Game Physics Tutorial 3

Rigid Bodies in 2D

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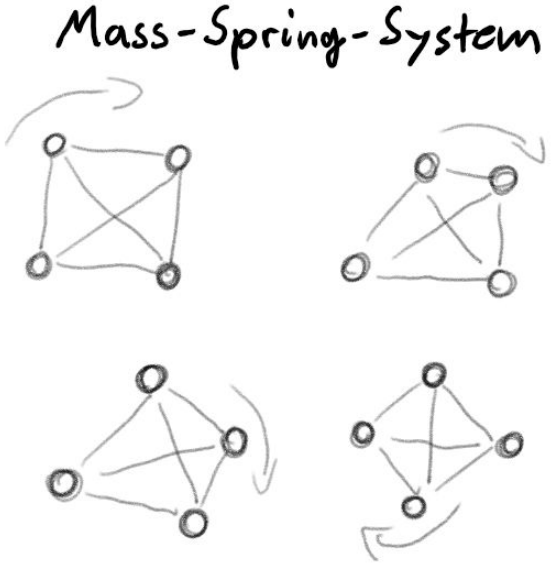
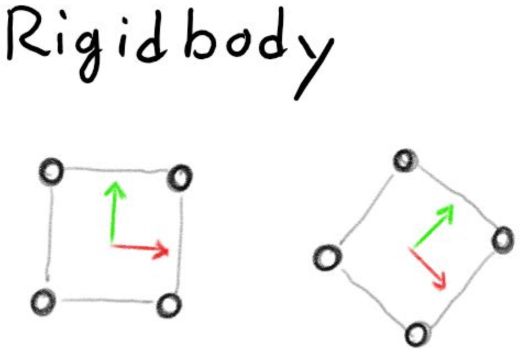




- Simulate each Mass Point individually
- If no deformation wanted: High stiffness for springs
  → High forces
  → Integration only stable for extremely small timesteps

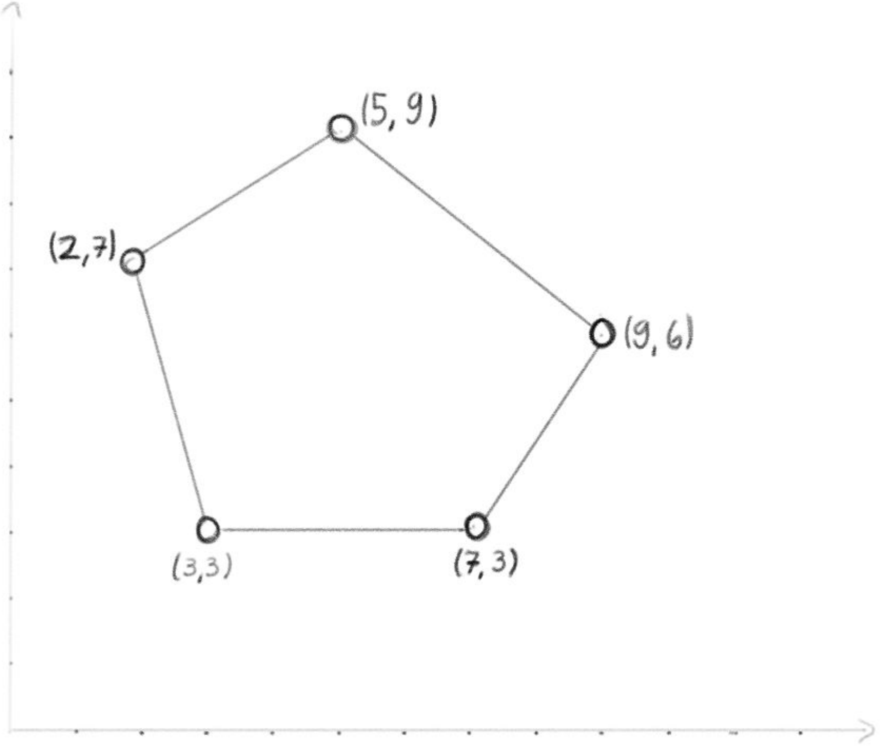
- Rigid = Points don't move relative to each other
  → Only simulate center of mass (Rotation, Translation)
  → points defined relative to center of mass (= local space)

Center of Mass

- Weighted average of Points
  1. Sum up points × their Mass
  2. Divide by Sum of Masses

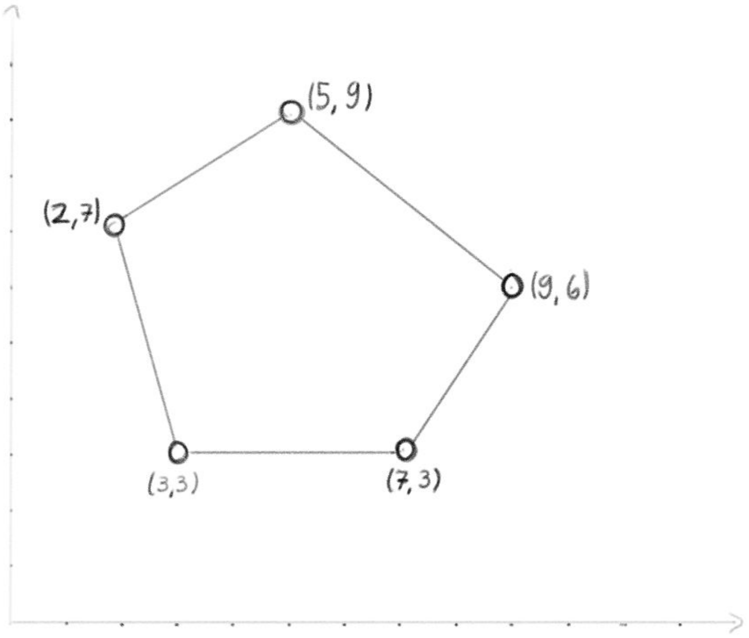
$$\mathbf{x}_{cm} = \frac{\sum_i m_i \mathbf{x_i}}{\sum_i m_i}$$

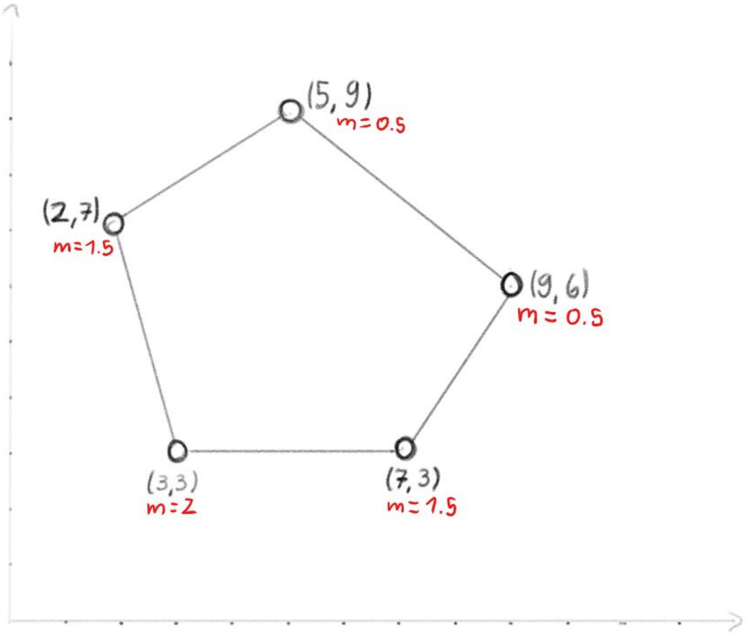
- If no mass specified, assume it to be 1



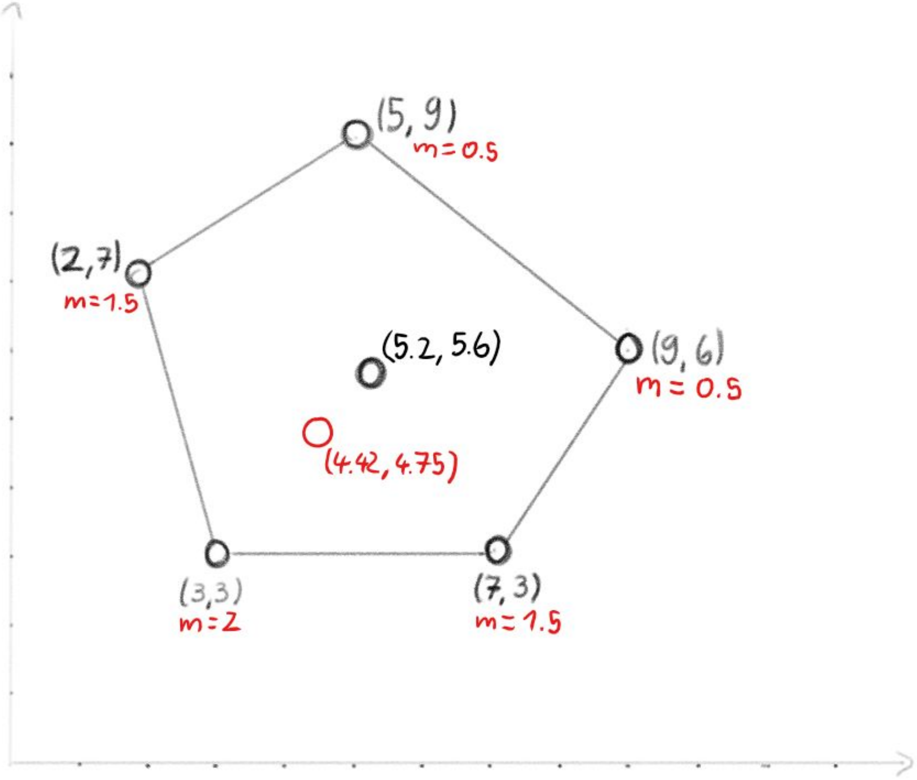
Exercise: Center of Mass

Calculate the Centers of Mass for the given setups!





Exercise: Center of Mass (Solution)



2D: Simulating a Rigidbody

- Translation the same as for Mass Points:
  position, **linear velocity**, **linear forces**, **mass** **+** Integration (e.g. Euler Step)

- Rotation (in 2D there is *only 1 rotation axis* → much easier):

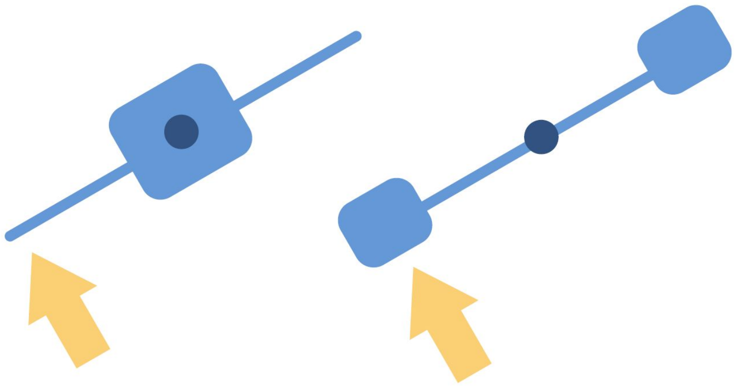
  - **Orientation *r***
    - current angle

  - **Angular velocity *w*** (& **angular momentum L**)
    - How quickly are we currently rotating around the z-axis?

  - **Torque *q***
    - rotational equivalent to forces (**q** changes L and **w**)

  - **Inertia tensor *i***
    - How easily does the Rigidbody rotate around the z-axis?

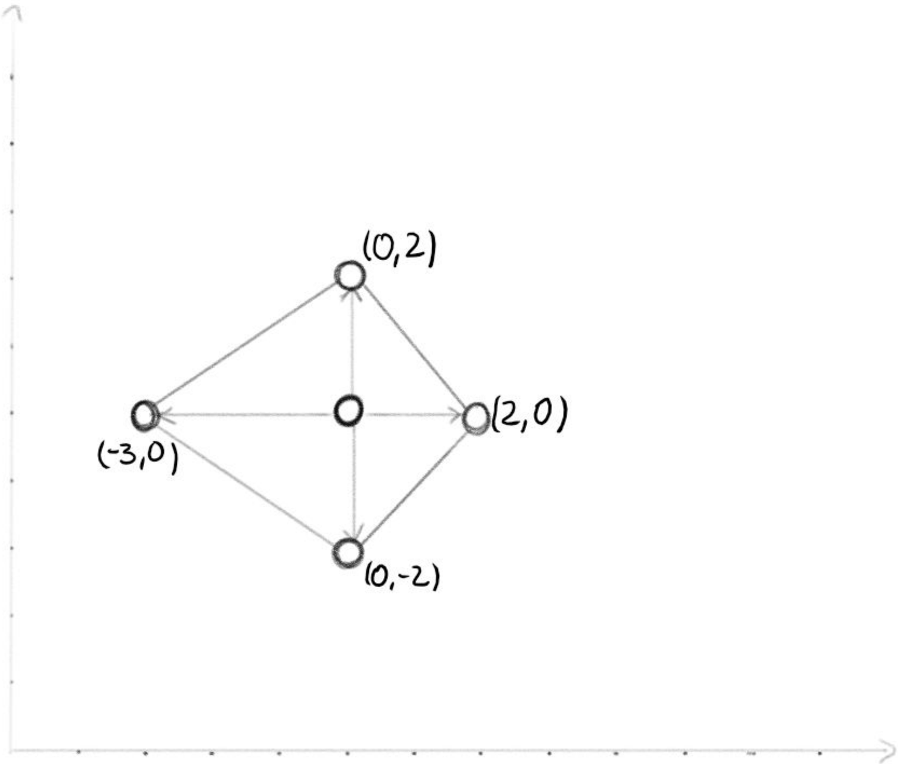
2D: Inertia Tensor *i*



- Different objects rotate more / less easily around different axes
- Inertia Tensor *i* models this behaviour
- In 2D, only rotation about Z-Axis possible
  $\longrightarrow$ *i* is simply a scalar
  $\longrightarrow$ "How easily does the Rigidbody rotate around the z-axis?"
- Can be precomputed:
  - Sum over all mass points:
  - Take the dot product of its local space position vector
  - Multiply the dot product by the mass of the point

$$i = \sum_n m_n \mathbf{x}_n \cdot \mathbf{x}_n$$

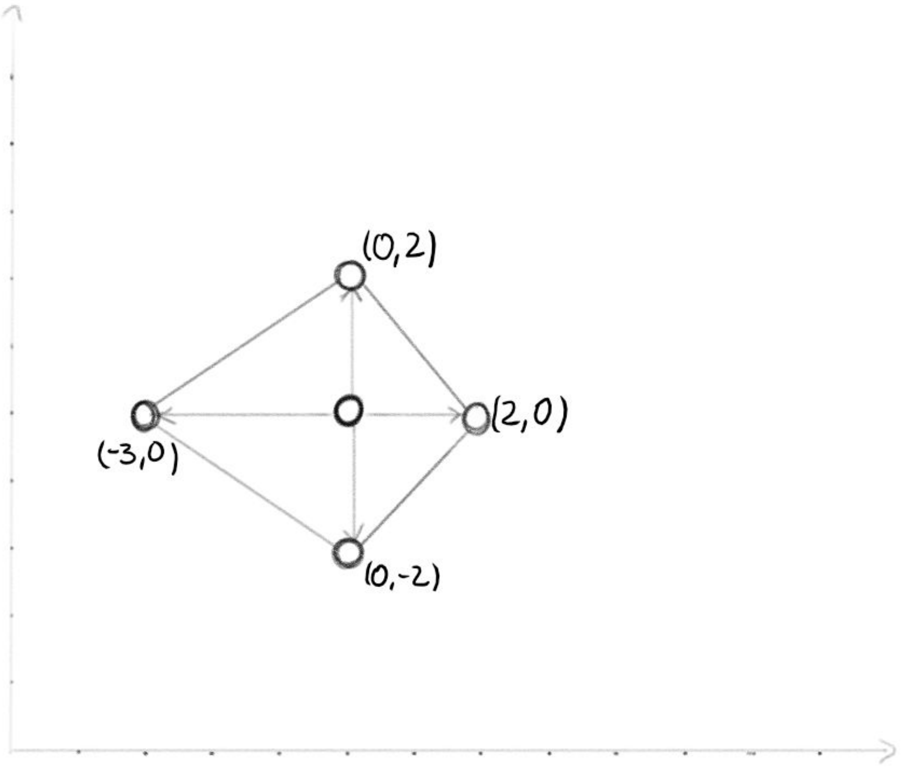
Exercise: 2D Inertia Tensor



$$i = \sum_n m_n \mathbf{x}_n \cdot \mathbf{x}_n$$

(assume mass = 1)

Exercise: 2D Inertia Tensor



$$\begin{pmatrix}0\\2\end{pmatrix}\cdot\begin{pmatrix}0\\2\end{pmatrix}+$$
$$\begin{pmatrix}2\\0\end{pmatrix}\cdot\begin{pmatrix}2\\0\end{pmatrix}+$$
$$\begin{pmatrix}-3\\0\end{pmatrix}\cdot\begin{pmatrix}-3\\0\end{pmatrix}+$$
$$\begin{pmatrix}0\\-2\end{pmatrix}\cdot\begin{pmatrix}0\\-2\end{pmatrix}=$$

$$4+$$
$$4+$$
$$9+$$
$$4=21=i$$

2D: Angular Momentum

- **angular momentum L** vs. **angular velocity w**:
  → Velocity = "How fast"
  → Momentum = "How fast & How 'heavy'?"
  → If something that is difficult to rotate rotates quickly, it has a lot of momentum

- **L** = **i** × **w**

- Momentum: "How much force would you need to stop the rotation?"

- In 2D, **i** is constant
  → We can ignore **L** for 2D and use the **torque q** to change **w** directly:

$$\mathbf{w}(t+h) = \mathbf{w}(t) + h\mathbf{q}/i$$

2D: Simulating Rigidbody Rotation

1. Calculate forces & convert them to **torque q**

$$\mathbf{q}(t) = \sum_i \mathbf{x}_i \times \mathbf{f}_i$$

Cross Product 2D:

$$\begin{pmatrix} x_a \\ y_a \end{pmatrix} \times \begin{pmatrix} x_b \\ y_b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_a \cdot y_b - x_b \cdot y_a \end{pmatrix} = x_a \cdot y_b - x_b \cdot y_a$$

$$\begin{matrix} 0 & & 0 \\ x_a & & x_b \\ y_a & & y_b \end{matrix}$$

2. Integrate the **orientation r** using the **angular velocity w**

$$\mathbf{r}_{cm} = \mathbf{r}_{cm} + h\mathbf{w}_{cm}$$

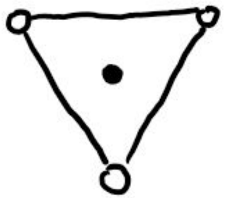
3. Integrate the **angular velocity w** using the calculated **torque q** and the **Inertia tensor i**

$$\mathbf{w}(t+h) = \mathbf{w}(t) + h\mathbf{q}/i$$

4. Update the World Space positions of the Points based on the new **orientation r**

$$\mathbf{x}_i^{world} = \mathbf{x}_{cm}(t) + \mathrm{Rot}_{r(t)}(\mathbf{x}_i)$$

Exercise: 2D Rigidbody Rotation



$r = 45$ deg $\quad h = 0.25$ s

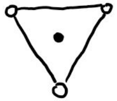
$w = 4 \frac{\text{deg}}{\text{s}}$

$i = 5 \frac{\text{kg} \cdot \text{m}}{\text{deg}}$

A force of $\begin{pmatrix} 0 \\ -20 \end{pmatrix} \frac{\text{kg} \cdot \text{m}}{\text{s}^2}$ is applied at point $\begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$.

Calculate the new angular velocity $w(t+h)$ and the new rotation $r(t+h)$!

Exercise: 2D Rigidbody Rotation



$\mathbf{r} = 45 \text{ deg}$ $\quad h = 0.25 \text{ s}$

$w = 4 \ \frac{\text{deg}}{\text{s}}$

$i = 5 \ \frac{\text{kg} \cdot \text{m}}{\text{deg}}$

A force of $\begin{pmatrix} 0 \\ -20 \end{pmatrix} \frac{\text{kg} \cdot \text{m}}{\text{s}^2}$ is applied at point $\begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$.

$$q(t) = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} \times \begin{pmatrix} 0 \\ -20 \end{pmatrix} \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = -20 \ \frac{\text{kg} \cdot \text{m}}{\text{s}^2}$$

$$r(t+h) = 45 \text{ deg} + 0.25 \text{ s} \cdot 4 \ \frac{\text{deg}}{\text{s}} = 46 \text{ deg}$$

$$w(t+h) = 4 \ \frac{\text{deg}}{\text{s}} + 0.25 \text{ s} \cdot -20 \ \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \Big/ 5 \ \frac{\text{kg} \cdot \text{m}}{\text{deg}}$$

$$= 4 \ \frac{\text{deg}}{\text{s}} + \frac{1}{4} \text{ s} \cdot -4 \ \frac{\text{deg}}{\text{s}^2}$$

$$= 3 \ \frac{\text{deg}}{\text{s}}$$

Impulse

- Sometimes, we want the **velocity** to change instantly (*no time step*)
- Example: On Collision
- Problem: **Forces** don't work here since they change the **velocity** in the *next* timestep (at the earliest)
- Solution: change **velocity** directly (we call this the Impulse J)

$$\mathbf{J} = m\Delta\mathbf{v} \quad \text{(no time step!)}$$

2D: Impulse Equation Explained

$$J = \frac{-(1+c)\mathbf{v}_{rel} \cdot \mathbf{n}}{\frac{1}{M_a} + \frac{1}{M_b} + (\mathbf{x}_a \times \mathbf{n})^2 / I_a + (\mathbf{x}_b \times \mathbf{n})^2 / I_b}$$

- **c**: “bounciness coefficient”, hard 0 .. 1 bouncy
- **n**: normal of the collision
- **v_rel**: relative velocity of the collision =
  [vel at collisionPoint from object A] - [vel at collisionPoint from object B]
  (important: calculate the velocities at these points, don’t just use the linear velocity of the rigid body!)
- **x_a / x_b**: Collision point relative to the center of mass of rigid body A / B

2D: Impulse Calculation

$$J = \frac{-(1+c)\mathbf{v}_{rel} \cdot \mathbf{n}}{\frac{1}{M_a} + \frac{1}{M_b} + (\mathbf{x}_a \times \mathbf{n})^2/I_a + (\mathbf{x}_b \times \mathbf{n})^2/I_b}$$

1. Calculate velocities at collision point for each rigidbody $\mathbf{v}_i = \mathbf{v}_{cm} + \mathbf{w} \times \mathbf{x}_i$

2. Calculate relative velocity

3. Fill in formula

4. Apply impulse (in according directions)

$$\mathbf{v}'_a = \mathbf{v}_a + J\mathbf{n}/M_a \qquad \mathbf{w}'_a = \mathbf{w}_a + (\mathbf{x}_a \times J\mathbf{n})/I_a$$
$$\mathbf{v}'_b = \mathbf{v}_b - J\mathbf{n}/M_b \qquad \mathbf{w}'_b = \mathbf{w}_b - (\mathbf{x}_b \times J\mathbf{n})/I_b$$

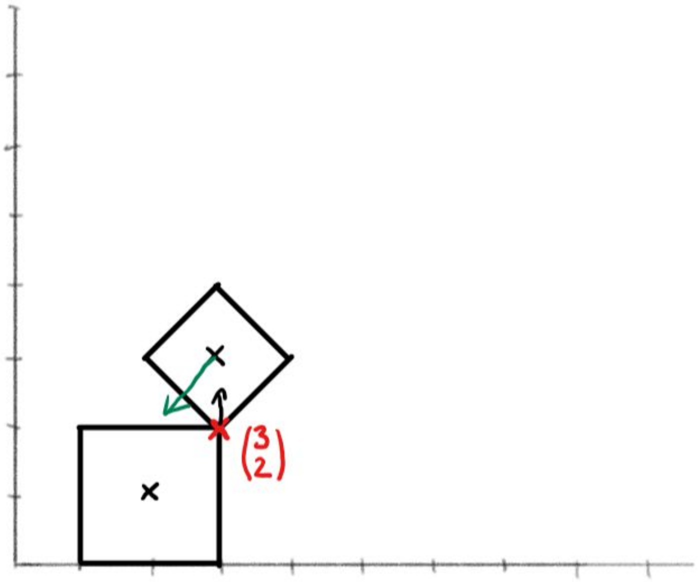
Cross Product 2D:

$$\begin{pmatrix} x_a \\ y_a \end{pmatrix} \times \begin{pmatrix} x_b \\ y_b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_a \cdot y_b - x_b \cdot y_a \end{pmatrix} = x_a \cdot y_b - x_b \cdot y_a$$

$$\begin{matrix} 0 \\ x_a \\ y_a \end{matrix} \qquad \begin{matrix} 0 \\ x_b \\ y_b \end{matrix}$$

2D: Impulse Exercise

Calculate the impulse for the following collision, then apply it to the rigidbodies:



$X_{cm,A} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $m_A = 4$ $I_A = 6$

$X_{cm,B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $m_B = 8$ $I_B = 8$

$V_{cm,A} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ $w_A = -4$

$V_{cm,B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $w_B = 0$

$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $c = 1$

2D: Impulse Exercise (Solution)

$$X_{cm,A} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad m_A = 4 \quad I_A = 6$$

$$X_{cm,B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad m_B = 8 \quad I_B = 8$$

$$V_{cm,A} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad w_A = -4$$

$$V_{cm,B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad w_B = 0$$

$$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad c = 1$$

1. Calculate velocities at collision point:

$$x_a = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad x_b = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_a = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$$

$$v_b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2. Calculate relative velocity:

$$v_{rel} = v_a - v_b = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$$

3. Calculate impulse strength:

$$J = \frac{-(1+1) \cdot \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\frac{1}{4} + \frac{1}{8} + \left(\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)^2 / 6 + \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)^2 / 8}$$

$$= \frac{-2 \cdot -1}{\frac{1}{4} + \frac{1}{8} + \frac{0}{6} + \frac{1}{8}} = 4$$

4. Apply Impulse:

$$V'_{cm,A} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} / 4 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$V'_{cm,B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} / 8 = \begin{pmatrix} 0 \\ -0.5 \end{pmatrix}$$

$$w'_A = -4 + \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} \times 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) / 6 = -4$$

$$w'_B = 0 - \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \times 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) / 8 = -\frac{1}{2}$$