# Simulating the Term Structure of Risk

FNCE 5321

Hang Bai

# Overview

- When the horizon of interest is longer than one day, we need to rely on simulation methods for computing *VaR* to compute entire term structure of risk
- First, we will consider simulating forward the univariate risk models by using Monte Carlo simulation and Filtered Historical Simulation
- Second, we simulate forward in time multivariate risk models with constant correlations across assets using Monte Carlo as well as FHS
- Third, we simulate multivariate risk models with dynamic correlations using the DCC model

# The Risk Term Structure in Univariate Models

- When portfolio returns are normally distributed with a constant variance, $\sigma^2_{PF}$, returns over the next $K$ days are also normally distributed, but with variance $K\sigma^2_{PF}$
- The *VaR* for returns over the next $K$ days calculated on day $t$ is

$$VaR^p_{t+1:t+K} = -\sqrt{K}\sigma_{PF}\Phi^{-1}_p = \sqrt{K}VaR^p_{t+1}$$

# The Risk Term Structure in Univariate Models

- The variance of the *K*-day return is in general:

$$\sigma^2_{t+1:t+K} \equiv E_t \left( \sum_{k=1}^{K} R_{t+k} \right)^2 = \sum_{k=1}^{K} E_t \left[ \sigma^2_{t+k} \right]$$

- where we have omitted the portfolio, *PF*, subscripts
- In the simple RiskMetrics variance model, where

$\sigma^2_{t+1} = \lambda \sigma^2_t + (1 - \lambda) R^2_t$ , we get

$$\sigma^2_{t+1:t+K} = \sum_{k=1}^{K} \sigma^2_{t+1} = K \sigma^2_{t+1}$$

- so that variances actually do scale by *K* in the Risk-Metrics model

# The Risk Term Structure in Univariate Models

- In the symmetric GARCH(1,1) model, where

  $\sigma_{t+1}^2 = \omega + \alpha R_t^2 + \beta\sigma_t^2$ , we get

$$\sigma_{t+1:t+K}^2 = K\sigma^2 + \sum_{k=1}^{K} (\alpha+\beta)^{k-1} \left(\sigma_{t+1}^2 - \sigma^2\right) \neq K\sigma_{t+1}^2$$

- where $\sigma^2 = \dfrac{\omega}{1-\alpha-\beta}$
- is the unconditional, or average, long-run variance
- In GARCH, the variance does mean revert and it does not scale by the horizon *K*, and the returns over the next *K* days are not normally distributed
- In GARCH, as *K* gets large, the return distribution does approach the normal distribution

# The Risk Term Structure in Univariate Models

- In Chapter 1 we discussed average daily return: First, that it is very difficult to forecast, and, second that it is very small relative to daily standard deviation
- At a longer horizon, it is difficult to forecast the mean but its relative importance increases with horizon
- Consider an example where daily returns are normally distributed with a constant mean and variance as in

$$R_{t+1} \overset{i.i.d.}{\sim} N\left(\mu, \sigma^2\right)$$

- The 1-day *VaR* is thus

$$VaR_{t+1}^{p} = -\left(\mu + \sigma\Phi_p^{-1}\right) \approx -\sigma\Phi_p^{-1}$$

# The Risk Term Structure in Univariate Models

- The *K*-day return in this case is distributed as

$$R_{t+1:t+K} \sim N\left(K\mu, K\sigma^2\right)$$

- and the *K*-day *VaR* is thus

$$VaR^{p}_{t+1:t+K} = -\left(K\mu + \sqrt{K}\sigma\Phi_p^{-1}\right) \not\approx -\sqrt{K}\sigma\Phi_p^{-1}$$

- As the horizon, *K*, gets large, the relative importance of the mean increases

# Monte Carlo Simulation

- Consider our GARCH(1,1)-normal model of returns

$$R_{t+1} = \sigma_{t+1} z_{t+1}, \text{ with } z_{t+1} \overset{i.i.d.}{\sim} N(0,1)$$

- and $\sigma_{t+1}^2 = \omega + \alpha R_t^2 + \beta \sigma_t^2$

- In the GARCH model, at the end of day *t* we obtain $R_t$ and we can calculate $\sigma^2_{t+1}$ ,tomorrow's variance
- Using random number generators, we can generate a set of artificial (or pseudo) random numbers drawn from the standard normal distribution, *N*(0,1)

$$\check{z}_{i,1}, \quad i = 1, 2, ..., MC$$

- *MC* denotes the number of draws around 10,000

# Monte Carlo Simulation

- QQ plot of the random numbers is constructed to confirm that the random numbers conform to the standard normal distribution
- From these random numbers we can calculate a set of hypothetical returns for tomorrow as

$$\check{R}_{i,t+1} = \sigma_{t+1}\check{z}_{i,1}$$

- Given these hypothetical returns, we can update the variance to get a set of hypothetical variances for the day after tomorrow, *t*+2, as follows:

$$\check{\sigma}^2_{i,t+2} = \omega + \alpha\check{R}^2_{i,t+1} + \beta\sigma^2_{t+1}$$

- Given a new set of random numbers drawn from the *N*(0,1) distribution

$$\check{z}_{i,2}, \qquad i = 1, 2, ..., MC$$

# Monte Carlo Simulation

- we can calculate hypothetical return on day *t*+2 as

$$\check{R}_{i,t+2} = \check{\sigma}_{i,t+2}\check{z}_{i,2}$$

- and the variance is now updated using

$$\check{\sigma}^2_{i,t+3} = \omega + \alpha \check{R}^2_{i,t+2} + \beta \check{\sigma}^2_{i,t+2}$$

- Graphically, we can illustrate the simulation of hypothetical daily returns from day *t*+1 to day *t*+*K* as

$$\begin{array}{cccccc}
 & & \check{z}_{1,1} \rightarrow \check{R}_{1,t+1} \rightarrow \check{\sigma}^2_{1,t+2} & \check{z}_{1,2} \rightarrow \check{R}_{1,t+2} \rightarrow \check{\sigma}^2_{1,t+3} & \cdots & \check{z}_{1,K} \rightarrow \check{R}_{1,t+K} \\
 & \nearrow & \check{z}_{2,1} \rightarrow \check{R}_{2,t+1} \rightarrow \check{\sigma}^2_{2,t+2} & \check{z}_{2,2} \rightarrow \check{R}_{2,t+2} \rightarrow \check{\sigma}^2_{2,t+3} & \cdots & \check{z}_{2,K} \rightarrow \check{R}_{2,t+K} \\
\sigma^2_{t+1} & \longrightarrow & \cdots & \cdots & \cdots & \cdots \\
 & \searrow & \cdots & \cdots & \cdots & \cdots \\
 & & \check{z}_{MC,1} \rightarrow \check{R}_{MC,t+1} \rightarrow \check{\sigma}^2_{MC,t+2} & \check{z}_{MC,2} \rightarrow \check{R}_{MC,t+2} \rightarrow \check{\sigma}^2_{MC,t+3} & \cdots & \check{z}_{MC,K} \rightarrow \check{R}_{MC,t+K}
\end{array}$$

# Monte Carlo Simulation

- Each row corresponds to a Monte Carlo simulation path, which branches out from $\sigma^2_{t+1}$ on the first day, but which does not branch out after that

- On each day a given branch gets updated with a new random number, which is different from the one used any of the days before

- We end up with *MC* sequences of hypothetical daily returns for day $t$+1 through day $t$+$K$

# Monte Carlo Simulation

- From these hypothetical future daily returns, we can easily calculate the hypothetical *K*-day return from each Monte Carlo path as

$$\check{R}_{i,t+1:t+K} = \sum_{k=1}^{K} \check{R}_{i,t+k}, \text{ for } i = 1, 2, \ldots, MC$$

- Collect these *MC* hypothetical *K*-day returns in a set $\left\{\check{R}_{i,t+1:t+K}\right\}_{i=1}^{MC}$, and calculate the *K*-day value at risk by calculating the 100*p*th percentile as in

$$VaR^{p}_{t+1:t+K} = -Percentile\left\{\left\{\check{R}_{i,t+1:t+K}\right\}_{i=1}^{MC}, 100p\right\}$$

# Monte Carlo Simulation

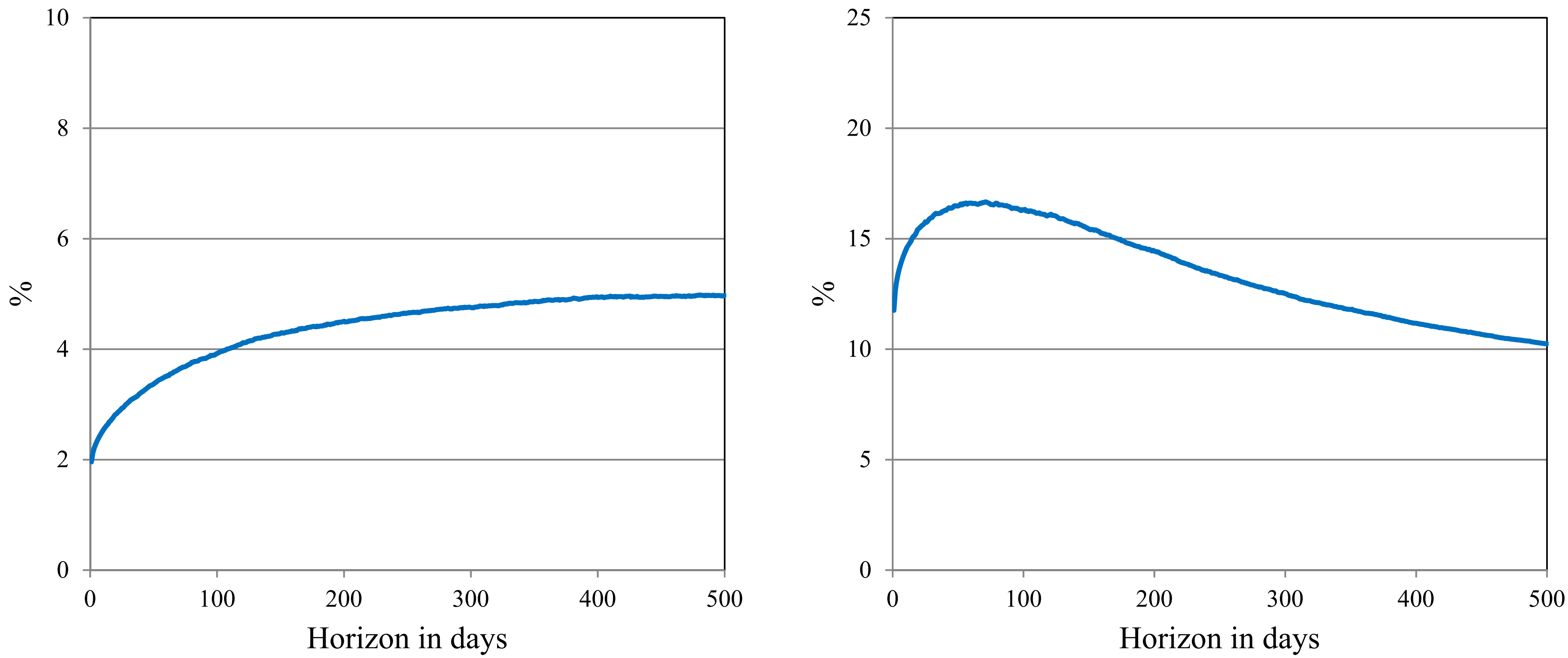
- where $\mathbf{1}(*)$ takes the value 1 if the argument is true and zero otherwise

- GARCH-MCS method builds on today's estimate of tomorrow's variance

- MCS can be used for any assumed distribution of standardized returns—normality is not required

- MCS technique can also be used for any fully specified dynamic variance model

# Figure 8.1: VaR Term Structures using NGARCH and Monte Carlo Simulation




Notes to Figure: The left panel shows the S&P 500 VaR per day across horizons when the current volatility is one half its long run value. The right panel assumes the current volatility is 3 times its long run value.

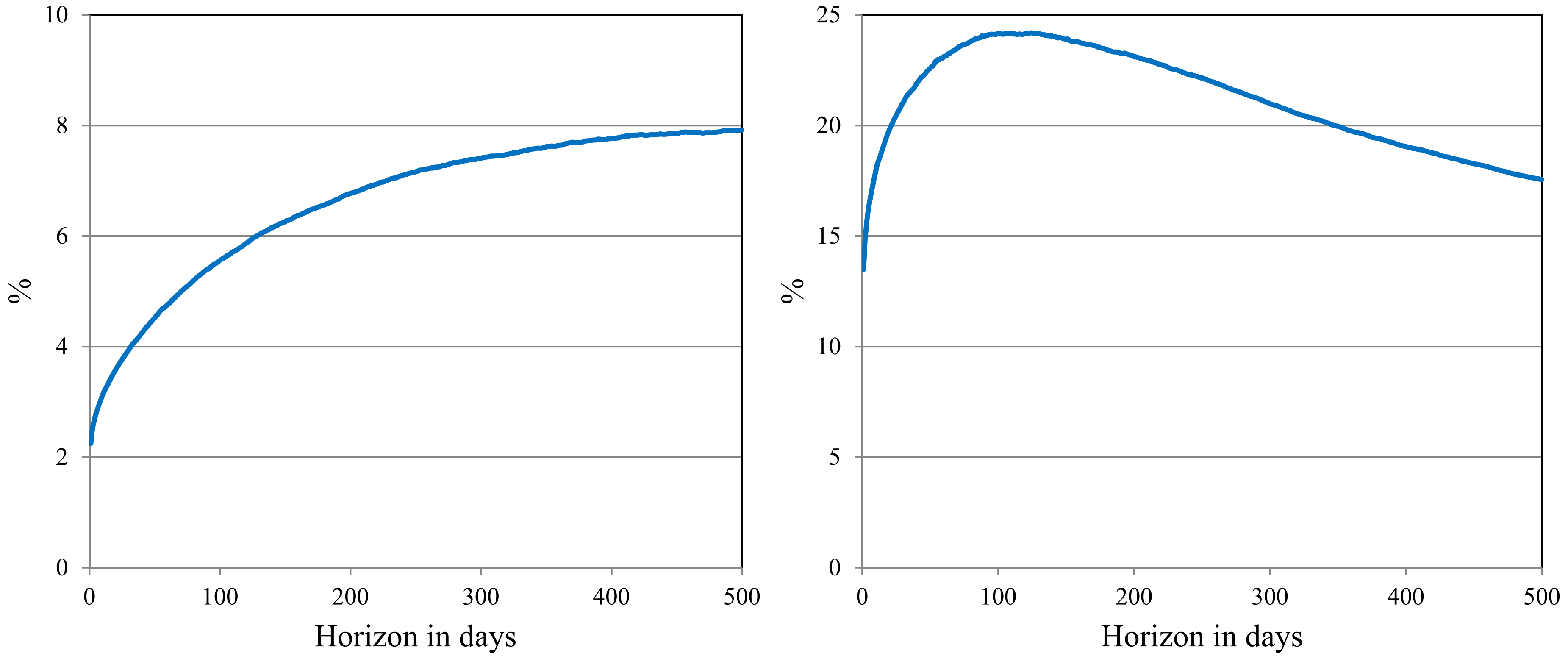
# Monte Carlo Simulation

- In Figure 8.1, the *VaR* is simulated using Monte Carlo on an NGARCH model
- We use MCS to construct *VaR* per day as a function of horizon *K* for two different values of $\sigma_{t+1}$
- In the left panel the initial volatility is one-half the unconditional level and in the right panel $\sigma_{t+1}$ is three times the unconditional level.
- The horizon goes from 1 to 500 trading days
- The *VaR* coverage level *p* is set to 1%

# Monte Carlo Simulation

- Figure 8.1 shows that the term structure of *VaR* is initially upward sloping both when volatility is low and when it is high
- The *VaR* term structure is also driven by the term structure of skewness and kurtosis and other moments
- Kurtosis is strongly increasing at short horizons and then decreasing for longer horizons
- This hump-shape in the term structure of kurtosis creates the hump in the *VaR* as seen in the right panel of Figure 8.1 when the initial volatility is high

# Figure 8.2: ES Term Structures using NGARCH and Monte Carlo Simulation




Notes to Figure: The left panel shows the S&P 500 ES per day across horizons when the current volatility is one half its long run value. The right panel assumes the current volatility is 3 times its long run value.

# Filtered Historical Simulation (FHS)

- FHS combines model-based methods of variance with model-free methods of the distribution of shocks
- Here we use the past returns data to tell us about the distribution without making any assumptions about the specific distribution
- Consider a GARCH(1,1) model

$$R_{t+1} = \sigma_{t+1} z_{t+1}$$

- where

$$\sigma_{t+1}^2 = \omega + \alpha R_t^2 + \beta \sigma_t^2$$

# Filtered Historical Simulation (FHS)

- Given a sequence of past returns, $\{R_{t+1-\tau}\}_{\tau=1}^{m}$, we can estimate the GARCH model and calculate past standardized returns from the observed returns and from the estimated standard deviations as

$$\hat{z}_{t+1-\tau} = R_{t+1-\tau}/\sigma_{t+1-\tau}, \text{ for } \tau = 1, 2, ..., m$$

- The number of historical observations, *m*, should be as large as possible
- In GARCH model, at the end of day *t* we obtain $R_t$ and we can calculate $\sigma^2_{t+1}$, which is day *t*+1's variance
- We draw random $\hat{z}$s with replacement from our own database of past standardized residuals, $\{\hat{z}_{t+1-\tau}\}_{\tau=1}^{m}$

# Filtered Historical Simulation (FHS)

- The random drawing can be operationalized by generating a discrete uniform random variable distributed from 1 to *m*.
- Each draw from the discrete distribution then tells us which $\tau$ and thus which $\hat{z}_{t+1-\tau}$ to pick from the set $\{\hat{z}_{t+1-\tau}\}_{\tau=1}^{m}$
- The distribution of hypothetical future returns:

$$
\begin{array}{cccccc}
 & & \hat{z}_{1,1} \rightarrow \hat{R}_{1,t+1} \rightarrow \hat{\sigma}^2_{1,t+2} & \hat{z}_{1,2} \rightarrow \hat{R}_{1,t+2} \rightarrow \hat{\sigma}^2_{1,t+3} & \ldots & \hat{z}_{1,K} \rightarrow \hat{R}_{1,t+K} \\
 & \nearrow & \hat{z}_{2,1} \rightarrow \hat{R}_{2,t+1} \rightarrow \hat{\sigma}^2_{2,t+2} & \hat{z}_{2,2} \rightarrow \hat{R}_{2,t+2} \rightarrow \hat{\sigma}^2_{2,t+3} & \ldots & \hat{z}_{2,K} \rightarrow \hat{R}_{2,t+K} \\
\sigma^2_{t+1} & \longrightarrow & \ldots & \ldots & \ldots & \ldots \\
 & \searrow & \ldots & \ldots & \ldots & \ldots \\
 & & \hat{z}_{FH,1} \rightarrow \hat{R}_{FH,t+1} \rightarrow \hat{\sigma}^2_{FH,t+2} & \hat{z}_{FH,2} \rightarrow \hat{R}_{FH,t+2} \rightarrow \hat{\sigma}^2_{FH,t+3} & \ldots & \hat{z}_{FH,K} \rightarrow \hat{R}_{FH,t+K}
\end{array}
$$

# Filtered Historical Simulation (FHS)

- where *FH* is the number of times we draw from the standardized residuals on each future date (ex:10000)
- *K* is horizon of interest measured in number of days
- We end up with *FH* sequences of hypothetical daily returns for day $t$+1 through day $t$+$K$.
- From these hypothetical daily returns, we calculate the hypothetical *K*-day returns as

$$\hat{R}_{i,t+1:t+K} = \sum_{k=1}^{K} \hat{R}_{i,t+k}, \text{ for } i = 1, 2, ..., FH$$

# Filtered Historical Simulation (FHS)

- If we collect the *FH* hypothetical *K*-day returns in a set $\left\{\hat{R}_{i,t+1:t+K}\right\}_{i=1}^{FH}$, then we can calculate the *K*-day Value-at-Risk by calculating the 100*p*th percentile as

$$VaR^{p}_{t+1:t+K} = -Percentile\left\{\left\{\hat{R}_{i,t+1:t+K}\right\}_{i=1}^{FH}, 100p\right\}$$

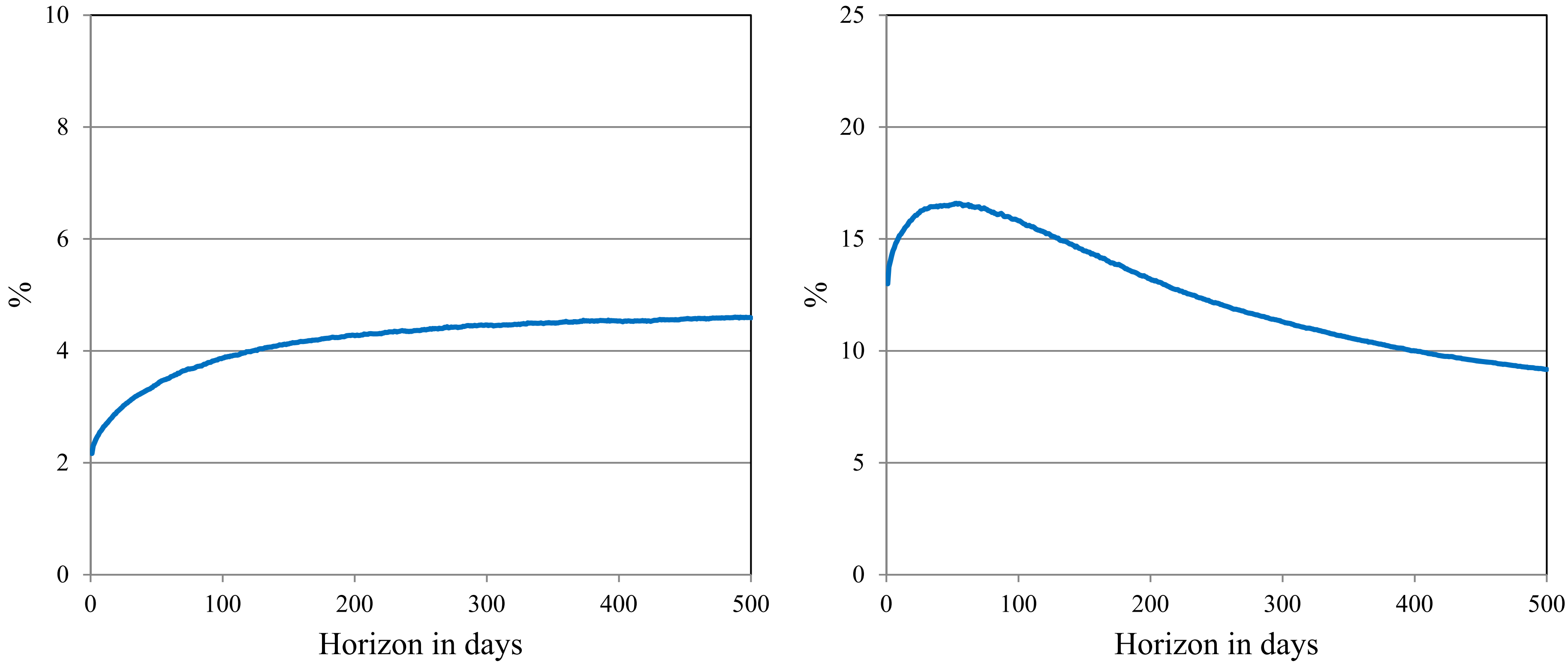
# Filtered Historical Simulation (FHS)

- FHS can generate large losses in the forecast period, even without having observed a large loss in the recorded past returns
- Consider the case where we have a relatively large negative $z$ in our database, which occurred on a relatively low variance day
- If this $z$ gets combined with a high variance day in the simulation period then the resulting hypothetical loss will be large

# Filtered Historical Simulation (FHS)

- In Figure 8.3, the *VaR* is simulated using FHS on an NGARCH model
- The VaR per day is plotted as a function of horizon *K* for two different values of $\sigma_{t+1}$
- In the top panel the initial volatility is one-half the unconditional level and in the bottom panel $\sigma_{t+1}$ is three times the unconditional level.
- The horizons goes from 1 to 500 trading days
- The *VaR* coverage level *p* is set to 1% again

# Figure 8.3: VaR Term Structures using NGARCH and Filtered Historical Simulation




Notes to Figure: The left panel shows the S&P 500 VaR per day across horizons when the current volatility is one half its long run value. The right panel assumes the current volatility is 3 times its long run value.

# The Risk Term Structure with Constant Correlations

- Multivariate risk models allow us to compute risk measures for different hypothetical portfolio allocations without having to re-estimate model parameters.
- Once the set of assets has been determined, the next step in the multivariate model is to estimate a dynamic volatility model for each of the *n* assets
- we can write the *n* asset returns in vector form as:

$$r_{t+1} = D_{t+1} z_{t+1}$$

- where $D_{t+1}$ is an n × n diagonal matrix containing the dynamic standard deviations on the diagonal, and zeros on the off diagonal

# The Risk Term Structure with Constant Correlations

- The $n \times 1$ vector $z_{t+1}$ contains the shocks from the dynamic volatility model for each asset
- The conditional covariance matrix of the returns is:

$$Var_t\left(r_{t+1}\right) = \Sigma_{t+1} = D_{t+1} \Upsilon D_{t+1}$$

- where $\Upsilon$ is a constant $n \times n$ matrix containing the base asset correlations on the off diagonals and ones on the diagonal
- When simulating the multivariate model forward we must ensure that the vector of shocks have the correct correlation matrix, $\Upsilon$

# The Risk Term Structure with Constant Correlations



- Random number generators provide uncorrelated random standard normal variables, $z^u{}_t$, which must be correlated before using them to simulate returns forward
- In the case of two uncorrelated shocks, we have

$$E\left[z_{t+1}^u \left(z_{t+1}^u\right)'\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- To create correlated shocks with the correlation matrix:

$$E\left[z_{t+1}\left(z_{t+1}\right)'\right] = \Upsilon = \begin{bmatrix} 1 & \rho_{1,2} \\ \rho_{1,2} & 1 \end{bmatrix}$$

# The Risk Term Structure with Constant Correlations

- We therefore need to find the matrix square root, $\Upsilon^{1/2}$, so that $\Upsilon^{1/2}\left(\Upsilon^{1/2}\right)' = \Upsilon$ and so that $z_{t+1} = \Upsilon^{1/2} z^u_{t+1}$ will give the correct correlation matrix, namely

$$E\left[z_{t+1}\left(z_{t+1}\right)'\right] = E\left[\Upsilon^{1/2} z^u_{t+1}\left(z^u_{t+1}\right)'\left(\Upsilon^{1/2}\right)'\right] = \Upsilon$$

- In the bivariate case we have that

$$\Upsilon^{1/2} = \begin{bmatrix} 1 & 0 \\ \rho_{1,2} & \sqrt{1-\rho^2_{1,2}} \end{bmatrix}$$

# The Risk Term Structure with Constant Correlations

- so that when multiplying out $z_{t+1} = \Upsilon^{1/2} z^u_{t+1}$ we get

$$\begin{aligned} z_{1,t+1} &= z^u_{1,t+1} \\ z_{2,t+1} &= \rho_{1,2} z^u_{1,t+1} + \sqrt{1-\rho^2_{1,2}} z^u_{2,t+1} \end{aligned}$$

- which implies that

$$\begin{aligned} E\left[z_{1,t+1}\right] &= E\left[z^u_{1,t+1}\right] = 0 \\ E\left[z_{2,t+1}\right] &= \rho_{1,2} E\left[z^u_{1,t+1}\right] + \sqrt{1-\rho^2_{1,2}} E\left[z^u_{2,t+1}\right] = 0 \end{aligned}$$

- and $$\begin{aligned} Var\left[z_{1,t+1}\right] &= Var\left[z^u_{1,t+1}\right] = 1 \\ Var\left[z_{2,t+1}\right] &= \rho^2_{1,2} Var\left[z^u_{1,t+1}\right] + \left(1-\rho^2_{1,2}\right) Var\left[z^u_{2,t+1}\right] = 1 \end{aligned}$$

- because $Var\left[z^u_{1,t+1}\right] = Var\left[z^u_{2,t+1}\right] = 1.$

# The Risk Term Structure with Constant Correlations



- Thus $z_{1,t+1}$ and $z_{2,t+1}$ will each have a mean of 0 and a variance of 1 as desired
- Now we have the correlation as follows:

$$E\left[z_{1,t+1}z_{2,t+1}\right] = \rho_{1,2}E\left[z^u_{1,t+1}z^u_{1,t+1}\right] + \sqrt{1-\rho^2_{1,2}}E\left[z^u_{1,t+1}z^u_{2,t+1}\right] = \rho_{1,2}$$

- To verify $\Upsilon^{1/2}$ matrix, multiply it by its transpose

$$\Upsilon^{1/2}\left(\Upsilon^{1/2}\right)' = \begin{bmatrix} 1 & 0 \\ \rho_{1,2} & \sqrt{1-\rho^2_{1,2}} \end{bmatrix} \begin{bmatrix} 1 & \rho_{1,2} \\ 0 & \sqrt{1-\rho^2_{1,2}} \end{bmatrix} = \begin{bmatrix} 1 & \rho_{1,2} \\ \rho_{1,2} & 1 \end{bmatrix} = \Upsilon$$

- If $n > 2$ assets we use a Cholesky decomposition or a spectral decomposition of $\Upsilon$ to compute $\Upsilon^{1/2}$

# Multivariate Monte Carlo Simulation

- The algorithm for multivariate Monte Carlo simulation is as follows
- First, draw a vector of uncorrelated random normal variables $\breve{z}^u_{i,1}$ with a mean of zero and variance of one
- Second, use the matrix square root $\Upsilon^{1/2}$ to correlate the random variables; this gives $\breve{z}_{i,t+1} = \Upsilon^{1/2} z^u_{i,1}$.
- Third, update the variances for each asset
- Fourth, compute returns for each asset
- Loop through these four steps from day $t$+1 until day $t$+$K$

# Multivariate Monte Carlo Simulation

- Now we can compute the portfolio return using the known portfolio weights and the vector of simulated returns on each day

- Repeating these steps $i = 1,2,\ldots,MC$ times gives a Monte Carlo distribution of portfolio returns

- From these *MC* portfolio returns we can compute *VaR* and *ES* from the simulated portfolio returns

# Multivariate Filtered Historical Simulation

- Assume constant correlations for Multivariate Filtered Historical Simulation
- First, draw a vector (across assets) of historical shocks from a particular day in historical sample of shocks, and use that to simulate tomorrow's shock, $\hat{z}_{i,1}$
- The vector of historical shocks from the same day will preserve the correlation across assets that existed historically if the correlations are constant over time
- Second, update the variances for each asset
- Third, compute returns for each asset
- Loop through these steps from day $t$+1 until day $t$+$K$

# Multivariate Filtered Historical Simulation

- Now we can compute the portfolio return using the known portfolio weights and the vector of simulated returns on each day as before

- Repeating these steps $i = 1,2,\ldots,FH$ times gives a simulated distribution of portfolio returns

- From these *FH* portfolio returns we can compute *VaR* and *ES* from the simulated portfolio returns

# The Risk Term Structure with Dynamic Correlations

- Consider the more complicated case where the correlations are dynamic as in the DCC model
- We have

$$r_{t+1} = D_{t+1} z_{t+1}$$

- where $D_{t+1}$ is an $n \times n$ diagonal matrix containing the GARCH standard deviations on the diagonal, and zeros on the off diagonal
- The $n \times 1$ vector $z_t$ contains the shocks from the GARCH models for each asset.

# The Risk Term Structure with Dynamic Correlations



- Now, we have

$$Var_t\left(r_{t+1}\right) = \Sigma_{t+1} = D_{t+1}\Upsilon_{t+1}D_{t+1}$$

- where $\Upsilon_{t+1}$ is an $n \times n$ matrix containing the base asset correlations on the off diagonals and ones on the diagonal.
- The elements in $D_{t+1}$ can be simulated forward but we now also need to simulate the correlation matrix forward.

# Monte Carlo Simulation with Dynamic Correlations

- Random number generators provide uncorrelated random standard normal variables, $\check{z}^u$, and we must correlate them before simulating returns forward
- At the end of day $t$ the GARCH and DCC models provide us with $D_{t+1}$ and $\Upsilon_{t+1}$
- Therefore a random return for day $t$+1 is

$$\check{r}_{i,t+1} = D_{t+1}\Upsilon_{t+1}^{1/2}\check{z}_{i,1}^{u} = D_{t+1}\check{z}_{i,t+1}$$

where $\check{z}_{i,t+1} = \Upsilon_{t+1}^{1/2}\check{z}_{i,1}^{u}$

- The new simulated shock vector, $\check{z}_{i,t+1}$ ,can update the volatilities and correlations using the GARCH models and the DCC model

# Monte Carlo Simulation with Dynamic Correlations

- Drawing a new vector of uncorrelated shocks, $\check{z}^u_{i,2}$, enables us to simulate the return for the second day ahead as

$$\check{r}_{i,t+2} = \check{D}_{i,t+2}\check{\Upsilon}^{1/2}_{i,t+2}\check{z}^u_{i,2} = \check{D}_{i,t+2}\check{z}_{i,t+2}$$

where $\check{z}_{i,t+2} = \check{\Upsilon}^{1/2}_{i,t+2}\check{z}^u_{i,2}$

- We continue this simulation from day *t*+1 through day *t*+*K*, and repeat it for *i* = 1,2,..,*MC* vectors of simulated shocks on each day.
- We can compute the portfolio return using the known portfolio weights and the vector of simulated returns on each day

# Monte Carlo Simulation with Dynamic Correlations

- From these *MC* portfolio returns we can compute *VaR* and *ES* from the simulated portfolio returns
- In dynamic correlation models If we want to construct a forecast for the correlation matrix two days ahead we can use

$$\Upsilon_{t+2|t} = \frac{1}{MC} \sum_{i=1}^{MC} \check{\Upsilon}_{i,t+2}$$

- where the Monte Carlo average is done element by element for each of the correlations in the matrix.

# Filtered Historical Simulation with Dynamic Correlations



- When correlations across assets are assumed to be dynamic then we need to ensure that the correlation dynamics are simulated forward but in FHS we still want to use the historical shocks
- In this case we must first create a database of historical dynamically uncorrelated shocks from which we can resample.
- We create the dynamically uncorrelated historical shock as

$$\hat{z}^{u}_{t+1-\tau} = \Upsilon^{-1/2}_{t+1-\tau}\hat{z}_{t+1-\tau}, \text{ for } \tau = 1, 2, ..., m$$

# Filtered Historical Simulation with Dynamic Correlations

- where $\hat{z}_{t+1-\tau}$ is the vector of standardized shocks on day $t$+1-τ and where $\Upsilon_{t+1-\tau}^{-1/2}$ is the inverse of the matrix square-root of the conditional correlation matrix $\Upsilon_{t+1-\tau}$
- When calculating the multiday conditional *VaR* and *ES* from the model, we need to simulate daily returns forward from today's (day $t$) forecast of tomorrow's matrix of volatilities, $D_{t+1}$ and correlations, $\Upsilon_{t+1}$
- From the database of uncorrelated shocks $\{\hat{z}_{t+1-\tau}^{u}\}_{\tau=1}^{m}$ we can draw a random vector of historical uncorrelated shocks, called $\hat{z}_{i,1}^{u}$. The entire vector of shocks represents the same day for all the assets

# Filtered Historical Simulation with Dynamic Correlations

- From this draw, we can compute a random return for day *t*+1 as

$$\hat{r}_{i,t+1} = D_{t+1}\Upsilon_{t+1}^{1/2}\hat{z}_{i,1}^{u} = D_{t+1}\hat{z}_{i,t+1}$$

where $$\hat{z}_{i,t+1} = \Upsilon_{t+1}^{1/2}\hat{z}_{i,1}^{u}$$

- Using the new simulated shock vector, $\hat{z}_{i,t+1}$, we can update the volatilities and correlations using the GARCH models and the DCC model.
- We thus obtain simulated $\hat{D}_{i,t+2}$ and $\hat{\Upsilon}_{i,t+2}$

# Filtered Historical Simulation with Dynamic Correlations

- Drawing a new vector of uncorrelated shocks, $\hat{z}^{u}_{i,2}$, enables us to simulate the return for second day as

$$\hat{r}_{i,t+2} = \hat{D}_{i,t+2}\hat{\Upsilon}^{1/2}_{i,t+2}\hat{z}^{u}_{i,2} = D_{t+2}\hat{z}_{i,t+2}$$

- Where $\hat{z}_{i,t+2} = \hat{\Upsilon}^{1/2}_{t+1}\hat{z}^{u}_{i,2}$.
- We continue this simulation for *K* days, and repeat it for *FH* vectors of simulated shocks on each day.
- We can compute the portfolio return using the known portfolio weights and the vector of simulated returns on each day.
- From these *FH* portfolio returns we can compute *VaR* and *ES* from the simulated portfolio returns

# Filtered Historical Simulation with Dynamic Correlations



- The advantages of the multivariate FHS approach tally with those of the univariate case:
  - It captures current market conditions by means of dynamic variance and correlation models.
  - It makes no assumption on the conditional multivariate shock distributions.
  - And, it allows for the computation of any risk measure for any investment horizon of interest.

# Summary

- Risk managers want to know the term structure of risk
- This chapter introduced Monte Carlo simulation and filtered Historical Simulation techniques used to compute the term structure of risk
- When simulating from dynamic risk models, we use all the relevant information available at any given time to compute the risk forecasts across future horizons