Limit

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§1.1 Limit

§1.1.1 Definition of Limit

Question 1.1. Why do we invent *limit*?

Example 1.2 (Achilles and the tortoise)

In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. - *Aristotle, Physics VI:9, 239b15*

In the paradox of Achilles and the tortoise, Achilles is in a footrace with the tortoise. Achilles allows the tortoise a head start of 100 meters, for example. Suppose that each racer starts running at some constant speed, one faster than the other. After some finite time, Achilles will have run 100 meters, bringing him to the tortoise's starting point. During this time, the tortoise has run a much shorter distance, say 2 meters. It will then take Achilles some further time to run that distance, by which time the tortoise will have advanced farther; and then more time still to reach this third point, while the tortoise moves ahead. Thus, whenever Achilles arrives somewhere the tortoise has been, he still has some distance to go before he can even reach the tortoise.

It seems to be counter-intuitive but convincing, but how do you know that it is actually a paradox?

Definition 1.3 (An Intuitive Definition of Limit)**.** We say $\lim_{x \to a} f(x) = L$ if $f(x)$ approaches L when x approaches a.

But what does it mean by *approaches* but not *reach*? This is an ambiguous definition of limit, even though this is what we studied in IB. Here we also introduce another rigorous definition of limit known as *the* $\delta - \varepsilon$ *definition of a limit*.

Definition 1.4 (The $\delta - \varepsilon$ Definition of A Limit)**.** Let $f(x)$ be a function defined on an open interval around x_0 $\big(f(x_0)$ need not be defined$\big)$. We say that the limit of $f(x)$ as x approaches x_0 is L, i.e.

$$\lim_{x \to x_0} f(x) = L,$$

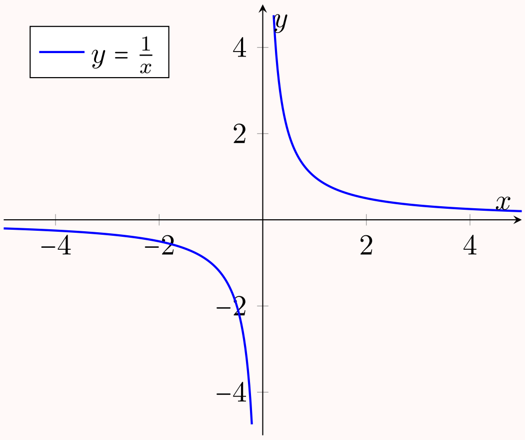
if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$

This is a better definition of limit. Why? It does not need complicated $\lim_{x \to x_0^+} f(x)$ or $\lim_{x \to x_0^-} f(x)$ as already included in the absolute value, and also it gives a correct definition of *approaches but not reaches.* Also, it implies that $L \in \mathbb{R}$ since that if L is not a real number, the difference between $f(x)$ and L cannot be compared.

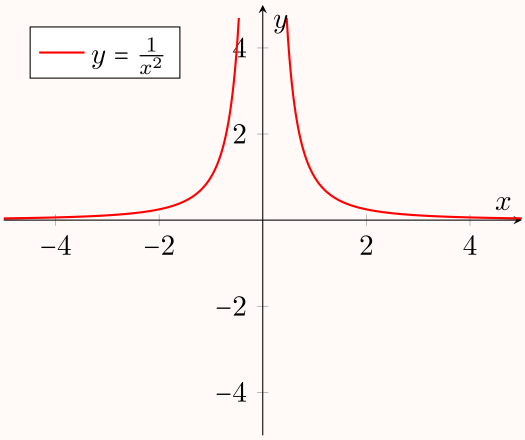
Example 1.5 (An example that shows the reason of using $\delta - \varepsilon$ definition)

Consider $f(x) = \dfrac{1}{x}$.



When x is approaching 0 from the right side, marked with $\lim\limits_{x\to 0^+} \dfrac{1}{x}$ should be $+\infty$ and $\lim\limits_{x\to 0^-} \dfrac{1}{x}$ should be $-\infty$ (even though as stated before, it is not that rigorous because we do not typically state that a limit is infinity). By $\lim\limits_{x\to 0^+} \dfrac{1}{x} \neq \lim\limits_{x\to 0^-} \dfrac{1}{x}$, we know that it does not exist.

However, what about $g(x) = \dfrac{1}{x^2}$?



If we just consider the intuitive definition, we say $\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = \lim\limits_{x\to 0^-} \dfrac{1}{x^2} = +\infty$, and this cause $\lim\limits_{x\to 0} \dfrac{1}{x^2} = +\infty$ but it's actually not true according to $\delta - \varepsilon$ definition since we cannot define a constant value $+\infty - f(x)$ as ε.

Remark 1.6. When the domain of a function $f(x)$ is (a, b), $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} f(x)$

§1.1.2 Existence of Limit

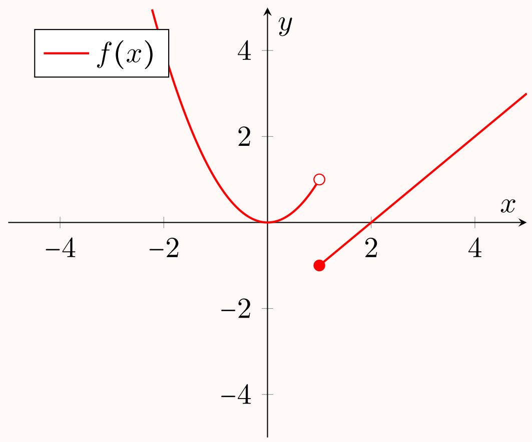
According to the definition of limit (either the intuitive one or $\delta - \varepsilon$ description), a few conditions can obviously considered where the limit does not exist.

The first condition is shown in Example 2.5, for both $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$. We say there exists a **break** when $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ does not exist, specifically when they approach $\pm\infty$.

When the left and right limit does exist, but are equal to different values, this is called a **jump**.

Example 1.7 (Example of Jump)

We define $h(x) = \begin{cases} x^2, & x < 1 \\ x - 2, & x \geq 1 \end{cases}$, and the diagram of $h(x)$ is shown below.

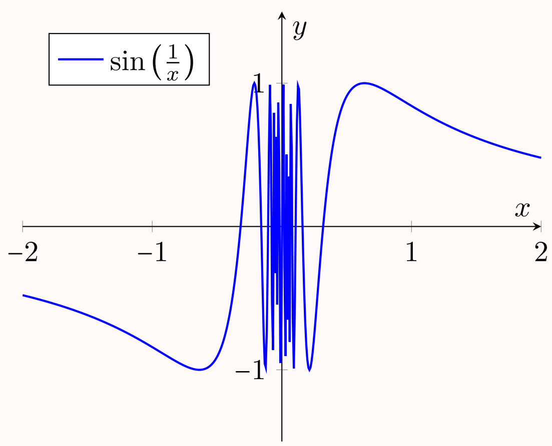


Here in the diagram, we know that $\lim_{x \to 1^+} h(x) = -1$ and $\lim_{x \to 1^-} h(x) = 1$ and they are different, therefore $\lim_{x \to 1} h(x)$ does not exist.

The 3rd possibility is **oscillation**

Example 1.8 (Oscillation)

Here we provide a function $k(x) = \sin\left(\frac{1}{x}\right)$, known as the *Notorious Oscillating Function* for its difficulty of plotting.



Intuitively, we know that $\lim_{x \to 0} k(x)$ does not exist.

Question 1.9. Can you explain it using $\delta - \varepsilon$ definition of limit?

§1.1.3 Limit Laws

There exist a few limit laws.

> **Remark 1.10.** For $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right)$, when l and m equals to ∞ or 0 at the same time, *L' Hospital Theorem* should be used.

Consider function $f(x)$ and $g(x)$ for which $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, where $a, l, m \in \mathbb{R}$.

- $\lim_{x\to a}(f(x) \pm g(x)) = l \pm m$
- $\lim_{x\to a} f(x)g(x) = lm$
- $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{l}{m}$ when provided $m \neq 0$

§1.2 Continuity

Example 1.11 (The Meaning of Continuity - A Classic Mathematics Modelling Question)
When a normal desk, with 4 feet and equal length is placed on uneven ground, it normally stands with 3 feet. However, it can be moved to where it can stand with 4 feet, why is that? Can you prove it?

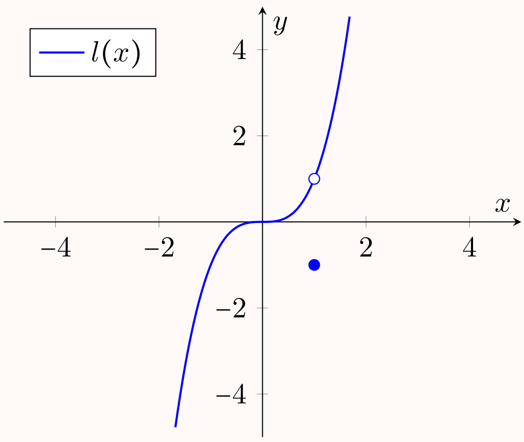
Definition 1.12 (Continuity)**.** We say a function is **continuous** when $\lim_{x\to a} f(x) = f(x)$.

Directly from the definition can we derive

Claim 1.13 — $\lim_{x\to a} f(x)$ DNE $\implies$ $f(x)$ is not continuous.

When $\lim_{x\to a} f(x)$ does exist but is not equal to $f(x)$, the discontinuity is defined as a **hole**.

Example 1.14 (Example of Hole)
We define $l(x) = \begin{cases} x^3, & x \neq 1 \\ -1, & x = 1 \end{cases}$



Both $\lim_{x\to 1^+} l(x)$ and $\lim_{x\to 1^-} l(x)$ are 1 while $l(1) = -1$.

Hole discontinuity is **removable**, which means can be changed to a continuous function just by changing $l(x)$ to $l'(x) = x^3, x \in \mathbb{R}$, while other discontinuity that is caused by undefined limit is irremovable.

§1.3 Two Special Limit

§1.3.1 $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (x in radians)

Theorem 1.15 (Sandwich Theorem (Squeeze Theorem))

Let I be an interval containing the point a. Let g, f and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have

$$g(x) \leq f(x) \leq h(x)$$

and also suppose that

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then $\lim_{x \to a} f(x) = L$.

First, try a few numerical values:

x	$\sin x$
1	0.84
0.8	0.72
0.6	0.56
0.4	0.39
0.2	0.199

it seems that when x approaches 0, $\sin x$ and x are closer and closer.

Proposition 1.16

When x approaches 0, $\frac{\sin x}{x}$ approaches 1, equivalent to

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

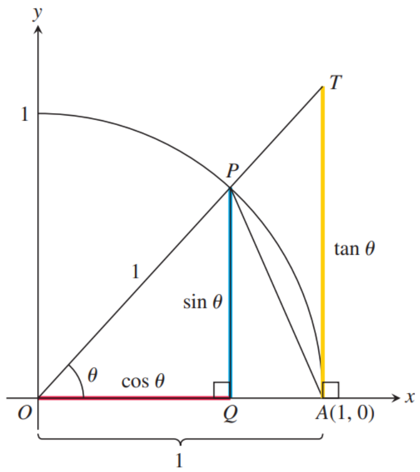


Figure 1: $\sin x / x$ proof using Sandwich Theorem

Proof. First, consider the Sandwich Theorem, we compute the areas of $\triangle OAP$, sector OAP, and $\triangle OAT$:

$$S_{\triangle OAP} = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin\theta = \frac{1}{2} \cdot \sin\theta$$

$$S_{Sector\ OAP} = \frac{1}{2} \cdot 1 \cdot 1 \cdot \theta = \frac{\theta}{2}$$

$$S_{\triangle OAT} = \frac{1}{2} \cdot 1 \cdot \tan\theta = \frac{1}{2} \cdot \tan\theta$$

Thus, referring to the diagram, we know $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$.

By dividing $\sin\theta$ on every part of the inequality and then taking reciprocals, we can get a new one with $\frac{\sin\theta}{\theta}$.

Since $\theta \to 0^+$, $\cos\theta < \frac{\sin\theta}{\theta} < 1$.

Since $\lim\limits_{\theta\to 0^+}\cos x = 1$, $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ by Sandwich (Squeeze) Theorem. □

Remark 1.17. The proof of $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ cannot use the L' Hospital Theorem, it will cause circular proof indeed.

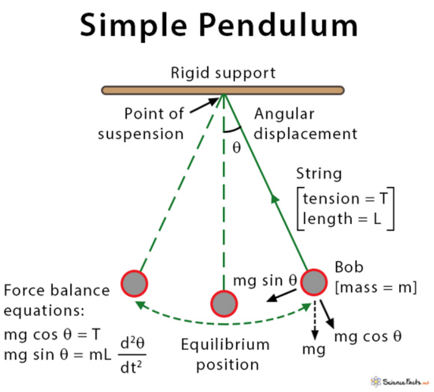


Figure 2: Application (Simple harmonic motion)

Exercise 1.18. $\lim\limits_{x\to 0}\frac{\sin Ax}{\sin Bx} = \frac{\sin Ax}{Ax}\cdot\frac{Bx}{\sin Bx}\cdot\frac{A}{B} = A/B$ (A and B are constants not equal to 0)

$\lim\limits_{x\to 0}\frac{\tan x}{5x} = \frac{\sin x}{\cos x \cdot 5x} = \frac{1}{5}$

§1.3.2 $\lim\limits_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$

Definition 1.19. The number e, also known as Euler's Number, is an irrational number, with a numerical value of 2.718281828459...

Example 1.20

A little story: Suppose you put 1 dollar in a bank. The annual interest rate is 100%, but if you take the money twice a year, the interest rate becomes 50%, and so on... Can you have infinite money?

n	$(1+\frac{1}{n})^n$
2	2.25
5	2.49
10	2.59
20	2.65
100	2.70

Your money will approach a value, which is e.

Remark 1.21. Another explanation of e:

$$e = \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \ldots$$

Proof. Binomial expansion is used in the proof.

Theorem 1.22 (Binomial Expansion)

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n$$

where $\binom{n}{r} = C_r^n = \dfrac{n!}{r!(n-r)!}$.

Notice that the Euler's Number e is the key to continuity! ☐

Exercise 1.23. Compute $\lim_{n\to\infty}\left(1+\frac{1}{kn}\right)^n = \left(1+\frac{1}{kn}\right)^{kn/k}$.

Solution.

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{1}{kn}\right)^n &= \left(1+\frac{1}{kn}\right)^{kn/k}\\ &= \boxed{e^{\frac{1}{k}}}\end{aligned}$$

☐

Exercise 1.24. Compute $\lim_{x\to\infty}\left(\frac{x+7}{x-7}\right)^x = \left(1+\frac{14}{x-7}\right)^x$.

Solution.

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x+7}{x-7}\right)^x &= \left(1+\frac{14}{x-7}\right)^x\\ &= \left(1+\frac{14}{x-7}\right)^{\frac{x-7}{14}\cdot\frac{14}{x-7}\cdot x}\\ &= e^{\lim_{x\to\infty}\frac{14x}{x-7}}\\ &= \boxed{e^{14}}\end{aligned}$$

☐

References and Extended Reading Materials

[1] Mathematics: Analysis and Approaches HL - Haese Mathematics

[2] Thomas' Calculus

[3] Epsilon-delta definition of continuity - Serlo

[4] Squeeze Theorem Wikipedia

[5] Why proving $\lim_{x\to 0}\frac{\sin x}{x} = 1$ using L' Hospital is circular

[6] Simple Pendulum

[7] Binomial Expansion