Root Finding

Goal: Solve $f(r) = 0$. The solution r is called a *root*.

Do we need methods for solving $f(x) = c$ for $c \neq 0$?

You've had the most experience solving polynomials like $x^2 - 5x - 6 = 0$. Name some strategies! What would work best on floating point numbers?

Since there's no way to immediately calculate the roots of an arbitrary function, we turn to *iterative* methods.

Definition 1. An iterative method consists of the following general process.

1. Start with a guess for the solution
2. Do something to get a new guess (hopefully better)
3. Repeat until a *stopping criteria* is met

If repeating this process gets you close to a correct answer, the method is called *convergent.*

Many common math problems can famously **only** be solved by iterative methods. General root finding is one of them. The eigenvalue problem from linear algebra is also one, which is a consequence of the root-finding problem: Eigenvalues are the roots of the characteristic equation (a polynomial)!

We are covering 4 iterative methods for root-finding: Bisection, Fixed Point Iteration, Newton's Method, and Secant Method. Why more than one? In short: there are differences in speed and accuracy that depend on the problem input.

Bisection Method

How do you even know if a function has a root?

Theorem 2 (Intermediate Value Theorem)**.** *Let f be a continuous function on $[a, b]$ satisfying $f(a)f(b) < 0$. Then f has a root between a and b; that is, there exists a number r satisfying $a < r < b$ and $f(r) = 0$.*

The intermediate value theorem is an example of a *sufficient* condition for having a root, but not a *necessary* condition.

Such an interval $[a, b]$ is said to *bracket* the root. The "best" guess for the root, given such an interval, is the midpoint $\frac{a+b}{2}$, where "best" means the smallest bound on the worst error. What is this maximum error for the midpoint?

Algorithm 1 Bisection Method Algorithm

Input: Function f, initial interval $[a, b]$, desired accuracy TOL

Check: $f(a)f(b) < 0$

while $\frac{b-a}{2} > TOL$ **do**

  New guess: $c = \frac{b+a}{2}$

  Compute $f(c)$

  if $f(a)f(c) < 0$ **then**

    $b = c$

  else if $f(a)f(c) > 0$ **then**

    $a = c$

  else

    c is a root! Stop immediately

Return $\frac{b+a}{2}$

Example 3. Let $f(x) = 2x^2 - 1$. Apply three steps of the bisection method on $[0, 4]$.

Example 4. If you perform n steps of the bisection method on an interval $[a, b]$, how many times do you evaluate the function? What is the maximum error of the result?

Definition 5. A solution is *correct within p decimal places* if the error is less than 0.5×10^{-p}.

Is the result in Example 3 correct within 1 decimal place? If not, how many additional iterations would be needed?

Definition 6. Let e_i denote the error at step i of an iterative method. If

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = S < 1,$$

the method is said to obey *linear convergence* with rate S.

Does the bisection method converge linearly?

Fixed-Point Iteration

Definition 7. A real number x is a *fixed point* of the function g if $g(x) = x$.

Can fixed points be found by the Bisection Method?

Algorithm 2 Fixed-point Iteration Algorithm (FPI)

Input: A function f, an initial guess x_0, number of iterations n

for $i = 1, \dots, n$ **do**

$x_i = g(x_{i-1})$

Return x_n

Example 8. Apply two steps of FPI to $f(x) = x^2$ with initial guess $\frac{1}{2}$. What fixed point are you approximating?

Draw a *cobweb diagram* showing the FPI process for this example.

Definition 9. An iterative method is called *locally convergent* to a solution if the method converges for initial guess sufficiently close to the solution. If it's convergent for any initial guess, the method is *globally convergent*.

FPI for Example 8 is locally convergent for the fixed point 0 as long as the initial guess is in $(-1, 1)$. It is not locally convergent for the fixed point 1, and it is not globally convergent.

Theorem 10. *Let $g(x)$ be a continuously differentiable function, r be a fixed point of g, and suppose that $|g'(r)| = S < 1$. Then Fixed-Point Iteration converges linearly with rate S to the fixed point r for initial guesses sufficiently close to r.*

Example 11. Use this theorem to predict the previously observed convergence regarding fixed points 0 and 1.

Example 12. What are the implications of Theorem 10 for sine and cosine?

In FPI, we gave as input a pre-set number of iterations for the stopping criteria. But there is one alternative: to go until the answer doesn't seem to change much anymore. In other words, if $|x_i - x_{i-1}| < TOL$.