Geometry

G1. A rectangular garden has a length that is twice its width. The dimensions are increased so that the perimeter is doubled and the new shape is a square with an area of 3600 square feet. What was the area of the original garden, in square feet?

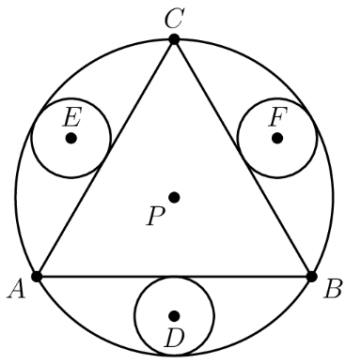
G2. In $\triangle XYZ$, we have $\angle X = 90^\circ$ and $\tan Z = 7$. If $YZ = 100$, then what is XY?

G3. A right cylinder with a base radius of 3 units is inscribed in a sphere of radius 5 units. The total volume, in cubic units, of the space inside the sphere and outside the cylinder is $W\pi$. Find W, as a common fraction.

G4. In triangle ABC, the angle bisectors are AD, BE, and CF, which intersect at the incenter I. If $\angle ACB = 38^\circ$, then find the measure of $\angle AIE$, in degrees.

G5. In triangle ABC, let I be the incenter of triangle ABC. The line through I parallel to BC intersects AB and AC at M and N, respectively. If $AB = 17$, $AC = 24$, and $BC = 33$, then find the perimeter of triangle AMN.

G6. Equilateral triangle ABC has side length 6 cm and is inscribed in circle P. Congruent smaller circles centered at D, E and F are inscribed in the three regions between an arc of circle P and a side of $\triangle ABC$, as shown. If segments AF, BE and CD all intersect P, what is the area of $\triangle DEF$? Express your answer as a common fraction in simplest radical form.



G7. Let $ABCDEF$ be a regular hexagon, and let P be a point inside quadrilateral $ABCD$. If the area of triangle PBC is 20, and the area of triangle PAD is 23, compute the area of hexagon $ABCDEF$.

Combinatorics

C1. Two cards are dealt at random from a standard deck of 52 cards. What is the probability that the first card is a ♡ and the second card is a ♣?

C2. What is the probability that when we roll 5 fair 6-sided dice, at most 4 of them will show a 1?

C3. Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

C4. Joan tries to solve a really hard problem once each day. She has a 1/4 probability of solving it each day. What is the probability that she will solve it before her sixth try?

C5. A manufacturer of airplane parts makes a certain engine that has a probability p of failing on any given fight. There are two planes that can be made with this sort of engine, one that has 3 engines and one that has 5. A plane crashes if more than half its engines fail. For what values of p do the two plane models have the **same probability** of crashing?

C6. A standard deck of cards contains 52 cards. These 52 cards are arranged in a circle, at random. Find the expected number of pairs of adjacent cards that are both hearts.

Algebra & Number Theory

A1. Find the maximum of $44 \cdot 46^x + 46 \cdot 44^x - 2024^x$ over all real numbers x.

A2. Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m. What is $x + y + m$?

A3. Find all positive integers n for which

$$\sqrt{\binom{n}{3}} - \sqrt{\binom{n}{2}} = 105$$

where $\binom{n}{k}$ denotes combinations of n taken k at a time.

A4. Pairwise distinct real numbers a, b, c satisfied the equality

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

Find abc.