Stationarity: Properties and Examples

- For a Weakly Stationary Process, $-1 \leq \rho(t) \leq 1$

  This is, of course, perfectly analogous to the property that $-1 \leq \rho \leq 1$ from elementary statistics. If you have had a linear algebra course, this may feel familiar (that is, if you showed that $|x^T y| \leq \|x\|_2 \|y\|_2$). We know variances are non-negative, so set up a linear combination

  $$V[a\, X_1 + b\, X_2] \geq 0$$

  In particular, in the spirit of autocorrelation, set up for lag spacing τ

  $$V[a\, X(t) + b\, X(t + \tau)] \geq 0$$

  Your probability teacher probably told you (time and time again) that

  $$V[X + Y] = V[X] + V[Y] + 2\, cov(X, Y)$$

  As well as

  $$V[aX] = a^2 V[X]$$

  So, immediately,

  $$V[a\, X(t) + b\, X(t + \tau)] = a^2 V[X(t)] + b^2 V[X(t + \tau)] + 2ab\, cov\big(X(t), X(t + \tau)\big) \geq 0$$

  We are assuming weak stationarity, so replace variance operators with a notation which suggests constants

  $$a^2 \sigma^2 + b^2 \sigma^2 + 2ab\, cov\big(X(t), X(t + \tau)\big) \geq 0$$

  Two special cases: (1) Let $a = b = 1$

  $$2\,\sigma^2 \geq -2cov\big(X(t), X(t + \tau)\big), \quad \sigma^2 \geq -cov\big(X(t), X(t + \tau)\big)$$

  $$1 \geq -\frac{cov\big(X(t), X(t + \tau)\big)}{\sigma^2} = -\frac{\gamma(\tau)}{\gamma(0)} = -\rho(\tau)$$

  This gives us

$$\rho(\tau) \geq -1$$

(2) Let $a = 1, b = -1$

It's your turn- take a moment to show

$$\rho(\tau) \leq 1$$

We have already seen a few simple models: noise, random walks, and moving averages. Can we now show that some of our simple models are, in fact, weakly stationary?

Examples

  White Noise

Is it obvious to you that Gaussian white noise is weakly stationary? Consider a discrete family of independent, identically distributed normal random variables

$$X_t \overset{iid}{\underset{\sim}{}} N(\mu, \sigma)$$

The mean function $\mu(t)$ is obviously constant, so look at

$$\gamma(t_1, t_2) = \begin{cases} 0 & t_1 \neq t_2 \\ \sigma^2 & t_1 = t_2 \end{cases}$$

And

$$\rho(t_1, t_2) = \begin{cases} 0 & t_1 \neq t_2 \\ 1 & t_1 = t_2 \end{cases}$$

We are evidently weakly stationary, and could even show strict stationarity if we wanted to.

  Random Walks

Simple random walks are obviously ***not*** stationary. Think of a walk with N steps built off of IID Z_t where $E[Z_t] = \mu$, $V[Z_t] = \sigma^2$. We would create

$$X_1 = Z_1$$
$$X_2 = X_1 + Z_2$$
$$X_3 = X_2 + Z_3 = X_1 + X_2 + X_3$$
$$\vdots$$
$$X_t = X_{t-1} + Z_t = \sum_{i=1}^{t} Z_i$$

For the mean, using the idea that "the mean of the sum is the sum of the means":

$$E[X_t] = E\left[\sum_{i=1}^{t} Z_i\right] = \sum_{i=1}^{t} E[Z_i] = \boxed{t \cdot \mu}$$

For the variance, using the idea that "the variance of the sum is the sum of the variances *when the random variables are independent*":

$$V[X_t] = V\left[\sum_{i=1}^{t} Z_i\right] = \sum_{i=1}^{t} V[Z_i] = \boxed{t \cdot \sigma^2}$$

(Independent random variables have variances which add. All random variables have means which add).

Even if $\mu = 0$ the variances will still increase along the time series.

Moving Average Processes, $MA(q)$

A moving average process will create a new set of random variables from an old set, just like the random walk does, but now we build them as, for IID Z_t with $E[Z_t] = 0$ and $V[Z_t] = \sigma_Z^2$

$$MA(q)\ process: \qquad X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}$$

The parameter q tells us how far back to look along the white noise sequence for our average. Since the Z_t are independent, we immediately have (using the usual linear operator results)

$$E[X_t] = \beta_0 E[Z_t] + \beta_1 E[Z_{t-1}] + \cdots + \beta_q E[Z_{t-q}] = 0$$

$$V[X_t] = \beta_0^2 V[Z_t] + \beta_1^2 V[Z_{t-1}] + \cdots + \beta_q^2 V[Z_{t-q}] = \sigma_Z^2 \sum_{i=0}^{q} \beta_i^2$$

The autocovariance isn't all that hard to find either. Consider random variables k steps apart and set up their covariance.

$$cov[X_t, X_{t+k}] = cov\big[\, \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}\,,$$
$$\beta_0 Z_{t+k} + \beta_1 Z_{t+k-1} + \cdots + \beta_q Z_{t+k-q}\big]$$

This is a little tricky, but please stay focused. There are two numbers to keep track of, the lag spacing k and the support of the MA process, q.

Now

$$cov[X_t, X_{t+k}] = E[X_t \cdot X_{t+k}] - E[X_t]E[X_{t+k}] = E[X_t \cdot X_{t+k}]$$

Since $E[X_t] = 0$ we really just need

$$E[X_t \cdot X_{t+k}] = E\big[\left(\beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}\right)$$
$$\cdot \left(\beta_0 Z_{t+k} + \beta_1 Z_{t+k-1} + \cdots + \beta_q Z_{t+k-q}\right)\big]$$

We can rely on matrix results concerning linear combinations of random variables or just work directly. The patient among us will write out

$$E[X_t \cdot X_{t+k}] = \beta_0\beta_0 E[Z_t \, Z_{t+k}] + \beta_0\beta_1 E[Z_t \, Z_{t+k-1}] + \cdots + \beta_0\beta_q E\big[Z_t \, Z_{t+k-q}\big]$$
$$+ \beta_1\beta_0 E[Z_{t-1} \, Z_{t+k}] + \beta_1\beta_1 E[Z_{t-1} \, Z_{t+k-1}] + \cdots + \beta_1\beta_q E\big[Z_{t-1} \, Z_{t+k-q}\big]$$
$$+ \cdots \cdot +$$
$$\beta_q\beta_0 E\big[Z_{t-q} \, Z_{t+k}\big] + \beta_q\beta_1 E\big[Z_{t-q} \, Z_{t+k-1}\big] + \cdots \beta_q\beta_q E\big[Z_{t-q} \, Z_{t+k-q}\big]$$

The key to simplifying this is to notice that, since the Z_t are independent, we can say that the expectation of the product is the product of the expectations and so we have

$$E\big[Z_i \cdot Z_j\big] = E[Z_i]E\big[Z_j\big] = \begin{cases} 0 & i \neq j \\ \sigma_Z^2 & i = j \end{cases}$$

When the lag spacing k is greater than the order of the process q then the subscripts can never be the same (there is no overlap on the underlying $Z_t's$) and we have $cov[X_t, X_{t+k}] = 0$. When the lag spacing is small enough to have contributions, that is if $q - k \geq 0$, you can visualize the sum like this (we just need to keep track of the $\beta's$):

Z_{t-q}	Z_{t-q+1}	...	$Z_{t-(q-k)}$	...	Z_{t-1}	Z_t	...		
β_q	β_{q-1}		β_{q-k}		β_1	β_0		0	0

			Z_{t+k-q}	...	$Z_{t+k-k+1}$	Z_{t+k-k}	...	Z_{t+k-1}	Z_{t+k}
0	0	0	β_q		β_{k+1}	β_k		β_1	β_0

This should make clear that, when $k \leq q$

$$E[X_t, X_{t+k} \,] = \sigma_Z^2 \cdot \sum_{i=0}^{q-k} \beta_i \, \beta_{i+k} \quad (no \; t \; dependence)$$

Summing up, then, we have found that

$$\gamma(t_1, t_2) = \gamma(k) = \begin{cases} 0 & k > q \\ \sigma_Z^2 \cdot \sum_{i=0}^{q-k} \beta_i \, \beta_{i+k} & k \leq q \end{cases}$$

We know that the mean function is constant, in fact $\mu(t) = 0$ and the autocovariance function has no t dependence, so we conclude that the *MA(q)* process is (weakly) stationary.

Let's finish this lecture by finding the autocorrelation function. In general

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$

Obviously, then $\rho(0) = 1$. It is easy to see that

$$\gamma(0) = \sigma_Z^2 \cdot \sum_{i=0}^{q} \beta_i \, \beta_i = \sigma_Z^2 \cdot \sum_{i=0}^{q} \beta_i^2$$

Finally

$$\rho(k) = \frac{\sum_{i=0}^{q-k} \beta_i \, \beta_{i+k}}{\sum_{i=0}^{q} \beta_i^2}$$

In the next lecture we will simulate an MA(q) process and validate these results numerically.