Interpolation Error

Example 1. Interpolate the function $f(x) = \sin(x)$ at 4 equally spaced points on $[0, \frac{\pi}{2}]$. Use 4 digits of accuracy.

First note that we do want to include the endpoints, so unlike in calculus where we would use a step size of $\frac{b-a}{n} = \frac{\pi}{4}$, we want to use $n-1$ in the denominator, so that we get 4 points instead of 5. So the points for interpolation are $0, \frac{\pi}{6}, \frac{\pi}{3}$, and $\frac{\pi}{2}$.

Then a divided difference triangle is

$$
\begin{array}{c|cccc}
0 & 0 & & & \\
 & & 0.9549 & & \\
\frac{\pi}{6} & \frac{1}{2} & & \text{-0.2443} & \\
 & & 0.6990 & & \text{-0.1139} \\
\frac{\pi}{3} & 0.8660 & & \text{-0.4232} & \\
 & & 0.2559 & & \\
\frac{\pi}{2} & 1 & & &
\end{array}
$$

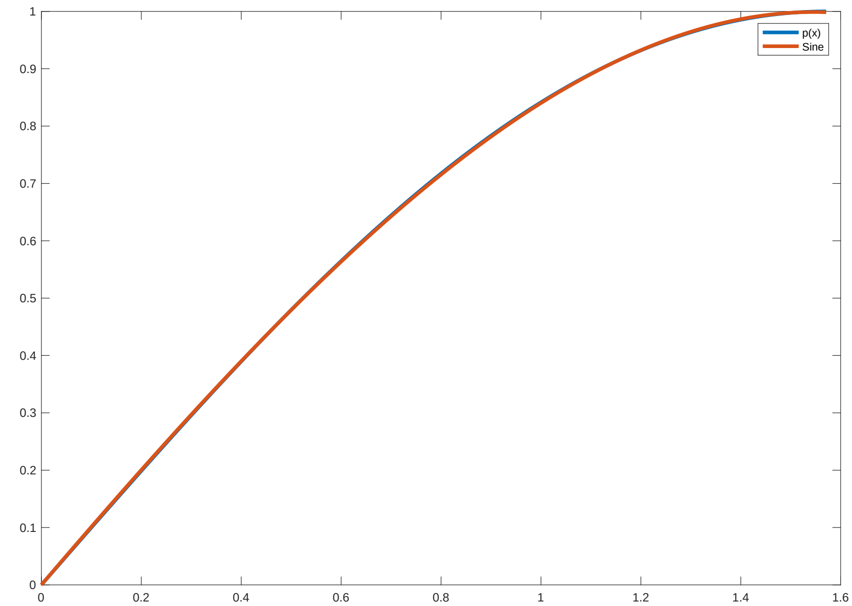
so the polynomial is

$$
\begin{aligned}
P_3(x) &= 0 + 0.9549x - 0.2443x\left(x - \frac{\pi}{6}\right) - 0.1139x\left(x - \frac{\pi}{6}\right)\left(x - \frac{\pi}{3}\right) \\
&= x\left(0.9549 + \left(x - \frac{\pi}{6}\right)\left(-0.2443 + \left(x - \frac{\pi}{3}\right)(-0.1139)\right)\right)
\end{aligned}
$$

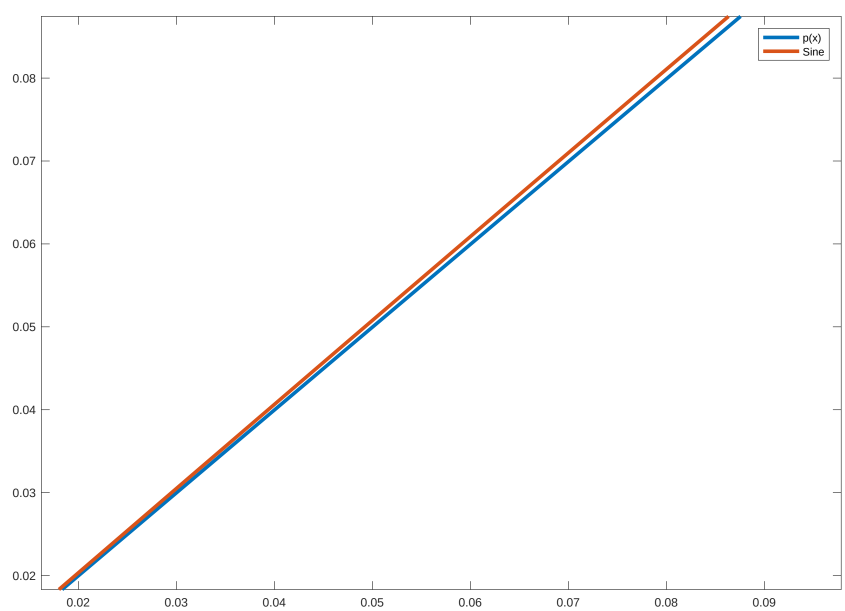
Example 2. How well would this polynomial serve as a sine key for a calculator?

Well, let's first address how inputs outside of $[0, \frac{\pi}{2}]$ would work. Because sine is periodic, if we can find all the values for $[0, 2\pi]$, then we would have enough information for the whole number line: to compute $\sin(x)$, find $p(y)$ where $y \equiv x \pmod{2\pi}$. Since sine is symmetric on $[0, \pi]$, if y is in $[\frac{\pi}{2}, \pi]$, we have$\sin(y) = \sin(\pi - y)$. And if y is in $[\pi, 2\pi]$, then $\sin(y) = -\sin(2\pi - y)$. So we can in fact find an approximation for any input with the accuracy that we have on $[0, \frac{\pi}{2}]$.

So is it close? Well, have a look at the graphs:



I... can't even see the approximating function it's so close. Let's zoom in a bit:



Ok, that's quite good. Here's also a table of values:

x	$\sin(x)$	$P_3(x)$	error
1	0.8415	0.8411	0.0004
2	0.9093	0.9102	0.0009
4	-0.7568	-0.7557	0.0011
1000	0.8269	0.8263	0.0006

So we're getting about 3 digits of accuracy. We'd like better, but that does seem like a good start!

Formally, the interpolation error is the difference between the true function and the interpolating function.

Definition 1. The *interpolation error* at x is $f(x) - P(x)$.

Theorem 2. *Assume that $P(x)$ is the degree at most $n-1$ interpolating polynomial fitting the n points $(x_1, y_1), \ldots, (x_n, y_n)$. The interpolation error is*

$$f(x) - P(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{n!} f^{(n)}(c)$$

where c lies between the smallest and largest of the numbers $x, x_1, \ldots, x_n$.

Example 3. Use Theorem 2 to compute a bound on the error at $x = 1$ using P_3 above.

First, note that all the derivatives of sine have a maximum value of 1. So we always have

$$|f(x) - P(x)| \leq |\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{n!}|(1).$$

Then for $n = 4$ on $[0, \frac{\pi}{2}]$, we have

$$|f(x) - P(x)| \leq |\frac{(x)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{4!}|$$

and at $x = 1$

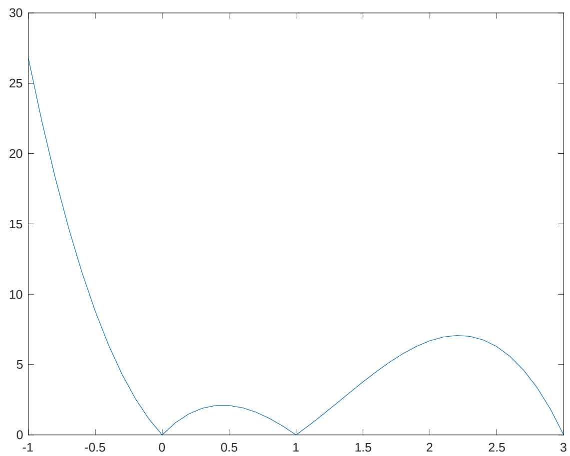
$$\frac{(1)(1-\frac{\pi}{6})(1-\frac{\pi}{3})(1-\frac{\pi}{2})}{24}| \approx 0.0005348$$

We observed an actual error of about 0.0004, which is less than the bound as expected.

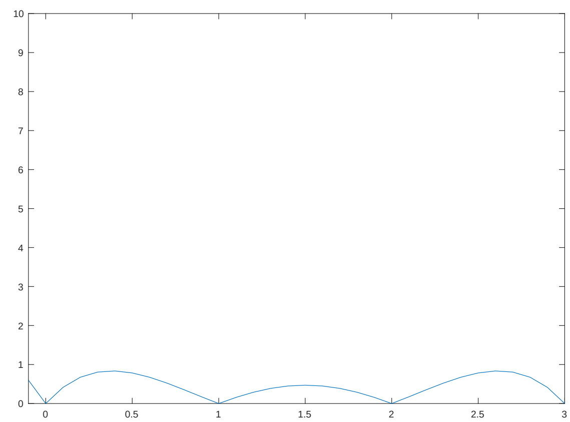
Example 4. Consider the function $f(x) = e^x$ and its interpolating polynomial using the values $x = 0, 1$ and 3. Find an expression for the error on the interval $[0, 3]$. At what x is it largest/smallest?

$$\begin{aligned}
|e^x - P_4(x)| &\leq |\frac{(x)(x-1)(x-3)}{3!}|e^3 \\
&= |\frac{e^3}{6}(x^3 - 4x^2 + 3x)|
\end{aligned}$$

The smallest error is going to occur at our interpolation points, 0, 1, and 3, since it should be exactly correct. Looking for the maximum, the derivative of this expression has roots at about 2.21 and 0.45, so those are our critical points. At $x = 0.45$, the error is about 2.11, and at $x = 2.21$, it's about 7.07, so the largest occurs about about 2.21.



If we included the missing point $x = 2$, the error picture changes to:

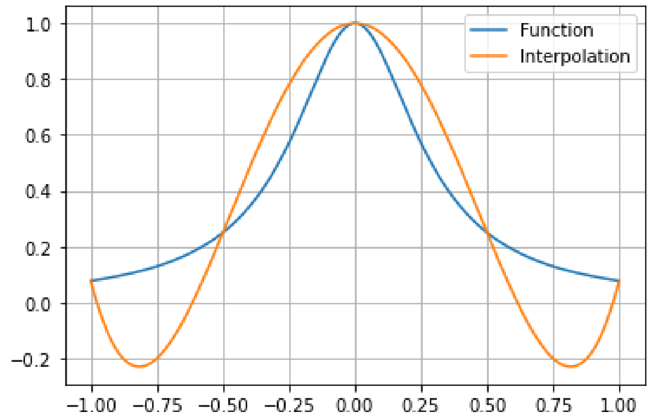


When using an even grid, it is almost always the case that the error is lowest in the middle and bigger at the ends of the interval.

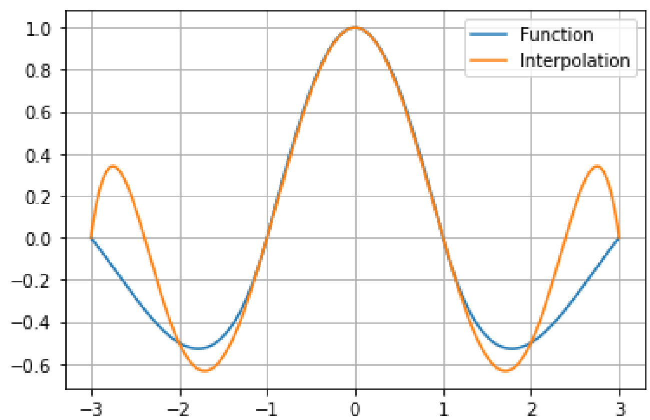
However, there's a real danger to using regularly spaced points for polynomial interpolation.

Example 5. Suppose you have a function like a trig function whose values are between -1 and 1. Also suppose that at your evenly spaced interpolating values, the function evaluates to 0 except at one point in the middle, where it is 1.

Look at $r(x) = \dfrac{1}{1+12x^2}$



$$\text{and } f(x) = \frac{\cos(\frac{\pi}{2}x)}{\max|x|, 1}$$



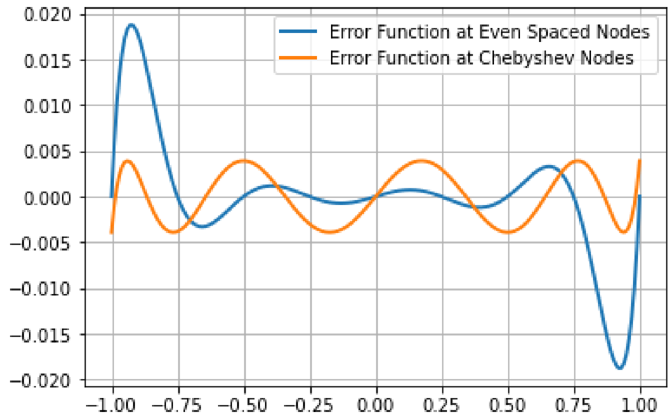
The more points you have (the smaller the step size), the more wiggle you see at the end of the interval. This interpolation behavior is called the *Runge phenomenon*. Basically, the Runge phenomenon describes the extreme amount of wiggle we observe with high-degree polynomial interpolation at evenly spaced points. The function $r(x)$ above is called the Runge example.

Question 3. What should we do to get better accuracy near the ends of the interval?

Basically, have more points specifically at the ends over in the middle. This isn't always possible - if you're working data where readings happen at a set time interval, you can't really do much about it. But if you're say, designing a sine key, then we are free to choose our interval steps however we like.

Then the next logical question is: is there an optimal way to pick n points in $[a, b]$ control the maximum value of $f(x) - P(x) = \dfrac{(x-x_1)(x-x_2)\dots(x-x_n)}{n!} f^{(n)}(c)$? This question is sometimes called the minmax problem of interpolation - and is the subject of the next lecture!

The picture of the polynomial part of the error, $(x-x_1)(x-x_2)\dots(x-x_n)$, on $[-1, 1]$ is below.



Question 4. A common way to approximate functions is using their Taylor polynomials. Have you observed in our discussions thus far any similarities between interpolating polynomials and Taylor polynomials? Why would you maybe choose one over the other?

Well, first, a Taylor expansion picks a single point a and results in a polynomial of degree n as

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)(x)}}{n!}(x-a)^n.$$

One reason you might choose an interpolating over a Taylor is simply that you may not have the knowledge about the derivatives to use a Taylor polynomial. But note that the error term is (for some c between x and a):

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

An interpolating polynomial for $n+1$ points (so that the result is the same degree) has an error formula of (for some c in the interval including x, x_1, etc.)

$$f(x) - P(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_{n+1})}{(n+1)!} f^{(n+1)}(c).$$

It isn't a coincidence that these are so similar. A Taylor polynomial is like the limit of an interpolating polynomial at a bunch of points super close to a, so that the derivatives match. The order of the approximation is essentially identical.