Some pretty big and amazing results have been coming our way:

- Given an *MA(q)* process, we have invertibility (and hence one to one mapping with ACF and the β's) when the roots of $\theta(B) = \theta_0 + \theta_1 B + \cdots + \theta_q B^q$ all lie outside the unit circle.
- Given an *AR(p)* process we define $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$ and claim that if $\phi(B)$ has all of its roots outside of the unit circle, then the process is stationary.
- Given an autoregressive *AR(p)* process, we can find an equivalent moving average $MA(q = \infty)$ process using the convenience of our shift operator B. The process then has autocorrelation $\rho(k) = {\sum_{i=0}^{\infty} \theta_i \, \theta_{i+k}} \Big/ {\sum_{i=0}^{\infty} \theta_i \, \theta_i}$
- Since the θ may be hard to find, we also have the Yule-Walker Equations
$$\rho(k) = \phi_1 \rho(k-1) + \cdots + \phi_p \rho(k-p)$$
- We know how to simulate moving averages and autoregressive processes.

Mixed *ARMA* Processes

We have several interesting and important case studies coming our way (tree rings, sunspots, etc.) so we need to be able to estimate the *parameters* of a process. That is, we'd like to estimate the *order* of the process, the *coefficients* of the process, *variability*, etc. It turns out that many "real world" examples are most efficiently modeled if we build a description with both moving average terms and autoregressive terms. We like efficiency not because we are lazy (or at least not only because we are lazy), but these simpler models provide better estimates and are easier to communicate and understand.

The easiest way to form a mixed process is to just bring together an *MA(q)* and an *AR(p)*

$$X_t = Noise + AutoRegressive\ Part + Moving\ Average\ Part$$

$$X_t = Z_t + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

This is called a mixed *ARMA(p,q)* process. Our hope (born out by quite a bit of experience) is that the phenomenon we are exploring will be more efficiently modelled (that is, will have fewer coefficients) by an *ARMA(p,q)* process than either an *MA(q)* or an *AR(p)* alone.

Let's tidy up the equations a bit with the usual notation from our lectures. For the *AR(p)* terms

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$

$$\phi(B)\, X_t = X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}$$

And, for the *MA(q)* terms,

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

$$\theta(B) Z_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

Thus for a mixed process we write

$$\theta(B)\, Z_t = \phi(B)\, X_t$$

Again, this is not just compact notation, but rather this representation allows us to work with our process simply and cleanly and will suggest results we wouldn't otherwise think about. For example, suppose you would like to express a mixed process as a moving average? Solve for X_t in terms of the innovations alone by defining:

$$\psi(B) \equiv \frac{\theta(B)}{\phi(B)}$$

We have called the ratio $\psi(B)$ and may express as an infinite order moving average

$$\text{ARMA(p, q) expressed as MA}(\infty): \quad \psi(B)\, Z_t = X_t\,, \quad \text{where} \quad \frac{\theta(B)}{\phi(B)} = \psi(B)$$

How about setting up the corresponding an autoregressive process? Bring in $\pi(B)$

$$\text{ARMA(p, q) expressed as AR}(\infty): \quad \pi(B) X_t = Z_t, \quad \text{where} \quad \frac{\phi(B)}{\theta(B)} = \pi(B)$$

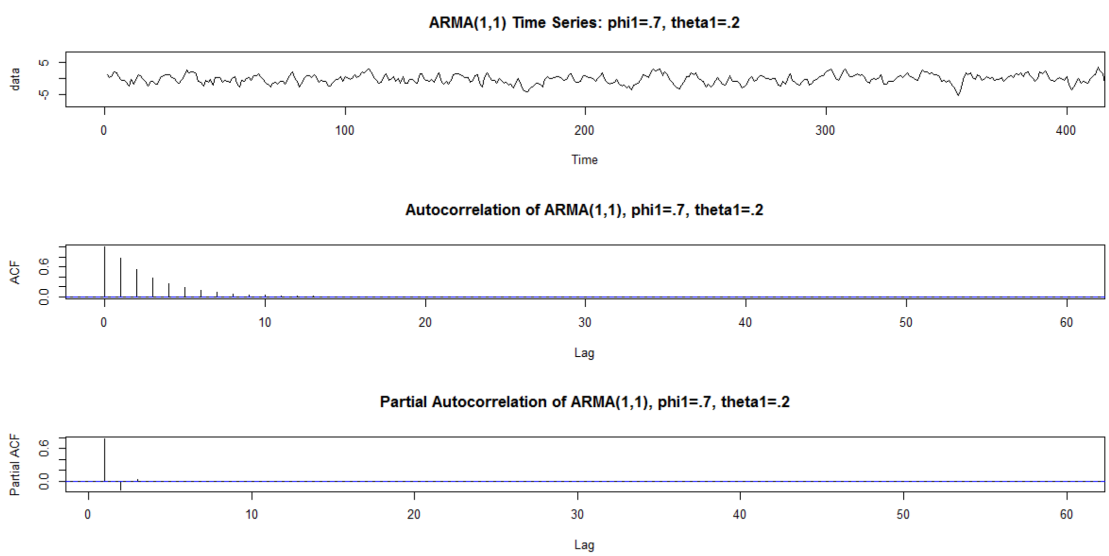
I'm sure an example would be most welcome at this point. Let's look at the simplest realistic case, an *ARMA(p=1, q=1)* process. We'll take a pretty large ϕ_1 value and moderate θ_1 value and define

$$X_t = 0.7\, X_{t-1} + Z_t + 0.2\, Z_{t-1}$$

Simulation

You should be able to simulate this almost trivially at this point. Here's what I did, and here's what I'm looking at. (Since we are comparing to theory I took a lot of terms. Hopefully your machine can do this as well).

```
rm(list=ls(all=TRUE))
set.seed(500)           # Beginning of Heptarchy: Kent, Essex, Sussex,
                        # Wessex, East Anglia, Mercia, and Northumbria.
data = arima.sim( list(order = c(1,0,1), ar =.7, ma=.2), n = 1000000)
par(mfcol = c(3,1 ))
plot(data, main="ARMA(1,1) Time Series: phi1=.7, theta1=.2", xlim=c(0,400)) #first terms
acf(data, main="Autocorrelation of ARMA(1,1), phi1=.7, theta1=.2")
acf(data, type="partial", main="Partial Autocorrelation of ARMA(1,1), phi1=.7, theta1=.2")
```



Mixed ARMA to Autoregressive

Can we find the equivalent π weights (i.e. the autoregressive process equivalent to the mixed process?) That will allow us to predict the autocorrelation function and see if everything is consistent. We rewrite the given process in our new notation. Just a little backward shift algebra and we have:

$$(1 - 0.7\,B)X_t = (1 + 0.2\,B)\,Z_t$$

$$\theta(B) = 1 + \beta_1 B = 1 + 0.2\,B$$

$$\phi(B) = 1 - \alpha_1 B = 1 - 0.7B$$

We want an autoregressive process, so take

$$\pi(B) = \frac{\phi(B)}{\theta(B)} = \frac{1 - 0.7B}{1 + 0.2\,B} = (1 - 0.7B)\,(1 + 0.2\,B)^{-1}$$

Work this out with the geometric series

$$\pi(B) = (1 - 0.7B)\,(1 + 0.2\,B)^{-1} = (1 - 0.7B)\,(1 - 0.2B + .04B^2 - .008B^3 + \cdots)$$

$$\pi(B) = 1 - .9\,B + .18\,B^2 - 0.036\,B^3 + \cdots$$

Is there a way to get a closed form expression for these π weights rather than the infinite series? Then we could express the autocorrelation function as a formula. Let's work a little more generally and develop a formula for a generic *ARMA(1,1)* process, then come back and apply it here.

We want a way of representing

$$\pi(B) = \frac{1 - \phi\,B}{1 + \theta\,B} = \text{some nice formula}$$

Writing this as a geometric series gives us

$$\begin{aligned}\pi(B) &= (1 - \phi\,B)(1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \theta^4 B^4 - \cdots)\\ &= 1 - (\phi + \theta)B + (\phi \cdot \theta + \theta^2)B^2 - (\phi \cdot \theta^2 + \theta^3)B^3 + (\phi\theta^3 + \theta^4)B^4 + \cdots\end{aligned}$$

We see a pattern here. Factor out the $\phi + \theta$ term

$$\pi(B) = 1 - (\phi + \theta)(B - \theta\,B^2 + \theta^2 B^3 - \theta^3 B^4) + \cdots$$

So, the coefficient of the term B^k is $(-1)^k\,(\phi + \theta)\,\theta^{k-1}$. Does this check out for us?

Write

$$\pi(B) = 1 - \sum_{i=1}^{\infty} \pi_i$$

Since we have $\phi = .7, \theta = .2$ (Watch your sign, please! We are subtracting the terms on B^k).

$$\pi_1 = .9,\ \ \pi_2 = -.18,\ \ \ \pi_3 = .036,\ \ \pi_4 = -\ 0.0072, \ldots$$

Mixed ARMA to Moving Average

Since we are warmed up, express the *ARMA(1,1)* process as a moving average. Now that we see the method, we can move a little more quickly here.

$$\psi(B) = \frac{\theta(B)}{\phi(B)} = \frac{1 + \theta\, B}{1 - \phi\, B} = (1 + \theta\, B)(1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \phi^4 B^4 + \cdots)$$

$$\psi(B) = 1 + (\phi + \theta) \cdot 1 \cdot B + (\phi + \theta) \cdot \phi \cdot\ B^2 + (\phi + \theta) \cdot \phi^2 \cdot B^3 + (\phi + \theta) \cdot \phi^3 \cdot B^4 + \cdots$$

Write $\psi(B) = \sum_{i=1}^{\infty} \psi_i B^i$. Then

$$\psi_i = (\phi + \theta)\phi^{i-1}, \quad\ i = 1, 2, 3, \ldots$$

With $\phi = .7,\ \ \theta = .2$

$$\psi_1 = 0.9,\ \ \psi_2 = .63,\ \ \psi_3 = .441,\ \ \psi_4 = .3087,\ \ \psi_5 = .21609,\ \ \psi_6 = .151263,\ \text{etc.}$$

Is this consistent with your autocorrelations? We can show that

$$\rho(1) = \frac{(1 + \phi\, \theta)\, (\phi + \theta)}{1 + \theta^2 + 2\, \phi\, \theta}$$

$$\rho(k) = \phi\ \ \rho(k-1)$$

This gives us theory and estimate (and some pretty terrific agreement! If you take fewer terms you should be close, though perhaps not this close.)

k	1	2	3	4	...
$\rho(k)$	0.7772727	0.5440909	0.3808636	0.2666045	...
r_k	0.777	0.544	0.380	0.266	