Finite Differences: Partial Derivatives and Boundary Value Problems

Suppose we now have $f(x, y)$, a function of two or more variables.

Question 1. What is the definition of the partial derivative with respect to x?

Question 2. Find a first order formula to approximate the partial derivative with respect to x.

The previous work gives approximations for second derivatives as well, especially when they are not mixed. For instance,

$$\frac{\partial^2 f}{\partial x^2}(a, b) \approx \frac{f(a-h, b) - 2f(a, b) + f(a+h, b)}{h^2}.$$

Question 3. Find an approximation formula for the mixed derivative $\frac{\partial^2 f}{\partial x \partial y}(a, b)$.

Because the notation is getting bulky, we use the coefficients on h as subscripts on f to write this as:

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) \approx \frac{f_{1,1} - f_{1,-1} - f_{-1,1} + f_{-1,-1}}{4h^2}$$

Example 1. For the function $f(x,y) = xy^2 + x$ and the point $(1,2)$, approximate the partial derivatives and the mixed second derivative of f using $h = 0.1$.

Boundary Value Problem

One of the other common uses of finite differences is to solve boundary value problems. We return to functions of one variable and examine the second-order boundary value problem

$$\begin{cases} y'' = f(t, y, y') \\ f(a) = y_a \\ f(b) = y_b \end{cases}$$

on some interval $a \leq t \leq b$.

Question 4. What is the difference between an initial value problem and a boundary value problem?

Question 5. Verify that the boundary value problem

$$\begin{cases} y'' = -y \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

has infinitely many solutions of the form $y = k \sin t$.

Finite Difference Methods

Example 2. Use finite differences to approximate solutions to the linear BVP for $n = 3$.

$$\begin{cases} y'' = 4y \\ y(0) = 1 \\ y(1) = 3 \end{cases}$$

Example 3. Use finite difference to approximate solutions to the linear BVP for $n = 7$.

$$\begin{cases} y'' = (2 + 4t^2)y \\ y(0) = 1 \\ y(1) = e \end{cases}$$

$$\begin{bmatrix} 1.0173 \\ 1.0675 \\ 1.15514 \\ 1.2890 \\ 1.48338 \\ 1.7603 \\ 2.15409 \end{bmatrix}$$

With the end points, we're comparing

$$w = \begin{bmatrix} 1.000000000000000 \\ 1.017346037963531 \\ 1.067477641603622 \\ 1.155137756081340 \\ 1.289048503370908 \\ 1.483383399255987 \\ 1.760289441388713 \\ 2.154089704238659 \\ 2.718281828459046 \end{bmatrix} \qquad y = \begin{bmatrix} 1.000000000000000 \\ 1.015747708586686 \\ 1.064494458917860 \\ 1.150992944691176 \\ 1.284025416687741 \\ 1.477904195411738 \\ 1.755054656960298 \\ 2.150337915952300 \\ 2.718281828459046 \end{bmatrix}$$