Historical Simulation, Value-at-Risk, and Expected Shortfall

FNCE 5321
Hang Bai

Plan

•Introduce the most commonly used method for computing VaR, namely Historical Simulation and discuss the pros and cons of this method.

•Discuss the pros and cons of the VaR risk measure

•Consider the Expected Shortfall, ES, alternative.

More specifically

- Introduction of the historical simulation (HS) method and its pros and cons.
- Comparison of the performance of HS and RiskMetrics during the 2008-2009 financial crisis.
- Then we simulate artificial return data and assess the HS VaR on this data.
- Compare the VaR risk measure with ES.

Defining Historical Simulation

- Let today be day t. Consider a portfolio of n assets. If we today own $N_{i,t}$ units or shares of asset i then the value of the portfolio today is

$$V_{PF,t} = \sum_{i=1}^{n} N_{i,t} S_{i,t}$$

- We use today's portfolio holdings but historical asset prices to compute yesterday's hypothetical portfolio value as

$$V_{PF,t-1} = \sum_{i=1}^{n} N_{i,t} S_{i,t-1}$$

Defining Historical Simulation

- This is a hypothetical value because the units of each asset held typically changes over time. The pseudo log return can now be defined as

$$R_{PF,t} = \ln\left(V_{PF,t}/V_{PF,t-1}\right)$$

- Consider the availability of a past sequence of m daily hypothetical portfolio returns, calculated using past prices of the underlying assets of the portfolio, but using today's portfolio weights, call it $\{R_{PF,t+1-\tau}\}^{m}_{\tau=1}$

Defining Historical Simulation

- Distribution of $R_{PF,t+1}$ is captured by the histogram of $\{R_{PF,t+1-\tau}\}^m_{\tau=1}$
- The VaR with coverage rate, p is calculated as 100pth percentile of the sequence of past portfolio returns.

$$VaR^p_{t+1} = -Percentile\left(\{R_{PF,t+1-\tau}\}^m_{\tau=1}, 100p\right)$$

- Sort the returns in $\{R_{PF,t+1-\tau}\}^m_{\tau=1}$ in ascending order
- Choose VaR^P_{t+1} such that only 100p% of the observations are smaller than the VaR^P_{t+1}
- Use linear interpolation to calculate the exact VaR number.

Pros and Cons of HS

Pros

- the ease with which it is implemented.
- its model-free nature.

Cons

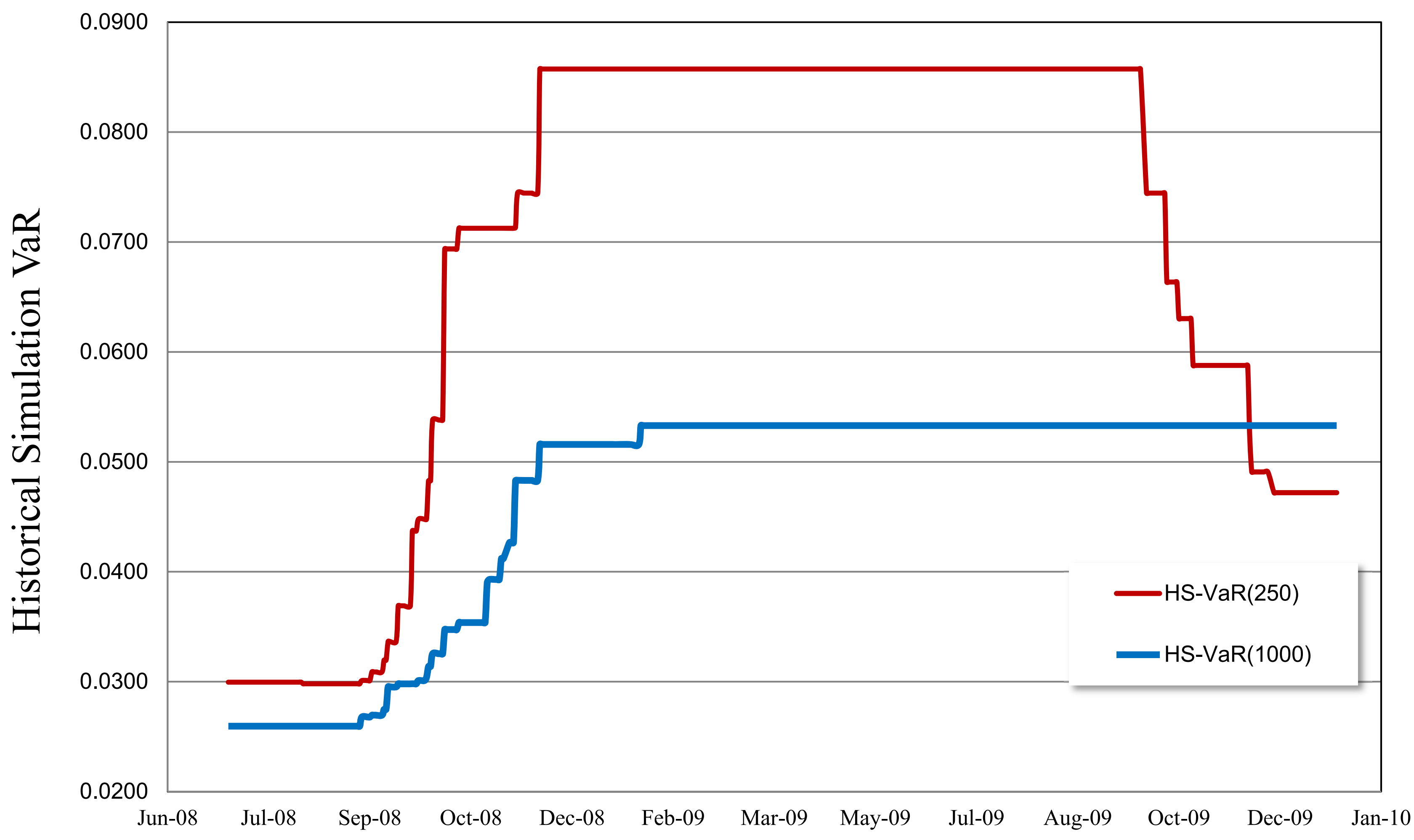
- Not clear how to choose the data sample length

Issues with model free nature of HS

How large should m be?

- If m is too large, then the most recent observations will carry very little weight, and the VaR will tend to look very smooth over time.
- If m is too small, then the sample may not include enough large losses to enable the risk manager to calculate VaR with any precision.
- To calculate 1% VaRs with any degree of precision for the next 10 days, HS technique needs a large m value

Figure 2.1:
VaRs from HS with 250 and 1,000 Return Days
Jul 1, 2008 - Dec 31, 2010



Advantages of Risk Metrics model

- It can pick up the increase in market variance from the crash regardless of whether the crash meant a gain or a loss
- In this model, returns are squared and losses and gains are treated as having the same impact on tomorrow’s variance and therefore on the portfolio risk.

Evidence from the 2008-2009 Crisis

- We consider the daily closing prices for a total return index of the S&P 500 starting in July 2008 and ending in December 2009.

- The index lost almost half its value between July 2008 and the market bottom in March 2009.

- The recovery in the index starting in March 2009 continued through the end of 2009.

Figure 2.4: S&P 500 Total Return Index: 2008-2009 Crisis Period

Evidence from the 2008-2009 Crisis

- The 10-day 1% HS VaR is computed from the 1-day VaR by simply multiplying it by $\sqrt{10}$

$$VaR^{.01,HS}_{t+1:t+10} = -\sqrt{10} \cdot Percentile\left(\left\{R_{PF,t+1-\tau}\right\}_{\tau=1}^{m}, 1\right), \text{with } m = 250$$

- Alternative to HS is the RiskMetrics variance model
- 10-day, 1% *VaR* computed from the Risk- Metrics model is as follows:

$$\begin{aligned} VaR^{.01,RM}_{t+1:t+10} &= -\sqrt{10} \cdot \sigma_{t+1} \cdot \Phi^{-1}_{.01} \\ &= -\sqrt{10} \cdot \sigma_{t+1} \cdot 2.33 \end{aligned}$$

Evidence from the 2008-2009 Crisis

- where the variance dynamics are driven by

$$\sigma^2_{PF,t+1} = 0.94\sigma^2_{PF,t} + 0.06R^2_{PF,t}$$

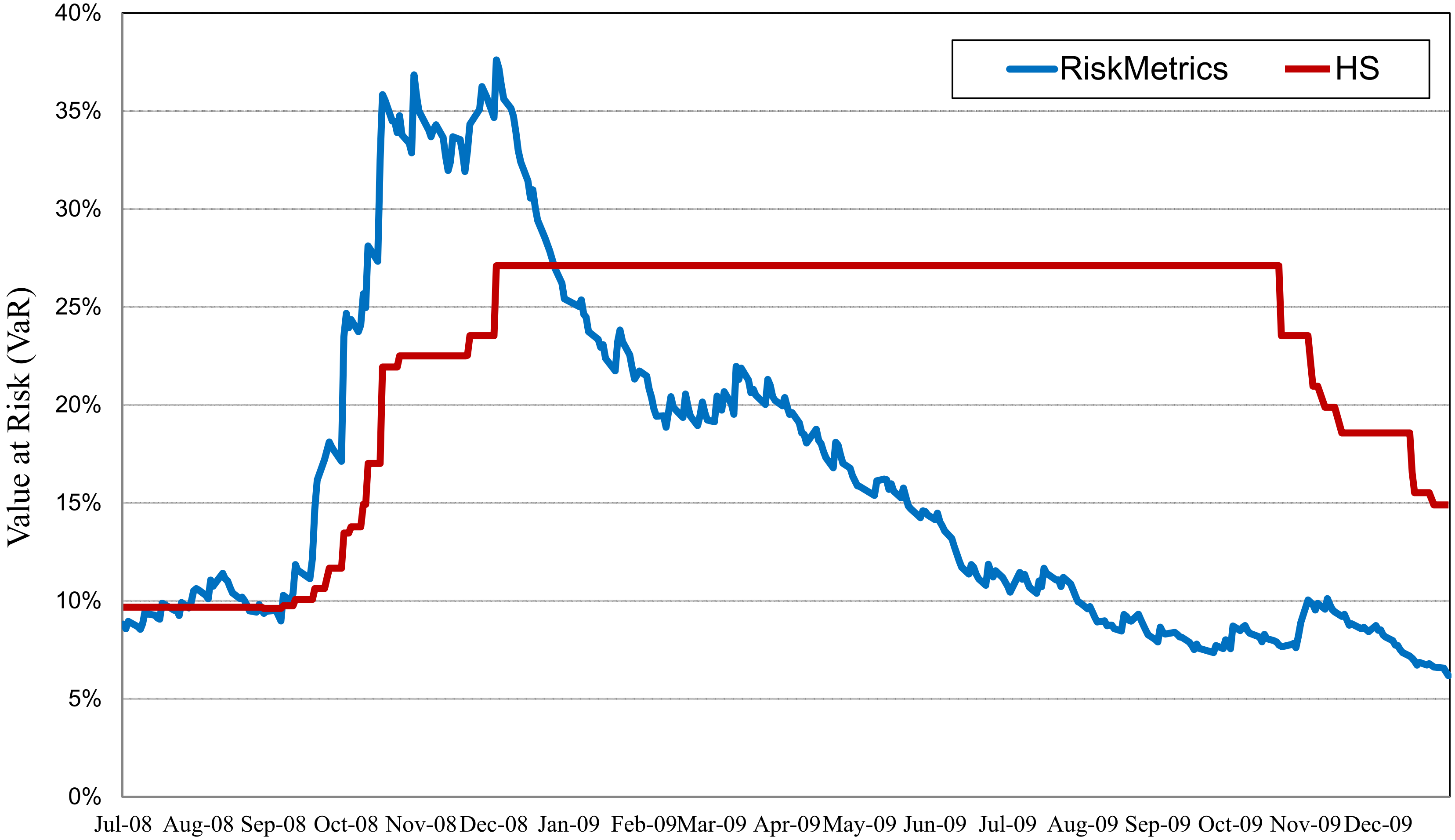
Difference between the HS and the RM *VaR*s

•The HS *VaR* rises much more slowly as the crisis gets underway in the fall of 2008

•The HS *VaR* stays at its highest point for almost a year during which the volatility in the market has declined considerably

•HS *VaR* will detect the brewing crisis quite slowly and will enforce excessive caution after volatility drops in the market

Figure 2.5: 10-day, 1% VaR from Historical Simulation and RiskMetrics During the 2008-2009 Crisis Period



Evidence from the 2008-2009 Crisis

- The units in figure above refer to the least percent of capital that would be lost over the next 10 days in the 1% worst outcomes.

- Let's put some dollar figures on this effect
- Assume that each day a trader has a 10-day, 1% dollar *VaR* limit of $100,000
- Thus each day he is therefore allowed to invest

$$\$Position_{t+1} \leq \frac{\$100,000}{VaR^{.01}_{t+1:t+10}}$$

Evidence from the 2008-2009 Crisis

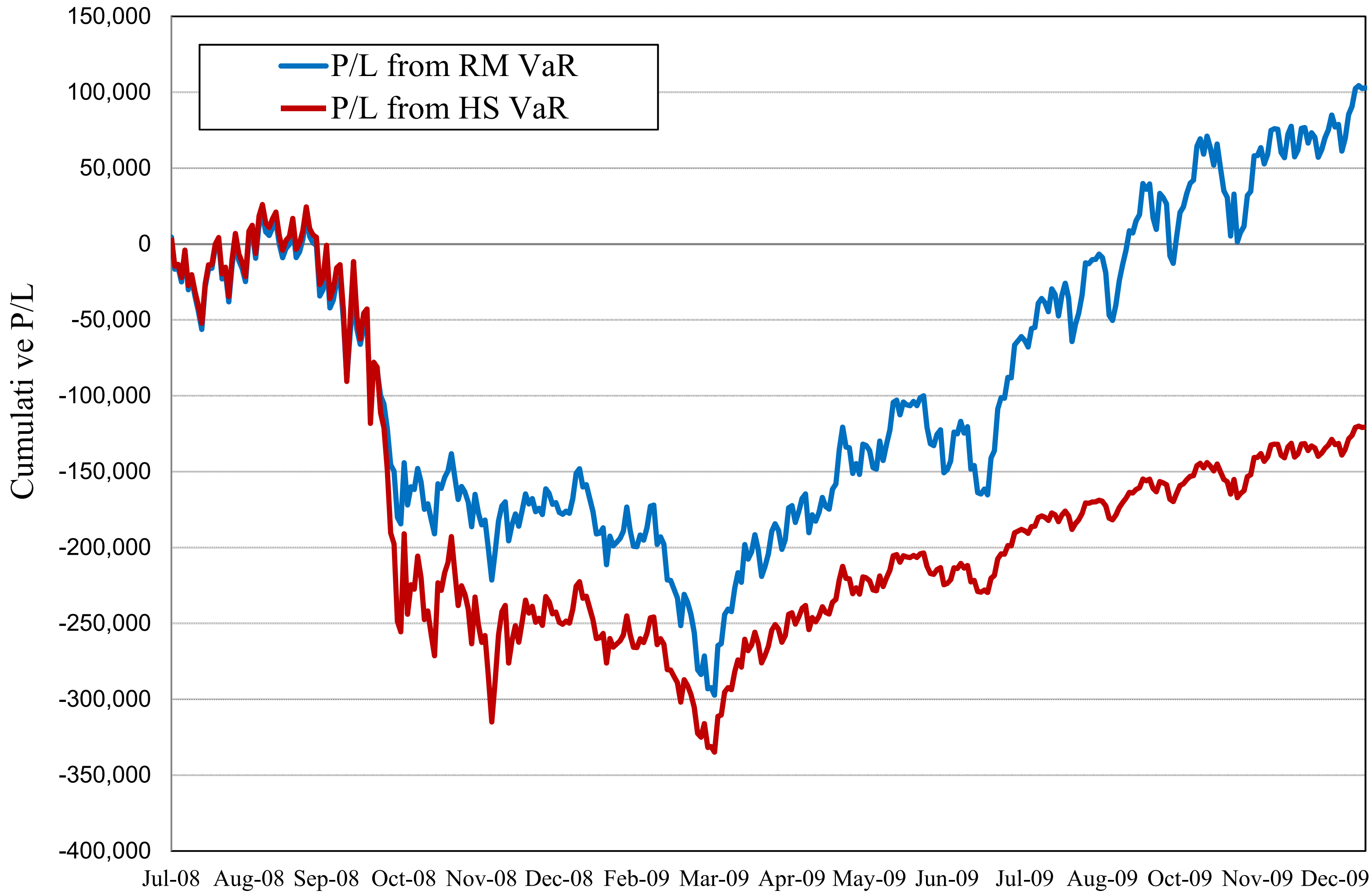
- Let's assume that the trader each day invests the maximum amount possible in the S&P 500

$$\$Position_{t+1} = \frac{\$100,000}{VaR^{.01}_{t+1:t+10}}$$

- The daily P/L is computed as

$$(P/L)_{t+1} = \$Position_{t+1}\,(S_{t+1}/S_t - 1)$$

Figure 2.6: Cumulative P/L from Traders with HS and RM VaRs



Evidence from the 2008-2009 Crisis

Performance difference between HS and RM *VaR*s

- The RM trader will lose less in the fall of 2008 and earn much more in 2009.
- The HS trader takes more losses in the fall of 2008 and is not allowed to invest sufficiently in the market in 2009
- The HS *VaR* reacts too slowly to increases in volatility as well as to decreases in volatility.

VaR with Extreme Coverage Rates

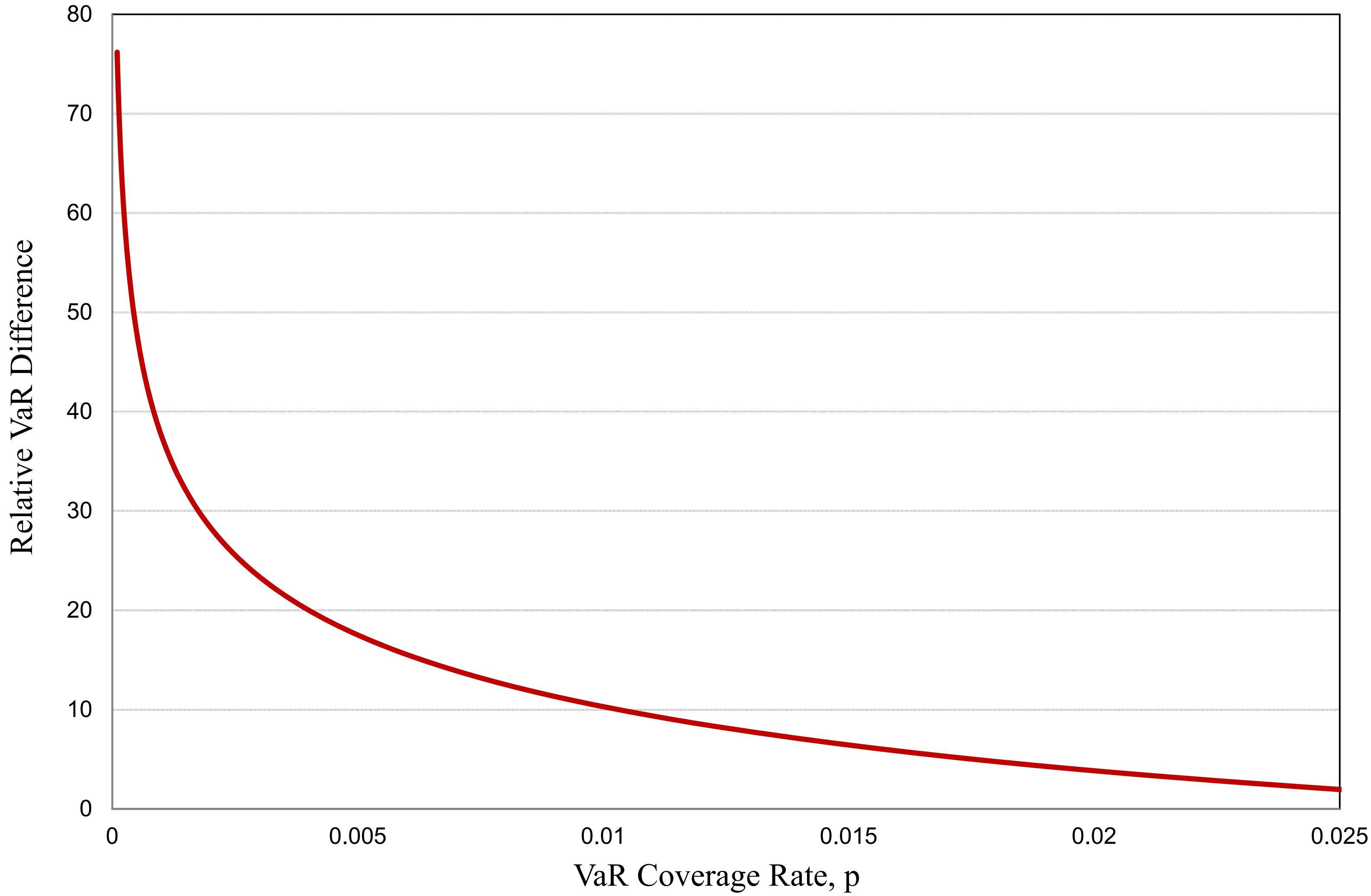
- The tail of the portfolio return distribution conveys information about the future losses.

- Reporting the entire tail of the return distribution corresponds to reporting *VaR*s for many different coverage rates

- Here *p* ranges from 0.01% to 2.5% in increments

- When using HS with a 250-day sample it is not possible to compute the *VaR* when $p < 1/250 = 0.4\%$

Figure 2.8: Relative Difference between Non-Normal (Excess Kurtosis=3) and Normal VaR



VaR with Extreme Coverage Rates

- Note that (from the above figure) as p gets close to zero the nonnormal *VaR* gets much larger than the normal *VaR*

- When $p = 0.025$ there is almost no difference between the two *VaR*s even though the underlying distributions are different

Expected Shortfall

- *VaR* is concerned only with the percentage of losses that exceed the *VaR* and not the magnitude of these losses.
- Expected Shortfall (*ES*), or TailVaR accounts for the magnitude of large losses as well as their probability of occurring
- Mathematically *ES* is defined as

$$ES^p_{t+1} = -E_t\left[R_{PF,t+1} \mid R_{PF,t+1} < -VaR^p_{t+1}\right]$$

Expected Shortfall

- The negative signs in front of the expectation and the *VaR* are needed because the *ES* and the *VaR* are defined as positive numbers
- The ES tells us the expected value of tomorrow's loss, conditional on it being worse than the *VaR*
- The Expected Shortfall computes the average of the tail outcomes weighted by their probabilities
- *ES* tells us the expected loss given that we actually get a loss from the 1% tail

Expected Shortfall

- To compute *ES* we need the distribution of a normal variable conditional on it being below the *VaR*
- The truncated standard normal distribution is defined from the standard normal distribution as

$$\phi_{Tr}(z|z \leq Tr) = \frac{\phi(z)}{\Phi(Tr)} \quad \text{with } E[z|z \leq Tr] = -\frac{\phi(Tr)}{\Phi(Tr)}$$

Expected Shortfall

- $\phi(\bullet)$ denotes the density function and $\Phi(\bullet)$ the cumulative density function of the standard normal distribution
- In the normal distribution case *ES* can be derived as

$$
\begin{aligned}
ES^{p}_{t+1} &= -E_t\left[R_{PF,t+1} | R_{PF,t+1} \leq -VaR^{p}_{t+1}\right] \\
&= -\sigma_{PF,t+1} E_t\left[z_{PF,t+1} | z_{PF,t+1} \leq -VaR^{p}_{t+1}/\sigma_{PF,t+1}\right] \\
&= \sigma_{PF,t+1} \frac{\phi\left(-VaR^{p}_{t+1}/\sigma_{PF,t+1}\right)}{\Phi\left(-VaR^{p}_{t+1}/\sigma_{PF,t+1}\right)}
\end{aligned}
$$

Expected Shortfall

- In the normal case we know that

$$VaR_{t+1}^{p} = -\sigma_{PF,t+1}\Phi_{p}^{-1}$$

- Thus, we have

$$ES_{t+1}^{p} = \sigma_{PF,t+1}\frac{\phi\left(\Phi_{p}^{-1}\right)}{p}$$

- The relative difference between *ES* and *VaR* is

$$\frac{ES_{t+1}^{p} - VaR_{t+1}^{p}}{VaR_{t+1}^{p}} = -\frac{\phi\left(\Phi_{p}^{-1}\right)}{p\Phi_{p}^{-1}} - 1$$

Expected Shortfall

- For example, when p =0.01, we have $\Phi_p^{-1} \approx -2.33$ and the relative difference is then

$$\frac{ES_{t+1}^{.01} - VaR_{t+1}^{.01}}{VaR_{t+1}^{.01}} \approx -\frac{(2\pi)^{-1/2}\exp(-(-2.33)^2/2)}{.01(-2.33)} - 1 \approx 15\%$$

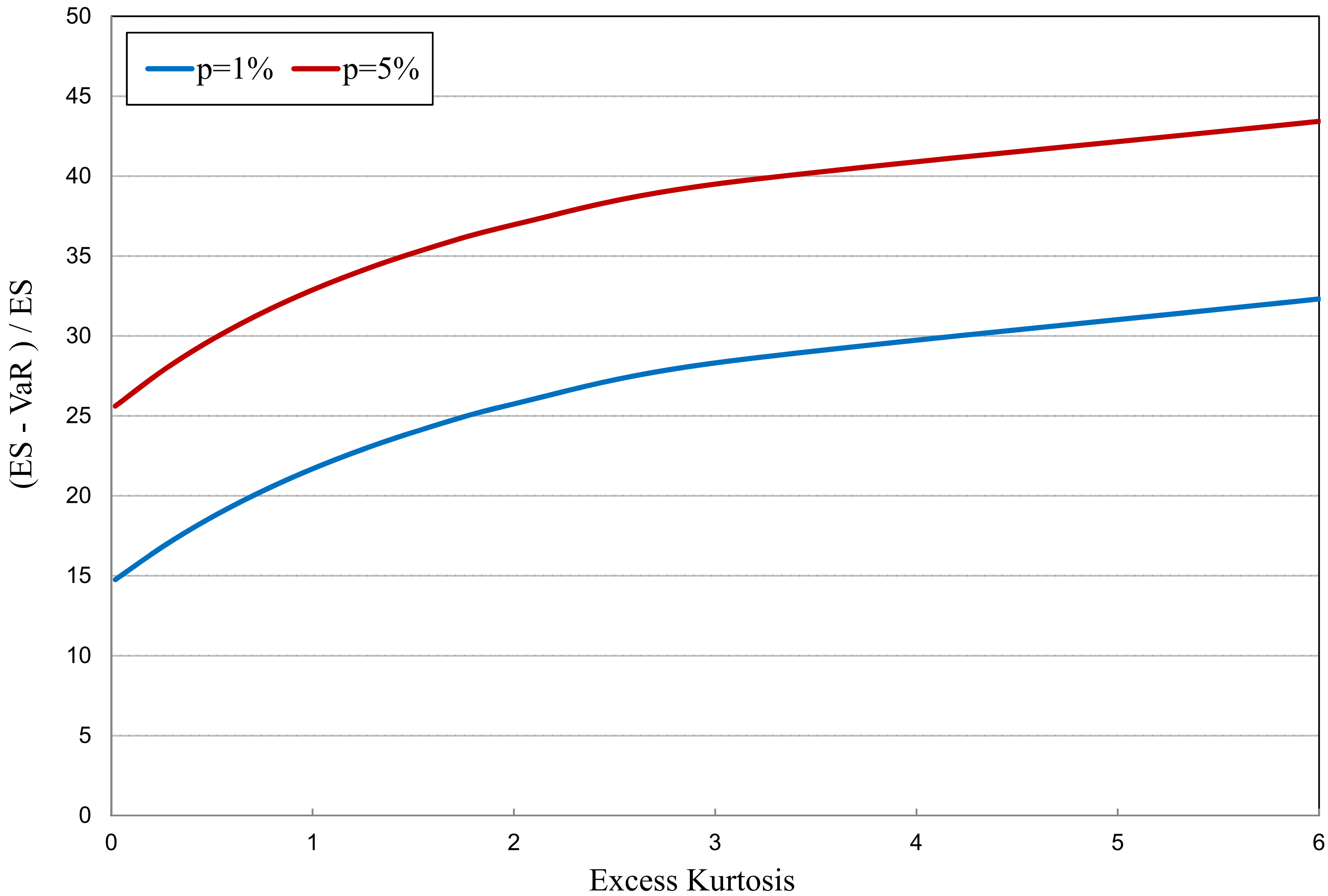
- In the normal case, as *p* gets close to zero, the ratio of the *ES* to the *VaR* goes to 1

- From the below figure, the blue line shows that when excess kurtosis is zero, the relative difference between the *ES* and *VaR* is 15%

Expected Shortfall

- The blue line also shows that for moderately large values of excess kurtosis, the relative difference between *ES* and *VaR* is above 30%
- From the figure, the relative difference between *VaR* and *ES* is larger when p is larger and thus further from zero
- When p is close to zero *VaR* and *ES* will both capture the fat tails in the distribution
- When p is far from zero, only the *ES* will capture the fat tails in the return distribution

Figure 2.9: ES vs VaR as a Function of Kurtosis



Summary

- *VaR* is the most popular risk measure in use
- HS is the most often used methodology to compute *VaR*
- *VaR* has some shortcomings and using HS to compute *VaR* has serious problems as well
- We need to use risk measures that capture the degree of fatness in the tail of the return distribution
- We need risk models that properly account for the dynamics in variance and models that can be used across different return horizons