Romberg Integration and Adaptive Quadrature

The goal with both Romberg Integration and Adaptive Quadrature is to increase the accuracy of integration methods while minimizing the amount of extra computations that need to be performed. We begin with Romberg Integration.

Romberg Integration

Romberg integration begins with the Trapezoid Rule:

$$\int_a^b f(x)dx = \frac{h}{2}(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)) + \text{ error terms.}$$

It can be shown (though we don't) that for an infinitely differentiable function f, the error terms are all constants times the even powers of h. Furthermore, the constants depend only on the derivatives of f at the endpoints (a and b) and not anywhere in the middle. So we have

Regarding minimizing the amount of extra computation, we first show that current trapezoid approximations can be used in producing the next approximation when cutting each step size in half. Define the various step sizes, starting with $n = 1$, as follows.

So each time we cut the step size in half, we are going from h_j to h_{j+1}. We're going to use the notation R_{j1} and $R_{j+1,1}$ to denote the Trapezoid approximations (the part without the error terms) using h_j and h_{j+1}. Looking at the trapezoid rules and rewriting in terms of previous approximations gives the following pattern.

This shows how the previous approximation, $R_{j-1,1}$, can serve as a term in R_{j1}.

The second part of the discussion involves extrapolation. Recall the definition of extrapolation.

Definition 1. The *Richardson extrapolation* or *extrapolation* formula for $F(h)$ is

$$Q \approx \frac{2^n F(\frac{h}{2}) - F(h)}{2^n - 1}.$$

Romberg Integration

$R_{11} = (b-a)\frac{f(a)+f(b)}{2}$

for $j = 2, 3, \ldots$ **do**

$h_j = \frac{b-a}{2^{j-1}}$

$R_{j1} = \frac{1}{2}R_{j-1,1} + h_j \sum_{i=1}^{2^{j-2}} f(a + (2i-1)h_j)$

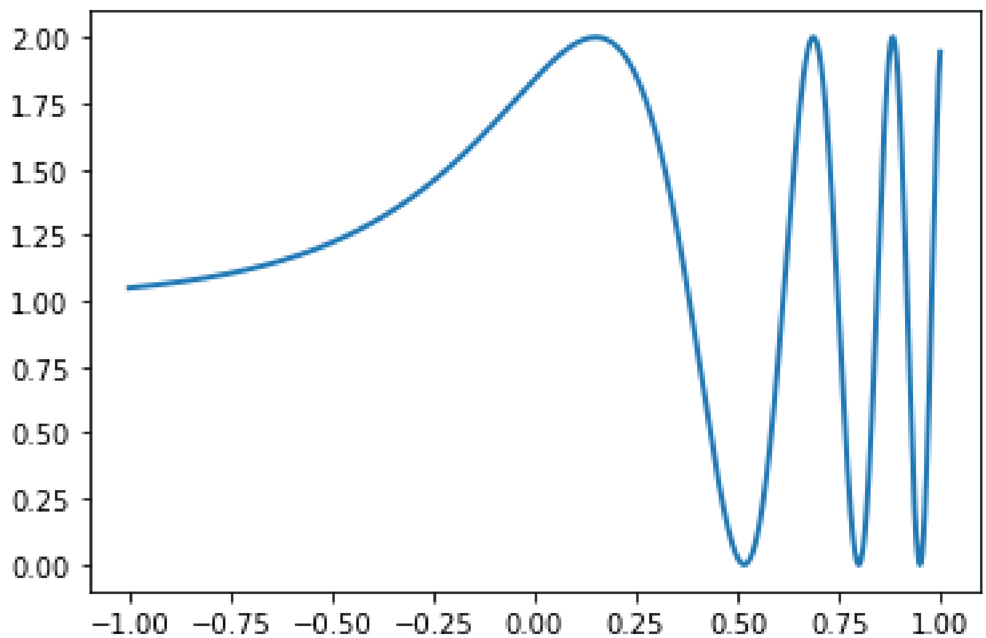
for k = 2, ..., j **do**

$R_{jk} = \frac{4^{k-1}R_{j,k-1} - R_{j-1,k-1}}{4^{k-1}-1}$

Example 1. Find R_{33} for $\int_0^1 xe^x dx$. Homework problem asks for R_{55}, so this gets you started.

Adaptive Quadrature

In general, a smaller step size means better approximations for numerical integration. However, some functions might only need a tiny step size in part of the interval, where a lot of change happens, and we can leave a larger step size for the rest. For example, look at the graph of $f(x) = 1 + \sin(e^{3x})$.



Adaptive Quadrature

$c = \frac{a+b}{2}$

$T_{a,b} = (b-a)\frac{f(a)+f(b)}{2}$

if $|T_{a,b} - T_{a,c} - T_{c,b}| < 3(TOL)(\frac{b-a}{b_b-a_a})$ **then**

accept $T_{a,c} + T_{c,b}$ as approximation over [a,b]

else

repeat above recursively for [a,c] and [c,b]

Example 2. Use Adaptive Quadrature to approximate the integral $\int_0^1 x^2 dx$ with TOL = 0.05.

The choice of trapezoid rule only really affects the scalar on the tolerance for the stopping criteria. If using Simpson's Rule instead, the stopping criteria should be

$$|S_{a,b} - S_{a,c} - S_{c,b}| < 15 * TOL,$$

although sometimes 15 is replaced with 10 to make the algorithm more conservative.