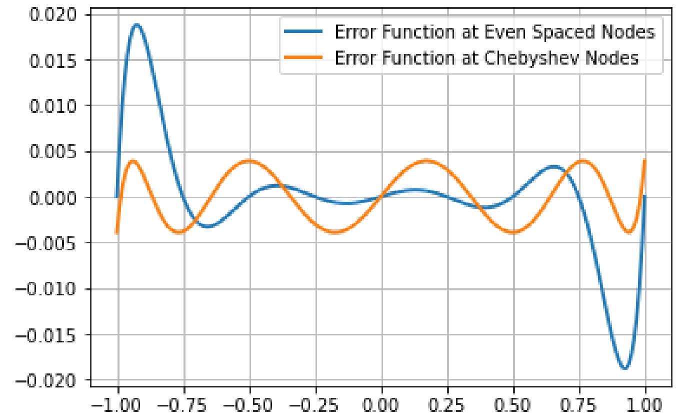
Chebyshev Interpolation

Is there an optimal way to pick n points in $[a, b]$ control the maximum value of $f(x) - P(x) = \dfrac{(x - x_1)(x - x_2)\ldots(x - x_n)}{n!} f^{(n)}(c)$? This question is sometimes called the *minmax problem of interpolation*.

Recall this plot: 

There IS a solution to the minmax problem of interpolation, called the *Chebyshev interpolation nodes*. When the Chebyshev nodes are used, we call the resulting polynomial the *Chebyshev interpolating polynomial*. However, these are different from another polynomial that we need to describe the interpolation nodes.

Definition 1. The n^{th} *Chebyshev polynomial* $C_n(x)$ is $\cos(n \arccos(x))$.

Question 2. What is the domain of a Chebyshev polynomial?

Question 3. Does that definition actually give polynomials?

Example 1. State $C_3(x)$ and $C_4(x)$.

Question 4. What degree is $C_n(x)$?

Question 5. What's the leading coefficient on $C_n(x)$ for $n \geq 1$?

Question 6. What is $C_n(1)$?

Question 7. What are the maximum/minimum values of $C_n(x)$?

Theorem 8. *All zeros of* $C_n(x)$ *lie in* $[-1, 1]$.

Specifically, the roots are $x_i = \cos\left(\frac{(2i-1)\pi}{2n}\right)$ for $i = 1, \ldots, n$, which are unique (all multiplicity one). This theorem means that each Chebyshev polynomial is going to alternate between positive and negative values a total of $n+1$ times.

Definition 9. A polynomial is called *monic* if its leading coefficient is 1.

So $\frac{1}{2^{n-1}}C_n(x)$ is a monic polynomial, which means it can be written in factored form as $(x-x_1)(x-x_2)\ldots(x_n)$. Which is the form we were looking for in the error formula.

To summarize, Chebyshev polynomials are polynomials who have favorable properties on the interval $[-1, 1]$.

- They have all their zeros in the interval.
- Their outputs in the interval are bounded by the same values.
- They have leading coefficients as large as possible.
- They are orthogonal. This isn't relevant to the current discussion, but it's an important enough property I wanted to mention it. Orthogonality for functions on $[-1, 1]$ means $\int_{-1}^{1} C_i(x)C_j(x)dx = 0$ for $i \neq j$, and this implies that they're independent and can serve as a basis for the polynomial function space etc.
- Their n roots are the Chebyshev interpolation nodes.

Theorem 10. *The choice of real numbers $-1 \leq x \leq 1$ that makes the value of*

$$\max_{-1\leq x\leq 1} |(x-x_1)\ldots(x-x_n)|$$

as small as possible is

$$x_i = \cos\left(\frac{(2i-1)\pi}{2n}\right) \text{ for } i = 1, \ldots, n,$$

and the minimum value is $\frac{1}{2^{n-1}}$. In fact, the minimum is achieved by

$$(x-x_1)\ldots(x-x_n) = \frac{1}{2^{n-1}}C_n(x)$$

where C_n is the degree n Chebyshev polynomial.

To prove Theorem 9, we need to verify that there is no function with smaller extreme values, noting that the extreme value of $\frac{1}{2^{n-1}}C_n(x)$ is $\frac{1}{2^{n-1}}$.

Example 2. Find the Chebyshev interpolation nodes for the Runge example on $[-1, 1]$ using $n = 9$.

Question 11. How would you find interpolation nodes on an interval $[a, b]$ that isn't $[-1, 1]$?

Definition 12. The *Chebyshev interpolation nodes* on the interval $[a, b]$ are given by

$$x_i = \frac{b+a}{2} + \frac{b-a}{2} \cos\left(\frac{(2i-1)\pi}{2n}\right).$$

The inequality

$$|(x-x_1)\ldots(x-x_n)| \leq \frac{(\frac{b-a}{2})^n}{2^{n-1}}$$

holds on $[a, b]$.

Example 3. Find the Chebyshev interpolation nodes for $\sin(x)$ on $[0, \frac{\pi}{2}]$ using $n = 4$ and find an upper bound on the Chebyshev interpolation error on the interval.