Properties of Determinants

Theorem 4. *A square matrix A is invertible if and only if $det(A) \neq 0$.*

Theorem 5. *If A is an $n \times n$ matrix, then $det(A^T) = det(A)$.*

Theorem 6. *If A and B are $n \times n$ matrices, then $det(AB) = det(A)det(B)$.*

Prove or reject the following statements.

1. Suppose $\det(A^3) = 0$. Then A is invertible.

2. Suppose A is invertible. Then $\det(A^{-1}) = \frac{1}{\det(A)}$.

3. Suppose U is a square matrix such that $U^T U = I$. Then $\det(U)$ is either 1 or -1.

Recall from Sections 1.8 and 1.9 that a matrix describes a transformation. In particular, when the coefficients are in certain patterns, the transformation behaves in very predictable ways, like reflection, stretches, rotations, shears, or projections. The determinant of a matrix also relates to how the matrix changes vectors.

Let $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The area of the rectangle formed by these two vectors is 2.

For each of the following matrices,

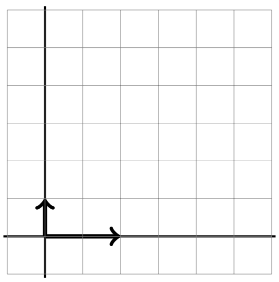
(a) Calculate $A\mathbf{u}$ and $A\mathbf{v}$.

(b) Find the area of the new parallelogram sketched out by $A\mathbf{u}$ and $A\mathbf{v}$. (area is base times height).

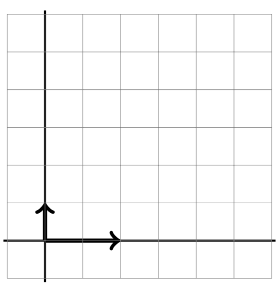
(c) Calculate $\det(A)$.

The area of the new parallelogram equals the original area (2) times what?

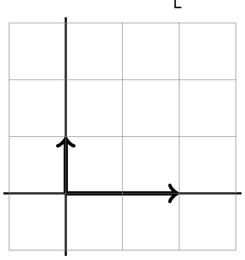
- A stretch transformation: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.



- A shear transformation: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.



- A projection: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ where x is the angle of rotation.



- A rotational transformation: $\begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}$ where x is the angle of rotation.