x^i\partial x^l}b^{ir}b^{lr}\varDelta\Delta\hat{W}^j \\ & - \sum_{j=1}^{m}\sum_{\substack{r=1\\ r\neq j}}^{m}(\beta_2^r\bar{\gamma}_r\bar{\gamma}_j + \beta_3^r\bar{\bar{\gamma}}_r\bar{\bar{\gamma}}_j)\sum_{i,l=1}^{d}\frac{\partial^2 b^{kr}}{\partial x^i\partial x^l}b^{ij}b^{lr}\varDelta\Delta\hat{W}^j. \end{aligned}$$

For each $j = 1,\dots,m$ system (46) is analogous to (38); so, we can deduce that (46) has the m-parametric family of solutions

$$\alpha_1 = \alpha_2 = \tfrac{1}{2}, \quad \gamma_j = \lambda = \mu = 1,$$

$$\beta_1^j = \tfrac{1}{2}, \quad \beta_2^j = \frac{1}{2 + 6\bar{\gamma}_j^2}, \quad \beta_3^j = \frac{3\bar{\gamma}_j^2}{2 + 6\bar{\gamma}_j^2}, \quad \bar{\lambda} = \bar{\bar{\lambda}} = 1, \quad \bar{\bar{\gamma}}_j = \frac{-1}{3\bar{\gamma}_j},$$

where $\bar{\gamma}_j \neq 0$, $j = 1,\dots,m$. By using the above values with (47) we obtain that $\bar{\gamma}_i^2 = \bar{\gamma}_j^2$, $i,j = 1,\dots,m$; if we denote $\gamma = \bar{\gamma}_1$, we can write $\bar{\gamma}_i = \sigma_i\gamma$, $\bar{\bar{\gamma}}_i = -\sigma_i/3\gamma$, where $\sigma_1 = 1$, $\sigma_i \in \{\pm 1\}$, $i = 2,\dots,m$, and then

$$\beta_2^j = \frac{1}{2 + 6\gamma^2}, \quad \beta_3^j = \frac{3\gamma^2}{2 + 6\gamma^2}, \qquad j = 1,\dots,m$$

and

$$\begin{aligned} R = {} & \frac{1}{2}\sum_{i,j=1}^{m}\left(\sum_{l=1}^{d} b^{li}\frac{\partial b^{kj}}{\partial x^l}\right)\left(\Delta\hat{W}^i\Delta\hat{W}^j + V_{ij}\right) \\ & + \frac{1}{6}\sum_{j=1}^{m}\sum_{\substack{r=1\\ r\neq j}}^{m}\sum_{i,l=1}^{d}\left(\frac{\partial^2 b^{kj}}{\partial x^i\partial x^l}b^{ir}b^{lr} - \sigma_r\sigma_j\frac{\partial^2 b^{kr}}{\partial x^i\partial x^l}b^{ij}b^{lr}\right)\varDelta\Delta\hat{W}^j. \end{aligned} \tag{48}$$

This family of solutions leads to the class of schemes

$$\bar{X}^k_{n+1} = \bar{X}^k_n + \sum_{j=1}^m \frac{1}{2}\left\{b^{kj} + \frac{1}{1+3\gamma^2} b^{kj}\left(t_n + \Delta, \bar{X}_n + a\Delta + \gamma \sum_{r=1}^m \sigma_r b^r \Delta\hat{W}^r\right)\right.$$

$$\left. + \frac{3\gamma^2}{1+3\gamma^2} b^{kj}\left(t_n + \Delta, \bar{X}_n + a\Delta - \frac{1}{3\gamma}\sum_{r=1}^m \sigma_r b^r \Delta\hat{W}^r\right)\right\}\Delta\hat{W}^j$$

$$+ \frac{1}{2}\left\{a^k + a^k\left(t_n + \Delta, \bar{X}_n + a\Delta + \sum_{r=1}^m b^r \Delta\hat{W}^r\right)\right\}\Delta + R,$$

where $\gamma \neq 0$, $\sigma_1 = 1$, $\sigma_2, \ldots, \sigma_m \in \{\pm 1\}$, R is given by (48) and the variables $\Delta\hat{W}^j$ and V_{ij} satisfy (13) and (17), respectively. They are 2-equivalent to the simplified order 2 Taylor scheme; with the same reasoning employed in the scalar case we can conclude that these schemes are of second order in the weak sense if a and b belong to $\mathscr{C}^6_{\mathrm{P}}$, verify a Lipschitz condition and, together with the partial derivatives $\partial b^{kj}/\partial x^i$, $\partial^2 b^{kj}/\partial x^i \partial x^l$, $i, l, k = 1, \ldots, d$, $j = 1, \ldots, m$, have at most linear growth. In particular, if $\sigma_2 = \cdots = \sigma_m = 1$ we have the one-parameter class of schemes

$$\bar{X}^k_{n+1} = \bar{X}^k_n + \sum_{j=1}^m \frac{1}{2}\left\{b^{kj} + \frac{1}{1+3\gamma^2} b^{kj}(t_n + \Delta, \bar{X}_n + a\Delta + \gamma b \Delta\hat{W})\right.$$

$$\left. + \frac{3\gamma^2}{1+3\gamma^2} b^{kj}\left(t_n + \Delta, \bar{X}_n + a\Delta - \frac{1}{3\gamma} b\Delta\hat{W}\right)\right\}\Delta\hat{W}^j$$

$$+ \tfrac{1}{2}\{a^k + a^k(t_n + \Delta, \bar{X}_n + a\Delta + b\Delta\hat{W})\}\Delta + R,$$

where $\gamma \neq 0$ and R is given by (48) with $\sigma_j = 1$, $j = 1, \ldots, m$.

7. Third order RK schemes

By the same technique employed in previous sections we will obtain now two third order RK schemes for scalar stochastic differential equations ($d = m = 1$) with constant diffusion coefficient $b(t,x) = b$. In this case a three-stage RK scheme of the form (27) can be written as

$$\bar{X}_{n+1} = \bar{X}_n + \{\alpha_1 a + \alpha_2 a(t_n + \mu_2\Delta, \eta_2) + \alpha_3 a(t_n + \mu_3\Delta, \eta_3)\}\Delta + \beta b \Delta\hat{W} + R, \tag{49}$$

where

$$\eta_2 = \bar{X}_n + \lambda_{21} a\Delta + \gamma_2 b\Delta\hat{W},$$

$$\eta_3 = \bar{X}_n + \lambda_{31} a\Delta + \lambda_{32} a(t_n + \mu_2\Delta, \eta_2)\Delta + \gamma_3 b\Delta\hat{W}.$$

Here we have to find out the parameters and R in such a way that (49) is 3-equivalent to the simplified order 3 Taylor scheme (18). And, as before, it suffices to choose them after replacing (49) by its third order truncated expansion.

We begin by evaluating the third order truncated expansion of $a(t_n + \mu_2\Delta, \eta_2)\Delta$. By using (30) and equivalences (26) we get

$$
\begin{aligned}
a(t_n + \mu_2\Delta, \bar{X}_n + \lambda_{21}a\Delta + \gamma_2 b\Delta\hat{W})\Delta &\overset{(3)}{\simeq} a\Delta + a_{01}b\gamma_2\Delta\Delta\hat{W} \\
&+ a_{10}\mu_2\Delta^2 + aa_{01}\lambda_{21}\Delta^2 + \frac{1}{2}a_{02}b^2\gamma_2^2\Delta(\Delta\hat{W})^2 + a_{11}b\mu_2\gamma_2\Delta^2\Delta\hat{W} \\
&+ aa_{02}b\lambda_{21}\gamma_2\Delta^2\Delta\hat{W} + \frac{1}{2}a_{03}b^3\gamma_2^3\Delta^2\Delta\hat{W} + aa_{11}\mu_2\lambda_{21}\Delta^3 + \frac{1}{2}a^2a_{02}\lambda_{21}^2\Delta^3 \\
&+ \left(a_{12} + \frac{1}{2}a_{04}b^2\right)\frac{b^2}{2}\mu_2\gamma_2^2\Delta^3 + \frac{1}{2}aa_{03}b^2\gamma_2^2\lambda_{21}\Delta^3 + \frac{1}{2}\left(a_{20} - \frac{b^4}{4}a_{04}\right)\mu_2^2\Delta^3. \qquad (50)
\end{aligned}
$$

Now let's calculate the third order truncated expansion of $a(t_n + \mu_3\Delta, \eta_3)\Delta$. For simplicity, we denote $M = \eta_3 - \bar{X}_n = \lambda_{31}a\Delta + \lambda_{32}a(t_n + \mu_2\Delta, \eta_2)\Delta + \gamma_3 b\Delta\hat{W}$. By (26) we have that $\Delta^4 \overset{(3)}{\simeq} 0 \overset{(3)}{\simeq} \Delta^3 \cdot M$, and then, by using (30) we get

$$
\begin{aligned}
a(t_n + \mu_3\Delta, \bar{X}_n + M)\Delta &\overset{(3)}{\simeq} a\Delta + a_{10}\mu_3\Delta^2 + a_{01}\Delta\cdot M + \left(a_{20} - \frac{b^4}{4}a_{04}\right)\frac{\mu_3^2\Delta^3}{2} \\
&+ a_{11}\mu_3\Delta^2\cdot M + a_{02}\frac{\Delta\cdot M^2}{2} + \left(a_{12} + \frac{b^2}{2}a_{04}\right)\frac{\mu_3\Delta^2M^2}{2} + a_{03}\frac{\Delta\cdot M^3}{6}. \qquad (51)
\end{aligned}
$$

The third order truncated expansion of M can be obtained from (50); and from its value, using (26), it's easy to show the equivalences

$$
\begin{aligned}
\Delta\cdot M &\overset{(3)}{\simeq} \gamma_3 b\Delta\Delta\hat{W} + (\lambda_{31} + \lambda_{32})a\Delta^2 + a_{01}b\lambda_{32}\gamma_2\Delta^2\Delta\hat{W} \\
&+ a_{10}\mu_2\lambda_{32}\Delta^3 + aa_{01}\lambda_{32}\lambda_{21}\Delta^3 + \tfrac{1}{2}a_{02}b^2\lambda_{32}\gamma_2^2\Delta^3, \qquad (52)
\end{aligned}
$$

$$\Delta^2\cdot M \overset{(3)}{\simeq} \gamma_3 b\Delta^2\Delta\hat{W} + (\lambda_{31} + \lambda_{32})a\Delta^3, \qquad (53)$$

$$
\begin{aligned}
\Delta\cdot M^2 &\overset{(3)}{\simeq} \gamma_3^2 b^2\Delta(\Delta\hat{W})^2 + 2ab\gamma_3(\lambda_{31} + \lambda_{32})\Delta^2\Delta\hat{W} \\
&+ (\lambda_{31} + \lambda_{32})^2a^2\Delta^3 + 2a_{01}b^2\lambda_{32}\gamma_2\gamma_3\Delta^3, \qquad (54)
\end{aligned}
$$

$$\Delta^2\cdot M^2 \overset{(3)}{\simeq} \gamma_3^2 b^2\Delta^3, \qquad (55)$$

$$\Delta\cdot M^3 \overset{(3)}{\simeq} \gamma_3^3 b^3\Delta(\Delta\hat{W})^3 + 3ab^2\gamma_3^2(\lambda_{31} + \lambda_{32})\Delta^3. \qquad (56)$$

Substituting (52)–(56) into (51) and using the obtained expression together with (50) we have that the scheme (49) shall be 3-equivalent to

$$
\begin{aligned}
\bar{X}_{n+1} = \bar{X}_n &+ \beta b\Delta\hat{W} + (\alpha_1 + \alpha_2 + \alpha_3)a\Delta + a_{01}b(\alpha_2\gamma_2 + \alpha_3\gamma_3)\Delta\Delta\hat{W} \\
&+ a_{10}(\alpha_2\mu_2 + \alpha_3\mu_3)\Delta^2 + aa_{01}(\alpha_2\lambda_{21} + \alpha_3(\lambda_{31} + \lambda_{32}))\Delta^2 \\
&+ \tfrac{1}{2}a_{02}b^2(\alpha_2\gamma_2^2 + \alpha_3\gamma_3^2)\Delta(\Delta\hat{W})^2 + ba_{01}^2\alpha_3\lambda_{32}\gamma_2\Delta^2\Delta\hat{W}
\end{aligned}
$$

$$
\begin{aligned}
&+ ba_{11}(\alpha_2\mu_2\gamma_2 + \alpha_3\mu_3\gamma_3)\Delta^2\Delta\hat{W} + aa_{02}b(\alpha_2\lambda_{21}\gamma_2 + \alpha_3(\lambda_{31} + \lambda_{32})\gamma_3)\Delta^2\Delta\hat{W} \\
&+ \tfrac{1}{2}a_{03}b^3(\alpha_2\gamma_2^3 + \alpha_3\gamma_3^3)\Delta^2\Delta\hat{W} + \tfrac{1}{2}(a_{12} + \tfrac{1}{2}a_{04}b^2)b^2(\alpha_2\mu_2\gamma_2^2 + \alpha_3\mu_3\gamma_3^2)\Delta^3 \\
&+ \tfrac{1}{2}aa_{03}b^2(\alpha_2\lambda_{21}\gamma_2^2 + \alpha_3(\lambda_{31} + \lambda_{32})\gamma_3^2)\Delta^3 + \tfrac{1}{2}a_{01}a_{02}b^2\alpha_3\lambda_{32}\gamma_2(\gamma_2 + 2\gamma_3)\Delta^3 \\
&+ \tfrac{1}{2}\left(a_{20} - \tfrac{1}{4}b^4a_{04}\right)(\alpha_2\mu_2^2 + \alpha_3\mu_3^2)\Delta^3 + aa_{11}(\alpha_2\mu_2\lambda_{21} + \alpha_3\mu_3(\lambda_{31} + \lambda_{32}))\Delta^3 \\
&+ \tfrac{1}{2}a^2a_{02}(\alpha_2\lambda_{21}^2 + \alpha_3(\lambda_{31} + \lambda_{32})^2)\Delta^3 + a_{01}a_{10}\alpha_3\mu_2\lambda_{32}\Delta^3 \\
&+ aa_{01}^2\alpha_3\lambda_{21}\lambda_{32}\Delta^3 + R. \qquad (57)
\end{aligned}
$$

Schemes (18) and (57) coincide if the constants satisfy the system

$$
\begin{array}{ll}
\alpha_1 + \alpha_2 + \alpha_3 = 1, & \beta = 1, \\
\alpha_2\mu_2 + \alpha_3\mu_3 = \frac{1}{2}, & \alpha_2\mu_2\gamma_2 + \alpha_3\mu_3\gamma_3 = \frac{1}{3}, \\
\alpha_2\lambda_{21} + \alpha_3(\lambda_{31} + \lambda_{32}) = \frac{1}{2}, & \alpha_2\lambda_{21}\gamma_2 + \alpha_3(\lambda_{31} + \lambda_{32})\gamma_3 = \frac{1}{3}, \\
\alpha_2\mu_2^2 + \alpha_3\mu_3^2 = \frac{1}{3}, & \alpha_2\gamma_2^2 + \alpha_3\gamma_3^2 = \frac{1}{3}, \\
\alpha_2\mu_2\lambda_{21} + \alpha_3\mu_3(\lambda_{31} + \lambda_{32}) = \frac{1}{3}, & \alpha_2\mu_2\gamma_2^2 + \alpha_3\mu_3\gamma_3^2 = \frac{1}{3}, \\
\alpha_2\lambda_{21}^2 + \alpha_3(\lambda_{31} + \lambda_{32})^2 = \frac{1}{3}, & \alpha_2\lambda_{21}\gamma_2^2 + \alpha_3(\lambda_{31} + \lambda_{32})\gamma_3^2 = \frac{1}{3}, \\
\alpha_3\mu_2\lambda_{32} = \frac{1}{6}, & \alpha_3\lambda_{32}\gamma_2(\gamma_2 + 2\gamma_3) = \frac{1}{2}, \\
\alpha_3\lambda_{21}\lambda_{32} = \frac{1}{6}, & \alpha_2\gamma_2^3 + \alpha_3\gamma_3^3 = \frac{1}{3}, \\
 & \alpha_3\lambda_{32}\gamma_2 = \frac{1}{6},
\end{array} \qquad (58)
$$

and if $R = ba_{01}\{\Delta Z - (\alpha_2\gamma_2 + \alpha_3\gamma_3)\Delta\Delta W\} + \frac{1}{12}b^2a_{02}\Delta^2$. The equations on the left column are the same which appear in the deterministic case. As it's known, see REA [13], they reduce to the system

$$
\begin{aligned}
&\alpha_1 + \alpha_2 + \alpha_3 = 1, \\
&\alpha_2\mu_2 + \alpha_3\mu_3 = \tfrac{1}{2}, \\
&\alpha_2\mu_2^2 + \alpha_3\mu_3^2 = \tfrac{1}{3}, \\
&\alpha_3\mu_2\lambda_{32} = \tfrac{1}{6}, \\
&\lambda_{21} = \mu_2, \\
&\lambda_{31} + \lambda_{32} = \mu_3,
\end{aligned} \qquad (59)
$$

Using (59), the equations on the right column of (58) reduce to

$$
\begin{aligned}
&\beta = 1, \\
&\gamma_2 = \mu_2, \\
&\gamma_3 = \mu_3,
\end{aligned}
$$

$$\alpha_2\mu_2^3 + \alpha_3\mu_3^3 = \tfrac{1}{3},$$
$$\mu_2 + 2\mu_3 = 3 \tag{60}$$

and $R = ba_{01}(\Delta Z - \frac{1}{2}\varDelta\Delta W) + \frac{1}{12}b^2a_{02}\varDelta^2$. The ordinary case system (59) has two one-parameter and one two-parameter families of solutions; see [2]. It's easy to see that none of the solutions of the one-parameter families (one of the families corresponds to the case $\mu_3 = 0$, $\mu_2 = 2/3$; the other one to the case $\mu_2 = \mu_3 = \frac{2}{3}$) can be a solution of (60). On the other hand, since we have in the two-parameter family that

$$\alpha_2 = \frac{\frac{1}{2}\mu_3 - \frac{1}{3}}{\mu_2(\mu_3 - \mu_2)}, \quad \alpha_3 = \frac{\frac{1}{3} - \frac{1}{2}\mu_2}{\mu_3(\mu_3 - \mu_2)}, \quad \mu_2 \neq \mu_3, \quad \mu_2 \neq 0, \quad \mu_3 \neq 0, \quad \mu_2 \neq \tfrac{2}{3},$$

by imposing on a solution of this family to verify (60) we get

$$6\mu_3^3 - 17\mu_3^2 + 15\mu_3 - 4 = 0,$$

the roots of this equation are $\mu_3 = 1$, $\mu_3 = 1/2$ and $\mu_3 = 4/3$. If $\mu_3 = 1$ then $\mu_2 = \mu_3$ and, as we have said, these solutions do not verify (60). Each of the other roots leads to a solution of system (59)–(60). If $\mu_3 = \frac{1}{2}$ we have the solution

$$\mu_2 = 2 = \gamma_2, \quad \gamma_3 = \tfrac{1}{2}, \quad \alpha_1 = \tfrac{1}{12}, \quad \alpha_2 = \tfrac{1}{36}, \quad \alpha_3 = \tfrac{8}{9}, \quad \lambda_{21} = 2, \quad \lambda_{31} = \tfrac{13}{32}, \quad \lambda_{32} = \tfrac{3}{32};$$

and if $\mu_3 = \frac{4}{3}$ we have

$$\mu_2 = \tfrac{1}{3} = \gamma_2, \quad \gamma_3 = \tfrac{4}{3}, \quad \alpha_1 = \tfrac{-1}{8}, \quad \alpha_2 = 1, \quad \alpha_3 = \tfrac{1}{8}, \quad \lambda_{21} = \tfrac{1}{3}, \quad \lambda_{31} = \tfrac{-8}{3}, \quad \lambda_{32} = 4.$$

Each solution defines a scheme 3-equivalent to the simplified order 3 Taylor scheme. The first one is given by

$$\begin{aligned}\bar{X}_{n+1} = {} & \bar{X}_n + b\Delta\hat{W}_n + \tfrac{1}{12}a\varDelta + \tfrac{1}{36}a(t_n + 2\varDelta, \bar{X}_n + 2b\Delta\hat{W}_n + 2a\varDelta)\varDelta \\ & + \frac{8}{9}a\left(t_n + \frac{\varDelta}{2}, \bar{X}_n + \frac{1}{2}b\Delta\hat{W}_n + \frac{13}{32}a\varDelta + \frac{3}{32}a(t_n + 2\varDelta, \bar{X}_n + 2b\varDelta\hat{W}_n + 2a\varDelta)\varDelta\right)\varDelta \\ & + ba_{01}\left(\Delta Z_n - \frac{1}{2}\varDelta\Delta\hat{W}_n\right) + \frac{1}{12}b^2a_{02}\varDelta^2,\end{aligned} \tag{61}$$

and the second one by

$$\begin{aligned}\bar{X}_{n+1} = {} & \bar{X}_n + b\Delta\hat{W}_n - \frac{1}{8}a\varDelta + a\left(t_n + \frac{1}{3}\varDelta, \bar{X}_n + \frac{1}{3}b\Delta\hat{W}_n + \frac{1}{3}a\varDelta\right)\varDelta \\ & + \frac{1}{8}a\left(t_n + \frac{4}{3}\varDelta, \bar{X}_n + \frac{4}{3}b\Delta\hat{W}_n - \frac{8}{3}a\varDelta + 4a\left(t_n + \frac{\varDelta}{3}, \bar{X}_n + \frac{1}{3}b\Delta\hat{W}_n + \frac{1}{3}a\varDelta\right)\varDelta\right)\varDelta \\ & + ba_{01}\left(\Delta Z_n - \frac{1}{2}\varDelta\Delta\hat{W}_n\right) + \frac{1}{12}b^2a_{02}\varDelta^2.\end{aligned} \tag{62}$$

Their Butcher arrays are, respectively,

$$
\begin{array}{c|ccc|c}
2 & 2 & & & 2 \\
\frac{1}{2} & \frac{13}{32} & \frac{3}{32} & & \frac{1}{2} \\
\hline
\mathrm{R} & \frac{1}{12} & \frac{1}{36} & \frac{8}{9} & 1
\end{array}
\qquad
\begin{array}{c|ccc|c}
\frac{1}{3} & \frac{1}{3} & & & \frac{1}{3} \\
\frac{4}{3} & \frac{-8}{3} & 4 & & \frac{4}{3} \\
\hline
\mathrm{R} & \frac{-1}{8} & 1 & \frac{1}{8} & 1
\end{array}
$$

Since it's immediate to show that these two schemes verify conditions (5) if a, $\partial a/\partial x$ and $\partial^2 a/\partial x^2$ grow at most linearly, we have the following.

Theorem 4. *Suppose that the diffusion coefficient b in the sde* (1) *is a constant. Suppose also that the drift coefficient a belongs to* $\mathscr{C}_{\mathrm{P}}^{8}$, *verifies a Lipschitz condition and, together with the partial derivatives* $\partial a/\partial x$ *and* $\partial^2 a/\partial x^2$, *have at most linear growth. Then the RK schemes* (61) *and* (62), *where* $\Delta\hat{W}_n$, ΔZ_n *are variables satisfying* (19), *have order* 3 *in the weak sense.*

Both schemes require at each step to evaluate a at three points, $\partial a/\partial x$ at one point and $\partial^2 a/\partial x^2$ at one point and to generate the variables $\Delta\hat{W}_n$, ΔZ_n verifying (19) (note that the simplified Taylor scheme also requires to evaluate a_{10}, a_{11}, a_{20}, a_{03}, a_{12}, a_{04}).

8. Numerical results

In this section, numerical results from the implementation of the second and third order schemes proposed in the paper are compared to those from the implementation of well-known schemes of the same order; we have also used in each example a lower order scheme to contrast with them.

The two-stage order 2 Runge–Kutta method proposed in (35) will be denoted by RK2-2st; we will denote by RK2-3st the particular three-stage order 2 Runge–Kutta method of the family (41) with $\bar{\gamma} = 1/3$; the order 3 Runge–Kutta method proposed in (61) will be denoted by RK3. In order to compare with them, we shall use the Euler method, the simplified order 2 Taylor method (14), denoted by Taylor2, a second order Runge–Kutta method proposed by Platen (see [7, p. 485]), denoted by RK2-PL, and the simplified order 3 Taylor scheme (18).

As test problems we have taken linear and nonlinear one-dimensional stochastic differential equations ($d = m = 1$) for which the exact solution X_t in terms of the Wiener process is known; our aim is to simulate the known value $E[g(X_T)]$, where we have chosen $g(x) = x$ or $g(x) = x^2$.

For each example we have used $N = 5000$ simulations for step sizes $\Delta = 2^{-1}, \ldots, 2^{-5}$ to compute the approximated value of the known expectation. The mean and the standard deviation of the errors for each considered scheme are summarized in Tables 1–4.

Example 1. Consider the nonautonomous linear equation

$$\mathrm{d}X_t = (t + X_t)\,\mathrm{d}t + t^2\,\mathrm{d}W_t,$$

$$X_0 = 1.$$

Table 1

	Euler		Taylor2		RK2-2st	
Δ	Error	St. dev.	Error	St. dev.	Error	St. dev.
2^{-1}	6.82499	1.05507	2.39532	2.63769	2.41669	3.4005
2^{-2}	4.69419	1.98406	0.897562	3.43742	0.897463	3.68215
2^{-3}	2.82498	2.72174	0.202145	3.73732	0.198247	3.81124
2^{-4}	1.61232	3.21608	0.0783977	3.82159	0.0783575	3.83989
2^{-5}	0.812689	3.51058	0.0364222	3.84255	0.0364302	3.84737
2^{-6}	0.457826	3.71088	0.0177697	3.88478	0.0177559	3.88594

Table 2

	Euler		RK2-PL		RK2-3st	
Δ	Error	St. dev.	Error	St. dev.	Error	St. dev.
2^{-1}	15.8098	2.46831	5.7853	6.22192	5.78145	6.23213
2^{-2}	10.3977	4.45399	2.02754	8.01811	2.02479	8.01739
2^{-3}	5.97178	6.41039	0.436488	8.96982	0.437138	8.96707
2^{-4}	3.33311	7.68503	0.174519	9.21052	0.174723	9.20959
2^{-5}	1.77872	8.52312	0.0914871	9.34292	0.0920636	9.34149
2^{-6}	0.833435	9.08408	0.049148	9.52737	0.04872236	9.52715

Table 3

	Euler		RK2-PL		RK2-3st	
Δ	Error	St. dev.	Error	St. dev.	Error	St. dev.
2^{-1}	714.328	62.3239	336.823	300.969	336.525	305.969
2^{-2}	539.347	168.103	130.174	465.036	130.043	467.544
2^{-3}	342.693	309.168	28.8318	556.84	28.9169	557.816
2^{-4}	201.517	435.735	9.94652	600.808	9.97494	601.18
2^{-5}	108.836	538.064	2.8628	633.786	2.9216	633.761
2^{-6}	58.333	585.088	2.28967	637.75	2.31216	637.75

Table 4

	Taylor2		Taylor3		RK3	
Δ	Error	St. dev.	Error	St. dev.	Error	St. dev.
2^{-1}	1.32851	0.129494	0.566487	0.142544	0.0925534	0.145892
2^{-2}	0.459119	0.137325	0.0976038	0.141423	0.0142057	0.141976
2^{-3}	0.124983	0.145034	0.00685451	0.14627	0.00540199	0.146353
2^{-4}	0.0366927	0.143237	0.00331031	0.143519	0.00165378	0.14353
2^{-5}	0.00940354	0.143731	0.000523082	0.143833	0.000307867	0.143834

We estimate the exact value $E[X_T]=2e^2-(1+T)$ at the point $T=2$. Since $\partial b/\partial x=0$ we can apply the second order RK2-2st scheme. In this case this scheme coincides with RK2-3st. We compare them with the Taylor2 scheme; the Euler scheme is also included. The values are summarized in Table 1. The results obtained using the RK2-2st scheme are similar to those from the second order Taylor scheme; the advantage of RK2-2st is that it is derivative free.

Example 2. Consider the nonlinear stochastic differential equation

$$\mathrm{d}X_t = \left(\frac{1}{3}X_t^{1/3} + 6X_t^{2/3}\right)\mathrm{d}t + X_t^{2/3}\,\mathrm{d}W_t,$$

$$X_0 = 1.$$

with solution $X_t=(2t+1+W_t/3)^3$. Since the equation is autonomous, we can use the second order RK-PL scheme to estimate the exact value $E[X_1]=28$ and compare the obtained values with those from the second order RK2-3st scheme. The results are summarized in Table 2.

Example 3. Consider the nonlinear equation given in Example 2 above. Here we approximate the value $E[X_1^2]=869+\frac{5}{3^5}$ using RK-PL and RK2-3st schemes; the results obtained are shown in Table 3.

In Examples 2 and 3 we observed practically no differences between the proposed second order Runge–Kutta scheme RK2-3st and the second order one proposed by Platen. RK-PL is completely derivative free; in opposition, RK2-3st has only one derivative, is simpler and valid for the nonautonomous case.

Example 4. Consider the linear nonautonomous stochastic differential equation

$$\mathrm{d}X_t = (tX_t + 10t)\,\mathrm{d}t + b\,\mathrm{d}W_t,$$

$$X_0 = 10,$$

with constant diffusion coefficient $b=0.1$. The exact value $E[X_1]=20e^{1/2}-10$ was estimated using Taylor3 and the RK3 schemes, both of third order. To simulate the Gaussian variables $\Delta\hat{W}_n$ and ΔZ_n satisfying (19) we have taken $\Delta\hat{W}=\sqrt{\Delta}U_1$ and $\Delta Z=\frac{1}{2}\Delta^{3/2}(U_1+\frac{1}{\sqrt{3}}U_2)$, where U_1 and U_2 are two independent Gaussian variables with distribution $\mathrm{N}(0,1)$. Although the results, summarized in Table 4, are not extensive enough to justify general conclusions, they suggest that RK3 is more efficient than Taylor3.

References

[1] L. Arnold, Stochastic Differential Equations, Wiley, New York, 1974.

[2] J.C. Butcher, The Numerical Analysis of Ordinary Differential Equations. Runge–Kutta and General Linear Methods, Wiley, Chichester, 1987.

[3] C.C. Chang, Numerical solution of stochastic differential equations with constant diffusion coefficients, Math. Comput. 49 (1987) 523–542.

[4] D.B. Hernandez, R. Spigler, A-stability of Runge–Kutta methods for systems with additive noise, BIT 32 (1992) 620–633.

[5] D.B. Hernandez, R. Spigler, Convergence and stability of implicit Runge–Kutta methods for systems with multiplicative noise, BIT 33 (1993) 654–669.
[6] J.R. Klauder, W.P. Petersen, Numerical integration of multiplicative-noise stochastic differential equations, SIAM J. Numer. Anal. 22 (1985) 1153–1166.
[7] P.E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1992.
[8] S. Mauthner, Step size control in the numerical solution of stochastic differential equations, J. Comput. Appl. Math. 100 (1998) 93–109.
[9] E.J. Mcshane, Stochastic Calculus and Stochastic Models, Academic Press, New York, 1974.
[10] G.N. Milstein, A method of second order accuracy integration of stochastic differential equations, Theory Probab. Appl. 23 (1978) 396–401.
[11] G.N. Milstein, Weak approximation of solutions of systems of stochastic differential equations, Theory Probab. Appl. 30 (1985) 750–766.
[12] G.N. Milstein, Numerical Integration of Stochastic Differential Equations, translated version of 1988 Russian work, Kluwer, Dordrecht, 1995.
[13] REA (Research, Education Association), Advanced Methods for Solving Differential Equations, New York, 1982.
[14] W. Rümelin, Numerical treatment of stochastic differential equations, SIAM J. Numer. Anal. 19 (1982) 604–613.
[15] Y. Saito, T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, SIAM J. Numer. Anal. 33 (1996) 2254–2267.
[16] D. Talay, Efficient numerical schemes for the approximation of expectations of functionals of the solution of a SDE and applications, in: H. Korezlioglu, G. Mazziotto, J. Szpirglas (Eds.), Lecture Notes in Control and Inform. Sci., Vol. 61, Springer, Berlin, 1984, pp. 294–313.
[17] A. Tocino, R. Ardanuy, Truncated Itô–Taylor expansions, Stochastic Anal. Appl., to appear.