Geometry

G1. A rectangular garden has a length that is twice its width. The dimensions are increased so that the perimeter is doubled and the new shape is a square with an area of 3600 square feet. What was the area of the original garden, in square feet? $\boxed{800}$. [Let w be the width of the original rectangular garden. The perimeter of the rectangle is $2(w+2w) = 6w$, so the perimeter of the square is $12w$. The dimensions of the square are $3w \times 3w$ and its area is $(3w)(3w) = 9w^2$, so set $9w^2 = 3600$ ft.^{2} to find $w^2 = 400$ square feet. The area of the original rectangle is $(2w)(w) = 2w^2 = 2 \cdot 400$ ft.$^2 = \boxed{800}$ square feet.]

G2. In $\triangle XYZ$, we have $\angle X = 90°$ and $\tan Z = 7$. If $YZ = 100$, then what is XY? $\boxed{70\sqrt{2}}$. [Since $\triangle XYZ$ is a right triangle with $\angle X = 90°$, we have $\tan Z = \frac{XY}{XZ}$. Since $\tan Z = 7$, we have $XY = 7k$ and $XZ = k$ for some value of k, as shown in the diagram. Applying the Pythagorean Theorem gives $(7k)^2 + k^2 = 100^2$, so $50k^2 = 100^2$, which gives $k^2 = 100^2/50 = 200$. Since k must be positive, we have $k = \sqrt{200} = 10\sqrt{2}$, so $XY = 7k = \boxed{70\sqrt{2}}$.]

G3. A right cylinder with a base radius of 3 units is inscribed in a sphere of radius 5 units. The total volume, in cubic units, of the space inside the sphere and outside the cylinder is $W\pi$. Find W, as a common fraction. $\boxed{284/3}$. [A diagonal drawn in the cylinder will have length 10, which is the diameter of the sphere. We can see that a 6-8-10 right triangle is formed by the height of the cylinder, the diameter of the sphere, and the diameter of the base of the cylinder. Now that we know the height of the cylinder, we have everything we need to compute the desired volume:

$$V_{sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3} \cdot \pi \cdot 5^3 = \frac{500\pi}{3}$$

$$V_{cylinder} = \pi r^2 \cdot h = \pi \cdot 3^2 \cdot 8 = 72\pi.$$

The volume inside the sphere and outside the cylinder is the difference of the above values:

$$V_{sphere} - V_{cylinder} = \frac{500\pi}{3} - 72\pi = \frac{500\pi - 216\pi}{3} = \boxed{\frac{284}{3}}\pi.$$

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G4. In triangle ABC, the angle bisectors are AD, BE, and CF, which intersect at the incenter I. If $\angle ACB = 38°$, then find the measure of $\angle AIE$, in degrees. $\boxed{71}$. [Since AD is an angle bisector, $\angle BAI = \angle BAC/2$. Since BE is an angle bisector, $\angle ABI = \angle ABC/2$. As an angle that is external to triangle ABI, $\angle AIE = \angle BAI + \angle ABI = \angle BAC/2 + \angle ABC/2$.

Since $\angle ACB = 38°$,

$$\angle AIE = \frac{\angle BAC + \angle ABC}{2} = \frac{180° - \angle ACB}{2} = \frac{180° - 38°}{2} = \boxed{71°}.$$

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G5. In triangle ABC, let I be the incenter of triangle ABC. The line through I parallel to BC intersects AB and AC at M and N, respectively. If $AB = 17$,

$AC = 24$, and $BC = 33$, then find the perimeter of triangle AMN. $\boxed{41}$. [Since MN is parallel to BC, $\angle MIB = \angle IBC$. But BI is an angle bisector, so $\angle IBC = \angle IBM$. Hence, triangle MIB is isosceles with $MI = MB$. By the same argument, triangle NIC is isosceles, with $NI = NC$.

Therefore, the perimeter of triangle AMN is simply

$$\begin{aligned} AM + AN + MN &= AM + AN + MI + NI \\ &= AM + AN + MB + NC \\ &= (AM + MB) + (AN + NC) \\ &= AB + AC \\ &= 17 + 24 \\ &= \boxed{41}. \end{aligned}$$

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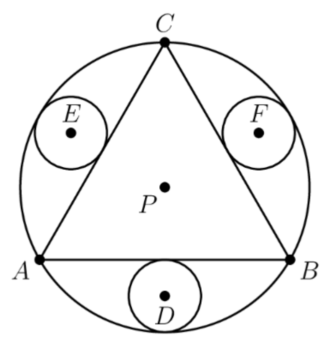
G6. Equilateral triangle ABC has side length 6 cm and is inscribed in circle P. Congruent smaller circles centered at D, E and F are inscribed in the three regions between an arc of circle P and a side of $\triangle ABC$, as shown. If segments AF, BE and CD all intersect P, what is the area of $\triangle DEF$? Express your answer as a common fraction in simplest radical form.



$\boxed{81\sqrt{3}/16}$. [Trivial by calculation, QED.]

G7. Let $ABCDEF$ be a regular hexagon, and let P be a point inside quadrilateral $ABCD$. If the area of triangle PBC is 20, and the area of triangle PAD is 23, compute the area of hexagon $ABCDEF$. $\boxed{189}$. [Hint: Suppose $AB = x$ and use the characteristic of regular hexagon and $AD = 2BC$, by bashing the area is 189.]

Combinatorics

C1. Two cards are dealt at random from a standard deck of 52 cards. What is the probability that the first card is a ♡ and the second card is a ♣? $\boxed{13/204}$. [The probability that the first card is a ♡ is $\frac{1}{4}$. The second card then has a probability of $\frac{13}{51}$ of being ♣. So the answer is $\frac{1}{4} \times \frac{13}{51} = \boxed{\frac{13}{204}}$.]

C2. What is the probability that when we roll 5 fair 6-sided dice, at most 4 of them will show a 1? $\boxed{7775/7776}$. [The only way that more than four can show 1 is if all 5 dice show 1, and the probability of that happening is $\frac{1}{6^5}$. Thus the answer is $1 - \frac{1}{6^5} = \boxed{\frac{7775}{7776}}$.]

C3. Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

$\boxed{62}$. [The first guide can take any combination of tourists except all the tourists or none of the tourists. Therefore the number of possibilities is

$$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} = 6 + 15 + 20 + 15 + 6 = 62.$$

OR

If each guide did not need to take at least one tourist, then each tourist could choose one of the two guides independently. In this case there would be $2^6 = 64$ possible arrangements. The two arrangements for which all tourists choose the same guide must be excluded, leaving a total of $64 - 2 = \boxed{62}$ possible arrangements.]

C4. Joan tries to solve a really hard problem once each day. She has a 1/4 probability of solving it each day. What is the probability that she will solve it before her sixth try?

$\boxed{781/1024}$. [We must find the probability that Joan can solve it at any time before the sixth try, so it is the sum of the probabilities that she will solve it on her first, second, third, fourth, and fifth tries. We could evaluate all those cases, but seeing all that casework, we wonder if it will be easier to find the probability that she fails to solve it before 6 tries, and subtract the result from 1.

In order for her to fail to solve it before her sixth try, she must fail 5 times. The probability of failure on each try is $1 - \frac{1}{4} = \frac{3}{4}$, so the probability that she fails on each of her first 5 tries is $\left(\frac{3}{4}\right)^5 = \frac{243}{1024}$. Therefore, the probability that she succeeds before her sixth try is

$$1 - \frac{243}{1024} = \frac{781}{1024}$$

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C5. A manufacturer of airplane parts makes a certain engine that has a probability p of failing on any given fight. There are two planes that can be made with this sort of engine, one that has 3 engines and one that has 5. A plane crashes if more than half its engines fail. For what values of p do the two plane models have the **same probability** of crashing? $\boxed{\text{0 or 1 or 1/2}}$. [Consider which ones fail when they crash, for 5 engine ones there exist 3 conditions, either 3, 4, 5 of the 5 engines failed. For the 3 engine one there exist 2 conditions, either 2 or 3 of the 3 engines failed. Therefore, the formula

$$\binom{5}{3}p^3(1-p)^2+\binom{5}{4}p^4(1-p)+\binom{5}{5}p^5=\binom{3}{2}p^2(1-p)+\binom{3}{3}p^3$$

expanding it and we can have

$$6p^5-15p^4+12p^3-3p^2=0,$$

therefore calculating the roots, either $p=1,0,$ or $\frac{1}{2}$.]

C6. A standard deck of cards contains 52 cards. These 52 cards are arranged in a circle, at random. Find the expected number of pairs of adjacent cards that are both hearts.

$\boxed{52/17}$. [The number of pairs of adjacent cards which are both hearts is equal to the number of hearts which have another heart card to their right. For each heart card, there is a $\frac{12}{51}$ chance that the card to its right is also hearts, giving 1 pair, and a $\frac{39}{51}$ chance that the card to its right is not hearts, giving 0 pairs. There are 13 hearts, so the expected value of the number of pairs of adjacent hearts is

$$13\left(\frac{12}{51}(1)+\frac{39}{51}(0)\right)=\boxed{\frac{52}{17}}.$$

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Algebra & Number Theory

A1. Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2-y^2=m^2$ for some positive integer m. What is $x+y+m$? $\boxed{154}$. [Let $x=10a+b$; $y=10b+a$. The given conditions imply $x>y$, which implies $a>b$, and they also imply that both a and b are nonzero. Then $x^2-y^2=(x-y)(x+y)=(9a-9b)(11a+11b)=99(a-b)(a+b)=m^2$.

Since this must be a perfect square, all the exponents in its prime factorization must be even. 99 factorizes into $3^2\cdot 11$, so $11\mid(a-b)(a+b)$. However, the maximum value of $a-b$ is $9-1=8$, so $11\mid a+b$. The maximum of $a+b$ is $9+8=17$, so $a+b=11$. Then we have $33^2(a-b)=m^2$, so $a-b$ is a perfect square, but the only perfect squares that are within our bound on $a-b$ are 1 and 4. We know $a+b=11$, and, for $a-b=1$, adding equations to eliminate b gives us $2a=12\implies a=6,b=5$. Testing $a-b=4$ gives us $2a=15\implies a=15/2,b=7/2$, which is impossible, as a and b must be digits. Therefore, $(a,b)=(6,5)$, and $x+y+m=65+56+33=154$.]

A2. Find the maximum of $44 \cdot 46^x + 46 \cdot 44^x - 2024^x$ over all real numbers x. $\boxed{2024}$. [

$$\begin{aligned} &44 \cdot 46^x + 44^x \cdot 46 - 2024^x \\ =&(46^x - 46)(44^x - 44) + 2024 \end{aligned}$$

$(46^x - 46)(44^x - 44) \geq 0$ and reaches 0 when $x = 1$, therefore, $44 \cdot 46^x + 46 \cdot 44^x - 2024^x$ reaches its maximum when $x = 1$, and $\max 44 \cdot 46^x + 46 \cdot 44^x - 2024^x = \boxed{2024}$.]

A3. Find all positive integers n for which

$$\sqrt{\binom{n}{3}} - \sqrt{\binom{n}{2}} = 105$$

where $\binom{n}{k}$ denotes combinations of n taken k at a time. $\boxed{50}$. [

$$\begin{aligned} \sqrt{\binom{n}{3}} - \sqrt{\binom{n}{2}} &= \sqrt{\frac{n!}{3!(n-3)!}} - \sqrt{\frac{n!}{2!(n-2)!}} \\ &= \sqrt{\frac{n(n-1)(n-2)}{6}} - \sqrt{\frac{n(n-1)}{2}} \\ &= \left(\sqrt{\frac{n-2}{3}} - 1\right)\sqrt{\frac{n(n-1)}{2}} = 105 \end{aligned}$$

$105 = 3 \cdot 5 \cdot 7$, consider the case where $\sqrt{\frac{n-2}{3}} - 1$ and $\sqrt{\frac{n(n-1)}{2}}$ are both positive integers, 105 can be factorized as

$$1 \cdot 105, 3 \cdot 35, 5 \cdot 21, 7 \cdot 15$$

Observe that $\frac{n(n-1)}{2} > (\frac{n-2}{3})^2$, $\sqrt{\frac{n(n-1)}{2}}$ should be much larger than $\sqrt{\frac{n-2}{3}} - 1$, we first try 1 and 105 which does not work through computation.

Then we try $\sqrt{\frac{n-2}{3}} - 1 = 3$, $\frac{n-2}{3} = 16, n = 50$, $\sqrt{\frac{50 \cdot 49}{2}} = 5 \cdot 7 = 35$ satisfied the condition.

Thus when $\sqrt{\binom{n}{3}} - \sqrt{\binom{n}{2}} = 105, n = \boxed{50}$.]

A4. Pairwise distinct real numbers a, b, c satisfied the equality

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

Find abc. $\boxed{\pm 1}$. [

$$\begin{aligned} a - b &= \frac{1}{c} - \frac{1}{b} = \frac{b-c}{bc} \\ b - c &= \frac{1}{a} - \frac{1}{c} = \frac{c-a}{ac} \\ c - a &= \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab} \\ \Longrightarrow (a-b)(b-c)(c-a) &= \frac{(b-c)(c-a)(a-b)}{(abc)^2}. \end{aligned}$$

As they are distinct real numbers, $a - b, b - c, c - a \neq 0 \implies (abc)^2 = 1, abc = \pm 1$]