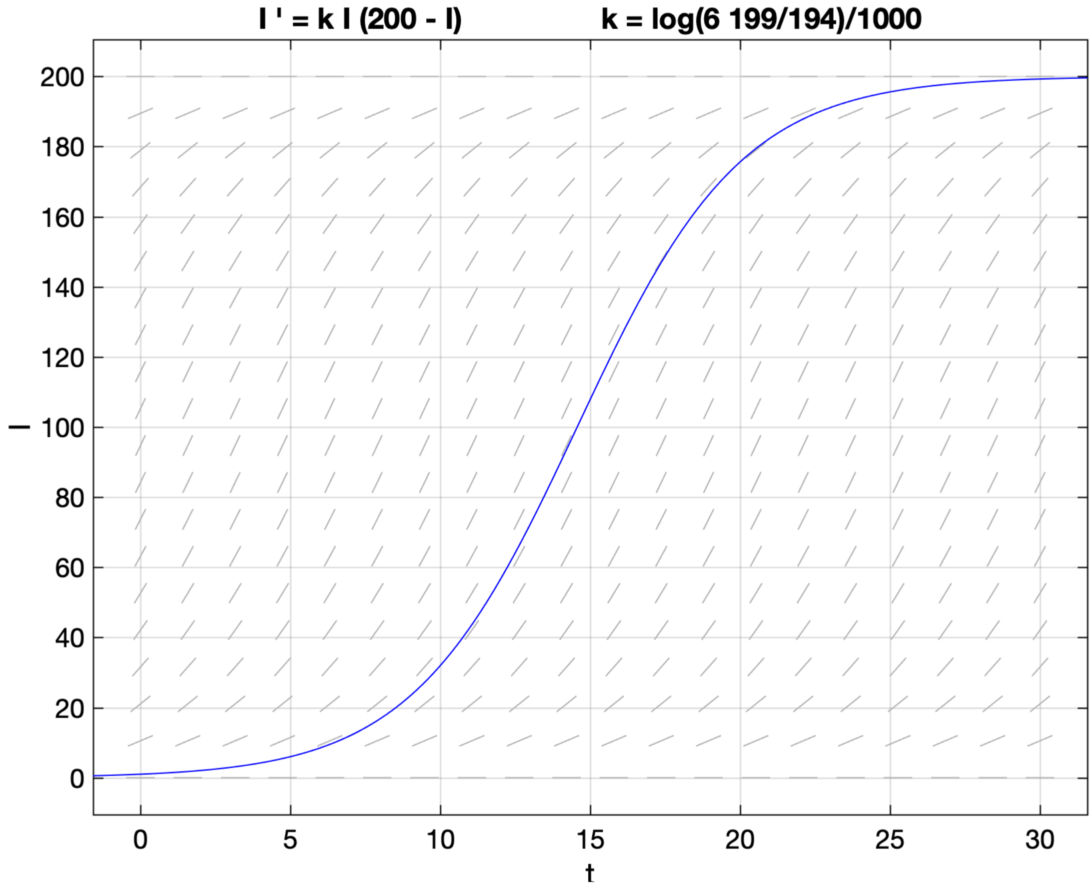
Initial Value Problems

An initial value problem is generally of the form

$$\begin{cases} y' = f(t, y) \\ y(0) = y_0 \end{cases} .$$

The first numerical method for solving initial value problems is one you've likely seen before: Euler's Method. Typically this is introduced by looking at the slope field. An approximate solution is produced by proceeding along the slope vectors by step length h, finding a new slope, and repeating.


I ' = k I (200 - I)
k = log(6 199/194)/1000
200
180
160
140
120
100
80
60
40
20
0
I
0
5
10
15
20
25
30
t

Definition 1 (Euler's Method)**.**

$$w_0 = y_0$$
$$w_{i+1} = w_i + hf(t_i, w_i)$$

Since the slope is $f(t_i, w_i)$, and $t_i = a + hi$, we can again find a list of approximate values $w_1, \ldots, w_n$ for y. Note that if we think of this in terms of finite difference formulas, what we have is

$$\frac{(w_{i+1} - w_i)}{h} = f(t_i, w_i)$$

and solving for w_{i+1} immediately gives Euler's Method.

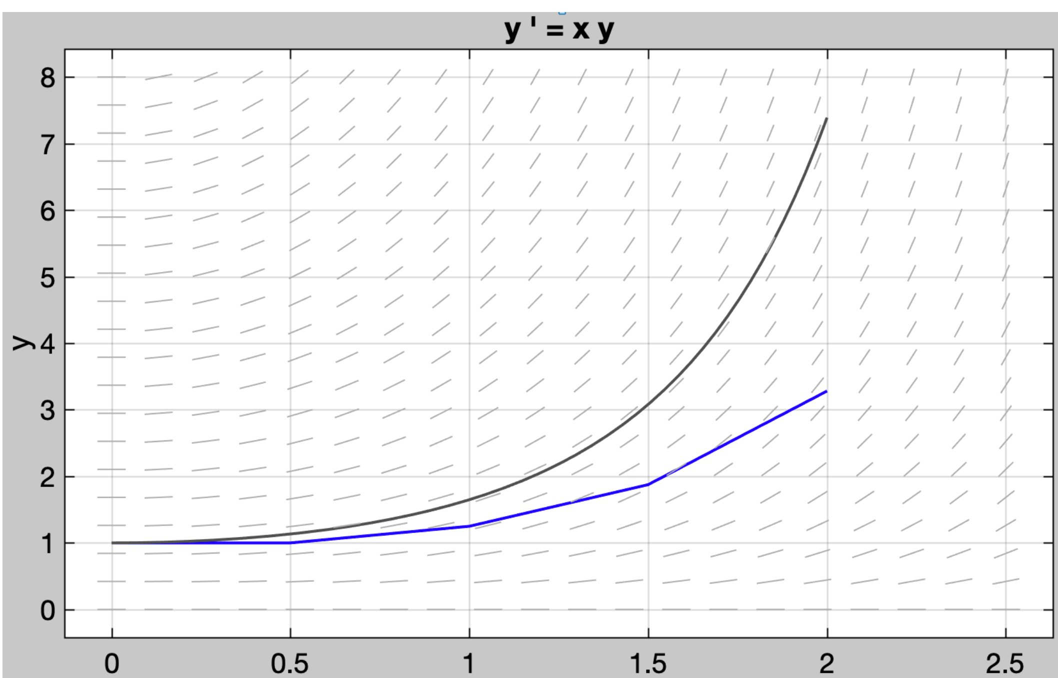
Example 1. Apply Euler's Method with $h = \frac{1}{2}$ to approximate $y(2)$ for the IVP.

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

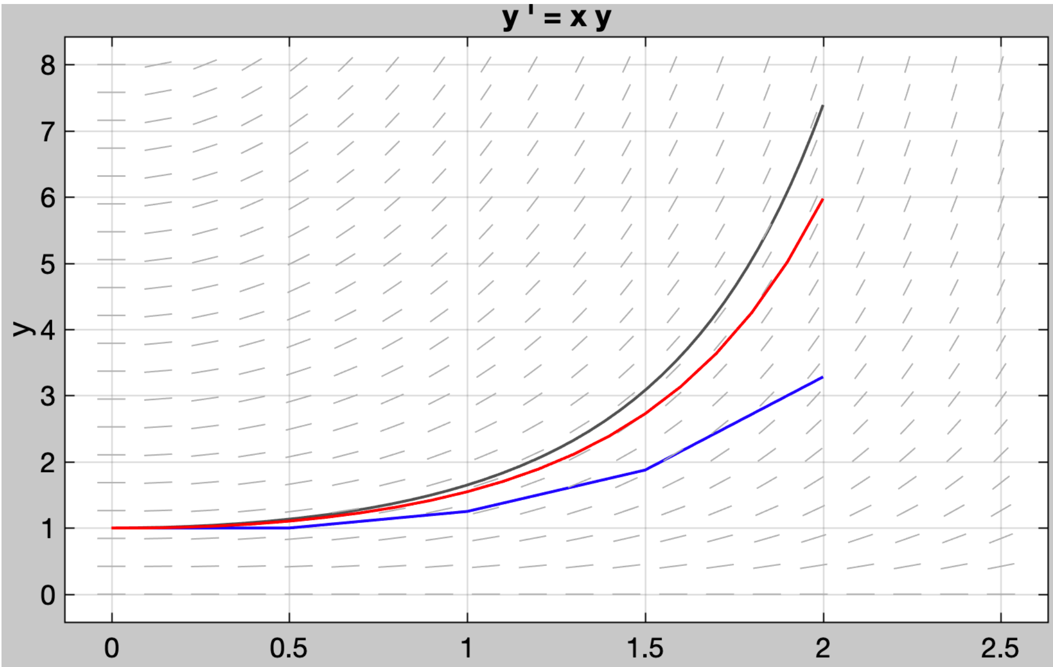
Example 2. Check that $y = e^{\frac{t^2}{2}}$ is the exact solution to the previous IVP.

Definition 2. The *global truncation error* at step i is $g_i = |w_i - y_i|$.

Example 3. Compute the global truncation error at $t = 2$.



Perhaps it's just that the step size was too large. Let's see what happens with $h = 0.1$.



Better. But still off by more than 1!

Euler's Method is honestly too simple and error prone to be used in practice, but the process does highlight the issues with IVP solvers in general.

Definition 3. Let z_{i+1} be the (exact) solution to the one-step initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_i) = w_i \end{cases} .$$

Then the *local truncation error* is the one-step error $e_{i+1} = |w_{i+1} - z_{i+1}|$.

Draw relationship between y, w, and z:

Example 4. Find a formula for the local truncation error for Euler's Method.

Existence and Uniqueness

Definition 4. A function $f(t, y)$ is *Lipschitz continuous* in the variable y on the rectangle $S = [a, b] \times [\alpha, \beta]$ if there exists a constant L (called the *Lipschitz constant*) satisfying

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

for each (t, y_1), (t, y_2) in S.

Example 5. Show that $f(t, y) = ty$ is Lipschitz continuous on $[0, b] \times (-\infty, \infty)$.

Question 5. What does $\frac{|f(t,y_1)-f(t,y_2)|}{|y_1-y_2|}$ remind you of?

Theorem 6. *Assume that $f(t, y)$ is Lipschitz continuous in the variable y on the set $[a, b] \times [\alpha, \beta]$ and that $\alpha < y_a < \beta$. Then there exists c between a and b such that the initial value problem*

$$\begin{cases} y' = f(t, y) \\ y(a) = y_a \\ t \in [a, c] \end{cases}$$

has exactly one solution $y(t)$. Moreover, if f is Lipschitz on $[a, b] \times (-\infty, \infty)$, then there exists exactly one solution on $[a, b]$.

Example 6. Does the initial value problem

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

have a unique solution for t in $[0, b]$?

Example 7. On which intervals $[0, c]$ does the initial value problem have a unique solution?

$$\begin{cases} y' = y^2 \\ y(0) = 1 \\ t \in [0, 2] \end{cases}$$

Theorem 7. *Assume that $f(t, y)$ is Lipschitz in the variable y on the set $S = [a, b] \times [\alpha, \beta]$. If $Y(t)$ and $Z(t)$ are solutions in S of the differential equation*

$$y' = f(t, y)$$

with initial conditions y_a and z_a respectively, then

$$|Y(t) - Z(t)| \leq e^{L(t-a)}|y_a - z_a|.$$

Question 8. What was the definition of a condition number?

We are ready to return to the global truncation error. We left off with $|z_2 - y_2|$.

Theorem 9. *Assume that $f(t, y)$ has a Lipschitz constant L for the variable y and that the value y_i of the solution of the initial value problem at t_i is approximated by w_i from a one-step ODE solver with local truncation error $e_i \leq Ch^{k+1}$ for some constant C and $k \geq 0$. Then for each $a < t_i < b$, the solver has global truncation error*

$$g_i = |w_i - y_i| \leq \frac{Ch^k}{L}(e^{L(t_i - a)} - 1).$$

Definition 10. If an ODE solver satisfies Theorem 9 as $h \to 0$, then we say the solver has *order k*.

Example 8. Find an error bound for Euler's Method applied to

$$\begin{cases} y' = ty \\ y(0) = 1 \\ t \in [0, 2] \end{cases}.$$