Null Space

The *null space* of an $m \times n$ matrix A, written Nul A, is the set of all solutions of $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Theorem 2. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.*

1. Prove Theorem 2 by verifying the three properties of a subspace.

2. Is $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ in the null space of $A = \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 0 & 2 & 3 \end{bmatrix}$?

3. Is Nul A ever an empty set?

4. Let H be the set of vectors in $\mathbb{R}^4$ where the third element is the sum of the first two and the last element is twice the second element. Show that H is a subspace by showing it is the set of solutions of a system of homogeneous linear equations.

Column Space

The *column space* of an $m \times n$ matrix A, written Col A, is the set of all linear combinations of the columns of A.

Theorem 3. *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.*

1. Let $A = \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 0 & 2 & 3 \end{bmatrix}$.

   (a) col A is a subspace of $\mathbb{R}^k$ for what k?

   (b) How do you know if Col A is all of $\mathbb{R}^k$?

   (c) State three unique nonzero vectors in Col A without doing any work- just look at A.

2. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

   (a) Is $\begin{bmatrix} 2 \\ 2 \\ -10 \end{bmatrix}$ in Nul A?

   (b) Is $\begin{bmatrix} 2 \\ 2 \\ -10 \end{bmatrix}$ in Col A?

A linear transformation T can be defined as going from any vector space to a vector space. That is, $T : V \to W$, satisfying the same two properties of linear transformations. The *kernel* of such a transformation is the set containing all vectors $\mathbf{u} \in V$ such that $T(\mathbf{u}) = \mathbf{0}$. The *range* of T is the set containing all vectors $\mathbf{v} \in W$ where there exists $\mathbf{u} \in V$ such that $T(\mathbf{u}) = \mathbf{v}$.

3. Suppose T is given as a matrix transformation, $T(\mathbf{x}) = A\mathbf{x}$. How do the kernel and range of T relate to the null space and column space of A?