Domain conversion (for spectral.m)

(This section shows the mathematics for function `gl_scale_xgrid`.)

- x represents the desired grid (`xgrid` in Matlab code)
- ϵ represents the unscaled chebyshev grid (`xh` in Matlab code)
- K a linear scaling factor (`L` in Matlab code)

Case 1 (infinite)

Reference: Stability and transition in shear flows. Applied mathematical sciences, Vol. 142, PJ Schmid and DS Henningson, Equation A.56

$$x = K\frac{\xi}{\sqrt{1-\xi^2}} \;\; ; \;\; \xi = \frac{x}{\sqrt{K^2+x^2}} \tag{0.1}$$

Case 2 (infinite)

Reference: Stability and transition in shear flows. Applied mathematical sciences, Vol. 142, PJ Schmid and DS Henningson, Equation A.57

$$x = K\tanh^{-1}(\xi) \;\; ; \;\; \xi = \tanh(x/K) \tag{0.2}$$

Case 3 (semi-infinite)

Reference: Chebyshev and Fourier Spectral Methods, John P. Boyd. U Michigan, 2000. Chapter 16.4 p 326-327.

$$x = K\frac{1+\xi}{1-\xi} \;\; ; \;\; \xi = \frac{x-K}{x+K} \tag{0.3}$$

$x : \{0 \to \infty\}$ and $\xi : \{\pm 1\}$. K is the scaling factor

Derivatives

(This section derives the mathematics for functions `gl_scaleD1` and `gl_scaleD2`.)

$$\frac{dq}{dx} = \frac{dq}{d\xi}\frac{d\xi}{dx} \tag{0.4}$$

$$\frac{d^2q}{dx^2} = \frac{d}{dx}\left[\frac{dq}{dx}\right] = \frac{d}{dx}\left[\frac{dq}{d\xi}\frac{d\xi}{dx}\right] = \frac{dq}{d\xi}\frac{d^2\xi}{dx^2} + \frac{d^2q}{d\xi dx}\frac{d\xi}{dx}$$

$$= \frac{dq}{d\xi}\frac{d^2\xi}{dx^2} + \frac{d^2q}{d\xi^2}(\frac{d\xi}{dx})^2 \tag{0.5}$$

Case 1

$$x = K\frac{\xi}{\sqrt{1-\xi^2}} \ \ ; \ \ \xi = \frac{x}{\sqrt{K^2+x^2}}$$

$$\frac{d\xi}{dx} = \frac{K^2}{(K^2+x^2)^{\frac{3}{2}}} = \frac{K^2}{(K^2+K^2\frac{\xi^2}{1-\xi^2})^{\frac{3}{2}}} = \frac{1}{K(\frac{1}{1-\xi^2})^{\frac{3}{2}}} = \frac{1}{K}(1-\xi^2)^{\frac{3}{2}} \tag{0.6}$$

$$\frac{d^2\xi}{dx^2} = -\frac{3\frac{\xi}{\sqrt{1-\xi^2}}}{K^2(\frac{1}{1-\xi^2})^{\frac{5}{2}}} = -\frac{3\xi}{K^2}(1-\xi^2)^2 \tag{0.7}$$

Substitute 0.6 and 0.7 into 0.4 and 0.5 to find the scaled differentiation term.

Case 2

$$x = K\tanh^{-1}(\xi) \ \ ; \ \ \xi = \tanh(\frac{x}{K})$$

$$\frac{d\xi}{dx} = \frac{1}{K}\text{sech}^2(\frac{x}{K}) = (1-\xi^2)/K \tag{0.8}$$

$$\frac{d^2\xi}{dx^2} = \frac{-2}{K^2}\text{sech}^2(\frac{x}{K})\tanh(\frac{x}{K}) = \frac{-2}{K^2}\xi(1-\xi^2) \tag{0.9}$$

Substitute 0.8 and 0.9 into 0.4 and 0.5 to find the scaled differentiation term.

Case 3

$$\frac{d\xi}{dx} = \frac{x+K-x+K}{(x+K)^2} = \frac{2K}{(x+K)^2} = \frac{2K}{K^2(\frac{1+\xi}{1-\xi}+1)^2} = \frac{2(1-\xi)^2}{K(1+\xi+1-\xi)^2} = \frac{(1-\xi)^2}{2K} \tag{0.10}$$

$$\frac{d^2\xi}{dx^2} = -\frac{4K}{(K+x)^3} = -\frac{4K}{(K+K\frac{1+\xi}{1-\xi})^3} = -\frac{4(1-\xi)^3}{K^2(1-\xi+1+\xi)^3} = -\frac{4(1-\xi)^3}{K^2(2)^3} = \frac{(\xi-1)^3}{2K^2} \tag{0.11}$$

Substitute 0.10 and 0.11 into 0.4 and 0.5 to find the scaled differentiation term.

Integration

(This section derives the mathematics for function gl_scale_w.)

We know that:

$$x = g(\xi), \tag{0.12}$$

where $g(\xi)$ is defined in 0.1,0.2 and 0.3.
We also know that:

$$\frac{d\xi}{dx} = P, \tag{0.13}$$

where P is defined in 0.6, 0.8 and 0.10.

To express the integral

$$F(x) = \int_{-\infty}^{\infty} f(x)dx$$

in terms of ξ we simply write:

$$F(x) = \int_{-\infty}^{\infty} f(x = g(\xi))\frac{1}{P}d\xi, \tag{0.14}$$

where

$$dx = \frac{1}{P}d\xi.$$