Initial Value Problems

An initial value problem is generally of the form

$$\begin{cases} y' = f(t, y) \\ y(0) = y_0 \end{cases}.$$

The first numerical method for solving initial value problems is one you've likely seen before: Euler's Method. Typically this is introduced by looking at the slope field. An approximate solution is produced by proceeding along the slope vectors by step length h, finding a new slope, and repeating.

Definition 1 (Euler's Method)**.**

$$w_0 = y_0$$
$$w_{i+1} = w_i + hf(t_i, w_i)$$

Since the slope is $f(t_i, w_i)$, and $t_i = a + hi$, we can again find a list of approximate values $w_1, \ldots, w_n$ for y. Note that if we think of this in terms of finite difference formulas, what we have is

$$\frac{(w_{i+1} - w_i)}{h} = f(t_i, w_i)$$

and solving for w_{i+1} immediately gives Euler's Method.

Example 1. Apply Euler's Method with $h = \frac{1}{2}$ to approximate $y(2)$ for the IVP.

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

$2_0 = 1, t_0 = 0$

$f(t_0, 2_0) = 0$ so $2_1 = 1 + 0 = 1$

$t_1 = \frac{1}{2}$ so $2_2 = 1 + \frac{1}{2}(\frac{1}{2}(1)) = \frac{5}{4}$

$t_2 = 1$ so $w_3 = \frac{5}{4} + \frac{1}{2}((1)(\frac{5}{4}) = \frac{15}{8}$

$t_3 = \frac{3}{2}$ so $w_4 = \frac{15}{8} + \frac{1}{2}(\frac{3}{2})(\frac{15}{8}) = \frac{105}{32}$

So $y(2) \approx \frac{105}{32} \approx 3.28$.

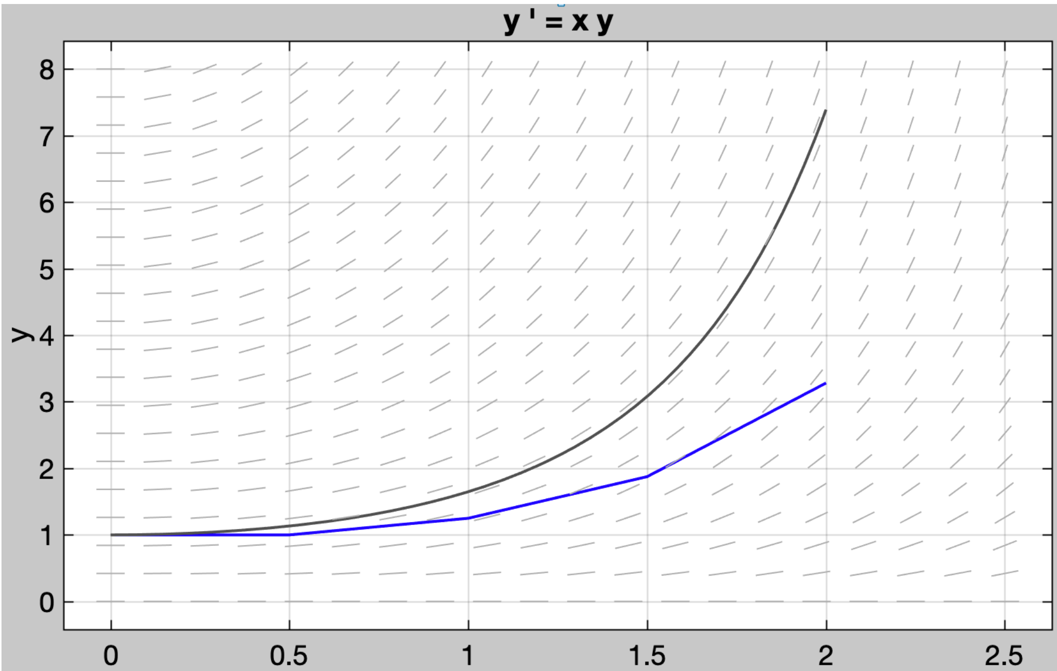
Example 2. Check that $y = e^{\frac{t^2}{2}}$ is the exact solution to the previous IVP.

$y' = e^{\frac{t^2}{2}}(\frac{1}{2})(2t) = te^{\frac{t^2}{2}} = ty$, and $y(0) = 0(1) = 0$.

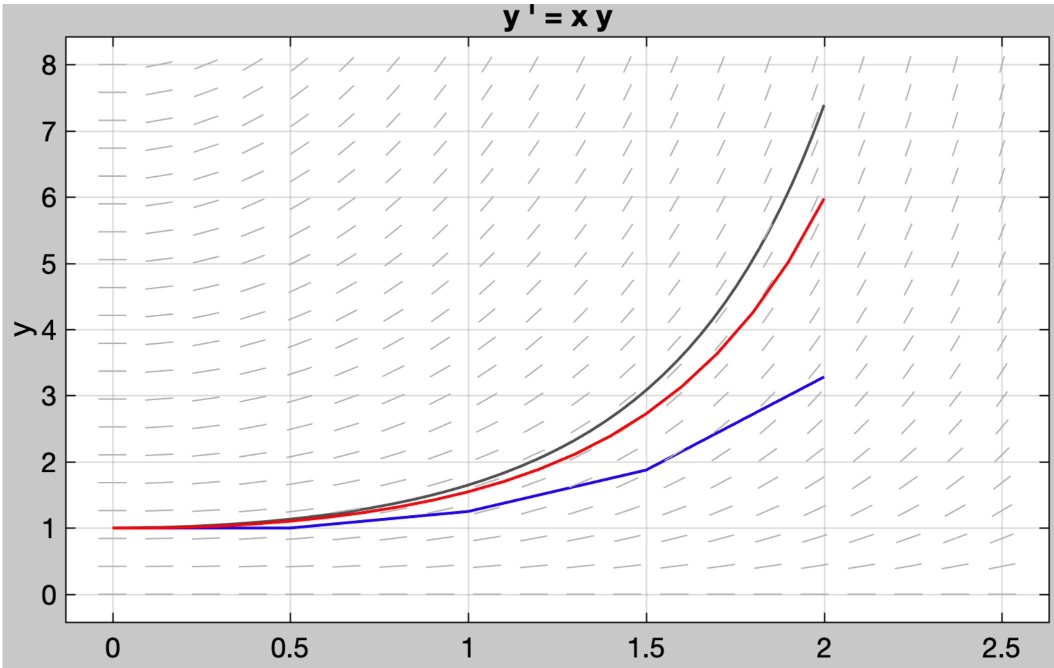
Definition 2. The *global truncation error* at step i is $g_i = |w_i - y_i|$.

Example 3. Compute the global truncation error at $t = 2$.

$|\frac{105}{32} - e^2| \approx 4.1078$. Ouch! What happened?



Perhaps it's just that the step size was too large. Let's see what happens with $h = 0.1$.



Better. But still off by more than 1!

Euler's Method is honestly too simple and error prone to be used in practice, but the process does highlight the issues with IVP solvers in general.

Definition 3. Let z_{i+1} be the (exact) solution to the one-step initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_i) = w_i \end{cases} .$$

Then the *local truncation error* is the one-step error $e_{i+1} = |w_{i+1} - z_{i+1}|$.

Example 4. Find the local truncation error for Euler's Method.

Assume y'' is continuous, so that we can use Taylor's Method.

$y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(c)$

$z_{i+1} = w_i + hf(t_i, w_i) + \frac{h^2}{2}y''(c)$

Subtracting,

$$e_{i+1} = |w_{i+1} - z_{i+1}| = \frac{h^2}{2}|y''(c)|.$$

So the truncation error on a single step is the same bound we've been having on first-order methods (two-point forward difference, for instance). If $y''(x) \le M$ on the interval of interest,

$$e_i \leq \frac{Mh^2}{2}.$$

On the first step, this same expression is the global error. But from then on, the global error is affected by both the previous global errors and the local truncation error.

So $g_2 = |w_2 - y_2| = |w_2 - z_2 + z_2 - y_2| \leq |w_2 - z_2| + |z_2 - y_2| = e_2 + |z_2 - y_2|$.

Now note that z_2 and y_2 are exact solutions to the same differential equation with differing initial values. We detour from our discussion of error to some theory about differential equations that will let us relate these two solutions.

Existence and Uniqueness

Definition 4. A function $f(t, y)$ is *Lipschitz continuous* in the variable y on the rectangle $S = [a, b] \times [\alpha, \beta]$ if there exists a constant L (called the *Lipschitz constant*) satisfying

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

for each (t, y_1), (t, y_2) in S.

Example 5. Show that $f(t, y) = ty$ is Lipschitz continuous on $[0, b] \times (-\infty, \infty)$.

Basically this problem is still looking at the differential equation

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

so t has a restriction, but y does not. The function of interest is the right-hand-side ty.

Then $f(t, y_1) - f(t, y2) = ty_1 - ty_2 = t(y_1 - y_2)$, so $|f(t, y_1) - f(t, y_2)| \leq |t||y_1 - y_2| \leq b|y_1 - y_2|$. So f is Lipschitz continuous with Lipschitz constant b.

Question 5. What does $\frac{|f(t,y_1)-f(t,y_2)|}{|y_1-y_2|}$ remind you of?

It's basically the finite difference formula for the partial derivative of f with respect to y. In fact, on any convex set (instead of a rectangle), the Lipschitz constant is the maximum of $\frac{\partial f}{\partial y}(t, c)$ on the set.

Theorem 6. *Assume that $f(t, y)$ is Lipschitz continuous in the variable y on the set $[a, b] \times [\alpha, \beta]$ and that $\alpha < y_a < \beta$. Then there exists c between a and b such that the initial value problem*

$$\begin{cases} y' = f(t, y) \\ y(a) = y_a \\ t \in [a, c] \end{cases}$$

has exactly one solution $y(t)$. Moreover, if f is Lipschitz on $[a, b] \times (-\infty, \infty)$, then there exists exactly one solution on $[a, b]$.

Example 6. Does the initial value problem

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

have a unique solution for t in $[0, b]$?

Yes it does. We've shown its Lipschitz on $[0, b] \times (-\infty, \infty)$, so using the second part of the theorem, there exists a unique solution. In particular, when we approximated $y(2)$, that solution is unique.

Example 7. On which intervals $[0, c]$ does the initial value problem have a unique solution?

$$\begin{cases} y' = y^2 \\ y(0) = 1 \\ t \in [0, 2] \end{cases}$$

$\frac{\partial f}{\partial y} = 2y$ so $L = 20$ on $0 \leq t \leq 2, -10 \leq y \leq 10$. However, this only gives that there is a solution starting at $t = 0$ and continuing to c for some c less than 2.

In fact, the unique solution is $y(t) = \frac{1}{1-t}$. Since $\lim_{t \to 1^-} \frac{1}{1-t} = \infty$, no solution exists at $t = 1$ and we only have a solution on $[0, c]$ for $c < 1$.

Theorem 7. *Assume that $f(t, y)$ is Lipschitz in the variable y on the set $S = [a, b] \times [\alpha, \beta]$. If $Y(t)$ and $Z(t)$ are solutions in S of the differential equation*

$$y' = f(t, y)$$

with initial conditions y_a and z_a respectively, then

$$|Y(t) - Z(t)| \leq e^{L(t-a)} |y_a - z_a|.$$

Question 8. What was the definition of a condition number?

It's a number that relates the forward error and backward error in this way:

$$\text{forward error} \lesssim \text{condition number} \cdot \text{backward error}.$$

For a fixed time t, we essentially have a condition number of $e^{L(t-a)}$.

We are ready to return to the global truncation error. We left off with $|z_2 - y_2|$. Assuming that the function f is Lipschitz with constant L and that $t_{i+1} - t_i = h$, we have

$$g_2 \leq e_2 + e^{Lh} |y_1 - z_1| = e_2 + e^{Lh} e_1.$$

Extending the pattern,

$$g_3 \leq e_3 + e^{Lh} e_2 + e^{2Lh} e_1.$$

We found earlier that the local truncation error is about h^2 for Euler's Method. More generally, assume local truncation error satisfies

$$e_i \leq Ch^{k+1}$$

for an integer k and a constant C. Then

$$\begin{aligned} g_i &\leq Ch^{k+1}(1 + e^{Lh} + e^{2Lh} + \cdots + e^{(i-1)Lh}) \\ &= Ch^{k+1}\frac{e^{iLh} - 1}{e^{Lh} - 1} \\ &\leq Ch^{k+1}\frac{e^{L(t_i-a)} - 1}{Lh} \\ &= \frac{Ch^k}{L}(e^{L(t_i-a)} - 1). \end{aligned}$$

Theorem 9. *Assume that $f(t, y)$ has a Lipschitz constant L for the variable y and that the value y_i of the solution of the initial value problem at t_i is approximated by w_i from a one-step ODE solver with local truncation error $e_i \leq Ch^{k+1}$ for some constant C and $k \geq 0$. Then for each $a < t_i < b$, the solver has global truncation error*

$$g_i = |w_i - y_i| \leq \frac{Ch^k}{L}(e^{L(t_i-a)} - 1).$$

Definition 10. If an ODE solver satisfies Theorem 9 as $h \to 0$, then we say the solver has *order k*.

So Euler's Method is order 1 by our previous work, specifically satisfying

$$g_i \leq \frac{Mh}{2L}(e^{L(t_i-a)} - 1)$$

if $y'' \leq M$ on $[a, b]$.

Example 8. Find an error bound for Euler's Method applied to

$$\begin{cases} y' = ty \\ y(0) = 1 \\ t \in [0, 2] \end{cases}.$$

We've found the Lipschitz constant to be 2, and since we know that $y = e^{\frac{t^2}{2}}$ is the exact solution, we can also find the second derivative, which is $y'' = e^{\frac{t^2}{2}}(t^2 + 1) \leq 5e^2$. So $g_i \leq \frac{5e^2h}{4}(e^4 - 1) \approx 495h$. Which predicts even a worse error than we saw! and it was pretty bad...

This work does tell us that as h gets smaller, so does the global error. We will next turn to methods with higher order that should make a big difference on accuracy.