1. Easy Mode

Problem 1.1. *What is the smallest r such that three disks of radius r can completely cover up a unit disk?*

$\boxed{\frac{\sqrt{3}}{2}}$. *[Consider the circumference, trivial.]*

Problem 1.2. *If* $\dfrac{\log_b 9}{\log_c 9} = 2023$*, compute* $\dfrac{b}{c} \mod 17$.

$\boxed{13}$. *[*$\frac{b}{c} = 9^{2022}, \operatorname{ord}_{17}(9) = 4, \frac{b}{c} \equiv 9^{2022} \equiv 9^2 \equiv 13 \pmod{17}$*.]*

Problem 1.3. *Let x and y be two-digit integers such that y is obtained by reversing the digit of x. Suppose that $x^2 - y^2 = m^2$ where m is some integer, find $x + y$.*

$\boxed{121}$. *[Suppose $x = 10a + b$ and $y = 10b + a$, we have $x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99(a + b)(a - b) = 3^2 \cdot 11 \cdot (a + b)(a - b)$. Therefore, $11 \mid (a + b)(a - b)$. Since a, b are integers from 1 to 9, the largest value of $a - b$ is 8 and $11 \mid a + b$. Also since $a + b < 18$, the only multiple of 11 is 11 itself. $a - b$ should be a square of integer, in its range the square can only be 1 and 4, while 4 is impossible because $a - b = 4$ implies a and b are both even or odd, therefore,*

$$\begin{cases} a + b & = 11 \\ a - b & = 1 \end{cases}$$

which reaches $a = 6, b = 5$.]

Problem 1.4. *Calculate the exact value of the angle expressed in the form with* arctan *between 2 $C - H$ bonds in CH_4 (methane) and give out the process of calculation.*

$\boxed{180^\circ - \arctan(2\sqrt{2})}$. *[Suppose the CH_4 molecule is a regular tetrahedron $ABC - D$, and the center point is O. If $AB = 2\sqrt{3}$, the segment OO' where O' is the projection of O on plane ABC is $\frac{1}{\sqrt{2}} \implies \angle DOA = 180^\circ - \arctan(2\sqrt{2}) \approx 109.5^\circ$.]*

Problem 1.5. *Given that $N = (20^{23} + 2^{915})^2 - (20^{23} - 2^{915})^2$, how many consecutive "0" are there at the end of N? (For example, 2480900 has 2 "0"s at the end.)*

$\boxed{23}$. *[$N = 2 \cdot 20^{23} \cdot 2 \cdot 2^{915} = 2^9 17 \cdot 20^{23} = 2^{963} \cdot 5^{23}$. Therefore, there exist 23 "0"s at the end of N.]*

Problem 1.6. *Positive integers a and b satisfy the condition*

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

$\boxed{881}$. *[*

$$\begin{aligned} \log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) &= 0 \\ \log_{2^a}(\log_{2^b}(2^{1000})) &= 1 \\ 2^a &= \log_{2^b}(2^{1000}) \\ 2^{b \cdot 2^a} &= 2^{1000} \\ b \cdot 2^a &= 1000 \end{aligned}$$

Therefore, the answer can be $a = 1, b = 500; a = 2, b = 250; a = 3, b = 125$. Sum them up and the final answer is $\boxed{881}$. *]*

2. Intermediate Mode

Problem 2.1. *In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9$, $DF = 8$, $\overline{BE} \parallel \overline{DF}$, $\overline{EF} \parallel \overline{AB}$, and line BE intersects segment $\overline{AD}$. What is the length of EF?*

$\boxed{3\sqrt{21} - 12}$. *[Let X and Y be the projections of E and F onto AB and CD, respectively. Let $EX = d$, then $FY = BC - EX = 10 - d$. Since triangles EBX and FDY both share a right angle, and because $EB||FD$, the two triangles are similar, so there is some k such that $XB = kd$ and $DY = k(10 - d)$. Also by similarity,*

$$\frac{d}{9} = \frac{10-d}{8} \Rightarrow d = \frac{90}{17}.$$

By the Pythagorean Theorem,

$$d\sqrt{k^2+1} = 9$$

$$\sqrt{k^2+1} = \frac{17}{10}$$

$$k = \frac{3\sqrt{21}}{10}.$$

Therefore $EF = DY + BX - AB = 10k - 12 = 3\sqrt{21} - 12$.]

Problem 2.2. *Triangle ABC has a right angle at C, and D is the foot of the altitude from C to AB. Points L, M and N are the midpoints of segments AD, DC and CA, respectively. If $CL = 9$ and $BM = 15$, compute BN^2.*

$\boxed{306}$. *[Note that CL, BM, BN are corresponding segments in the similar triangles $\triangle ACD \sim \triangle CBD \sim \triangle ABC$. So we have*

$$CL : BM : BN = AD : CD : AC.$$

Since $AD^2 + CD^2 = AC^2$, we also have $CL^2 + BM^2 = BN^2$, giving an answer of $9^2 + 15^2 =$ $\boxed{306}$. *]*

Problem 2.3. *Alice is thinking of a positive real number x, and Bob is thinking of a positive real number y. Given that $x^{\sqrt{y}} = 27$ and $(\sqrt{x})^y = 9$, compute xy.*

$\boxed{16\sqrt[4]{3}}$. *[Note that*

$$\begin{aligned} & 27^{\sqrt{y}} \\ =& (x^{\sqrt{y}})^{\sqrt{y}} \\ =& x^y = (\sqrt{x})^{2y} = 81 \end{aligned}$$

Therefore, $\sqrt{y} = \log_{27}(81) = \frac{4}{3}, y = \frac{16}{9}$. Follows that $x = 9\sqrt[4]{3}$. Multiplying them and we can get $xy =$ $\boxed{16\sqrt[4]{3}}$*.]*

Problem 2.4. *Find the smallest integer n such that*

$$\sqrt{n+99} - \sqrt{n} < 1$$

$\boxed{2402}$. *[This is equivalent to*

$$\sqrt{n+99} < 1 + \sqrt{n}$$

By squaring both sides,

$$n + 99 < 1 + n + 2\sqrt{n}$$

Therefore, $49 < \sqrt{n} \implies$ *the smallest n with this property is* $49^2 + 1 = 2402$. *]*

Problem 2.5. *Which number among the set $\{0, 1, 2, 3, 4\}$ stands for the imaginary number i mod 5? [* $\text{mod } n$ *stands for modulo arithmetic, which gives out the residue when divided by n. For example, $9 \equiv 1 \pmod 4$ means that 9's residue is 1 when divided by 4.]*

$\boxed{\textit{2 and 3}}$. *[$-1 \equiv 4 \pmod 5, i = \sqrt{-1} \equiv \pm 2 \pmod 5$. Therefore, 2 and 3 are equivalent to i mod 5.]*

Problem 2.6. *Points $A(6, 13)$ and $B(12, 11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?*

$\boxed{\frac{85\pi}{8}}$. *[Use geometry bash, or Geogebra.]*

3. Difficult Mode

Problem 3.1. *Let (a, b, c) be the real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.*
$\boxed{158}$. *[Add the three equations to get $a^3 + b^3 + c^3 = 28 + 3abc$. Now, let $abc = p$. $a = \sqrt[3]{p+2}$, $b = \sqrt[3]{p+6}$ and $c = \sqrt[3]{p+20}$, so $p = abc = (\sqrt[3]{p+2})(\sqrt[3]{p+6})(\sqrt[3]{p+20})$. Now cube both sides; the p^3 terms cancel out. Solve the remaining quadratic to get $p = -4, -\frac{15}{7}$. To maximize $a^3 + b^3 + c^3$ choose $p = -\frac{15}{7}$ and so the sum is $28 - \frac{45}{7} = \frac{196-45}{7}$ giving $151 + 7 = \boxed{158}$.]*

Problem 3.2. *How many digits are in the base two representation of $10!$ (factorial)?*

$\boxed{21}$. *[$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. The number of digits (base 2) of $10!$ is equal to $\lfloor log_2 10! \rfloor = 8 + log_2(3^4 \cdot 5^2 \cdot 7)$. Since $2^{13} < 3^4 \cdot 5^2 \cdot 7 < 2^{14}$, the number of digit in $10!$ should be $8 + 13 = \boxed{21}$.]*

Problem 3.3. *If $a_1 = 1, a_2 = 0$, and $2a_{n+1} = 2a_n + a_{n+2}$ for all $n \geq 1$, compute $log_2(-a_{2024})$*

$\boxed{1012}$. *[By writing out the first few terms, we find that $a_{n+4} = -4a_n$. Indeed, $a_{n+4} = 2(a_{n+3} - a_{n+2}) = 2(a_{n+2} - 2a_{n+1}) = 2(-2a_n) = -4a_n$. Then, by induction, we get $a_{4k} = (-4)^k$ for all positive integers k, and setting $k = 506$ gives the answer.]*

Problem 3.4. *Given that* $\log_{10}\sin x+\log_{10}\cos x=-1$ *and that* $\log_{10}(\sin x+\cos x)=\frac{1}{2}(\log_{10}n-1)$*, find* n*.*

$\boxed{12}$. *[By logarithm we know* $\sin x\cos x=\frac{1}{10}$*.*
Then manipulate the second equation to

$$\begin{aligned}\log_{10}(\sin x+\cos x)&=\log_{10}(\sqrt{\frac{n}{10}})\\ \sin x+\cos x&=\sqrt{\frac{n}{10}}\\ \sin^2 x+2\sin x\cos x+\cos^2 x&=\frac{n}{10}\end{aligned}$$

Therefore, $n=\boxed{12}$*.]*

Problem 3.5. *What is*

$$\sum_{n=1}^{2023}\frac{\lfloor\sqrt{n}+\sqrt{n+1}\rfloor}{\lfloor\sqrt{4n+2}\rfloor}$$

and prove it.

$\boxed{2023}$. *[By trying a few numbers, we want to prove that*

$$\lfloor\sqrt{n}+\sqrt{n+1}\rfloor=\sqrt{4n+2}$$

By squaring both sides it is easy to see that

$$\sqrt{4n+1}<\sqrt{n}+\sqrt{n+1}<\sqrt{4n+2}$$

($2n<2\sqrt{n(n+1)}<2(n+1)\implies 4n+1<n+(n+1)+2\sqrt{n(n+1)}<4n+3\implies\sqrt{4n+1}<\sqrt{n}+\sqrt{n+1}<\sqrt{4n+3}$*.)*

Since $\sqrt{4n+1}$ *and* $\sqrt{4n+3}$ *are neither squares,* $\lfloor\sqrt{4n+1}\rfloor=\lfloor\sqrt{4n+3}\rfloor=\lfloor\sqrt{n}+\sqrt{n+1}\rfloor$*.*

Therefore, $\lfloor\sqrt{4n+2}\rfloor=\lfloor\sqrt{n}+\sqrt{n+1}\rfloor$*.]*

Problem 3.6. *If a DP1 student chooses 5 HL (high level) courses out of 6 and needs to take 6 classes each week for each HL course, what is the probability that he has 2 2-period classes for different HL courses in a day?*