Game Physics Tutorial 1

Movement Basics & Mass Spring Systems

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Movement Basics

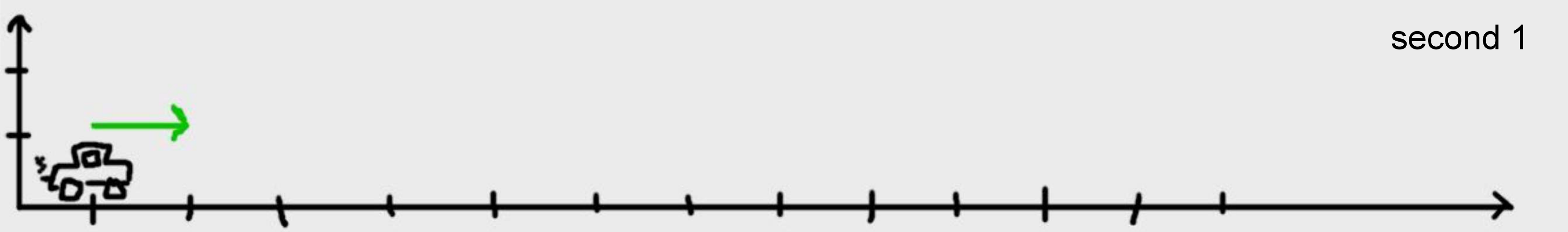
How does our position change over time?

> The **position**	is influenced by	the **velocity** (“speed and direction”).
The velocity	is influenced by	the **acceleration** (“= Beschleunigung”).
The acceleration	is influenced by	**forces**.

Let’s start with an easy example: A moving car!

Movement Basics

Our car is at **position (1, 0)**. It's **velocity is (1, 0)** (aka the "speed and direction").
Its weight is 2 tonnes (**mass = 2**).
Where is our car in the next second?



velocity = (1, 0)
position = (1, 0)

Movement Basics

Our car is at **position (1, 0)**. It's **velocity is (1, 0)** (aka the "speed and direction").
Its weight is 2 tonnes (**mass = 2**).
Where is our car in the next second?

velocity		= (1, 0)
position	= oldPosition + velocity × 1s	= (2, 0)

Good. But now we want to **accelerate**!

Movement Basics

How does our position change over time?

The **position**	is influenced by	the **velocity** (“speed and direction”).
> The velocity	is influenced by	the **acceleration** (“= Beschleunigung”).
The acceleration	is influenced by	**forces**.

Movement Basics

Now our car accelerates with acceleration (1, 0)!

acceleration			= (1, 0)
velocity	= oldVelocity + acceleration	× 1s	= ?
position	= oldPosition + velocity	× 1s	= ?

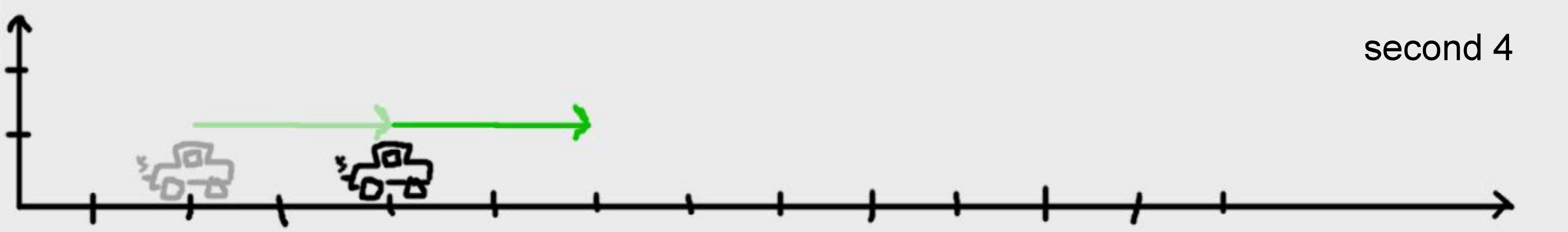
Movement Basics

Now our car accelerates with acceleration (1, 0)!

acceleration			= (1, 0)
velocity	= oldVelocity + acceleration	× 1s	= (2, 0)
position	= oldPosition + velocity	× 1s	= ?

Movement Basics

Now our car accelerates with acceleration (1, 0)!



acceleration			= (1, 0)
velocity	= oldVelocity + acceleration	× 1s	= (2, 0)
position	= oldPosition + velocity	× 1s	= (4, 0)

Movement Basics

We continue driving 1 second without accelerating more!

acceleration			= (0, 0)
velocity	= oldVelocity + acceleration	× 1s	= (2, 0)
position	= oldPosition + velocity	× 1s	= (6, 0)

Now a bomb explodes behind us! It pushes us with its force!

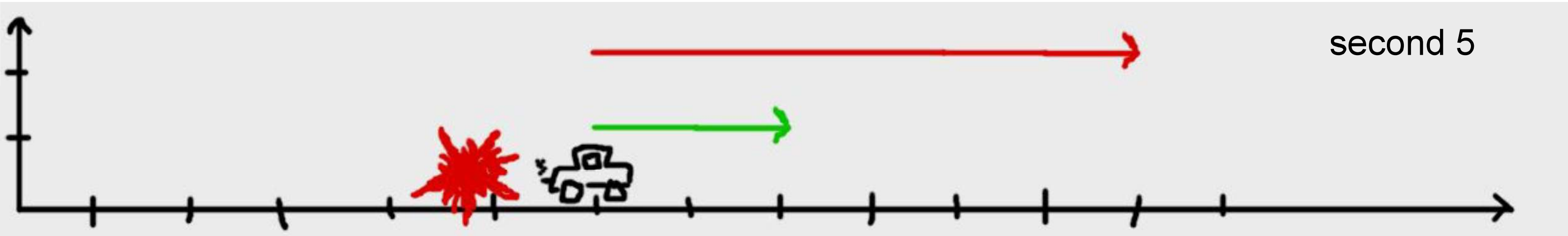
Movement Basics

How does our position change over time?

The **position**	is influenced by	the **velocity** (“speed and direction”).
The velocity	is influenced by	the **acceleration** (“= Beschleunigung”).
> The acceleration	is influenced by	**forces**.

Movement Basics

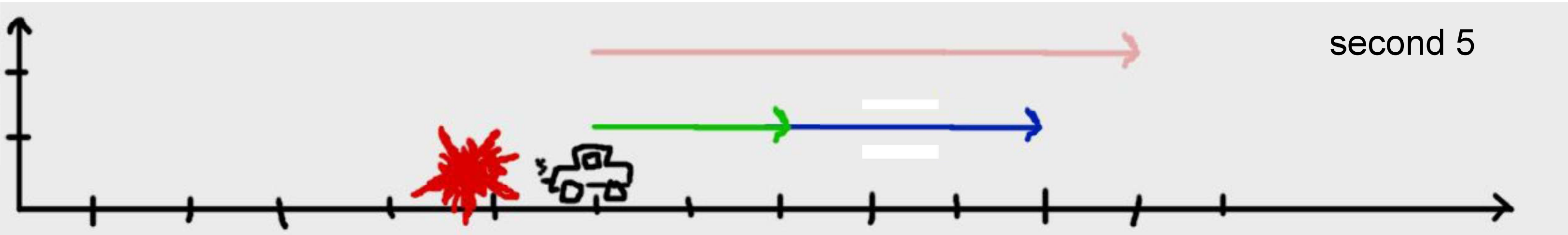
Explosion behind us, resulting in a force (6, 0) at our car!



force		= (6, 0)
acceleration	= force / mass	= ?
velocity	= oldVelocity + acceleration × 1s	= ?
position	= oldPosition + velocity × 1s	= ?

Movement Basics

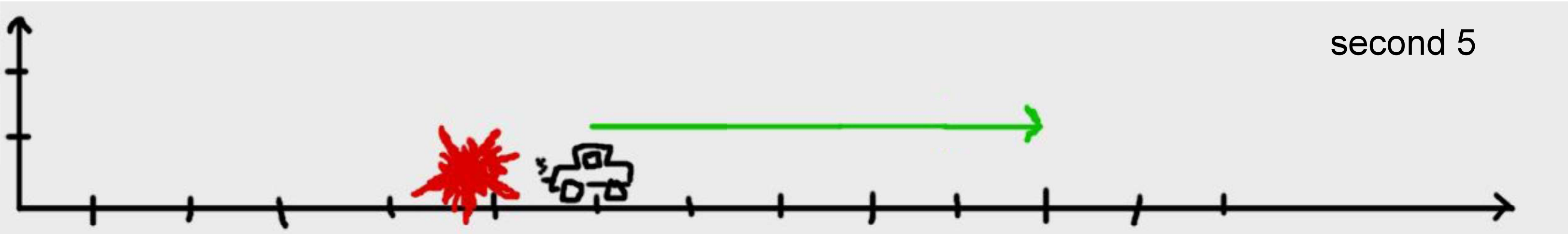
Explosion behind us, resulting in a force (6, 0) at our car!



force		= (6, 0)
acceleration	= force / mass	= (3, 0)
velocity	= oldVelocity + acceleration × 1s	= ?
position	= oldPosition + velocity × 1s	= ?

Movement Basics

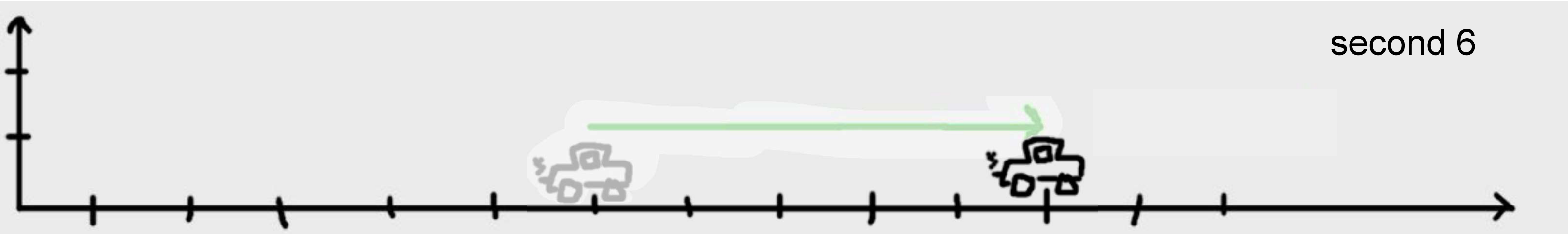
Explosion behind us, resulting in a force (6, 0) at our car!



force			= (6, 0)
acceleration	= force / mass		= (3, 0)
velocity	= oldVelocity + acceleration	× 1s	= (5, 0)
position	= oldPosition + velocity	× 1s	= ?

Movement Basics

Explosion behind us, resulting in a force (6, 0) at our car!



force		= (6, 0)
acceleration	= force / mass	= (3, 0)
velocity	= oldVelocity + acceleration × 1s	= (5, 0)
position	= oldPosition + velocity × 1s	= (11, 0)

Movement Basics

How does our position change over time?

The **position** is influenced by the **velocity** (“speed and direction”).

The velocity is influenced by the **acceleration** (“= Beschleunigung”).

The acceleration is influenced by **forces**.

Forces come in various forms: counter-force (when crashing), explosions, …

Small note: We cheated on the acceleration part: We cannot “just accelerate”. Actually, our motor was using a force on our car, which accelerated it.

Movement Basics

- Position *p*

  E.g. (1.5 m, -0.5 m, 2.0 m)

- Velocity *v*

  E.g. (1.0 m/s, -1.0 m/s, 0 m/s)

  ⟶ change of **position** over time

  (= derivative of position with respect to time)

$$\frac{\partial p}{\partial t}$$

An ODE ("Ordinary Differential Equation") is an equation which contains a function and its derivative(s).

Therefore, any equation involving e.g. position and velocity is an ODE!

- Acceleration *a*

  E.g. (2.0 m/s², -0.5 m/s², 0 m/s²)

  ⟶ change of **velocity** over time

  (= derivative of velocity with respect to time)

$$\frac{\partial v}{\partial t}$$

Movement Basics - Explicit Euler

- Usually: Acceleration / Velocity known, new Position wanted
- In Theory:
  - since v = $\frac{\partial p}{\partial t}$ and a = $\frac{\partial v}{\partial t}$ we would need to integrate…
  - Can be approximated!

- One way of approximating: Explicit Euler
  - newPos = oldPos + velocity × timestep
  - newVelocity = oldVelocity + acceleration × timestep

  - *“New Attribute”* = *“Old Attribute”* + *“How does Attribute change over time”* × timestep

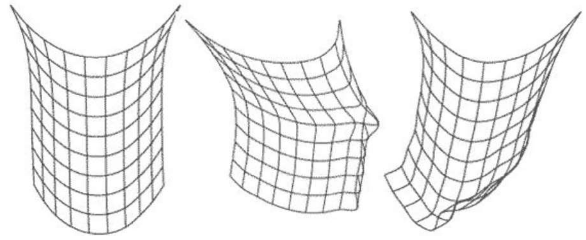
- More on this next Tutorial

Forces

- Force = mass × acceleration → "same force & heavier object → less acceleration"
  → forces and acceleration are essentially the same
  (difference just a constant linear factor)

- Forces = Influence of Physics on object's motion
  (e.g. Collision, attached Spring)

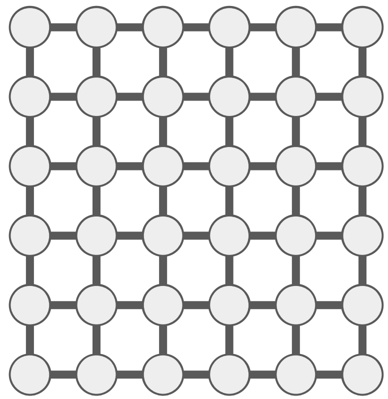
Mass Spring Systems



- Imagine we want to simulate a piece of cloth

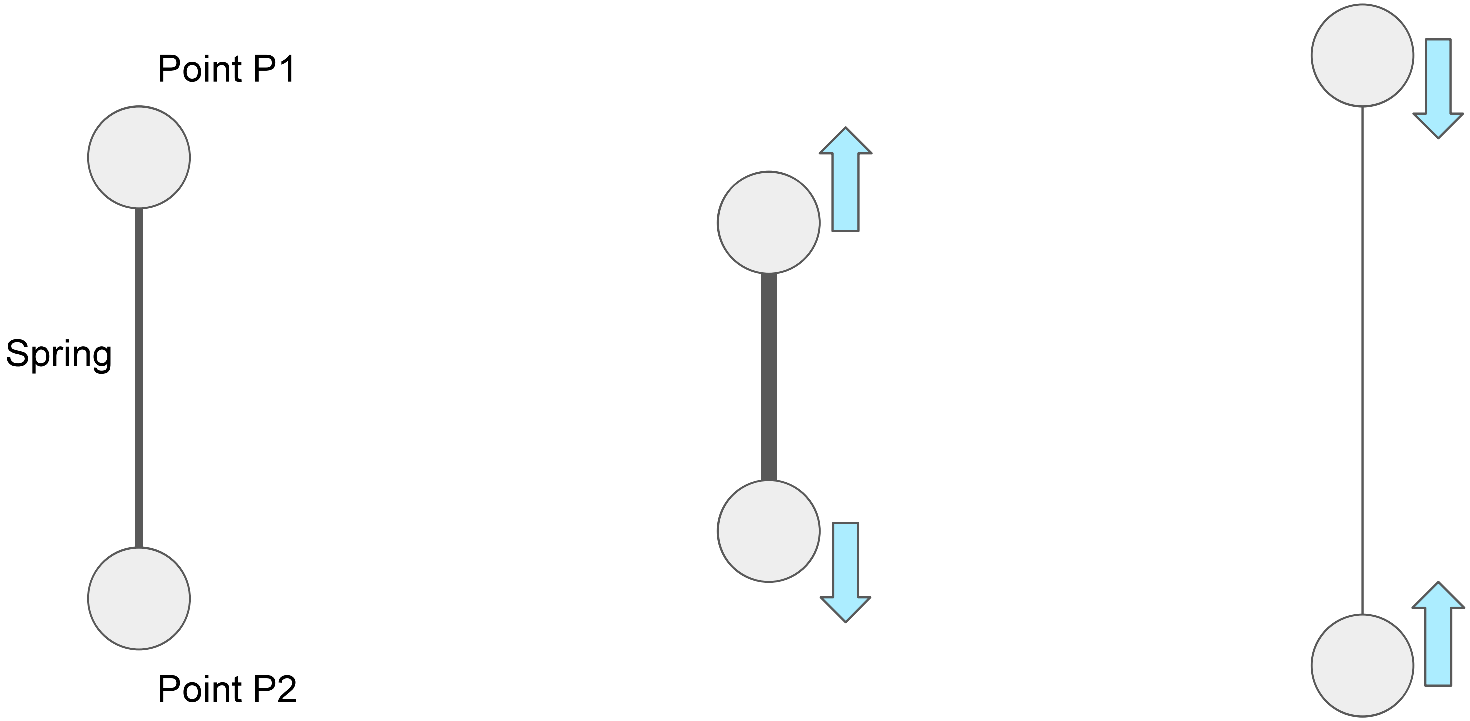
- Model it as a grid of mass points
  - Each mass point has a position p and mass m
  - Points do not have a volume / size !

- To simulate the forces holding the cloth together, we connect the points with “springs”



simplified piece of cloth

Mass Spring Systems - Springs



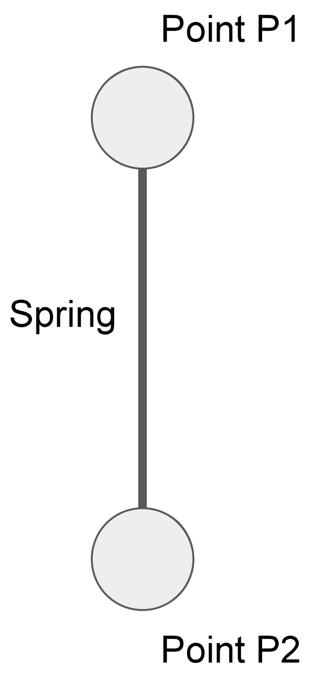
Mass Spring Systems - Hooke's Law

- Spring connecting two mass points has:
    - rest length ("normal" length of the spring)
    - stiffness

- Hooke's law:
    - Spring Force proportional to displacement
    - i.e. the closer the points are together the stronger the force pushing them apart

    - **F** = - stiffness × (||**P1** - **P2**|| - rest_length) × direction
    - direction(**P1**) = norm(**P1** - **P2**), direction(**P2**) = norm(**P2** - **P1**)

- Spring applies force to both points
  (in opposite directions)



Mass Spring Systems - Calculate one timestep

- **Update position** with velocity

- Calculate spring forces (Hooke's Law)
- Sum up all forces acting on point
- Update acceleration with accumulated forces (Newton's second law)
- (Add gravity to acceleration)
- **Update velocity** with acceleration
- Clear forces

Force calculation based on **old** position!

- Profit.

  (This is Explicit Euler!)

Mass Spring Systems - Exercise 1

- <u>Calculate the positions of P1 & P2 after two time steps!</u>

  The initial positions are

  P1.x0 = (2, 4) and **P2.x0 = (1, -3)**.

  Their initial velocities and accelerations are zero,

  their **masses** are **1**.

  P1 and P2 are connected by a spring with **rest length 4**

  and **stiffness 2**. The **timestep is 0.5**.

Mass Spring Systems - Exercise 1

initial positions:

$P_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\quad$ $P_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

masses:

$m_1 = 1$ $\quad$ $m_2 = 1$

No initial velocities or accelerations!

Spring: $s = 2$ $\quad$ $d_r = 4$ $\qquad$ Timestep: $h = 0.5$

Mass Spring Systems - Exercise 1

initial positions: $P_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

masses: $m_1 = 1$ $m_2 = 1$

No initial velocities or accelerations!

Spring: $s = 2$ $d_r = 4$ Timestep: $h = 0.5$

t=0:

$P_1(t=0.5) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $P_2(t=0.5) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$$d(t=0) = \sqrt{(2-1)^2 + (4+3)^2} = 5\sqrt{2}$$

$$\begin{aligned} f(t=0) &= -s \cdot (d - d_r) \cdot \vec{d}.\text{normalized} \\ &= -2 \cdot (5 \cdot \sqrt{2} - 4) \cdot \vec{d}.\text{normalized} \\ &= (8 - 10\sqrt{2}) \cdot \vec{d}.\text{normalized} \end{aligned}$$

$\rightsquigarrow P_1: f(t=0) = -6.14 \cdot \begin{pmatrix} 2-1 \\ 4+3 \end{pmatrix} \cdot \frac{1}{5 \cdot \sqrt{2}} = -6.14 \cdot \begin{pmatrix} \frac{\sqrt{2}}{10} \\ \frac{7 \cdot \sqrt{2}}{10} \end{pmatrix}$

$\rightsquigarrow P_2: f(t=0) = -6.14 \cdot \begin{pmatrix} 1-2 \\ -3-4 \end{pmatrix} \cdot \frac{1}{5 \cdot \sqrt{2}} = 6.14 \cdot \begin{pmatrix} \frac{\sqrt{2}}{10} \\ \frac{7 \cdot \sqrt{2}}{10} \end{pmatrix}$

$a(t=0) = f(t=0) \cdot \frac{1}{1}$

$$V_1(t=0.5) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0.5 \cdot (-6.14) \cdot \begin{pmatrix} \sqrt{2}/10 \\ 7 \cdot \sqrt{2}/10 \end{pmatrix} = \begin{pmatrix} -0.434 \\ -3.04 \end{pmatrix}$$

$$V_2(t=0.5) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0.5 \cdot 6.14 \cdot \begin{pmatrix} \sqrt{2}/10 \\ 7 \cdot \sqrt{2}/10 \end{pmatrix} = \begin{pmatrix} 0.434 \\ 3.04 \end{pmatrix}$$

Mass Spring Systems - Exercise 1

initial positions: masses:

$P_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\quad P_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\quad m_1 = 1 \quad m_2 = 1$

No initial velocities or accelerations!

Spring: $s = 2 \quad d_r = 4$ $\quad$ Timestep: $h = 0.5$

t=0:

$$P_1(t=0.5) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \qquad P_2(t=0.5) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$d(t=0) = \sqrt{(2-1)^2 + (4+3)^2} = 5\sqrt{2}$$

$$\begin{aligned} f(t=0) &= -s \cdot (d - d_r) \cdot \vec{d}.\text{normalized} \\ &= -2 \cdot (5 \cdot \sqrt{2} - 4) \cdot \vec{d}.\text{normalized} \\ &= (8 - 10\sqrt{2}) \cdot \vec{d}.\text{normalized} \end{aligned}$$

$$\left.\begin{aligned} \rightsquigarrow P_1 &: f(t=0) = -6.14 \cdot \begin{pmatrix} 2-1 \\ 4+3 \end{pmatrix} \cdot \frac{1}{5 \cdot \sqrt{2}} = -6.14 \cdot \begin{pmatrix} \frac{\sqrt{2}}{10} \\ \frac{7 \cdot \sqrt{2}}{10} \end{pmatrix} \\ \rightsquigarrow P_2 &: f(t=0) = -6.14 \cdot \begin{pmatrix} 1-2 \\ -3-4 \end{pmatrix} \cdot \frac{1}{5 \cdot \sqrt{2}} = 6.14 \cdot \begin{pmatrix} \frac{\sqrt{2}}{10} \\ \frac{7 \cdot \sqrt{2}}{10} \end{pmatrix} \end{aligned}\right\} a(t=0) = f(t=0) \cdot \frac{1}{1}$$

$$V_1(t=0.5) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0.5 \cdot (-6.14) \cdot \begin{pmatrix} \sqrt{2}/10 \\ 7 \cdot \sqrt{2}/10 \end{pmatrix} = \begin{pmatrix} -0.434 \\ -3.04 \end{pmatrix}$$

$$V_2(t=0.5) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0.5 \cdot 6.14 \cdot \begin{pmatrix} \sqrt{2}/10 \\ 7 \cdot \sqrt{2}/10 \end{pmatrix} = \begin{pmatrix} 0.434 \\ 3.04 \end{pmatrix}$$

t=0.5:

$$P_1(t=1) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} -0.434 \\ -3.04 \end{pmatrix} = \begin{pmatrix} 1.783 \\ 2.48 \end{pmatrix}$$

$$P_2(t=1) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 0.434 \\ 3.04 \end{pmatrix} = \begin{pmatrix} 1.217 \\ -1.48 \end{pmatrix}$$

Mass Spring Systems - Exercise 2

- <u>Calculate the positions of P1 & P2 after two time steps!</u>

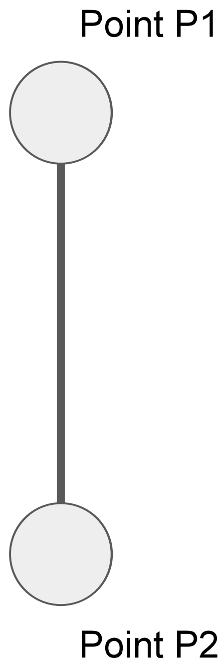
  Initial positions **P1.x0 = (-2, 2)** **P2.x0 = (3, 2)**.

  Initial velocities **P1.v0 = (-1, -1.5)** **P2.v0 = (1, -1.5)**

  Masses **P1.m = 2** **P2.m = 3**

  The points are affected by **gravity (0, -10)**.

  P1 and P2 are connected by a spring with **rest length 6** and **stiffness 4**. The **timestep is 0.5**.



Mass Spring Systems - Exercise 2

<u>$t=0$:</u>

$P_1(t=0.5) = \begin{pmatrix}-2\\2\end{pmatrix} + 0.5 \cdot \begin{pmatrix}-1\\-1.5\end{pmatrix} = \begin{pmatrix}-2.5\\1.25\end{pmatrix}$

$P_2(t=0.5) = \begin{pmatrix}3\\2\end{pmatrix} + 0.5 \cdot \begin{pmatrix}1\\-1.5\end{pmatrix} = \begin{pmatrix}3.5\\1.25\end{pmatrix}$

$d = 5$

$f(t=0) = -4 \cdot (5-6) \cdot \vec{d}$.normalized

P_1: $f(t=0) = 4 \cdot \begin{pmatrix}-1\\0\end{pmatrix}$

P_2: $f(t=0) = 4 \cdot \begin{pmatrix}1\\0\end{pmatrix}$

P_1: $a(t=0) = \begin{pmatrix}-4\\0\end{pmatrix} \cdot \frac{1}{2} + \begin{pmatrix}0\\-10\end{pmatrix} = \begin{pmatrix}-2\\-10\end{pmatrix}$

P_2: $a(t=0) = \begin{pmatrix}4\\0\end{pmatrix} \cdot \frac{1}{3} + \begin{pmatrix}0\\-10\end{pmatrix} = \begin{pmatrix}4/3\\-10\end{pmatrix}$

$V_1(t=0.5) = \begin{pmatrix}-1\\-1.5\end{pmatrix} + 0.5 \cdot \begin{pmatrix}-2\\-10\end{pmatrix} = \begin{pmatrix}-2\\-6.5\end{pmatrix}$

$V_2(t=0.5) = \begin{pmatrix}1\\-1.5\end{pmatrix} + 0.5 \cdot \begin{pmatrix}4/3\\-10\end{pmatrix} = \begin{pmatrix}5/3\\-6.5\end{pmatrix}$

<u>$t=0.5$:</u>

$P_1(t=1) = \begin{pmatrix}-2.5\\1.25\end{pmatrix} + 0.5 \cdot \begin{pmatrix}-2\\-6.5\end{pmatrix} = \begin{pmatrix}-3.5\\-2\end{pmatrix}$

$P_2(t=1) = \begin{pmatrix}3.5\\1.25\end{pmatrix} + 0.5 \cdot \begin{pmatrix}5/3\\-6.5\end{pmatrix} = \begin{pmatrix}13/3\\-2\end{pmatrix}$