Linear Independence

1. Classify each of the following sets of vectors in $\mathbb{R}^n$ as independent or dependent using row reduction.

   (a) $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$

   (b) $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$

   (c) $\{\begin{bmatrix}1\\3\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}\}$

2. Let $v \in \mathbb{R}^n$. For what vector is the set $\{v\}$ linearly dependent? Does there exist a linearly independent set of any size which contains this vector? Explain.

3. Classify each of the following pairs of vectors as independent or dependent. Can you do so without using row reduction?

   (a) $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\}$

   (b) $\{\begin{bmatrix}3\\3\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\}$

   (c) $\{\begin{bmatrix}3\\3\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\}$

4. Consider the following dependent sets of vectors.

(a) $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$

(b) $\{\begin{bmatrix}1\\3\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}\}$

(c) $\{\begin{bmatrix}1\\3\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}-3\\1\\-1\\0\end{bmatrix}\}$

(d) $\{v_1, v_2, v_3, v_4, v_5, v_6 : v \in \mathbb{R}^3\}$.

What do all of these sets have in common? Is there a way to observe that these sets must be dependent?

5. Here are eight vectors in $\mathbb{R}^3$.

- $v = \begin{bmatrix}1\\0\\0\end{bmatrix}$
- $x = \begin{bmatrix}1\\1\\1\end{bmatrix}$
- $z = \begin{bmatrix}0\\0\\0\end{bmatrix}$
- $s = \begin{bmatrix}0\\3\\0\end{bmatrix}$
- $w = \begin{bmatrix}0\\1\\0\end{bmatrix}$
- $y = \begin{bmatrix}3\\-1\\7\end{bmatrix}$
- $u = \begin{bmatrix}1\\1\\0\end{bmatrix}$
- $t = \begin{bmatrix}2\\1\\1\end{bmatrix}$

(a) Create three independent sets containing exactly one vector, such as $\{v\}$.

(b) Create three independent sets containing exactly two vectors, such as $\{v, w\}$.

(c) Create a dependent set of two vectors without using z.

(d) Create three independent sets containing exactly three vectors.

(e) Can you create independent sets of any other sizes? Why or why not?

(f) Do there exist independent sets containing w and s? v and u? What about v, x, and t?

6. Let $u = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $v = \begin{bmatrix}0\\1\\0\end{bmatrix}$. Explain why a vector w is in Span$\{u, v\}$ if and only if $\{u, v, w\}$ is linearly dependent.

Fact: A set containing one vector is linearly independent if and only if the vector is not the _____.

Theorem 9: If a set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ contains the _____, then S is linearly dependent.

Fact: A set containing two vectors is linearly independent if and only the two vectors are not _____ of each other.

Theorem 8: Any set $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if _____ is greater than _____.

Theorem 7: An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

Warning: This is a "there exists," not a "for all," theorem. Some vectors are usually not linear combinations of others.

Ex: $\left\{ \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ Which one of these vectors is not a linear combination of the other two?