Lagrangian Interpolation

The mathematical problem that we call interpolation is essentially a game of connect the dots. Any function that evaluates to the same values as the ones in the known points is a valid interpolating function.

Definition 1. The function $y = P(x)$ *interpolates* the data points $(x_1, y_1), \ldots (x_n, y_n)$ if $P(x_i) = y_i$ for each $1 \leq i \leq n$.

Interpolation is the opposite of evaluation: Evaluation is computing points given a curve, and interpolation is computing a curve given some points.

Example 1. Draw three different interpolating functions through the points $(1, 1)$, $(2, 2)$, and $(3, 5)$.

The most common are a linear spline (a piecewise linear function connecting them), a parabola, and a cubic. I also demonstrated the nearest neighbor function (which is essentially a step function).

Question 2. Why might you prefer a polynomial interpolating function over a piecewise-defined function?

Well, the sheer amount of experience we've had with them certainly counts for something. They're easy to evaluate (for both us and the computer). All our previous work on root finding will apply. They're both differentiable and integrable. And as we will also see, there's straightforward theory for when one exists and with what degree.

Polynomial Interpolating Functions

Question 3. Given n points $(x_1, y_1), \ldots, (x_n, y_n)$, does there exist a polynomial $p(x)$ where for all x_i, $i = 1, \ldots, n$, $p(x_i) = y_i$?

Definitely. When there's no restriction on how high a degree the polynomial can be, you can draw infinitely many polynomials through the points.

The *Lagrange interpolating polynomial* will specifically give us an interpolating polynomial of degree at most $n - 1$. For $n = 3$, the Lagrange interpolating polynomial is:

$$P_2(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_2)(x_3 - x_1)}$$

Example 2. Check that $P_2(x_1) = y_1, P_2(x_2) = y_2$, and $P_2(x_3) = y_3$.

When you substitute a particular x_i, all the terms except for the one scaled by y_i are zero by design. Then the remaining fraction terms are identical in numerator and denominator and thus simplify to 1, leaving us with y_i.

Example 3. Extend the pattern observed to write down the formula for a Langange interpolating polynomial for $n = 4$.

$$\begin{aligned}P_2(x) = y_1\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + y_2\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\ + y_3\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_2)(x_3-x_1)(x_3-x_4)} + y_4\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}.\end{aligned}$$

Formally, the Lagrange interpolating polynomial for n points is

$$P_{n-1}(x) = \sum y_i L_i(x) \text{ where } L_i(x) = \prod_{k\neq i}\frac{x-x_k}{x_i-x_k}.$$

Example 4. Find an interpolating polynomial for $(1,1)$, $(2,2)$, and $(3,5)$.

$$\begin{aligned}P_2(x) &= 1\frac{(x-2)(x-3)}{(1-2)(1-3)} + 2\frac{(x-1)(x-3)}{(2-1)(2-3)} + 5\frac{(x-1)(x-2)}{(3-1)(3-2)} \\ &= \frac{(x-2)(x-3)}{(2)} - 2(x-1)(x-3) + 5\frac{(x-1)(x-2)}{2} \\ &= x^2 - 2x + 2 = (x-1)^2 + 1\end{aligned}$$

The last form just makes it easy to go back and check the sketch of a parabola that we did at the start.

Theorem 4. *Let $(x_1, y_1), \ldots, (x_n, y_n)$ be n points in the plane with distinct x_i. Then there exists one and only one polynomial P of degree $n-1$ or less that satisfies $P(x_i) = y_i$ for $i = 1, \ldots, n$.*

In short: we have existence and uniqueness. To complete this proof, we need the following.

Theorem 5. *The Fundamental Theorem of Algebra A degree d polynomial can have at most d zeros, unless it is the identically zero polynomial.*

Proof. The existence comes from the formula, as given any n points we can generate a Lagrange interpolating polynomial. To show that there cannot be another, suppose to the contrary that we have two interpolating polynomials $P(x)$ and $Q(x)$ with degree at most $n-1$. Define a third polynomial $R(x) = P(x) - Q(x)$. Then the maximum degree of R is also $n-1$.

Then $R(x_1) = R(x_2) = \ldots = R(x_n) = 0$. So R has n distinct zeros. By the Fundamental Theorem of Algebra, R must be identically zero, and so $P(x) = Q(x)$. □

Example 5. Suppose you are given 6 points with distinct x coordinates that lie on the curve $y = x^2$. Does there exist a degree 4 interpolating polynomial for these points?

No, there does not. If such a polynomial existed, there would be both a degree 2 and a degree 4 interpolating polynomial for the 6 points, violating the uniqueness part of the theorem.

Follow-up: What is the maximum number of points from the set of 6 for which there exists a degree 4 interpolating polynomial?

Answer: 4. If $n = 4$, then the theorem applies up to degree 3. So there's a unique polynomial that is degree 1, 2, or 3, but there could be infinitely many others of degree 4 and higher.

Newton's Divided Differences

The issue with the formula for the Lagrange interpolating polynomial is its complexity. It was not easy to even write down...

And in practice, that formula is seldom used for that very reason. There exist less computationally complex versions of the formula that give the same answer, but are less obviously correct.

The process goes like this: Start with your data points in a table. Demonstrating the process with $n = 4$:

$$\begin{array}{c|c}
x_1 & y_1 \\
 & \\
x_2 & y_2 \\
 & \\
x_3 & y_3 \\
 & \\
x_4 & y_4
\end{array}$$

A divided difference is the change in y's over change in x's between lines of the table.

$$\begin{array}{c|cc}
x_1 & y_1 & \\
 & & \dfrac{y_2 - y_1}{x_2 - x_1} \\
x_2 & y_2 & \\
 & & \dfrac{y_3 - y_2}{x_3 - x_2} \\
x_3 & y_3 & \\
 & & \dfrac{y_4 - y_3}{x_4 - x_3} \\
x_4 & y_4 &
\end{array}$$

For simplicity, I am now going to call these new numbers a's. Then we create the next line by repeating the process, difference in a's over difference in x's. In the denominator, we're using the largest and smallest x's that have been used thus far.

$$
\begin{array}{c|cccc}
x_1 & y_1 & & \\
 & & a_1 & \\
 & & & \dfrac{a_2 - a_1}{x_3 - x_1} \\
x_2 & y_2 & & \\
 & & a_2 & \\
 & & & \dfrac{a_3 - a_2}{x_4 - x_2} \\
x_3 & y_3 & & \\
 & & a_3 & \\
x_4 & y_4 & &
\end{array}
$$

Calling the new fractions b's, we repeat the process one last time:

$$
\begin{array}{c|cccc}
x_1 & y_1 & & & \\
 & & a_1 & & \\
x_2 & y_2 & & b_1 & \\
 & & a_2 & & \dfrac{b_2 - b_1}{x_4 - x_1} \\
x_3 & y_3 & & b_2 & \\
 & & a_3 & & \\
x_4 & y_4 & & &
\end{array}
$$

$$
\begin{array}{c|cccc}
x_1 & y_1 & & & \\
 & & a_1 & & \\
x_2 & y_2 & & b_1 & \\
 & & a_2 & & c_1 \\
x_3 & y_3 & & b_2 & \\
 & & a_3 & & \\
x_4 & y_4 & & &
\end{array}
$$

Then the coefficients of the polynomial are the top edge of the triangle. Each coefficient is multiplied by the terms $(x - x_i)$, with one term for each row you've gone down in the table. So $P_3(x) = y_1 + a_1(x - x_1) + b_1(x - x_1)(x - x_2) + c_1(x - x_1)(x - x_2)(x - x_3)$.

Example 6. Find an interpolating polynomial for $(-2, -7), (-1, 0), (0, 1), (1, 2)$.

$$
\begin{array}{c|cc}
-2 & -7 & \\
 & & \dfrac{0-(-7)}{-1-0} \\
-1 & 0 & \\
 & & \dfrac{1-0}{0-(-1)} \\
0 & 1 & \\
 & & \dfrac{2-1}{1-0} \\
1 & 2 & \\
 & & \dfrac{28-2}{3-1} \\
3 & 28 &
\end{array}
$$

$$
\begin{array}{c|ccccc}
-2 & -7 & & & & \\
 & & 7 & & & \\
-1 & 0 & & \text{-3} & & \\
 & & 1 & & 1 & \\
0 & 1 & & 0 & & 0 \\
 & & 1 & & 1 & \\
1 & 2 & & 4 & & \\
 & & 13 & & & \\
3 & 28 & & & &
\end{array}
$$

So the interpolating polynomial is $-7+7(x+2)-3(x+2)(x+1)+(x+2)(x+1)(x)$, which is degree 3. For ourselves, we can simplify:

$$
\begin{aligned}
P_3(x) &= -7+7x+14-3(x^2+3x+2)+(x^2+3x+2)x \\
&= 7+7x-3x^2-9x-6+x^3+3x^2+2x \\
&= 1+x^3
\end{aligned}
$$

Example 7. Add in the point $(3,28)$. How much of the previous work can be reused?

However, the computer will not do this. Instead, it is naturally in nested form.

Example 8. Count the number of operations required to evaluate $f(x)=2x^4+3x^3-3x^2+5x-1$ in three ways.

- As $2(x)(x)(x)(x)+3(x)(x)(x)-3(x)(x)+5(x)-1$.

  10 multiplications + 4 additions = 14 operations.

- First, save $x_2 = x*x$, then save $x_3 = x*x_2$, and save $x_4 = x*x_3$. Then evaluate $2x_4+3x_3-3x_2+5x-1$.

  3 pre-multiplications + 4 multiplications + 4 additions = 11 operations.

- As $-1 + x(5 + x(-3 + x(3 + x(2))))$.
  1 mult then 1 add, 4 times = 8 operations.

The final way is called *nested multiplication* or *Horner's method*.