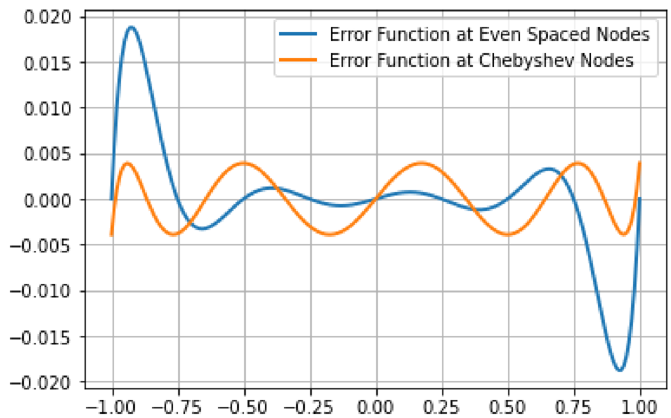
Chebyshev Interpolation

Is there an optimal way to pick n points in $[a, b]$ control the maximum value of $f(x) - P(x) = \dfrac{(x - x_1)(x - x_2)\ldots(x - x_n)}{n!} f^{(n)}(c)$? This question is sometimes called the *minmax problem of interpolation*.



There IS a solution to the minmax problem of interpolation, called the *Chebyshev interpolation nodes*. When the Chebyshev nodes are used, we call the resulting polynomial the *Chebyshev interpolating polynomial*. However, these are different from another polynomial that we need to describe the interpolation nodes.

Definition 1. The n^{th} *Chebyshev polynomial* $C_n(x)$ is $\cos(n \arccos(x))$.

Question 2. What is the domain of a Chebyshev polynomial?

The same as the domain of $\arccos(x)$, which is $[-1, 1]$.

Question 3. Does that definition actually give polynomials?

It sure isn't obvious that it does. As defined, I would say the answer is no. However, we can convert it to a polynomial form. Here are the first few:

$$
\begin{aligned}
C_0(x) &= \cos(0) = 1 \\
C_1(x) &= \cos(1 \arccos(x)) = x \\
C_2(x) &= \cos(2 \arccos(x)) = 2\cos^2(\arccos(x)) - 1 = 2x^2 - 1
\end{aligned}
$$

To prove that all have a polynomial form, we'll first show there is a recursion relation for the Chebyshev polynomials. Using the trig identity that $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$ and letting y represent $\arccos(x)$, we have:

$$
\begin{aligned}
C_{n-1}(x) &= \cos((n-1)y) = \cos(ny)\cos(y) + \sin(ny)\sin(y) \\
C_{n+1}(x) &= \cos((n+1)y) = \cos(ny)\cos(y) - \sin(ny)\sin(y)
\end{aligned}
$$

Adding these together gives

$$C_{n+1}(x) + C_{n-1}(x) = 2\cos(ny)\cos(y) = 2\cos(ny)x = 2xC_n(x)$$

Rearranging,

$$C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x).$$

The claim that C_n is a polynomial then follows by induction.

Example 1. State $C_3(x)$ and $C_4(x)$.

$C_3(x) = 2xC_2(x) - C_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$
$C_4(x) = 2xC_3(x) - C_2(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 6x^2 - 2x^2 + 1 = 8x^4 - 8x^2 + 1$

Question 4. What degree is $C_n(x)$?

Well, we certainly hoped that it is n for the notation to be consistent. And in fact, it is. We can see from the opening few ($n = 1, 2, 3$) that the degree of the polynomial is n. Then those later that we are getting from the recursion relation have degree $1 + n$ since x^n is multiplied by x and only terms of degree $n - 1$ and lower might be subtracted.

Question 5. What's the leading coefficient on $C_n(x)$ for $n \geq 1$?

We start with 1, then 2, and then multiply by 2 for each polynomial after that. So again by induction, we get 2^{n-1}.

Question 6. What is $C_n(1)$?

We can see $C_1(1) = C_2(1) = 1$. Then $2(1)(1) - 1 = 1$, so they must all evaluate to 1.

Similarly, $C_n(-1)$ is left for homework.

Question 7. What are the maximum/minimum values of $C_n(x)$?

Since cosine always lies between -1 and 1, so must $C_n(x)$.

Theorem 8. *All zeros of $C_n(x)$ lie in $[-1, 1]$.*

This theorem is true even if you let the polynomial forms have all numbers as the domain.

Proof. First note that $\cos(y) = 0$ only if y is an odd multiple of $\frac{\pi}{2}$. So let $n(\arccos(x)) = k(\frac{\pi}{2})$ for an odd integer k. Solving for x,

$$x = \cos\left(\frac{k\pi}{2n}\right).$$

Since, once again, the outputs of cosine are in $[-1, 1]$, no zeros can lie outside the interval. $\square$

Specifically, the roots are $x_i = \cos\left(\frac{(2i-1)\pi}{2n}\right)$ for $i = 1, \ldots, n$, which are unique (all multiplicity one). This theorem means that each Chebyshev polynomial is going to alternate between positive and negative values a total of $n+1$ times.

Definition 9. A polynomial is called *monic* if its leading coefficient is 1.

So $\frac{1}{2^{n-1}}C_n(x)$ is a monic polynomial, which means it can be written in factored form as $(x-x_1)(x-x_2)\ldots(x_n)$. Which is the form we were looking for in the error formula.

To summarize, Chebyshev polynomials are polynomials who have favorable properties on the interval $[-1,1]$.

- They have all their zeros in the interval.
- Their outputs in the interval are bounded by the same values.
- They have leading coefficients as large as possible.
- They are orthogonal. This isn't relevant to the current discussion, but it's an important enough property I wanted to mention it. Orthogonality for functions on $[-1,1]$ means $\int_{-1}^{1} C_i(x)C_j(x)dx = 0$ for $i \neq j$, and this implies that they're independent and can serve as a basis for the polynomial function space etc.
- Their n roots are the Chebyshev interpolation nodes.

Theorem 10. *The choice of real numbers $-1 \leq x \leq 1$ that makes the value of*

$$\max_{-1\leq x\leq 1} |(x-x_1)\ldots(x-x_n)|$$

as small as possible is

$$x_i = \cos\left(\frac{(2i-1)\pi}{2n}\right) \text{ for } i = 1, \ldots, n,$$

and the minimum value is $\frac{1}{2^{n-1}}$. In fact, the minimum is achieved by

$$(x-x_1)\ldots(x-x_n) = \frac{1}{2^{n-1}}C_n(x)$$

where C_n is the degree n Chebyshev polynomial.

To prove Theorem 9, we need to verify that there is no function with smaller extreme values, noting that the extreme value of $\frac{1}{2^{n-1}}C_n(x)$ is $\frac{1}{2^{n-1}}$.

Proof. Suppose to the contrary that $P_n(x)$ is a degree n monic polynomial with a smaller extreme value than $\frac{1}{2^{n-1}}$, that is, $|P_n(x)| < \frac{1}{2^{n-1}}$ for all $x \in [-1,1]$. Define $D(x) = P_n(x) - \frac{1}{2^{n-1}}C_n(x)$ on $[-1,1]$. Then D is degree at most $n-1$, since we are subtracting monic polynomials.

Let $y_1, \ldots, y_{n+1}$ be the points where $C_n(x)$ equals -1 or 1 (which it alternates between). Then for $i = 1, \ldots, n+1$, $D(y_i)$ is alternately positive and negative, so D must have n roots, a contradiction. $\square$

Example 2. Find the Chebyshev interpolation nodes for the Runge example on $[-1,1]$ using $n=9$.

They're $\cos(\frac{\pi}{18}), \cos(\frac{3\pi}{18}), \cos(\frac{5\pi}{18}), \cos(\frac{7\pi}{18}), \cos(\frac{9\pi}{18}), \cos(\frac{11\pi}{18}), \cos(\frac{13\pi}{18}), \cos(15\frac{\pi}{18}), \cos(17\frac{\pi}{18})$.

Example 3. Find the Chebyshev interpolation nodes for $\sin(x)$ on $[0, \frac{\pi}{2}]$ using $n=4$ and find an upper bound on the Chebyshev interpolation error on the interval.

Hm, well, thus far we only have the nodes on $[-1,1]$. We need a way to stretch these to an arbitrary interval $[a,b]$. First, let's stretch the interval. To get a width of $b-a$, we should divide by the original width of 2 and multiply by the new width of $b-a$.

Then we should shift the interval by $\frac{b+a}{2}$ to move the center of the interval from 0 to the new midpoint. The result is as follows.

Definition 11. The *Chebyshev interpolation nodes* on the interval $[a,b]$ are given by

$$x_i = \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{(2i-1)\pi}{2n}\right).$$

The inequality

$$|(x-x_1)\dots(x-x_n)| \leq \frac{(\frac{b-a}{2})^n}{2^{n-1}}$$

holds on $[a,b]$.

Ok, now we can work our example. Let $a=0$, $b=\frac{\pi}{2}$. Then the base points are

$$\frac{\frac{\pi}{2}}{2} + \frac{\frac{\pi}{2}}{2}\cos\left(\frac{(2i-1)\pi}{2(4)}\right)$$

$$= \frac{\pi}{4} + \frac{\pi}{4}\cos\left(\frac{(2i-1)\pi}{8}\right)$$

So the points are $x_1 = \frac{\pi}{4} + \frac{\pi}{4}\cos\left(\frac{\pi}{8}\right)$, $x_2 = \frac{\pi}{4} + \frac{\pi}{4}\cos\left(\frac{3\pi}{8}\right)$, $x_3 = \frac{\pi}{4} + \frac{\pi}{4}\cos\left(\frac{5\pi}{8}\right)$, and $\frac{\pi}{4} + \frac{\pi}{4}\cos\left(\frac{7\pi}{8}\right)$, which is about $0.05978488, 0.4848393, 1.08595703, 1.51101145$.

The worst-case error is bounded by

$$\frac{(\frac{\frac{\pi}{2}-0}{2})^4}{4!(2^3)}(1) \approx 0.00198.$$