Calculus Crash Course

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Contents

§1 October 8 - Derivative

§1.1 Derivative Function

Definition 1.1 (Derivative Function)**.** Gradient function, gradient of the tangent for the original function, of $y = f(x)$ is called its derivative function and is labelled $f'(x)$ or $\frac{dy}{dx}$

Exercise 1.2. What is the derivative function of $y = 3$ and $y = 2x$?

§1.2 First principle

Question 1.3. What is the gradient of a line if A $(a, f(a))$ and B $(a+h, f(a+h))$ are on the line?

Claim 1.4 — When A and B gets infinitely close, the gradient is the gradient of the tangent for $y = f(x)$ where $x = a$.

Definition 1.5 (First principle)**.** The derivative function is defined as: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

Exercise 1.6. Compute $y = 2x$, $y = 3x^2$ using first principle.

Exercise 1.7. Prove that $\frac{d}{dx} x^n = nx^{n-1}$ using first principle.

Exercise 1.8. Prove that if $f(x) = cu(x)$, then $f'(x) = cu'(x)$ using first principle.

Exercise 1.9. Prove that if $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$ using first principle.

§1.3 Differentiability

Definition 1.10. If the limit $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists, $f(x)$ is differentiable at $x = a$.

Claim 1.11 — If f is differentiable at $x = a$, then f is also continuous at $x = a$.

Proof.

$$
\begin{aligned}
&\lim_{h\to 0} f(a+h) - f(a) \\
=&\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \times h \\
=&\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \times \lim_{h\to 0} h \qquad \{\text{by the limit laws, since both limits exist}\} \\
=&f'(a) \times 0 \\
=&0
\end{aligned}
$$

Therefore, $\lim_{h\to 0} f(a+h) = f(a)$

Letting $x = a + h$, this is equivalent to $\lim_{x\to a} f(x) = f(a)$.

Therefore, f is continuous at $x = a$. □

So we can conclude the way to test for differentiability:

Proposition 1.12 (Test for Differentiability)
A function f with domain D is **differentiable at $\boldsymbol{x = a}$**, $a \in D$, if:

1. f is continuous at $x = a$, and
2. $f'_-(a) = \lim_{h \to 0^-} \dfrac{f(a+h) - f(a)}{h}$ and $f'_+(a) = \lim_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}$ both exist and are equal.

§1.4 Fundamental rules of differentiation

We have learned from former exercise that if $f(x) = cu(x)$, then $f'(x) = cu'(x)$, and if $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.

Then we can start thinking about the $f'(x)$ when $f(x) = u(x)v(x)$ or $f(x) = \dfrac{u(x)}{v(x)}$. Try to deduce the formula by using first principle.

Theorem 1.13 (The Product Rule)
If $f(x) = u(x)v(x)$, then $f'(x) = u'(x)v(x) + u(x)v'(x)$. Alternatively, if $y = uv$ where u and v are functions of x, then

$$\frac{dy}{dx} = u'v + uv' = \frac{du}{dx}v + u\frac{dv}{dx}$$

Theorem 1.14 (The Quotient Rule)
If $Q(x) = \dfrac{u(x)}{v(x)}$, then $Q'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$. Alternatively, if $y = \dfrac{u}{v}$ where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

The rules about calculations between simple functions are all listed and the next and maybe the most important rule is the chain rule.

Definition 1.15 (Chain rule)**.** Version 1: If $y = g(u)$ where $u = f(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
Version 2: If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$

Proof.

$$\begin{aligned}
\frac{dy}{du} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta u}\frac{\delta u}{\delta x} \\
&= (\lim_{\delta x \to 0} \frac{\delta y}{\delta u})(\lim_{\delta x \to 0} \frac{\delta u}{\delta x}) \\
&= (\lim_{\delta u \to 0} \frac{\delta y}{\delta u})(\lim_{\delta x \to 0} \frac{\delta u}{\delta x}) \\
&= \frac{dy}{du}\frac{du}{dx}
\end{aligned}$$

□