Lagrangian Interpolation

The mathematical problem that we call interpolation begins with a set of points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$. The goal is to find a function that evaluates to the same values at those points. It is essentially a game of connect the dots: Any function that evaluates to the same values as the ones in the known points is a valid interpolating function.

Definition 1. The function $y = P(x)$ *interpolates* the data points $(x_1, y_1), \ldots (x_n, y_n)$ if $P(x_i) = y_i$ for each $1 \leq i \leq n$.

Interpolation is the opposite of evaluation: Evaluation is computing points given a curve, and interpolation is computing a curve given some points.

Example 1. Draw three different interpolating functions through the points $(1, 1)$, $(2, 2)$, and $(3, 5)$.

Question 2. Why might you prefer a polynomial interpolating function over a piecewise-defined function?

Polynomial Interpolating Functions

Question 3. Given n points $(x_1, y_1), \ldots, (x_n, y_n)$, does there exist a polynomial $p(x)$ where for all x_i, $i = 1, \ldots, n$, $p(x_i) = y_i$?

The *Lagrange interpolating polynomial* will specifically give us an interpolating polynomial of degree at most $n-1$. For $n=3$, the Lagrange interpolating polynomial is:

$$P_2(x) = y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_2)(x_3-x_1)}.$$

Example 2. Check that $P_2(x_1) = y_1, P_2(x_2) = y_2$, and $P_2(x_3) = y_3$.

Example 3. Extend the pattern observed to write down the formula for a Langange interpolating polynomial for $n = 4$.

Formally, the Lagrange interpolating polynomial for n points is

$$P_{n-1}(x) = \sum y_i L_i(x) \text{ where } L_i(x) = \prod_{k \neq i} \frac{x - x_k}{x_i - x_k}.$$

Example 4. Find an interpolating polynomial for $(1, 1)$, $(2, 2)$, and $(3, 5)$.

Theorem 4. *Let $(x_1, y_1), \ldots, (x_n, y_n)$ be n points in the plane with distinct x_i. Then there exists one and only one polynomial P of degree $n-1$ or less that satisfies $P(x_i) = y_i$ for $i = 1, \ldots, n$.*

In short: we have existence and uniqueness. To complete this proof, we need the following.

Example 5. What is the maximum number of roots (real or complex) that a degree n polynomial can have?

Proof of the Main Theorem of Polynomial Interpolation.

Example 6. Suppose you are given 6 points with distinct x coordinates that lie on the curve $y = x^2$. Does there exist a degree 4 interpolating polynomial for these points?

Newton's Divided Differences

The issue with the formula for the Lagrange interpolating polynomial is its complexity. It was not easy to even write down...

And in practice, that formula is seldom used for that very reason. There exist less computationally complex versions of the formula that give the same answer, but are less obviously correct.

The process goes like this: Start with your data points in a table. Demonstrating the process with $n = 4$:

$$\begin{array}{c|c} x_1 & y_1 \\ \\ x_2 & y_2 \\ \\ x_3 & y_3 \\ \\ x_4 & y_4 \end{array}$$

A divided difference is the change in y's over change in x's between lines of the table.

$$\begin{array}{c|cc} x_1 & y_1 & \\ & & \dfrac{y_2 - y_1}{x_2 - x_1} \\ x_2 & y_2 & \\ & & \dfrac{y_3 - y_2}{x_3 - x_2} \\ x_3 & y_3 & \\ & & \dfrac{y_4 - y_3}{x_4 - x_3} \\ x_4 & y_4 & \end{array}$$

For simplicity, I am now going to call these new numbers a's. Then we create the next line by repeating the process, difference in a's over difference in x's. In the denominator, we're using the largest and smallest x's that have been used thus far.

$$\begin{array}{c|ccc} x_1 & y_1 & & \\ & & a_1 & \\ & & & \dfrac{a_2 - a_1}{x_3 - x_1} \\ x_2 & y_2 & & \\ & & a_2 & \\ & & & \dfrac{a_3 - a_2}{x_4 - x_2} \\ x_3 & y_3 & & \\ & & a_3 & \\ x_4 & y_4 & & \end{array}$$

Calling the new fractions b's, we repeat the process one last time:

$$
\begin{array}{c|cccc}
x_1 & y_1 & & & \\
 & & a_1 & & \\
x_2 & y_2 & & b_1 & \\
 & & a_2 & & \dfrac{b_2 - b_1}{x_4 - x_1} \\
x_3 & y_3 & & b_2 & \\
 & & a_3 & & \\
x_4 & y_4 & & &
\end{array}
$$

$$
\begin{array}{c|cccc}
x_1 & y_1 & & & \\
 & & a_1 & & \\
x_2 & y_2 & & b_1 & \\
 & & a_2 & & c_1 \\
x_3 & y_3 & & b_2 & \\
 & & a_3 & & \\
x_4 & y_4 & & &
\end{array}
$$

Then the coefficients of the polynomial are the top edge of the triangle. Each coefficient is multiplied by the terms $(x - x_i)$, with one term for each row you've gone down in the table. So $P_3(x) = y_1 + a_1(x - x_1) + b_1(x - x_1)(x - x_2) + c_1(x - x_1)(x - x_2)(x - x_3)$.

Example 7. Find an interpolating polynomial for $(-2,-7),(-1,0),(0,1),(1,2)$.

Example 8. Include an additional point $(3,28)$. How much of the previous work can be reused?

The computer will not simplify since the result is naturally in nested form.

Example 9. Count the number of operations required to evaluate $f(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$ in three ways.

- As $2(x)(x)(x)(x) + 3(x)(x)(x) - 3(x)(x) + 5(x) - 1$.
- First, save $x_2 = x * x$, then save $x_3 = x * x_2$, and save $x_4 = x * x_3$. Then evaluate $2x_4 + 3x_3 - 3x_2 + 5x - 1$.
- As $-1 + x(5 + x(-3 + x(3 + x(2))))$.

The final way is called *nested multiplication* or *Horner's method.*