



Series and series representation

PRACTICAL TIME SERIES ANALYSIS

THISTLETON AND SADIGOV

Objectives

- ▶ Recall infinite series and their convergence
- ▶ Examine geometric series
- ▶ Represent rational functions as a geometric series

Sequence and series

- ▶ Sequence $\{a_n\}$ is list of numbers in definite order

$$a_1, a_2, a_3, \dots a_n, \dots$$

- ▶ If the limit of the sequence exists, i.e.,

$$\lim_{n \rightarrow \infty} a_n = a$$

then we say the sequence is convergent.

Examples

► $a_n = \frac{n}{n+1}$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \rightarrow 1$$

► $a_n = 3^n$

$$3, 9, 27, \dots, 3^n, \dots$$

► $a_n = \sqrt{n}$

$$1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots$$

► $a_n = \frac{1}{n^2}$

$$1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots \rightarrow 0$$

Partial sums

- ▶ Partial sums of a sequence $\{a_n\}$ are defined as

$$s_n = a_1 + a_2 + \cdots + a_n$$

- ▶ $s_1 = a_1$
- ▶ $s_2 = a_1 + a_2$
- ▶ $s_3 = a_1 + a_2 + a_3$
- ▶ .
- ▶ .
- ▶ .

Series

- ▶ If the partial sums $\{s_n\}$ is convergent to a number s , then we say

the infinite series $\sum_{k=1}^{\infty} a_k$ is convergent, and is equal to s .

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (a_1 + a_2 + \cdots + a_n) = s$$

- ▶ Otherwise, we say $\sum_{k=1}^{\infty} a_k$ is divergent.

Some convergent series

▶ $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$

▶ $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

▶ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2)$

Some divergent series

▶ $\sum_{k=1}^{\infty} 3^k$

▶ $\sum_{k=1}^{\infty} (2k + 1)$

▶ $\sum_{k=1}^{\infty} \frac{1}{k}$

Absolute convergence

- ▶ Series is absolutely convergent if

$$\sum_{k=1}^{\infty} |a_k|$$

is convergent.

- ▶ Absolute convergence implies convergence.

Convergence tests

- ▶ Integral test
- ▶ Comparison test
- ▶ Limit comparison test
- ▶ Alternating series test
- ▶ Ratio test
- ▶ Root test

Geometric series

- ▶ Geometric sequence

$$\{ar^{n-1}\}_{n=1}^{\infty} = \{a, ar, ar^2, ar^3, \dots\}$$

- ▶ Geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1.$$

- ▶ $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$
since $a = \frac{1}{2}, r = \frac{1}{2}$.

Series representation

- ▶ Series representation for $\frac{1}{1-x}$ where $a = 1, r = x$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

if $|x| < 1$.

Series representation cont.

- Series representation for $\frac{1}{(1-x)\left(1-\frac{x}{2}\right)}$

$$\frac{1}{(1-x)\left(1-\frac{x}{2}\right)} = \frac{2}{1-x} + \frac{-1}{1-\frac{x}{2}} = \sum_{k=0}^{\infty} \left(2 - \frac{1}{2^k}\right) x^k$$

If $|x| < 1$ and $\left|\frac{x}{2}\right| < 1$, i.e., if $|x| < 1$.

Complex functions

Assume z is a complex number

$$\frac{a}{1-z} = a + az + az^2 + \cdots = \sum_{k=1}^{\infty} az^{k-1}$$

if $|z| < 1$.

What We've Learned

- ▶ The definition of infinite series and their convergence
- ▶ Geometric series is convergent if the multiplier has norm less than 1
- ▶ How to represent some rational functions as a geometric series



Backward shift operator

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Objectives

- ▶ Define and utilize backward shift operator

Definition

- ▶ X_1, X_2, X_3, \dots
- ▶ Backward shift operator is defined as

$$BX_t = X_{t-1}$$

- ▶ $B^2X_t = BBX_t = BX_{t-1} = X_{t-2}$
- ▶ $B^kX_t = X_{t-k}$

Example – Random Walk

$$X_t = X_{t-1} + Z_t$$

$$X_t = BX_t + Z_t$$

$$(1 - B)X_t = Z_t$$

$$\phi(B)X_t = Z_t$$

where

$$\phi(B) = 1 - B$$

Example – MA(2) process

$$X_t = Z_t + 0.2Z_{t-1} + 0.04Z_{t-2}$$

$$X_t = Z_t + 0.2BZ_t + 0.04B^2Z_t$$

$$X_t = (1 + 0.2B + 0.04B^2) Z_t$$

$$X_t = \beta(B)Z_t$$

where

$$\beta(B) = 1 + 0.2B + 0.04B^2$$

Example – AR(2) process

$$X_t = 0.2X_{t-1} + 0.3X_{t-2} + Z_t$$

$$X_t = 0.2BX_t + 0.3B^2X_t + Z_t$$

$$(1 - 0.2B - 0.3B^2) X_t = Z_t$$

$$\phi(B)X_t = Z_t$$

where

$$\phi(B) = 1 - 0.2B - 0.3B^2$$

MA(q) process (with a drift)

$$X_t = \mu + \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q},$$

Then,

$$X_t = \mu + \beta_0 Z_t + \beta_1 B^1 Z_t + \cdots + \beta_q B^q Z_t,$$

$$X_t - \mu = \beta(B) Z_t,$$

where

$$\beta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q.$$

AR(p) process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

Then,

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t$$


$$X_t - \phi_1 B X_t - \phi_2 B^2 X_t - \cdots - \phi_p B^p X_t = Z_t$$

$$\phi(B) X_t = Z_t,$$

Where

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$

What We've Learned

- ▶ The definition of the Backward shift operator 
- ▶ How to utilize backward shift operator to write $MA(q)$ and $AR(p)$ processes



Introduction to Invertibility

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Objectives

- ▶ Learn invertibility of a stochastic process

Two MA(1) models

- ▶ Model 1

$$X_t = Z_t + 2Z_{t-1}$$

- ▶ Model 2

$$X_t = Z_t + \frac{1}{2}Z_{t-1}$$

Theoretical Auto Covariance Function of Model 1

$$\gamma(k) = \text{Cov}[X_{t+k}, X_t] = \text{Cov}[Z_{t+k} + 2Z_{t+k-1}, Z_t + 2Z_{t-1}]$$

If $k > 1$, then $t + k - 1 > t$, so all Z 's are uncorrelated, thus $\gamma(k) = 0$.

If $k = 0$, then

$$\gamma(0) = \text{Cov}[Z_t + 2Z_{t-1}, Z_t + 2Z_{t-1}] =$$

$$\text{Cov}[Z_t, Z_t] + 4\text{Cov}[Z_{t-1}, Z_{t-1}] = \sigma_Z^2 + 4\sigma_Z^2 = 5\sigma_Z^2.$$

If $k = 1$, then

$$\gamma(1) = \text{Cov} [Z_{t+1} + 2Z_t, Z_t + 2Z_{t-1}] = \text{Cov} [2Z_t, Z_t] = 2\sigma_Z^2$$

If $k < 0$, then

$$\gamma(k) = \gamma(-k)$$

Auto Covariance Function and ACF of Model 1

$$\gamma(k) = \begin{cases} 0, & k > 1 \\ 2\sigma_Z^2, & k = 1 \\ 5\sigma_Z^2, & k = 0 \\ \gamma(-k), & k < 0 \end{cases}$$

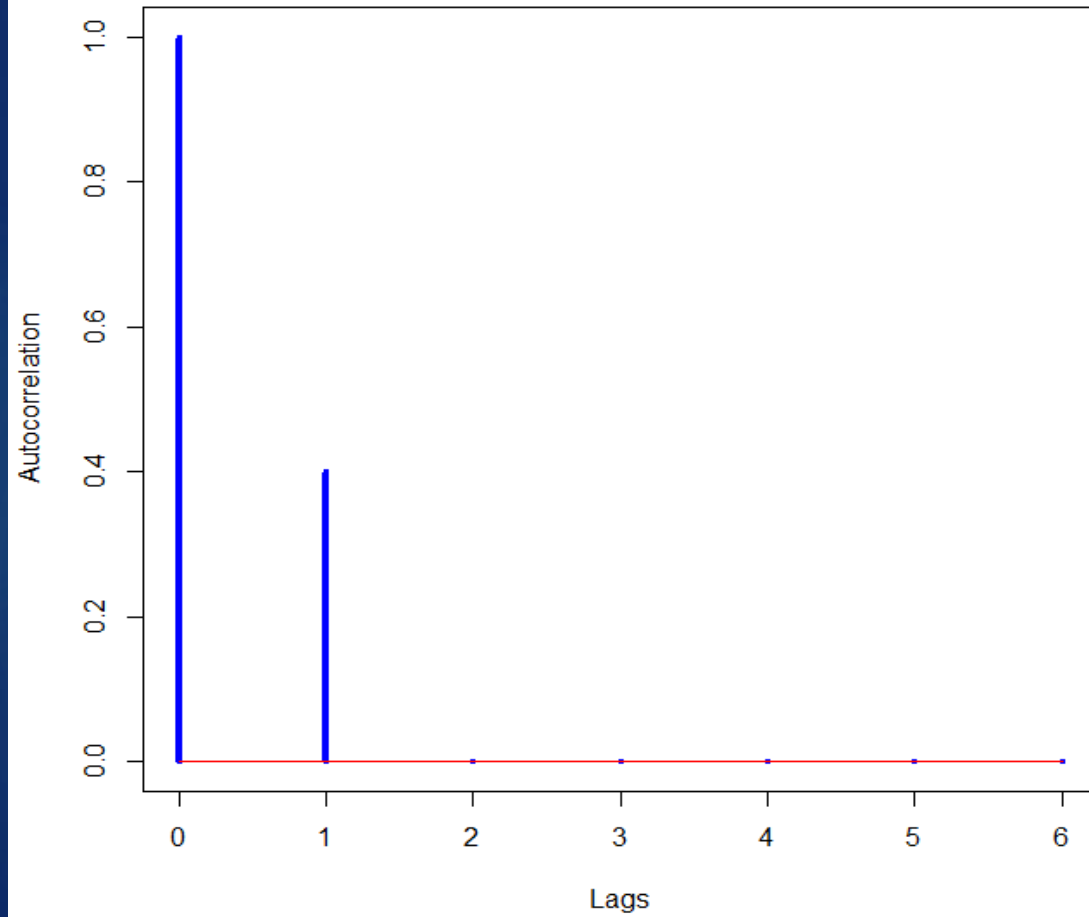
Then, since $\rho(k) = \frac{\gamma(k)}{\gamma(0)}$,

$$\rho(k) = \begin{cases} 0, & k > 1 \\ \frac{2}{5}, & k = 1 \\ 1, & k = 0 \\ \rho(-k), & k < 0 \end{cases}$$



ACF

ACF of Model 1



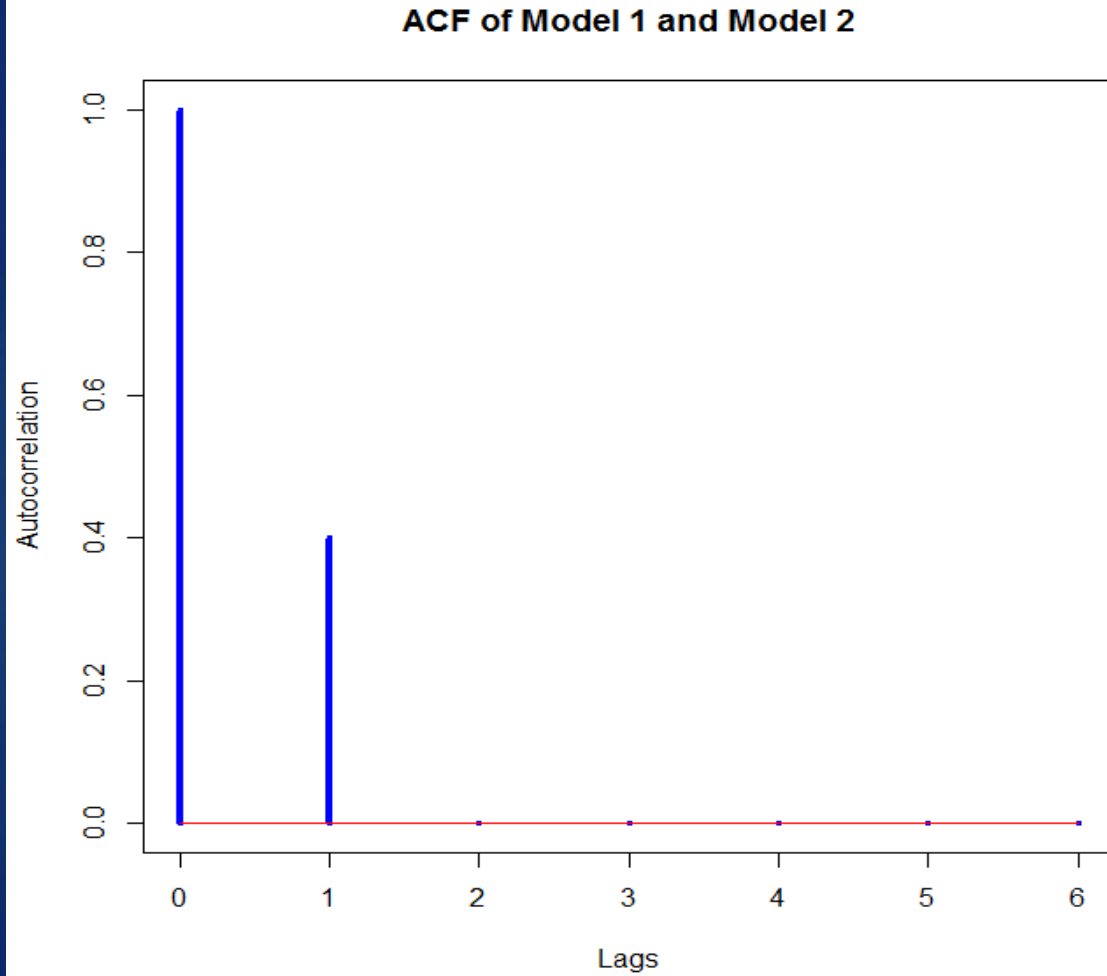
ACF of Model 2

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\text{Cov}\left[Z_{t+1} + \frac{1}{2}Z_t, Z_t + \frac{1}{2}Z_{t-1}\right]}{\text{Cov}\left[Z_t + \frac{1}{2}Z_{t-1}, Z_t + \frac{1}{2}Z_{t-1}\right]} = \frac{\frac{1}{2}}{1 + \frac{1}{4}} = \frac{2}{5}.$$

Thus we obtain the same ACF:

$$\rho(k) = \begin{cases} 0, & k > 1 \\ \frac{2}{5}, & k = 1 \\ 1, & k = 0 \\ \rho(-k), & k < 0 \end{cases}$$

ACFs are same!



Inverting through backward substitution

MA(1) process

$$X_t = Z_t + \beta Z_{t-1},$$

$$Z_t = X_t - \beta Z_{t-1} = X_t - \beta(X_{t-1} - \beta Z_{t-2}) = X_t - \beta X_{t-1} + \beta^2 Z_{t-2}$$

In this manner,

$$Z_t = X_t - \beta X_{t-1} + \beta^2 X_{t-2} - \beta^3 X_{t-3} + \dots$$

i.e.,

$$X_t = Z_t + \beta X_{t-1} - \beta^2 X_{t-2} + \beta^3 X_{t-3} - \dots$$



We 'inverted' MA(1) process to AR(∞).

Inverting using Backward shift operator

$$X_t = \beta(B)Z_t$$

where

$$\beta(B) = 1 + \beta B$$

Then, we find Z_t by inverting the polynomial operator $\beta(B)$:

$$\beta(B)^{-1}X_t = Z_t$$

Inverse of $\beta(B)$

$$\beta(B)^{-1} = \frac{1}{1 + \beta B} = 1 - \beta B + \beta^2 B^2 - \beta^3 B^3 + \dots$$

Here we expand the inverse of the polynomial operator as a 'rational function where βB is a complex number'.

Thus we obtain,

$$\beta(B)^{-1}X_t = 1 - \beta X_{t-1} + \beta^2 X_{t-2} - \beta^3 X_{t-3} + \dots$$

$$Z_t = \sum_{n=0}^{\infty} (-\beta)^n X_{t-n}$$



In order to make sure that the sum on the right is convergent (in the mean-square sense), we need $|\beta| < 1$.

There is an optional reading titled “Mean-square convergence” where we explain this result.

Invertibility - Definition

Definition:

$\{X_t\}$ is a stochastic process.

$\{Z_t\}$ is innovations, i.e., random disturbances or white noise.

$\{X_t\}$ is called invertible, if $Z_t = \sum_{k=0}^{\infty} \pi_k X_{t-k}$ where $\sum_{k=0}^{\infty} |\pi_k|$ is convergent.



Model 1 vs Model 2

- ▶ Model 1 is not invertible since

$$\sum_{k=0}^{\infty} |\pi_k| = \sum_{k=0}^{\infty} 2^k, \quad \text{Divergent}$$

- ▶ Model 2 is invertible since

$$\sum_{k=0}^{\infty} |\pi_k| = \sum_{k=0}^{\infty} \frac{1}{2^k}, \quad \text{Geometric Series, Convergent}$$

Model choice

- ▶ For 'invertibility' to hold, we choose Model 2, since $\left|\frac{1}{2}\right| < 1$.
- ▶ This way, ACF uniquely determines the MA process.

What We've Learned

- ▶ Definition of invertibility of a stochastic process
- ▶ Invertibility condition guarantees unique MA process corresponding to observed ACF



Invertibility and stationarity conditions

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Objectives

- ▶ Articulate invertibility condition for $MA(q)$ processes
- ▶ Discover stationarity condition for $AR(p)$ processes
- ▶ Relate MA and AR processes through duality

MA(q) process

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots \beta_q Z_{t-q}$$

Using Backward shift operator,

$$X_t = (\beta_0 + \beta_1 B + \cdots + \beta_q B^q) Z_t = \beta(B) Z_t$$

We obtain innovations Z_t in terms of present and past values of X_t ,

$$Z_t = \beta(B)^{-1} X_t = (\alpha_0 + \alpha_1 B + \alpha_2 B^2 + \cdots) X_t$$

For this to hold, “complex roots of the polynomial $\beta(B)$ must lie outside of the unit circle where B is regarded as complex variable”.

Invertibility condition for MA(q)

MA(q) process is invertible if the roots of the polynomial

$$\beta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q$$

all lie outside the unit circle, where we regard B as a complex variable (not an operator).

(Proof is done using mean-square convergence, see optional reading)

EX: MA(1) process

- ▶ $X_t = Z_t + \beta Z_{t-1}$
- ▶ $\beta(B) = 1 + \beta B$
- ▶ In this case only one (real) root $B = -\frac{1}{\beta}$
- ▶ $\left| -\frac{1}{\beta} \right| > 1 \Rightarrow |\beta| < 1.$
- ▶ Then, $Z_t = \sum_{k=0}^{\infty} (-\beta)^k B^k X_t = \sum_{k=0}^{\infty} (-\beta)^k X_{t-k}$

Example – MA(2) process

$$X_t = Z_t + \frac{5}{6}Z_{t-1} + \frac{1}{6}Z_{t-2}$$

Then,

$$X_t = \beta(B)Z_t$$

Where

$$\beta(B) = 1 + \frac{5}{6}B + \frac{1}{6}B^2$$

Example cont.

$$1 + \frac{5}{6}z + \frac{1}{6}z^2 = 0$$

$$z_1 = 2, z_2 = 3$$

Example cont.

$$\beta(B)^{-1} = \frac{1}{1 + \frac{5}{6}B + \frac{1}{6}B^2} = \frac{3}{1 + \frac{1}{2}B} - \frac{2}{1 + \frac{1}{3}B} \quad \text{💬}$$

$$\beta(B)^{-1} = \sum_{k=0}^{\infty} \left[3 \left(-\frac{1}{2} \right)^k - 2 \left(-\frac{1}{3} \right)^k \right] B^k \quad \text{💬}$$

$$Z_t = \sum_{k=0}^{\infty} \left[3 \left(-\frac{1}{2} \right)^k - 2 \left(-\frac{1}{3} \right)^k \right] B^k X_t$$

$$Z_t = \sum_{k=1}^{\infty} \pi_k B^k X_t = \sum_{k=1}^{\infty} \pi_k X_{t-k}$$

Where

$$\pi_k = 3 \left(-\frac{1}{2} \right)^k - 2 \left(-\frac{1}{3} \right)^k$$

MA(2) process \Rightarrow AR(∞) process 

Stationarity condition for AR(p)

AR(p) process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

is (weakly) stationary if the roots of the polynomial

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$

all lie outside the unit circle, where we regard B as a complex variable (not an operator).

AR(1) process

$$X_t = \phi_1 X_{t-1} + Z_t \implies (1 - \phi_1 B)X_t = Z_t$$

$$\phi(B) = 1 - \phi_1 B$$

$$\phi(z) = 1 - \phi_1 z = 0 \implies z = \frac{1}{\phi_1}$$

$$|z| = \left| \frac{1}{\phi_1} \right| > 1 \Rightarrow |\phi_1| < 1$$

Thus, when $|\phi_1| < 1$, the AR(1) process is stationary.

$$X_t = \frac{1}{1 - \phi_1 B} Z_t = (1 + \phi_1 B + \phi_1 B^2 + \dots) Z_t = \sum_{k=0}^{\infty} \phi_1^k Z_{t-k}$$

Another look at ϕ_1

Take Variance from both side,

$$Var[X_t] = Var\left[\sum_{k=0}^{\infty} \phi_1^k Z_{t-k}\right] = \sum_{k=0}^{\infty} \phi_1^{2k} \sigma_Z^2 = \sigma_Z^2 \sum_{k=0}^{\infty} \phi_1^{2k}$$

which is a convergent geometric series if $|\phi_1^2| < 1$, i.e.,

$$|\phi_1| < 1.$$

$AR(p)$ process \Rightarrow $MA(\infty)$ process

Duality between AR and MA processes



Under invertibility condition of $MA(q)$,

$$MA(q) \Rightarrow AR(\infty)$$

Under stationarity condition of $AR(p)$

$$AR(p) \Rightarrow MA(\infty)$$

What We've Learned

- ▶ Invertibility condition for $MA(q)$ processes
- ▶ Stationarity condition for $AR(p)$ processes
- ▶ Duality MA and AR processes



Mean Square Convergence

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Objectives

- ▶ Learn mean-square convergence
- ▶ Formulate necessary and sufficient condition for invertibility of $MA(1)$ process

Mean-square convergence

Let

$$X_1, X_2, X_3, \dots$$

be a sequence of random variables (i.e. a stochastic process).

We say X_n converge to a random variable X in the mean-square sense if

$$E[(X_n - X)^2] \rightarrow 0 \text{ as } n \rightarrow \infty$$

MA(1) model

We inverted MA(1) model

$$X_t = Z_t + \beta Z_{t-1}$$

as

$$Z_t = \sum_{k=0}^{\infty} (-\beta)^k X_{t-k}$$

Infinite sum above is convergent in mean-square sense under some condition on β .

Auto covariance function

$$\gamma(k) = \begin{cases} 0, & k > 1 \\ \beta\sigma_Z^2, & k = 1 \\ (1 + \beta^2)\sigma_Z^2, & k = 0 \\ \gamma(-k), & k < 0 \end{cases}$$

Series convergence

Lets find β 's that partial sum

$$\sum_{k=0}^n (-\beta)^k X_{t-k}$$

converges to Z_t in mean-square sense.

$$\begin{aligned}
E \left[\left(\sum_{k=0}^n (-\beta)^k X_{t-k} - Z_t \right)^2 \right] &= E \left[\left(\sum_{k=0}^n (-\beta)^k X_{t-k} \right)^2 \right] - 2E \left[\sum_{k=0}^n (-\beta)^k X_{t-k} Z_t \right] + E[Z_t^2] \\
&= E \left[\sum_{k=0}^n \beta^{2k} X_{t-k}^2 \right] + 2E \left[\sum_{k=0}^{n-1} (-\beta)^{2k+1} X_{t-k} X_{t-k+1} \right] - 2E[X_t Z_t] + \sigma_Z^2 \\
&= \sum_{k=0}^n \beta^{2k} E[X_{t-k}^2] - 2 \sum_{k=0}^{n-1} \beta^{2k+1} E[X_{t-k} X_{t-k+1}] - 2E[Z_t^2]
\end{aligned}$$

To get

$$E \left[\left(\sum_{k=0}^n (-\beta)^k X_{t-k} - Z_t \right)^2 \right] \rightarrow 0 \text{ as } n \rightarrow \infty$$

We need

$$\sigma_Z^2 \beta^{2n+2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Thus, $|\beta| < 1$.

i.e.,

$$\left| -\frac{1}{\beta} \right| > 1$$

i.e., zero of the polynomial

$$\beta(B) = 1 + \beta B$$

Lies outside of the unit circle.

What We've Learned

- ▶ Definition of the mean square convergence
- ▶ Necessary and sufficient condition for invertibility of $MA(1)$ process

Difference equations

PRACTICAL TIME SERIES ANALYSIS

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Objectives

- ▶ Recall and solve difference equations

Difference equation

- ▶ General term of a sequence is given, ex: $a_n = 2n + 1$. So,

$$3, 5, 7, \dots$$

- ▶ General term not given, but a relation is given, ex:

$$a_n = 5a_{n-1} - 6a_{n-2}$$

- ▶ This is a difference equation (recursive relation)

How to solve difference equations?

- ▶ We look for a solution in the format

$$a_n = \lambda^n$$

- ▶ For the previous problem,

$$\lambda^n = 5\lambda^{n-1} - 6\lambda^{n-2}$$

We simplify

$$\lambda^2 - 5\lambda + 6 = 0$$

- ▶ Auxiliary equation or characteristic equation.

- ▶ $\lambda = 2, \lambda = 3$
- ▶ $a_n = c_1 2^n + c_2 3^n$
- ▶ With some initial conditions, say $a_0 = 3, a_1 = 8$.

We get

$$\begin{cases} c_1 + c_2 = 3 \\ 2c_1 + 3c_2 = 8 \end{cases}$$

Thus,

$$c_1 = 1, c_2 = 2.$$

Solution

$$a_n = 2^n + 2 \cdot 3^n$$

Is the solution of 2nd order difference equation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

k -th order difference equation

$$a_n = \beta_1 a_{n-1} + \beta_2 a_{n-2} + \cdots + \beta_k a_{n-k}$$

Its characteristic equation

$$\lambda^k - \beta_1 \lambda^{k-1} - \cdots - \beta_{k-1} \lambda - \beta_k = 0$$

Then we look for the solutions of the characteristic equation. Say, all k solutions are distinct real numbers, $\lambda_1, \lambda_2, \dots, \lambda_k$, then

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots + c_k \lambda_k^n$$

Coefficients c_j 's are determined using initial values.

Example - Fibonacci sequence

Fibonacci sequence is defined as follows:

1, 1, 2, 3, 5, 8, 13, 21, ...

i.e., every term starting from the 3rd term is addition of the previous two terms.

Question: What is the general term, a_n , of the Fibonacci sequence?

Formulation

We are looking for a sequence $\{a_n\}_{n=0}^{\infty}$, such that

$$a_n = a_{n-1} + a_{n-2}$$

where $a_0 = 1, a_1 = 1$.

Characteristic equation becomes

$$\lambda^2 - \lambda - 1 = 0$$

Then $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$.

Thus

$$a_n = c_1 \left(\frac{1-\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1+\sqrt{5}}{2} \right)^n$$

Use initial data

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 \left(\frac{1-\sqrt{5}}{2} \right) + c_2 \left(\frac{1+\sqrt{5}}{2} \right) = 1 \end{cases}$$

General term of Fibonacci sequence

We obtain

$$c_1 = \frac{5 - \sqrt{5}}{10} = -\frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)$$

$$c_2 = \frac{5 + \sqrt{5}}{10} = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)$$

$$a_n = -\frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} + \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1}$$

Relation to differential equations

k –th order linear ordinary equation

$$y^{(k)} = \beta_1 y^{(k-1)} + \cdots \beta_{k-1} y + \beta_k$$

Solution format $y = e^{\lambda t}$ gives characteristic equation

$$\lambda^k - \beta_1 \lambda^{k-1} - \cdots - \beta_{k-1} \lambda - \beta_k = 0$$

Then we solve the characteristic equation.

What We've Learned

- ▶ Definition of difference equations and how to solve them

Yule-Walker Equations

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Objectives

- ▶ Introduce Yule – Walker equations
- ▶ Obtain ACF of AR processes using Yule – Walker equations

Procedure

- ▶ We assume stationarity in advance (a priori assumption)
- ▶ Take product of the AR model with X_{n-k}
- ▶ Take expectation of both sides
- ▶ Use the definition of covariance, and divide by $\gamma(0) = \sigma_X^2$
- ▶ Get difference equation for $\rho(k)$, ACF of the process
- ▶ This set of equations is called Yule-Walker equations
- ▶ Solve the difference equation

Example

We have an AR(2) process

$$X_t = \frac{1}{3}X_{t-1} + \frac{1}{2}X_{t-2} + Z_t \dots (*)$$

Polynomial

$$\phi(B) = 1 - \frac{1}{3}B - \frac{1}{2}B^2$$

has real roots $\frac{-2 \pm \sqrt{76}}{6}$ both of which has magnitude greater than 1, so roots are outside of the unit circle in \mathbb{R}^2 . Thus, this AR(2) process is a stationary process.

Example cont.

Note that if $E(X_t) = \mu$, then

$$\begin{aligned}E(X_t) &= \frac{1}{3}E(X_{t-1}) + \frac{1}{2}E(X_{t-2}) + E(Z_t) \\ \mu &= \frac{1}{3}\mu + \frac{1}{2}\mu \\ \mu &= 0\end{aligned}$$

Multiply both side of (*) with X_{t-k} , and take expectation

$$E(X_{t-k}X_t) = \frac{1}{3}E(X_{t-k}X_{t-1}) + \frac{1}{2}E(X_{t-k}X_{t-2}) + E(X_{t-k}Z_t)$$

Example cont.

Since $\mu = 0$, and assume $E(X_{t-k}Z_t) = 0$,

$$\gamma(-k) = \frac{1}{3}\gamma(-k+1) + \frac{1}{2}\gamma(-k+2)$$

Since $\gamma(k) = \gamma(-k)$ for any k ,

$$\gamma(k) = \frac{1}{3}\gamma(k-1) + \frac{1}{2}\gamma(k-2)$$

Divide by $\gamma(0) = \sigma_X^2$

$$\rho(k) = \frac{1}{3}\rho(k-1) + \frac{1}{2}\rho(k-2)$$

This set of equations is called Yule-Walker equations.

Solve the difference equation

We look for a solution in the format of $\rho(k) = \lambda^k$.

$$\lambda^2 - \frac{1}{3}\lambda - \frac{1}{2} = 0$$

Roots are $\lambda_1 = \frac{2+\sqrt{76}}{12}$ and $\lambda_2 = \frac{2-\sqrt{76}}{12}$, thus

$$\rho(k) = c_1 \left(\frac{2 + \sqrt{76}}{12} \right)^k + c_2 \left(\frac{2 - \sqrt{76}}{12} \right)^k$$

Finding c_1, c_2

Use constraints to obtain coefficients

$$\rho(0) = 1 \Rightarrow c_1 + c_2 = 1$$

And for $k = p - 1 = 2 - 1 = 1$,

$$\rho(k) = \rho(-k)$$

Thus,

$$\rho(1) = \frac{1}{3}\rho(0) + \frac{1}{2}\rho(-1) \Rightarrow \rho(1) = \frac{2}{3} \Rightarrow c_1 \left(\frac{2 + \sqrt{76}}{12} \right) + c_2 \left(\frac{2 - \sqrt{76}}{12} \right) = \frac{2}{3}$$

Solve the system for c_1, c_2

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 \left(\frac{2 + \sqrt{76}}{12} \right) + c_2 \left(\frac{2 - \sqrt{76}}{12} \right) = \frac{2}{3} \end{cases}$$

Then,

$$c_1 = \frac{4 + \sqrt{6}}{8} \text{ and } c_2 = \frac{4 - \sqrt{6}}{8}$$

ACF of the AR(2) model

For any $k \geq 0$,

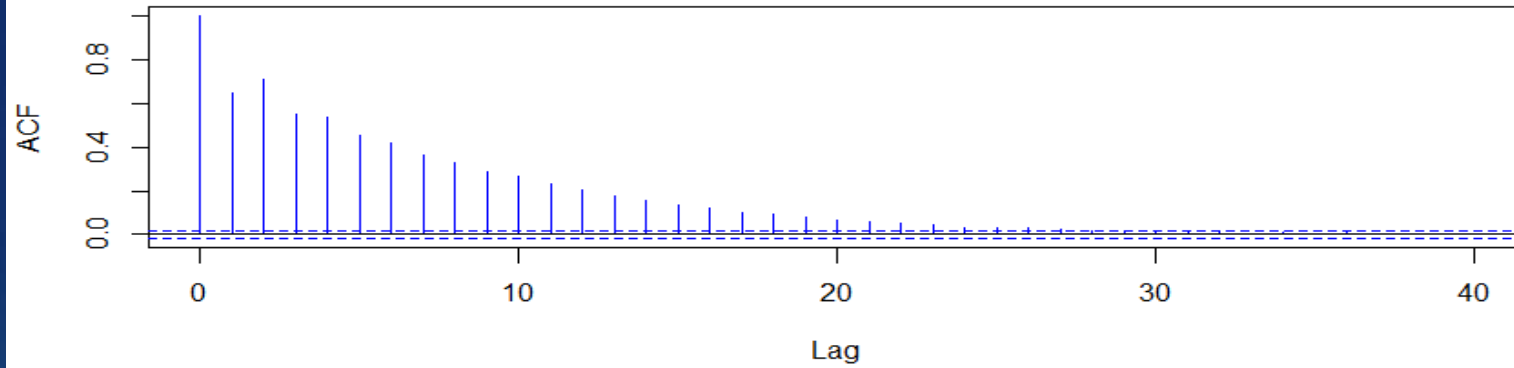
$$\rho(k) = \frac{4 + \sqrt{6}}{8} \left(\frac{2 + \sqrt{76}}{12} \right)^k + \frac{4 - \sqrt{6}}{8} \left(\frac{2 - \sqrt{76}}{12} \right)^k$$

And

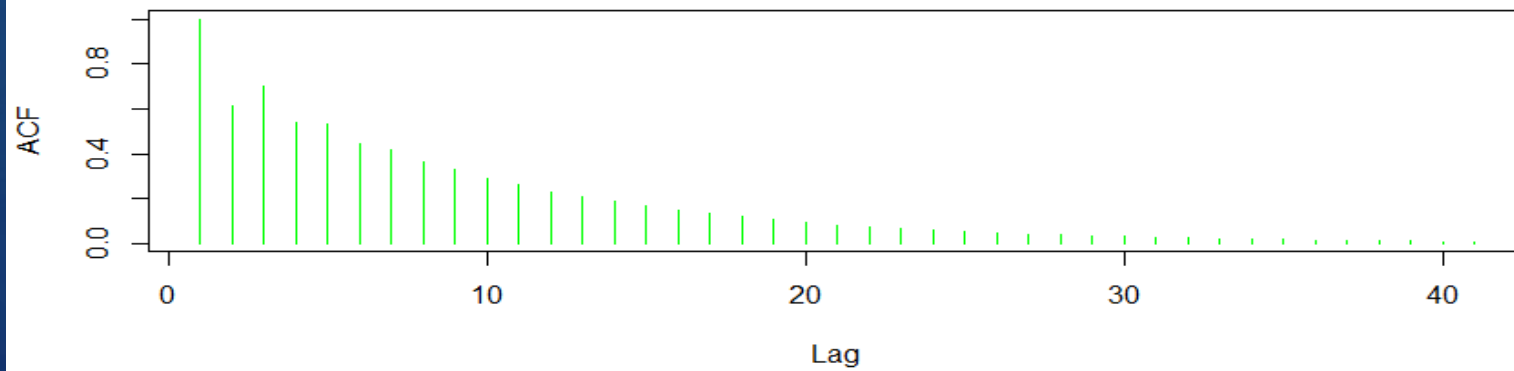
$$\rho(k) = \rho(-k)$$

Simulation

ACF of a simulation of the AR(2) model



$\rho(k)$ plot



What We've Learned

- ▶ Yule- Walker equations is set of difference equations governing ACF of the underlying AR process
- ▶ How to find the ACF of an AR process using Yule-Walker equations