

# Runge-Kutta Methods

The Euler Method and the Explicit Trapezoid Method are in the family of solvers called Runge-Kutta Methods. Runge-Kutta Methods of all orders exist, and often more than one of a particular order. To develop the overall approach of a Runge-Kutta method, we begin with examining another method within this family of solvers: the Midpoint Method.

**Definition 1** (Midpoint Method).

$$w_0 = y_0$$

$$w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i))$$

**Example 1.** Apply the midpoint method with  $h = \frac{1}{2}$  to approximate  $y(2)$

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

Table 1:  $h = 0.5$

t	ye	yt	yta	ym	y
0	1	1	1	1	1
0.5	1	1.125	1.125	1.125	1.13314845
1	1.25	1.6171875	1.58203125	1.59960938	1.64872127
1.5	1.88	2.93115234	2.76855469	2.8493042	3.08021685
2	3.28	6.59509277	5.96969604	6.27737331	7.3890561

Local Truncation error:

We're going to use the same expansion:

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2} \left( \frac{\partial f}{\partial t}(t_i, y_i) + f(t_i, y_i) \frac{\partial f}{\partial y}(t_i, y_i) \right) + \mathcal{O}(h^3).$$

We can again write:

$$w_{i+1} = y_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i))$$

$$= y_i + h \left( f(t_i, y_i) + \frac{h}{2} \frac{\partial f}{\partial t}(t_i, y_i) + \frac{h}{2} f(t_i, y_i) \frac{\partial f}{\partial y}(t_i, y_i) + \mathcal{O}(h^2) \right)$$

$$= y_i + hf(t_i, y_i) + \frac{h^2}{2} \frac{\partial f}{\partial t}(t_i, y_i) + \frac{h^2}{2} f(t_i, y_i) \frac{\partial f}{\partial y}(t_i, y_i) + \mathcal{O}(h^3)$$

Therefore  $y_{i+1} - w_{i+1}$  is  $\mathcal{O}h^3$  and the Midpoint Method is order 2. Thus we have two order two Runge-Kutta Methods: the Trapezoid Method and the Midpoint Method.

Each function evaluation of the right-hand side of the differential equation is called a *stage* of the method. The Trapezoid Method and the Runge-Kutta Methods are also both two-stage methods. They both have the form:

$$w_{i+1} = w_i + h \left( 1 - \frac{1}{2a} \right) f(t_i, w_i) + \frac{h}{2a} f(t_i + \alpha h, w_i + \alpha h f(t_i, w_i))$$

$\alpha = \frac{2}{3}$  is called the Ralston Method.

## Runge-Kutta

The family of (explicit) Runge-Kutta Methods are given by

$$y_{i+1} = y_i + h \sum_{j=1}^s b_j k_j$$

where the  $k_j$ 's are given by evaluations of  $f$ :

$$\begin{aligned} k_1 &= f(t_i, y_i) \\ k_2 &= f(t_i + c_2 h, y_i + h(a_{21} k_1)) \\ k_3 &= f(t_i + c_3 h, y_i + h(a_{31} k_1 + a_{32} k_2)) \\ &\vdots \\ k_s &= f(t_i + c_s h, y_i + h(a_{s1} k_1 + a_{s2} k_2 + \dots + a_{s,s-1} k_{s-1})) \end{aligned}$$

The number of stages is  $s$ , and these methods are consistent only if  $\sum_{j=1}^s b_j = 1$ .

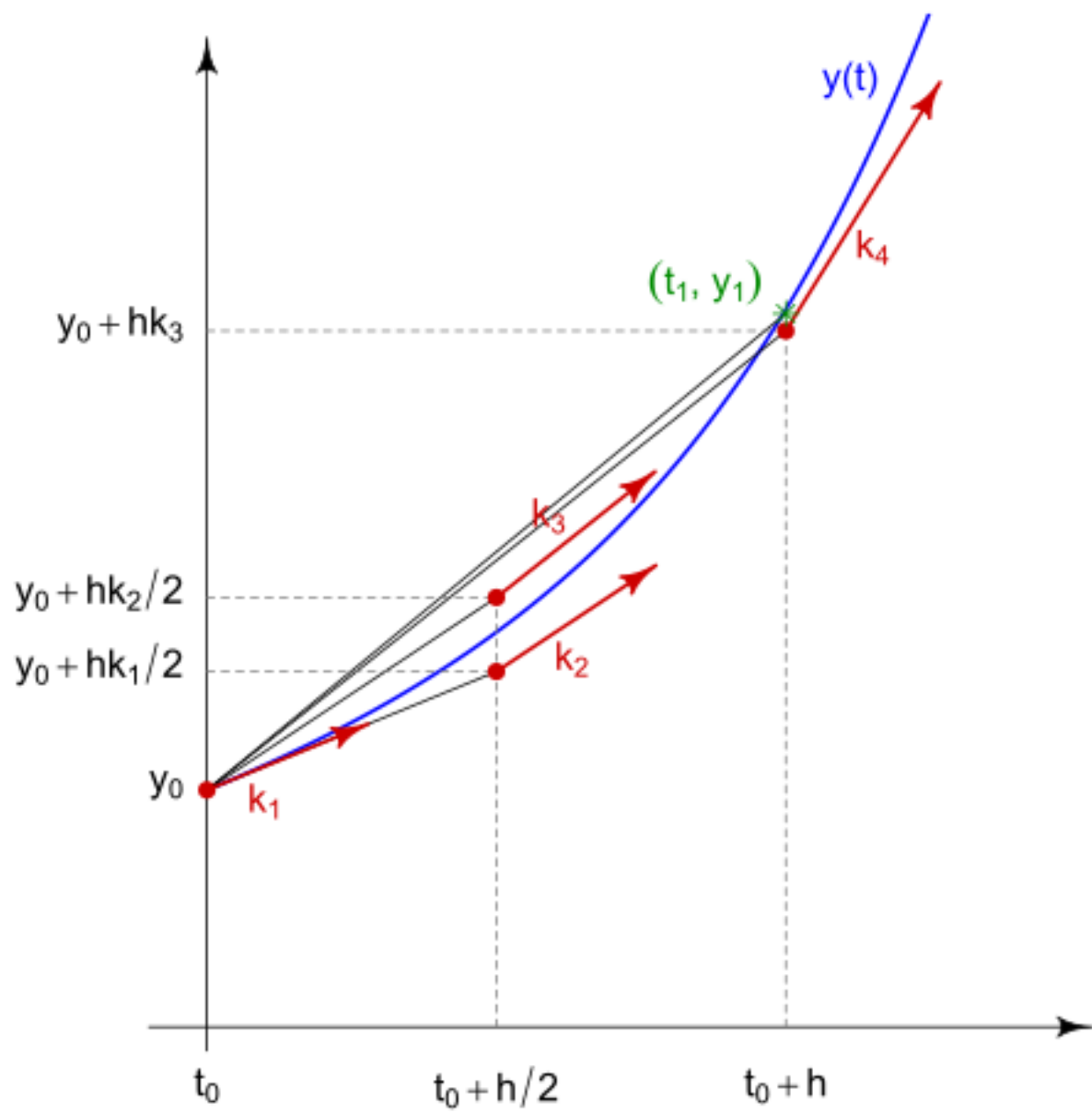
**Definition 2** (Runge-Kutta Method of Order 4).

$$w_0 = y_0$$

$$w_{i+1} = w_i + \frac{h}{6} (s_1 + 2s_2 + 2s_3 + s_4)$$

where

$$\begin{aligned} s_1 &= f(t_i, w_i) \\ s_2 &= f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} s_1\right) \\ s_3 &= f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} s_2\right) \\ s_4 &= f(t_i + h, w_i + h s_3) \end{aligned}$$



**Example 2.** Apply RK4 ...

Table 2:  $h=0.5$ 

<b>t</b>	<b>ye</b>	<b>yt</b>	<b>ytay</b>	<b>ym</b>	<b>yrk4</b>	<b>y</b>
0	1	1	1	1	1	1
0.5	1	1.125	1.125	1.125	1.13313802	1.13314845
1	1.25	1.6171875	1.58203125	1.59960938	1.6485277	1.64872127
1.5	1.88	2.93115234	2.76855469	2.8493042	3.07797616	3.08021685
2	3.28	6.59509277	5.96969604	6.27737331	7.36680329	7.3890561

Table 3:  $h = 0.1$ 

<b>t</b>	<b>ye</b>	<b>yt</b>	<b>ytay</b>	<b>ym</b>	<b>yrk4</b>	<b>y</b>
0.0	1	1	1	1	1	1
0.1	1	1.005	1.005	1.005	1.00501252	1.00501252
0.2	1.01	1.0201755	1.02012525	1.02015038	1.02020134	1.02020134
0.3	1.0302	1.04598594	1.04583241	1.04590917	1.04602786	1.04602786
0.4	1.061106	1.08322304	1.08290717	1.0830651	1.08328706	1.08328707
0.5	1.10355024	1.1330513	1.13250431	1.13277778	1.13314845	1.13314845
0.6	1.15872775	1.1970687	1.19620768	1.19663813	1.19721735	1.19721736
0.7	1.22825142	1.27739201	1.27611435	1.27675305	1.27762128	1.27762131
0.8	1.31422902	1.37677311	1.37494941	1.37586101	1.37712769	1.37712776
0.9	1.41936734	1.4987552	1.49621995	1.49748712	1.49930236	1.4993025
1.0	1.5471104	1.64788135	1.64442053	1.64615016	1.64872101	1.64872127
1.1	1.70182144	1.82997223	1.82530679	1.82763821	1.83125172	1.83125221
1.2	1.8890218	2.05249686	2.04626018	2.04937642	2.05443232	2.05443321
1.3	2.11570441	2.32506844	2.31677578	2.32091879	2.32797624	2.32797781
1.4	2.39074598	2.6601108	2.64911726	2.65460889	2.66445351	2.66445624
1.5	2.72545042	3.07375803	3.05920061	3.06647146	3.08021217	3.08021685
1.6	3.13426799	3.58707562	3.56779272	3.57742227	3.59663183	3.59663973
1.7	3.63575086	4.22772733	4.20214626	4.21491892	4.24183896	4.24185214
1.8	4.25382851	5.03226384	4.99824287	5.01522665	5.0530685	5.05309032
1.9	5.01951764	6.04928436	6.00388934	6.0265471	6.0799356	6.07997145
2.0	5.973226	7.34383122	7.28301796	7.31336557	7.38899753	7.3890561