Non-Normal Distributions

FNCE 5321

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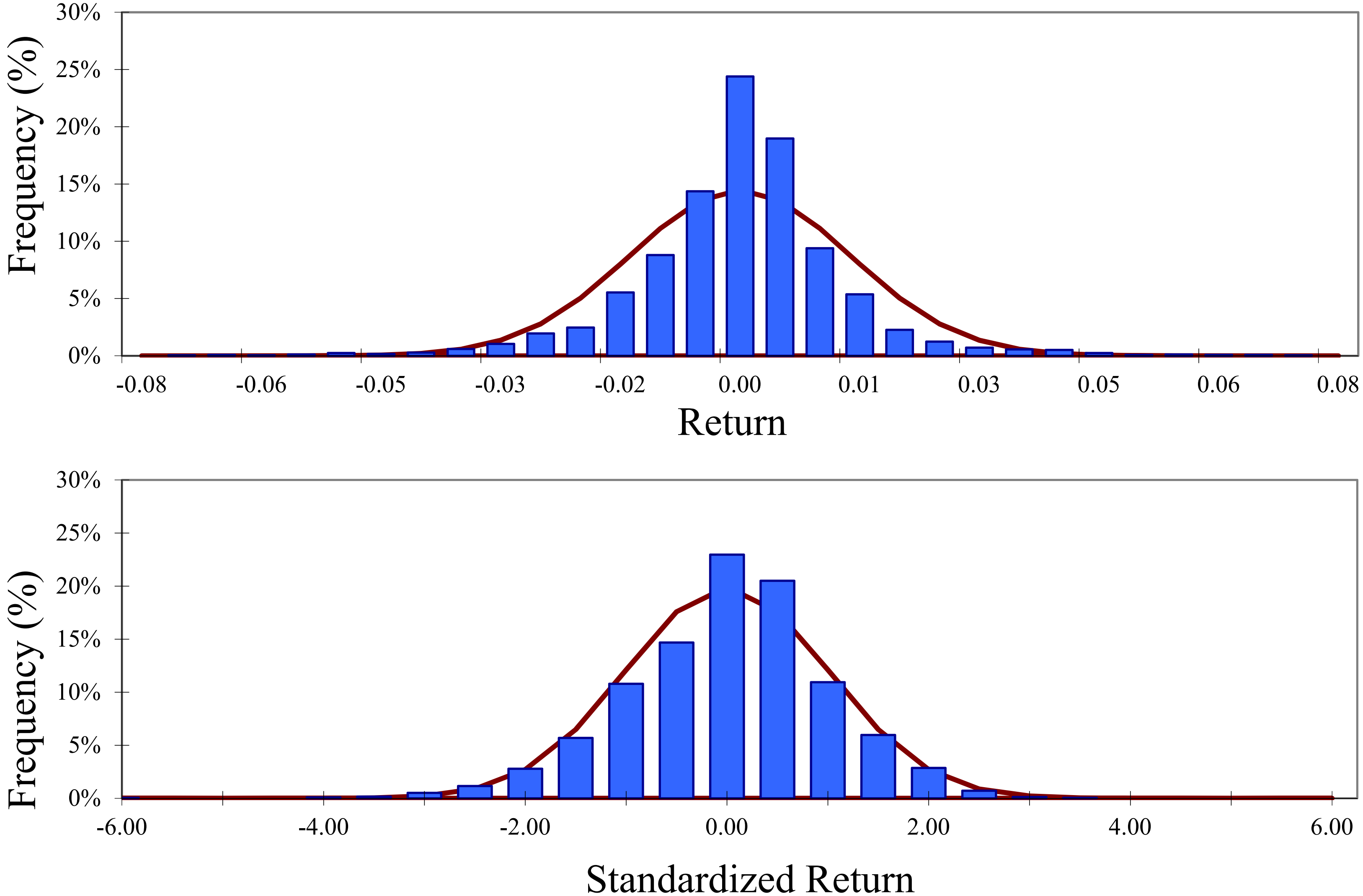
Learning Objectives

- We introduce the quantile-quantile (QQ) plot, which is a graphical tool better at describing tails of distributions than the histogram.
- We define the Filtered Historical Simulation approach which combines GARCH with historical simulation.
- We consider the standardized Student’s t distribution and discuss the estimation of it.

Overview

- Third part of the Stepwise Distribution Modeling (SDM) approach: accounting for conditional nonnormality in portfolio returns.
- Returns are conditionally normal if the dynamically standardized returns are normally distributed.
- Fig.6.1 illustrates how histograms from standardized returns typically do not conform to normal density
- The top panel shows the histogram of the raw returns superimposed on the normal distribution and the bottom panel shows the histogram of the standardized returns superimposed on the normal distribution

Figure 6.1: Histogram of Daily S&P 500 Returns and Histogram of GARCH Shocks



Visualising Non-normality Using QQ Plots

- Consider a portfolio of n assets with $N_{i,t}$ units or shares of asset i then the value of the portfolio today is

$$V_{PF,t} = \sum_{i=1}^{n} N_{i,t} S_{i,t}$$

- Yesterday's portfolio value would be

$$V_{PF,t-1} = \sum_{i=1}^{n} N_{i,t} S_{i,t-1}$$

- The log return can now be defined as

$$R_{PF,t} = \ln\left(V_{PF,t} / V_{PF,t-1}\right)$$

Visualising Non-normality Using QQ Plots

- Allowing for a dynamic variance model we can say

$$R_{PF,t} = \sigma_{PF,t} z_t, \text{ with } z_t \overset{iid}{\sim} D(0,1)$$

- where $\sigma_{PF,t}$ is the conditional volatility forecast
- So far, we have relied on setting *D(0,1) to N(0,1)*, but we now want to assess the problems of the normality assumption

Visualising Non-normality Using QQ Plots

- QQ (Quantile-Quantile) plot: Plot the quantiles of the calculated returns against the quantiles of the normal distribution.
- Systematic deviations from the 45 degree angle signals that the returns are not well described by normal distribution.
- QQ Plots are particularly relevant for risk managers who care about VaR, which itself is essentially a quantile.

Visualising Non-normality Using QQ Plots

- 1) Sort all standardized returns in ascending order and call them z_i
- 2) Calculate the empirical probability of getting a value below the value *i* as *(i-.5)/T*
- 3) Calculate the standard normal quantiles as $\Phi^{-1}_{(i-0.5)/T}$
- 4) Finally draw scatter plot

$$\{X_i, Y_i\} = \left\{\Phi^{-1}_{(i-0.5)/T}, z_i\right\}$$

- If the data were normally distributed, then the scatterplot should conform to the 45-degree line.

Filtered Historical Simulation Approach

- We have seen the pros and cons of both data-based and model-based approaches.
- The Filtered Historical Simulation (FHS) attempts to combine the best of the model-based with the best of the model-free approaches in a very intuitive fashion.
- FHS combines model-based methods of variance with model-free method of distribution in the following fashion.

Filtered Historical Simulation Approach

- Assume we have estimated a GARCH-type model of our portfolio variance.
- Although we are comfortable with our variance model, we are not comfortable making a specific distributional assumption about the standardized returns, such as a Normal or a $\tilde{t}(d)$ distribution.
- Instead we would like the past returns data to tell us about the distribution directly without making further assumptions.

Filtered Historical Simulation Approach

- To fix ideas, consider again the simple example of a GARCH(1,1) model

$$R_{PF,t+1} = \sigma_{PF,t+1} z_{t+1}$$

- where

$$\sigma^2_{PF,t+1} = \omega + \alpha R^2_{PF,t} + \beta \sigma^2_{PF,t}$$

- Given a sequence of past returns, $\{R_{PF,t+1-\tau}\}_{\tau=1}^{m}$ we can estimate the GARCH model.
- Next we calculate past standardized returns from the observed returns and from the estimated standard deviations as

$$\hat{z}_{t+1-\tau} = R_{PF,t+1-\tau} / \sigma_{PF,t+1-\tau}, \text{ for } \tau = 1, 2, ..., m$$

Filtered Historical Simulation Approach

- We will refer to the set of standardized returns as
  $\{\hat{z}_{t+1-\tau}\}_{\tau=1}^{m}$

- To calculate the 1-day *VaR* using the percentile of the database of standardized residuals

$$VaR_{t+1}^{p} = -\sigma_{PF,t+1} Percentile\left\{\{\hat{z}_{t+1-\tau}\}_{\tau=1}^{m}, 100p\right\}$$

- Expected shortfall (*ES*) for the 1-day horizon is

$$ES_{t+1}^{p} = -E_t\left[R_{PF,t+1} | R_{PF,t+1} < -VaR_{t+1}^{p}\right]$$

- The *ES* is calculated from the historical shocks via

$$ES_{t+1}^{p} = -\frac{\sigma_{PF,t+1}}{p \cdot m} \sum_{i=1}^{m} \hat{z}_{t+1-\tau} \cdot \mathbf{1}\left(\hat{z}_{t+1-\tau} < -Percentile\left\{\left\{\hat{z}_{t+1-\tau}\right\}_{\tau=1}^{m}, 100p\right\}\right)$$

Filtered Historical Simulation Approach

- where the indicator function $\mathbf{1}(*)$ returns a 1 if the argument is true and zero if not
- FHS can generate large losses in the forecast period even without having observed a large loss in the recorded past returns
- FHS deserves serious consideration by any risk management team

The Standardized *t* distribution

- The Student's *t* distribution is defined by

$$f_{t(d)}(x;d) = \frac{\Gamma((d+1)/2)}{\Gamma(d/2)\sqrt{d\pi}}(1+x^2/d)^{-(1+d)/2}, \text{ for } d > 0$$

□ $\Gamma(*)$ notation refers to the gamma function

- the distribution has only one parameter *d*
- In the Student's *t* distribution we have the following first two moments

$$\begin{aligned} E[x] &= 0, \text{ when } d > 1 \\ Var[x] &= d/(d-2) \text{ when } d > 2 \end{aligned}$$

The Standardized *t* distribution

- Define Z by standardizing x so that,

$$z = \frac{x - E\left[x\right]}{\sqrt{Var[x]}} = \frac{x}{\sqrt{d/\left(d-2\right)}}$$

- The Standardized *t* distribution, $\tilde{t}(d)$, is then defined as

$$f_{\tilde{t}(d)}(z; d) = C\left(d\right)\left(1 + z^2/(d-2)\right)^{-(1+d)/2}, \text{ for } d > 2$$

- where

$$C\left(d\right) = \frac{\Gamma((d+1)/2)}{\Gamma(d/2)\sqrt{\pi(d-2)}}$$

The Standardized *t* distribution

- In standardized *t* distribution random variable *z* has mean equal to zero and a variance equal to 1
- Note also that the parameter *d* must be larger than two for standardized distribution to be well defined
- In $\tilde{t}(d)$ distribution, the random variable, *z*, is taken to a power, rather than an exponential, which is the case in the standard normal distribution where

$$f(z) = (2\pi)^{-1/2} \exp\left(-z^2/2\right)$$

- The power function driven by *d* will allow for $\tilde{t}(d)$ distribution to have fatter tails than the normal

The Standardized *t* distribution

- The $\tilde{t}(d)$ distribution is symmetric around zero, and the mean μ, variance σ^2, skewness ζ_1, and excess kurtosis ζ_2 of the distribution are

$$
\begin{aligned}
\mu &\equiv E[z] = 0, \\
\sigma^2 &\equiv E[(z - E[z])^2] = 1, \\
\zeta_1 &\equiv E[z^3]/\sigma^3 = 0 \\
\zeta_2 &\equiv E[z^4]/\sigma^4 - 3 = 6/(d-4)
\end{aligned}
$$

The Standardized *t* distribution

- Note that *d* must be higher than 4 for the kurtosis to be well defined.

- Note also that for large values of *d* the distribution will have an excess kurtosis of zero, and we can show that it converges to the standard normal distribution as *d* goes to infinity.

- For values of *d* above 50, the $\tilde{t}(d)$ distribution is difficult to distinguish from the standard normal distribution.