Functions

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Contents

§1 Sept 11 - Functions

§1.1 Definition of Function

First, remember that we have commonly learned functions in middle school with x as the independent variable and y as the dependent variable.

Definition 1.1 (Function)**.** A function, sometimes called a mapping, is a reaction in which no two different ordered pairs have the same x-coordinate or first component.

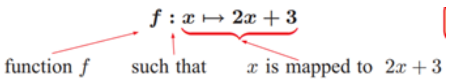


Figure 1: Example of mapping

§1.2 Identity of Function

Definition 1.2 (Domain)**.** The *domain* of a relation is the set of values of x in the relation.

Definition 1.3 (Range)**.** The *range* of a relation is the set of values of y in the relation.

§1.3 Inverse Function

Definition 1.4 (One-to-one function)**.** A **one-to-one** function is any function where:

- for each x there is only one value of y and
- for each y there is only one value of x.

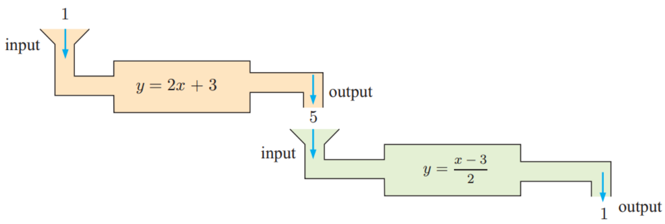


Figure 2: idea of inverse function

For example, $f : y = 5x + 2$, becomes $f^{-1} : x = 5y + 2$, thus $f^{-1} : y = \dfrac{x-2}{5}$.

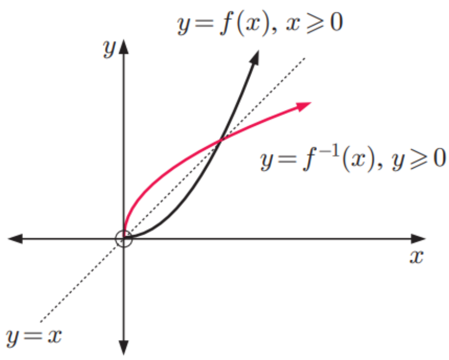


Figure 3: Example of an inverse function

Claim 1.5 — The domain of f^{-1} is equal to the range of f and the range of f^{-1} is equal to the domain of f.

§1.4 Transformation of Functions

- Horizontal Shift

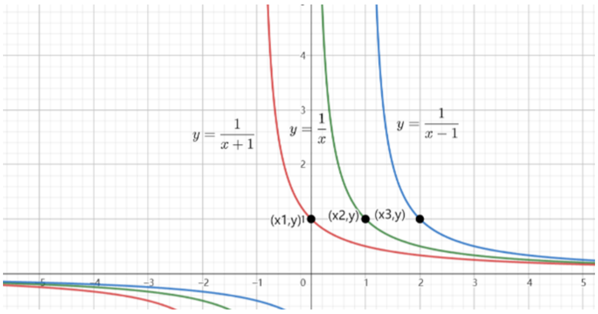


Figure 4: Example of horizontal shift

$$f(x) \longrightarrow f(x-a), a > 0; \text{ Moving rightward } a \text{ unit}$$
$$f(x) \longrightarrow f(x+a), a > 0; \text{ Moving leftward } a \text{ unit}$$

- Vertical Shift

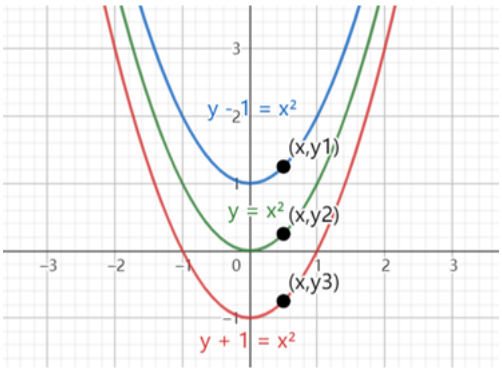


Figure 5: Example of vertical shift

$$f(x) \longrightarrow f(x) + a, a > 0; \text{ Moving upward } a \text{ unit;}$$
$$f(x) \longrightarrow f(x) - a, a > 0; \text{ Moving downward } a \text{ unit.}$$

- Shift of Equation

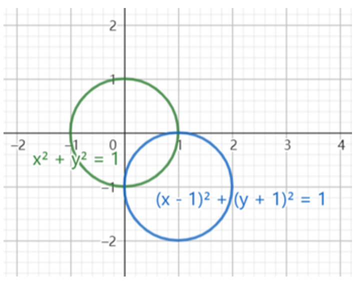


Figure 6: Example of shift of equation

$$f(x,y) \longrightarrow f(x-a, y-b), a, b > 0; \text{ Moving rightward } a \text{ unit and upward } b \text{ unit.}$$

§1.5 Even and Odd Function

Definition 1.6 (Even Function)**.** A function $f(x)$ is *even* if $f(x) = f(-x)$ for all x in the domain of f

Definition 1.7 (Odd Function)**.** A function $f(x)$ is *odd* if $f(x) = -f(-x)$ for all x in the domain of f

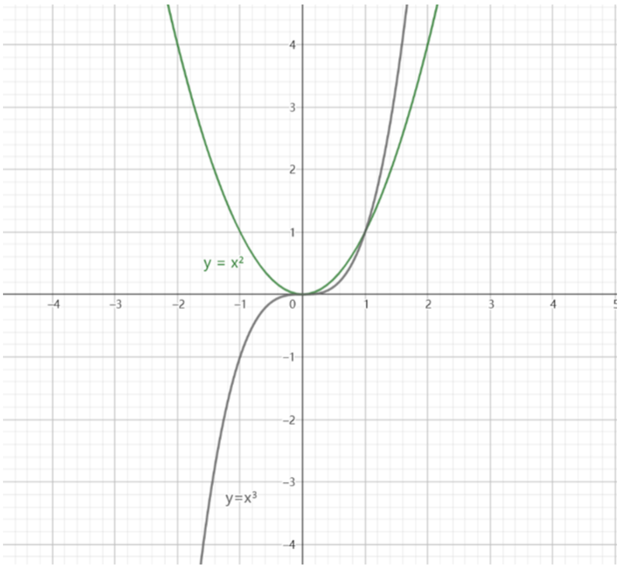


Figure 7: Example of even function in green and odd function in black.

§1.6 Power Function *

Now let's talk about some basic functions that commonly appear when doing calculus.

Definition 1.8 (Power Function)**.** Power functions are functions in the form

$$f(x) = x^a, a \in \mathbb{R}$$

where a is a constant.

§1.6.1 Exploration of a's value's impact on diagram

By observing Figure 8, 9 and 10, what patterns can you find?

Remark 1.9. When considering power function, the thing that we need to think carefully about is the domain of the function when $a < 1$.

Observing Figure 9 and 10, we can see that the functions all have different domains. For example, $f(x) = \sqrt{x}$ where $a = \frac{1}{2}$, the function only exists on $\mathbb{R}$ when $x \in [0, +\infty)$. However, for $f(x) = \sqrt[3]{x} = x^{1/3}$, the domain is $\mathbb{R}$.

Also, for $a \in (-\infty, 0)$, it is better to change $f(x) = x^a$ where $a < 0$ to $\frac{1}{x^{-a}}$.

Question 1.10. Is $f(x) = x^{2/4}$ the same as $f(x) = x^{1/2}$? If not, what is the difference?

[Hint: When solving this kind of question, the best way is to change it in the form with square root, for this one, change it into $f(x) = \sqrt[4]{x^2}$. Now you should be clear.]

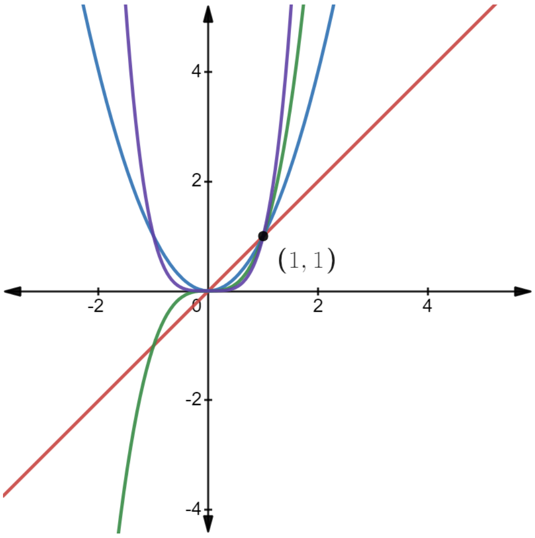


Figure 8: Graphs of power functions with $a \in [1, +\infty)$ including $y = x, x^2, x^3, x^4$. Can you justify them?

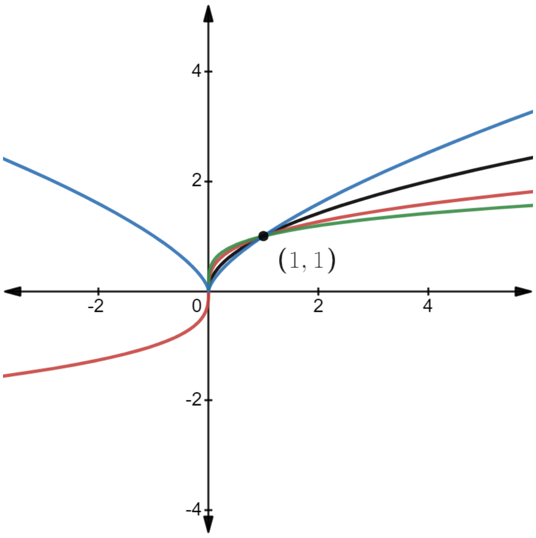


Figure 9: Graphs of power functions with $a \in (0, 1)$ including $y = x^{1/2}, x^{2/3}, x^{1/4}, x^{1/3}$. Can you justify them?

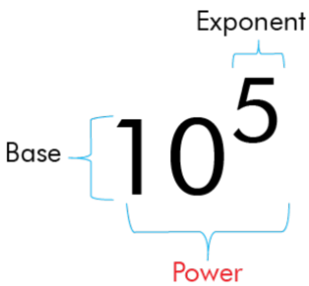
§1.7 Exponential Function

Definition 1.11 (Exponential Function)**.** Exponential functions are functions in the form

$$f(x) = a^x, a \in \mathbb{R}^+/\{1\} \ (a \in \mathbb{R}^+ \text{ and } a \neq 1).$$

where a is a constant.

Remark 1.12. Power functions and exponential functions are easy to mess up. Remember that for **exponential functions**, the independent variable x is the exponent, and the dependent variable $f(x)$ is the power while for **power functions**, the base is the independent variable and the power is the dependent variable $f(x)$.



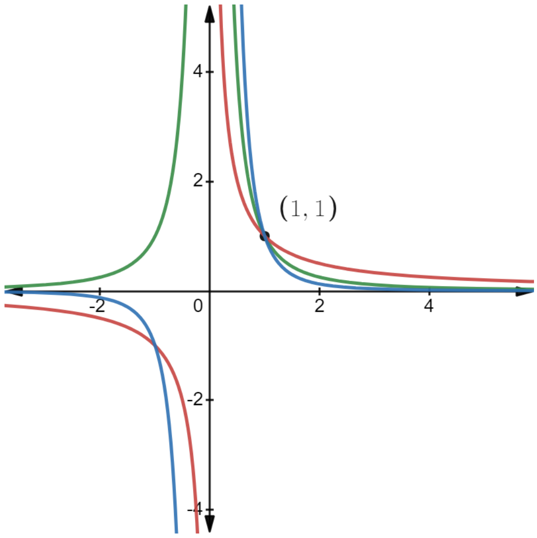


Figure 10: Graphs of power functions with $a \in (-\infty, 0)$ including $y = x^{-1}, x^{-2}, x^{-3}$. Can you justify them?

Some formulas for computing exponents:

$$a^\alpha \cdot a^\beta = a^{\alpha+\beta},$$
$$(a^\alpha)^\beta = a^{\alpha\cdot\beta},$$
$$(ab)^\alpha = a^\alpha b^\alpha$$

§1.7.1 Exploration of a's value's impact on diagram

By observing Figure 11 and 12, what patterns can you find?

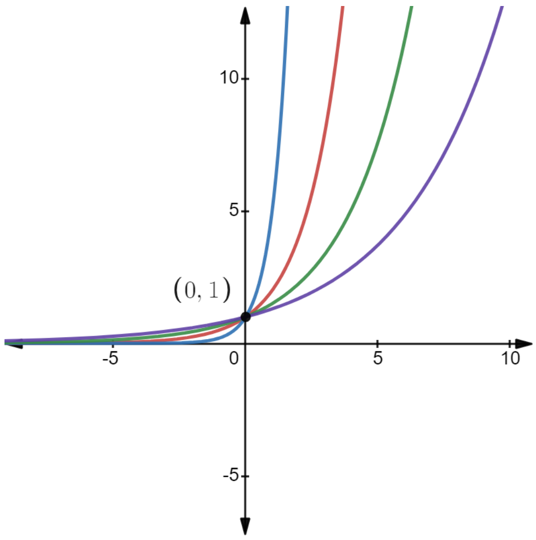


Figure 11: Graph of exponential functions with $a \in (1, +\infty)$ including $y = 1.3^x, 1.5^x, 2^x, 5^x$. Can you justify them?

Yes! I think you find out that $f(x) = 0.2^x$ and $f(x) = 5^x$, $f(x) = 0.5^x$ and $f(x) = 2^x$ are symmetric about y axis, why?

Proposition 1.13

$f(x) = a^x$ and $g(x) = \frac{1}{a}^x \iff$ exponential functions $f(x)$ and $g(x)$ are symmetric about y axis.

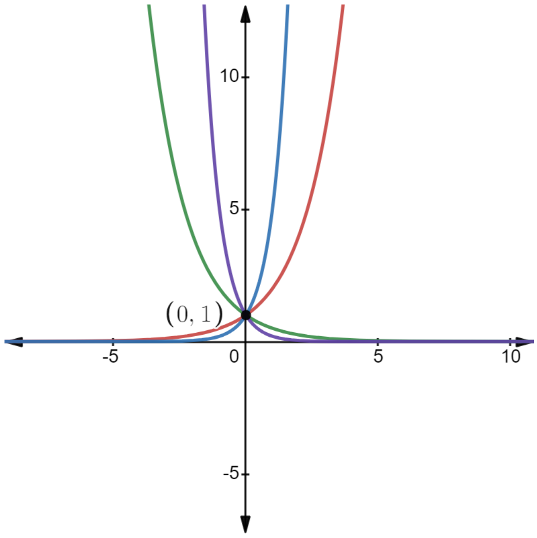


Figure 12: Graph of exponential functions $y = 0.2^x, 0.5^x, 2^x, 5^x$
. Can you justify them? Besides, what do you discover?

Exponential Function: $f(x) = a^x$		
	$0 < a < 1$	$a > 1$
Properties	$x \in \mathbb{R}, f(x) \in (0, +\infty)$	
	$a^0 = 1 \implies x = 0, y = 1$, all functions pass through (0,1).	
	$a^1 = a \implies x = 1, y = a$	
	monotonically decreasing	monotically increasing
	$\begin{cases} a^x > 1, & x < 0 \\ 0 < a^x < 1, & x > 0 \end{cases}$	$\begin{cases} 0 < a^x < 1, & x < 0 \\ a^x > 1, & x > 0 \end{cases}$
	neither odd nor even function	

§1.8 Logarithmic Function

§1.8.1 Basic Logarithm

Definition 1.14 (Logarithm)**.** logarithm is the inverse function to exponentiation. Suppose we have $y = a^x$, then $x = \log_a(y)$. Logarithm operation is used to solve the exponent when the base and power are given.

Some formulas for computing exponents:

$$\begin{aligned}
\log_a(MN) &= \log_a M + \log_a N \\
\log_a \frac{M}{N} &= \log_a M - \log_a N \\
\log_a M^\alpha &= \alpha \log_a M \\
a^{\log_a N} &= N \\
\log_{a^n} M &= \frac{1}{n} \log_a M \\
\log_a M^b &= b \log_a M \\
\log_a M &= \frac{\log_c M}{\log_c a}, c \in \mathbb{R}^+/\{1\} \\
\log_a b &= \frac{1}{\log_b a}
\end{aligned}$$

Since logarithm is the inverse function to exponentiation, we know that the diagram of logarithmic function and exponential function are symmetric about $y = x$ according to the content discussed above in §Inverse Function.

Definition 1.15 (Logarithmic Function)**.** Logarithmic functions are functions in the form

$$f(x) = \log_a(x), a \in \mathbb{R}^+/\{1\}$$

where a is a constant.

Can you explore the impact of different a values on the diagram from what is given for the exponential function? This is left as an exercise for readers. [Answer is on the next page, do not look until you think you get the answer.]

<table>
<tr><td colspan="3">Logarithmic Function: $f(x) = \log_a(x)$</td></tr>
<tr><td></td><td>$0 < a < 1$</td><td>$a > 1$</td></tr>
<tr><td rowspan="6">Properties</td><td colspan="2">$x \in (0, +\infty), f(x) \in \mathbb{R}$</td></tr>
<tr><td colspan="2">$\log_a(1) = 0 \implies x = 1, y = 0$, all functions pass through (1,0).</td></tr>
<tr><td colspan="2">$\log_a(a) = 1 \implies x = a, y = 1$</td></tr>
<tr><td>monotonically decreasing</td><td>monotically increasing</td></tr>
<tr><td>$\begin{cases} \log_a(x) > 0, & 0 < x < 1 \\ \log_a(x) \le 0, & x \ge 1 \end{cases}$</td><td>$\begin{cases} \log_a(x) < 0, & 0 < x < 1 \\ \log_a(x) \ge 0, & x \ge 1 \end{cases}$</td></tr>
<tr><td colspan="2">neither odd nor even function</td></tr>
</table>

§1.9 Trigonometry

§1.9.1 Unit circle and the trigonometric ratios

In middle school, we have learned the basic definition of sin, cos, and tan of an acute angle, and we can memorize the value of these for specific angles like 30°, 45°, and 60°.

In high school, we need to extend the scope from acute angles to all angles. When we are calculating the sin of an acute angle, we put it into a right-angled triangle and calculate the ratio of the opposite side to the hypotenuse. Consider if we can use this method in the plane coordinate system.

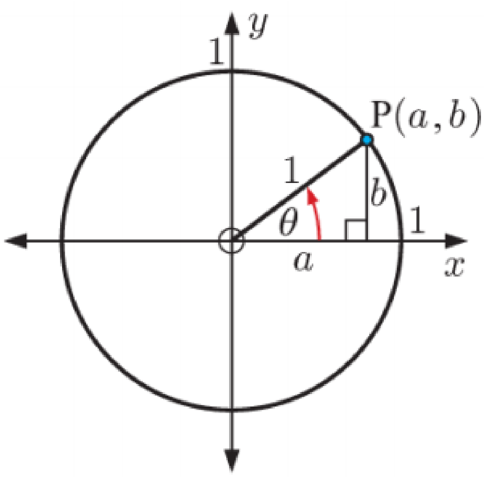


Figure 13: Unit Circle

Suppose point P lies on the circle so that segment OP makes angle θ with the positive x-axis, and θ is always measured in the anticlockwise direction.

In figure 13, it is obvious that $\sin\theta = b/1 = b$ and $\cos\theta = a/1 = a$ while the coordinate of point P is (a, b).

In fact, for any angle θ: $sin\theta$ is the y coordinate of P, $cos\theta$ is the x coordinate of P, $tan\theta$ is the slope of OP.

As a result, we can determine the sign of basic trigonometric ratio for any angle.

§1.9.2 Degree and Radian

Definition 1.16 (Radian)**.** A radian is a measurement of an angle based on the radius of a circle. 1 radian is the angle that is subtended by an arc that has a length equal to the radius of the circle.

Therefore, the angle of π radians has an arc that has a length equal to π times the radius of the circle, meaning that π radians=180°.

Suggestion: Degrees barely appear in calculus. Remember always to use radians.

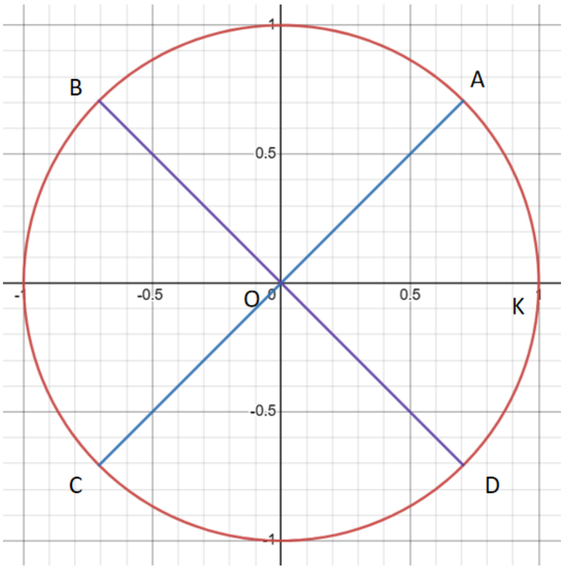


Figure 14: Can you write the sin, cos, and tan of 135°, 225°, 315°?(sin $AOK = \sqrt{2}/2$, cos $AOK = \sqrt{2}/2$)

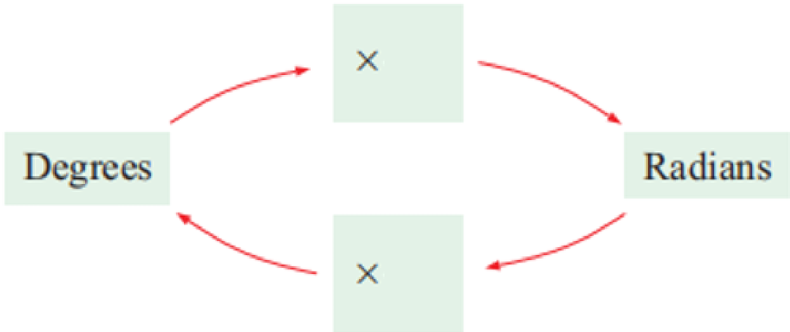


Figure 15: Can you convert degrees to radians and radians to degrees?

§1.9.3 Trigonometric function

According to the transformation of functions, try to answer the following information of a general sine function: $y = a\sin(b(x-c)) + d, b > 0$

Amplitude:

Period:

Principle axis: y=

Maximum point:

Minimum point:

Now, try to draw the following functions on the graph:

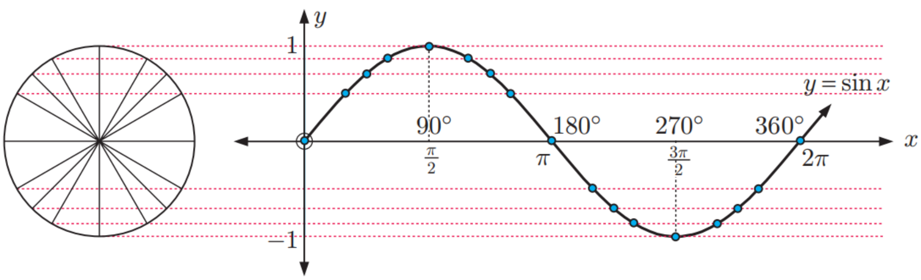


Figure 16: Imagine why the sine function look like this.

The wave oscillates about a horizontal line called the **principal axis** or **mean line** which has equation $\boldsymbol{y = \frac{\max + \min}{2}}$.

A **maximum point** occurs at the top of a crest, and a **minimum point** at the bottom of a trough.

The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

$$\textbf{amplitude} = \frac{\textbf{max} - \textbf{min}}{\textbf{2}}$$

Figure 17: Definitions of elements in periodic wave function

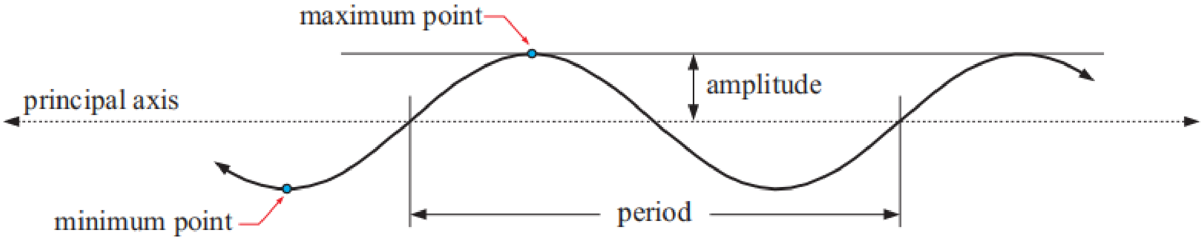


Figure 18: Diagram of periodic wave function

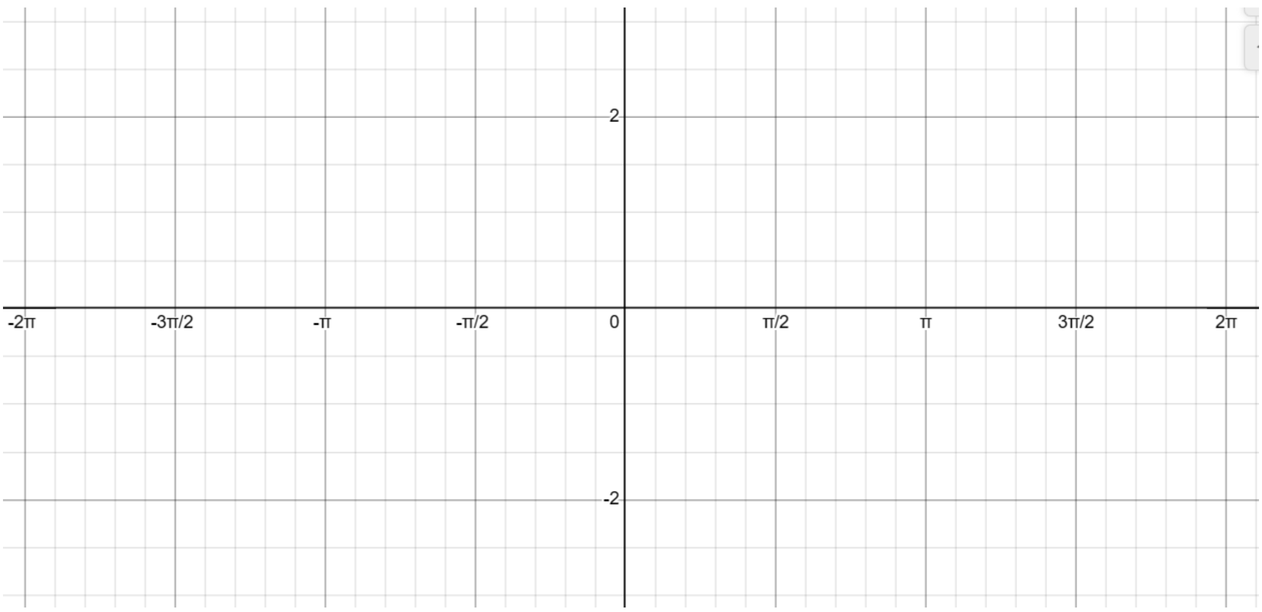


Figure 19: $y = \sin x$, $y = \cos x$, and $y = \tan x$

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

Figure 20: Definition of the reciprocal trigonometric functions

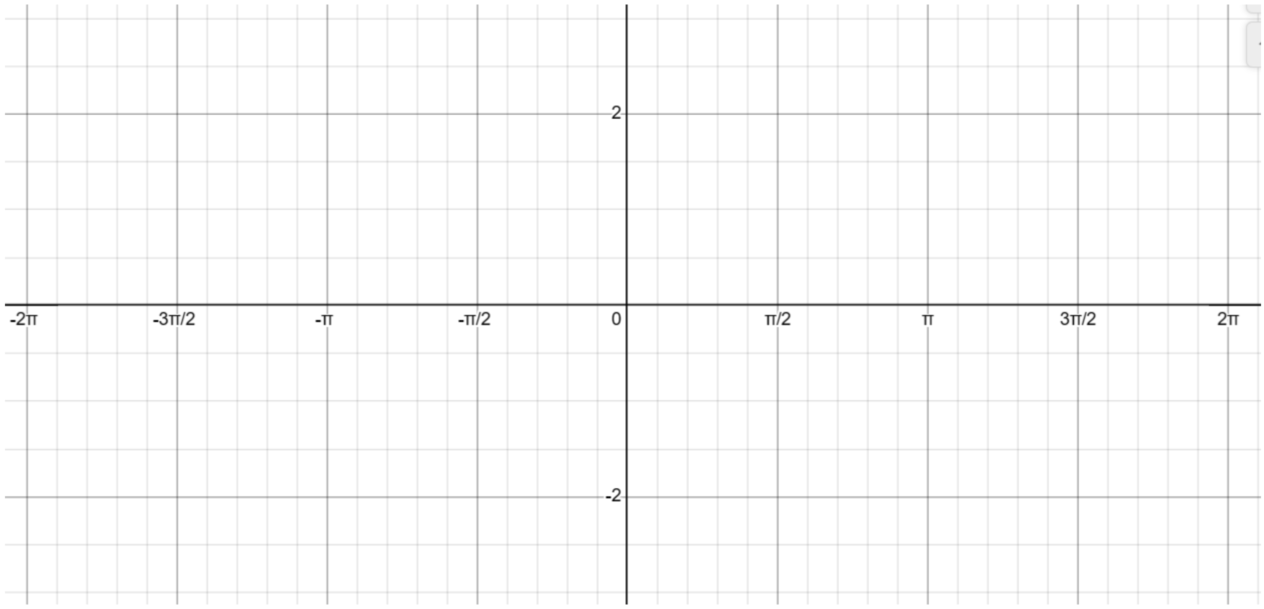


Figure 21: $y = \csc x$, $y = \sec x$, and $y = \cot x$

Definition 1.17 (Inverse Trigonometric Functions)**.**

$$\arcsin x = \sin^{-1} x, \arccos x = \cos^{-1} x, \arctan x = \tan^{-1} x$$

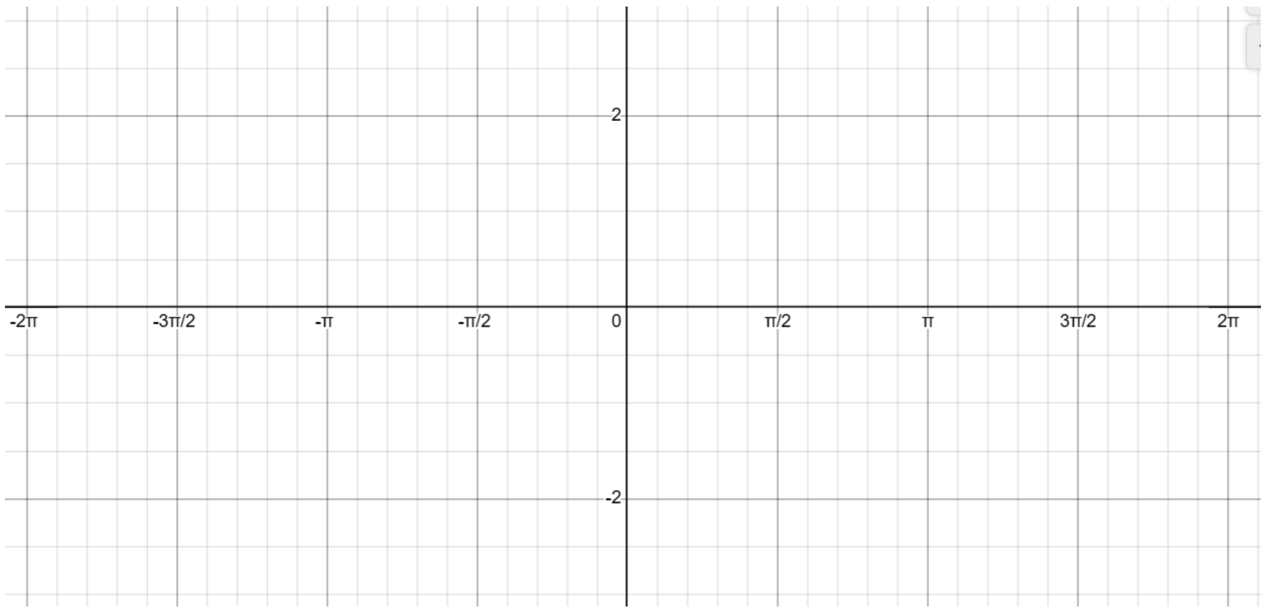


Figure 22: $y = \arcsin x$, $y = \arccos x$, and $y = \arctan x$

§1.9.4 Trigonometric Identities

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\tan^2(x) + 1 &= \sec^2(x) \\
\cot^2(x) + 1 &= \csc^2(x) \\
\sin(x) &= \cos(\pi/2 - x) \\
\tan(x) &= \cot(\pi/2 - x) \\
\sec(x) &= \csc(\pi/2 - x) \\
\sin(A + B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \\
\cos(A + B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\
\tan(A + B) &= \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}
\end{aligned}$$

Try to prove these identities using learned information, especially unit circle.
Try to deduce the following identities.

$$\begin{aligned}
\sin(2A) &= 2\sin(A)\sin(B) \\
\cos(2A) &= \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A) \\
\tan(2A) &= \frac{2\tan(A)}{1 - \tan^2(A)}
\end{aligned}$$

$$\begin{aligned}
\sin\alpha\cos\beta &= \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2} \\
\cos\alpha\sin\beta &= \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2} \\
\cos\alpha\cos\beta &= \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2} \\
\sin\alpha\sin\beta &= -\frac{\cos(\alpha+\beta) - \cos(\alpha-\beta)}{2}
\end{aligned}$$

$$\begin{aligned}
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha - \sin\beta &= 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}
\end{aligned}$$

References

[1] Mathematics for the International Student 10E (MYP 5 Extended) - Haese Mathematics
[2] Mathematics: Analysis and Approaches HL - Haese Mathematics