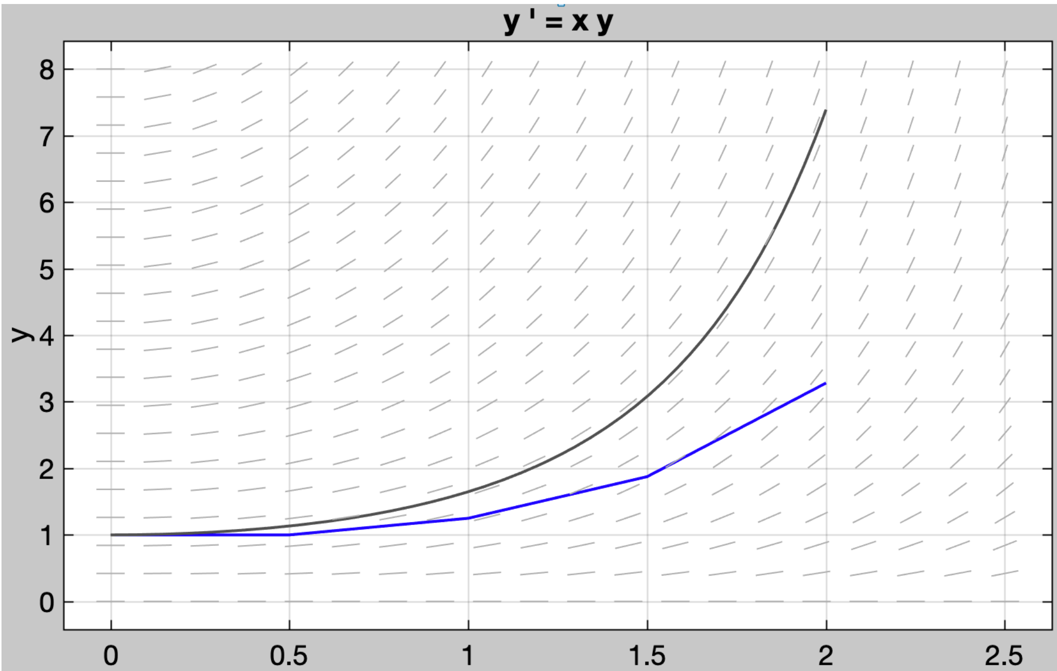
# Initial Value Problems

We motivate our first improvement on Euler's method by returning to the slope field.




So average together the slope at the current approximation, and the slope at what would be the next approximation from Euler's method.

**Definition 1** (Explicit Trapezoid Method)**.**

$$w_0 = y_0$$

$$w_{i+1} = w_i + h(\frac{f(t_i, w_i) + f(t_i + h, w_i) + hf(t_i, w_i)}{2})$$

**Example 1.** Apply the explicit trapezoid method with $h = \frac{1}{2}$ to approximate $y(2)$ for the IVP.

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

Table 1: h = 0.5

| **t** | **ye** | **yt** | **y** |
|---:|---:|---:|---:|
| 0 | 1 | 1 | 1 |
| 0.5 | 1 | 1.125 | 1.13314845 |
| 1 | 1.25 | 1.6171875 | 1.64872127 |
| 1.5 | 1.88 | 2.93115234 | 3.08021685 |
| 2 | 3.28 | 6.59509277 | 7.3890561 |

Table 2: h = 0.1

| t | ye | yt | y |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 0.1 | 1 | 1.005 | 1.00501252 |
| 0.2 | 1.01 | 1.0201755 | 1.02020134 |
| 0.3 | 1.0302 | 1.04598594 | 1.04602786 |
| 0.4 | 1.061106 | 1.08322304 | 1.08328707 |
| 0.5 | 1.10355024 | 1.1330513 | 1.13314845 |
| 0.6 | 1.15872775 | 1.1970687 | 1.19721736 |
| 0.7 | 1.22825142 | 1.27739201 | 1.27762131 |
| 0.8 | 1.31422902 | 1.37677311 | 1.37712776 |
| 0.9 | 1.41936734 | 1.4987552 | 1.4993025 |
| 1 | 1.5471104 | 1.64788135 | 1.64872127 |
| 1.1 | 1.70182144 | 1.82997223 | 1.83125221 |
| 1.2 | 1.8890218 | 2.05249686 | 2.05443321 |
| 1.3 | 2.11570441 | 2.32506844 | 2.32797781 |
| 1.4 | 2.39074598 | 2.6601108 | 2.66445624 |
| 1.5 | 2.72545042 | 3.07375803 | 3.08021685 |
| 1.6 | 3.13426799 | 3.58707562 | 3.59663973 |
| 1.7 | 3.63575086 | 4.22772733 | 4.24185214 |
| 1.8 | 4.25382851 | 5.03226384 | 5.05309032 |
| 1.9 | 5.01951764 | 6.04928436 | 6.07997145 |
| 2 | 5.973226 | 7.34383122 | 7.3890561 |

**Question 2.** Why is it called a Trapezoid Method?

**Question 3.** So what's the local truncation error this time?

Then revisiting this theorem from the last lecture:

**Theorem 4.** *Assume that $f(t, y)$ has a Lipschitz constant $L$ for the variable $y$ and that the value $y_i$ of the solution of the initial value problem at $t_i$ is approximated by $w_i$ from a one-step ODE solver with local truncation error $e_i \leq Ch^{k+1}$ for some constant $C$ and $k \geq 0$. Then for each $a < t_i < b$, the solver has global truncation error*

$$g_i = |w_i - y_i| \leq \frac{Ch^k}{L}(e^{L(t_i - a)} - 1).$$

**Definition 5.** If an ODE solver satisfies Theorem 4 as $h \to 0$, then we say the solver has *order $k$*.

What is the order of the Explicit Trapezoid Method?

## Methods of Higher Orders

In this section, we show that methods of all orders exist. These methods are called Taylor Methods - because, of course, they continue using the Taylor expansions.

The Taylor expansion of $y(t)$, assuming it is at least $k+1$ times continuously differentiable, is

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \cdots + \frac{h^k}{k!}y^{(k)}(t) + \frac{h^{k+1}}{(k+1)!}y^{(k+1)}(c)$$

**Definition 6** (Taylor Method of Order $k$)**.**

$$w_0 = y_0$$

$$w_{i+1} = w_i + hf(t_i, w_i) + \frac{h^2}{2}f'(t_i, w_i) + \cdots + \frac{h^k}{k!}f^{(k-1)}(t_i, w_i)$$

Here the prime notations refer to the total derivative of $f$. For example,

$$f'(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)y'(t) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial t}(t,y)f(t,y).$$

By design, the local truncation error is the error term $\frac{h^{k+1}}{(k+1)!}y^{(k+1)}(c)$ and so the method is order $k$.

**Example 2.** Write out the first order and second order Taylor Method formulas.

**Example 3.** Apply the second-order Taylor Method with $h = \frac{1}{2}$ to

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

Table 3: $h = 0.5$

| t | ye | yt | ytay | y |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 0.5 | 1 | 1.125 | 1.125 | 1.13314845 |
| 1 | 1.25 | 1.6171875 | 1.58203125 | 1.64872127 |
| 1.5 | 1.88 | 2.93115234 | 2.83007813 | 3.08021685 |
| 2 | 3.28 | 6.59509277 | 6.32029724 | 7.3890561 |

**Example 4.** Find the order three Taylor method formula for the differential equation $y' = y - t^2 + 1$.

**Question 7.** What's the biggest drawback of a Taylor Method verses the previous two?