Game Physics Tutorial 4

Rigid Bodies in 3D

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Simulating a Rigidbody (in 3D)

- Translation works the same (with an added dimension to the vectors)
- What changes regarding Rotation? *No longer only 1 rotation axis!*
  - **Orientation r**
    - not just an angle anymore
  - **Angular velocity w**
    - not just a scalar speed but also which axis to rotate around
  - **Inertia Tensor I**
    - now a Matrix and relative to current **orientation**

3D: **Angular velocity w**

- In 2D, the angular velocity was *actually* a vector too!
  → **w** = (0, 0, “speed of rotation around z-axis”)

- In 3D: **w** is vector
  - Direction = Axis of Rotation
  - Magnitude = Speed of Rotation around that axis

3D: Orientation r

- In 3D, we need a Matrix or a Quaternion to describe rotations
(1 angle isn't enough!)

- Quaternions are vectors with 4 components $\mathbf{r} = (s, xi, yj, zk)$
→ They describe rotations very well
→ However, we can't really interpret / understand the components well
→ Therefore, convert between Matrix and Quaternion

- To integrate **Orientation r** using **angular velocity w**:

$$\mathbf{r}' = \mathbf{r} + h/2 \begin{pmatrix} 0 \\ \mathbf{w} \end{pmatrix} \mathbf{r}$$

(0, w) is just short for **(0, w.x, w.y, w.z)** !

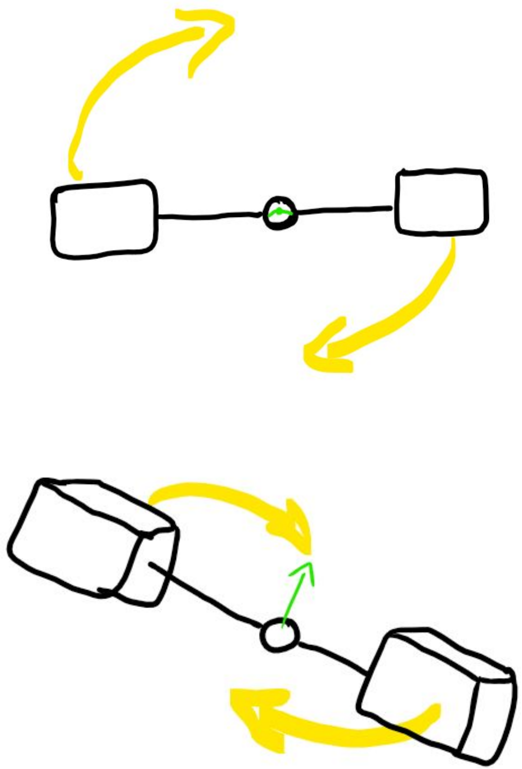
3D: Inertia Tensor I

- In 3D, the Inertia Tensor I:
  - is a 3x3 Matrix
  - depends on the **current orientation r**

- Rotating the Rigidbody will also change which axes it can rotate well around

- Therefore:
  - Precompute Inertia Tensor at resting state I_0
  - Use I_0 & **current orientation** to easily get current Inertia Tensor I

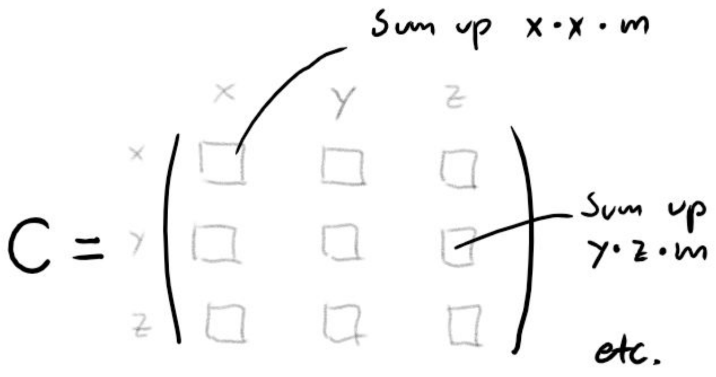
$$\mathbf{I}_{current} = \mathrm{Rot}_{\mathbf{r}}\ \mathbf{I}_0\ \mathrm{Rot}_{\mathbf{r}}^{-1} = \mathrm{Rot}_{\mathbf{r}}\ \mathbf{I}_0\ \mathrm{Rot}_{\mathbf{r}}^{T}$$



3D: Precomputing I_0

1. Compute “Covariance Matrix” C of the Point’s local space positions

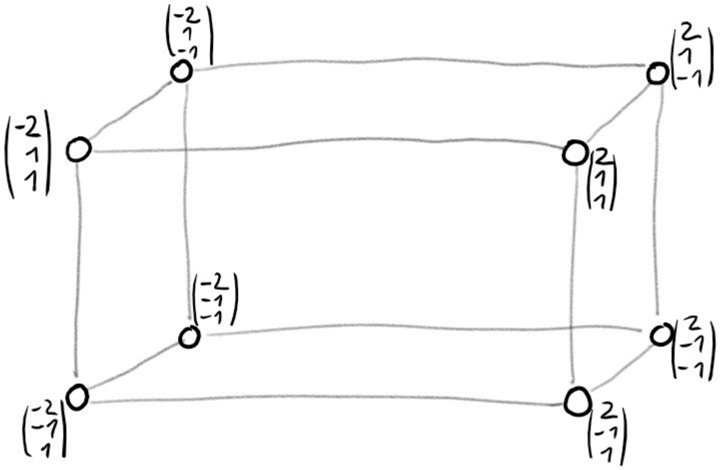
$$C_{j,k} = \sum_n m_n \, \mathbf{x}_{n,j} \, \mathbf{x}_{n,k}$$

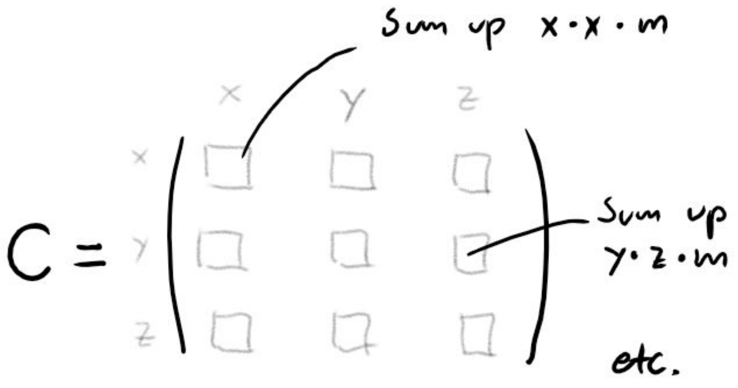


2. Compute trace(C) = $C_{1,1} + C_{2,2} + C_{3,3}$
   (= Sum up diagonal entries)

3. Compute I_0 = Identity * trace(C) - C

Exercise: Precomputing I_0



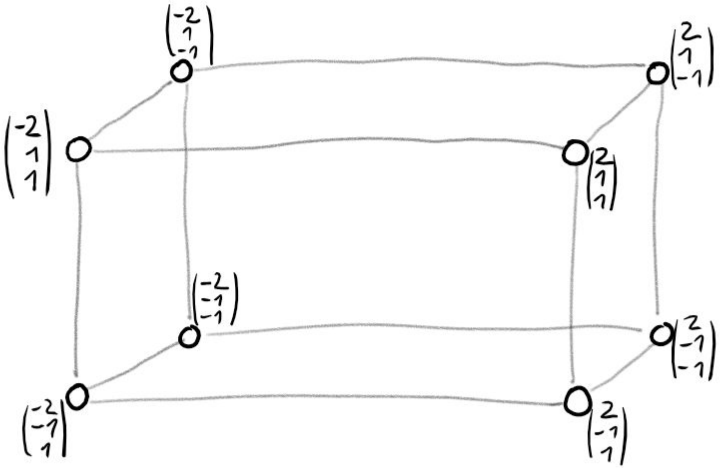


$$C_{j,k} = \sum_n m_n \, \mathbf{x}_{n,j} \, \mathbf{x}_{n,k}$$

$$\operatorname{trace}(\mathbf{A}) = a_{1,1} + a_{2,2} + a_{3,3}$$

$$\mathbf{I} = \mathbf{Id} \, \operatorname{trace}(\mathbf{C}) - \mathbf{C}$$

Exercise: Precomputing I_0



$$C = \begin{pmatrix} 32 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$\mathrm{trace}(C) = 48$$

$$I_0 = \begin{pmatrix} 48 & 0 & 0 \\ 0 & 48 & 0 \\ 0 & 0 & 48 \end{pmatrix} - \begin{pmatrix} 32 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 16 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{pmatrix}}}$$

3D: Angular Momentum

- **L** = **I** * **w**
- In 3D, **I** is not constant anymore!
  → we can't use **Torque q** to directly integrate **w** since it is no longer just **L** with a constant linear factor!
- We have to integrate **L** instead of **w**

$$\mathbf{L}(t+h) = \mathbf{L}(t) + h\mathbf{q}$$

- We can then use **L** and the inverse **I** to calculate **w**

$$\mathbf{w}(t+h) = \mathbf{I}^{-1}\,\mathbf{L}(t+h)$$

3D: Simulating Rigidbody Rotation

1. Calculate forces & convert them to **torque q**

$$\mathbf{q}(t) = \sum_i \mathbf{x}_i \times \mathbf{f}_i$$

2. Integrate the **orientation r** using the **angular velocity w**

$$\mathbf{r}' = \mathbf{r} + h/2 \begin{pmatrix} 0 \\ \mathbf{w} \end{pmatrix} \mathbf{r}$$

3. Integrate **angular momentum L**

$$\mathbf{L}(t+h) = \mathbf{L}(t) + h\mathbf{q}$$

4. Update (inverse) **Inertia Tensor I**

$$\mathbf{I}^{-1} = \mathrm{Rot}_{\mathbf{r}}\ \mathbf{I}_0^{-1}\ \mathrm{Rot}_{\mathbf{r}}^T$$

5. Update **angular velocity w** using **I** and **L**

$$\mathbf{w}(t+h) = \mathbf{I}^{-1}\ \mathbf{L}(t+h)$$

6. Update the World Space positions of the Points based on the new **orientation r**

$$\mathbf{x}_i^{world} = \mathbf{x}_{cm}(t) + \mathrm{Rot}_{r(t)}(\mathbf{x}_i)$$

Exercise: Integrating Orientation

$$h = 0.5 \qquad r = \begin{pmatrix} 0 \\ 0.7 \\ 0 \\ 0.7 \end{pmatrix} \qquad w = \begin{pmatrix} 0 \\ -10 \\ 0 \end{pmatrix}$$

$$r' = r + \frac{h}{2} \cdot \begin{pmatrix} 0 \\ w \end{pmatrix} \bullet r$$

Quaternion Multiplication!

$$q_1 \bullet q_2 = (\underset{\text{scalar}}{s_1}, \underset{\text{vector}}{v_1}) \cdot (s_2, v_2)$$

$$= (s_3, v_3)$$

new scalar

$$s_3 = s_1 \cdot s_2 - v_1 \cdot v_2$$

$$v_3 = s_1 \cdot v_2 + s_2 \cdot v_1 + v_1 \times v_2$$

new vector

Exercise: Integrating Orientation

$$r' = \begin{pmatrix} 0 \\ 0.7 \\ 0 \\ 0.7 \end{pmatrix} + \frac{1}{4} \cdot \begin{pmatrix} 0 \\ 0 \\ -10 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0.7 \\ 0 \\ 0.7 \end{pmatrix}$$

$$r' = \begin{pmatrix} 0 \\ 0,7 \\ 0 \\ 0.7 \end{pmatrix} + \frac{1}{4} \cdot \begin{pmatrix} 0 \\ -7 \\ 0 \\ 7 \end{pmatrix}$$

$$r' = \begin{pmatrix} 0 \\ -1.05 \\ 0 \\ 2.45 \end{pmatrix}$$

scalar vector

$$q_1 \bullet q_2 = (s_1, v_1) \cdot (s_2, v_2)$$

$$= (s_3, v_3)$$

new scalar

$$s_3 = s_1 \cdot s_2 - v_1 \cdot v_2$$

$$v_3 = s_1 \cdot v_2 + s_2 \cdot v_1 + v_1 \times v_2$$

new vector

Exercise: 3D Angular Velocity Update

Assume that:

- we have integrated r so that the object is rotated 90 degrees about the Y-axis now
- $I_0^{-1} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{pmatrix}$
- we have integrated the angular momentum already so that $L(t+h) = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$

Calculate the new angular velocity $w(t+h)$!

Exercise: 3D Angular Velocity Update

Calculate Rotation Matrix of $\mathbf{r}$:

$$Rot_{\mathbf{r}} = \begin{pmatrix} \cos 90^\circ & 0 & \sin 90^\circ \\ 0 & 1 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$Rot_{\mathbf{r}}^{T} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Exercise: 3D Angular Velocity Update

Calculate Rotation Matrix of $\mathbf{r}$:

$$Rot_{\mathbf{r}} = \begin{pmatrix} \cos 90^\circ & 0 & \sin 90^\circ \\ 0 & 1 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$Rot_{\mathbf{r}}^T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Calculate I^{-1}:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & -10 \\ 0 & 20 & 0 \\ 5 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

Exercise: 3D Angular Velocity Update

Calculate Rotation Matrix of **r**:

$$Rot_r = \begin{pmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$Rot_r^T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Calculate $w(t+h)$:

$$w(t+h) = I^{-1} L(t+h)$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 10 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -100 \\ 0 \end{pmatrix}$$

Calculate I^{-1}:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & -10 \\ 0 & 20 & 0 \\ 5 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

3D: Impulse Calculation

Similar to 2D!
Only new formula and
apply to the momentum (L)
instead of the velocity (w).

$$J = \frac{-(1+c)\mathbf{v}_{rel} \cdot \mathbf{n}}{\frac{1}{M_a} + \frac{1}{M_b} + [(\mathbf{I}_a^{-1}(\mathbf{x}_a \times \mathbf{n})) \times \mathbf{x}_a + (\mathbf{I}_b^{-1}(\mathbf{x}_b \times \mathbf{n})) \times \mathbf{x}_b] \cdot \mathbf{n}}$$

1. Calculate velocities at collision point for each rigidbody $\mathbf{v}_i = \mathbf{v}_{cm} + \mathbf{w} \times \mathbf{x}_i$
2. Calculate relative velocity
3. Fill in formula
4. Apply impulse (in according directions)

$$\mathbf{v}'_a = \mathbf{v}_a + J\mathbf{n}/M_a \qquad \mathbf{L}'_a = \mathbf{L}_a + (\mathbf{x}_a \times J\mathbf{n})$$
$$\mathbf{v}'_b = \mathbf{v}_b - J\mathbf{n}/M_b \qquad \mathbf{L}'_b = \mathbf{L}_b - (\mathbf{x}_b \times J\mathbf{n})$$

This is a lot to calculate, so you probably won't have to calculate this in the exam...