Root Finding - Newton Method and Secant Method

Root finding iterative methods are all centered around nonlinear equations; note that linear equations are much easier to solve. The next two methods both center around the idea of using a good, approximate line to find an exact root.

Example 1. Find the equation of the tangent line to $f(x) = 2 - x^2$ at $x = 1$ to approximate a root. Draw a graph with the function and tangent liine.

The formula for Newton's Method is:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

Question 2. Can you derive the Newton's Method formula from the equation of a tangent line?

Algorithm 1 Newton's Method

Input: A function f, its derivative f', an initial guess x_0, number of iterations n

for $i = 0, \dots, n-1$ **do**

$\quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

Return x_n

Example 3. Apply two steps of Newton's Method to $f(x) = 3x^2 - 6x$ with initial guess $x = 3$.

Question 4. When can you not use Newton's Method at all?

Question 5. How fast does Newton's Method converge?

Definition 6. Let e_i denote the error after step i of an iterative method. The iteration is *quadratically convergent* if

$$M = \lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} < \infty.$$

Theorem 7. *Let $f(x)$ be twice continuously differentiable and let r be an element of the domain such that $f(r) = 0$. If $f'(r) \neq 0$, Newton's Method is locally and quadratically convergent to r, and $e_{i+1} \approx e_i^2 \left|\frac{f''(r)}{2f'(r)}\right|$. If $f'(r) = 0$, Newton's Method is locally and linearly convergent to r,and $e_{i+1} \approx e_i \left(\frac{m-1}{m}\right)$ where m is the multiplicity of the root.*

Secant Method

Well, before you learned about tangent lines... you learned about secant lines!

That's really the idea in this last method: replace the derivative with slope between two points. One downside to this idea: you now need two different initial guesses.

Algorithm 2 Secant Method

Input: A function f, initial guesses x_0 and x_1, number of iterations n
for $i = 1, \ldots, n$ **do**
  $x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$
Return x_{n+1}

Example 8. Apply the first step of the Secant Method to $f(x) = (x-1)(x-2)(x-5)$ with initial guesses $x_0 = 3$ and $x_1 = 4$.

How fast is the Secant Method? For simple roots, we observe superlinear convergence.

Definition 9. Let e_i denote the error after step i of an iterative method. The iteration is *superlinearly convergent* if

$$\lim_{i\to\infty} \frac{e_{i+1}}{e_i^\alpha} = S$$

for some $1 < \alpha < 2$.

Just as a note - this limit is really generalizing all of what we're looking for with rates of convergence. When $\alpha = 1$, that's linear convergence. Similarly, $\alpha = 2$ is quadratic, $\alpha = 3$ is cubic, and so on. We are always interested in $\alpha > 0$

To find such an α, we start with the fact that for simple roots ($f'(r) \neq 0$), a very similar argument to Newton's Method gives that $e_{i+1} \approx \left|\frac{f''(r)}{2f'(r)}\right| e_i e_{i-1}$. For legibility, let $M = \left|\frac{f''(r)}{2f'(r)}\right|$.

To evaluate the limit $\lim_{i\to\infty} \frac{e_{i+1}}{e_i^\alpha} = S$, we're looking to find S and α such that $e_{i+1} \approx S e_i^\alpha$. Using our approximations together, we have $S e_i^\alpha = M e_i e_{i-1}$. Thus:

$$\begin{aligned} S e_i^\alpha &\approx M e_i e_{i-1} \\ e_i^{\alpha-1} &\approx \frac{M}{S} e_{i-1} \\ e_i &\approx \frac{M}{S}^{\frac{1}{\alpha-1}} e_{i-1}^{\frac{1}{\alpha-1}}. \end{aligned}$$

A direct comparison to $e_{i+1} \approx S e_i^\alpha$ gives that $S = \frac{M}{S}^{\frac{1}{\alpha-1}}$ (so $S = \left|\frac{f''(r)}{2f'(r)}\right|^{\alpha-1}$) and $\alpha = \frac{1}{\alpha-1}$. Therefore the α for the Secant Method is $\frac{1+\sqrt{5}}{2}$. Ever seen that number before???

So at the end of the day, for simple roots, the fastest convergence is Newton's Method, but that requires the derivative. Evaluating the Secant Method's formula is often faster that the Newton's Method formula per step, and is still superlinear (so it should converge faster than Bisection and FPI). Only the Bisection Method is guaranteed to converge; the others require "good" guesses.

One other note: the fact that the Bisection Method brackets the root is also an advantage. It gives you a sense of zooming in on the root and at every step you gain knowledge about the root. By contrast, Newton's Method and the Secant Method do NOT bracket the root, but there do exist modifications that can be made to the Secant Method so that it will.