Runge-Kutta Methods

The Euler Method and the Explicit Trapezoid Method are in the family of solvers called Runge-Kutta Methods. Runge-Kutta Methods of all orders exist, and often more than one of a particular order. To develop the overall approach of a Runge-Kutta method, we begin with examining another method within this family of solvers: the Midpoint Method.

Definition 1 (Midpoint Method)**.**

$$w_0 = y_0$$
$$w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i))$$

Example 1. Apply the midpoint method with $h = \frac{1}{2}$ to approximate $y(2)$

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

Table 1: h = 0.5

t	ye	yt	ytay	ym	y
0	1	1	1	1	1
0.5	1	1.125	1.125	1.125	1.13314845
1	1.25	1.6171875	1.58203125	1.59960938	1.64872127
1.5	1.88	2.93115234	2.76855469	2.8493042	3.08021685
2	3.28	6.59509277	5.96969604	6.27737331	7.3890561

Local Truncation error:

Each function evaluation of the right-hand side of the differential equation is called a *stage* of the method. The Trapezoid Method and the Runge-Kutta Methods are also both two-stage methods. They both have the form:

$$w_{i+1} = w_i + h\left(1 - \frac{1}{2a}\right) f(t_i, w_i) + \frac{h}{2a} f(t_i + \alpha h, w_i + \alpha h f(t_i, w_i))$$

$\alpha = \frac{2}{3}$ is called the Ralston Method.

Runge-Kutta

The family of (explicit) Runge-Kutta Methods are given by

$$y_{i+1} = y_i + h \sum_{j=1}^{s} b_j k_j$$

where the k_j's are given by evaluations of f:

$$\begin{aligned}
k_1 &= f(t_i, y_i) \\
k_2 &= f(t_i + c_2 h, y_i + h(a_{21} k_1)) \\
k_3 &= f(t_i + c_3 h, y_i + h(a_{31} k_1 + a_{32} k_2)) \\
&\vdots \\
k_s &= f(t_i + c_s h, y_i + h(a_{s1} k_1 + a_{s2} k_2 + \vdots + a_{s,s-1} k_{s-1}))
\end{aligned}$$

The number of stages is s, and these methods are consistent only if $\sum_{j=1}^{s} b_j = 1$.

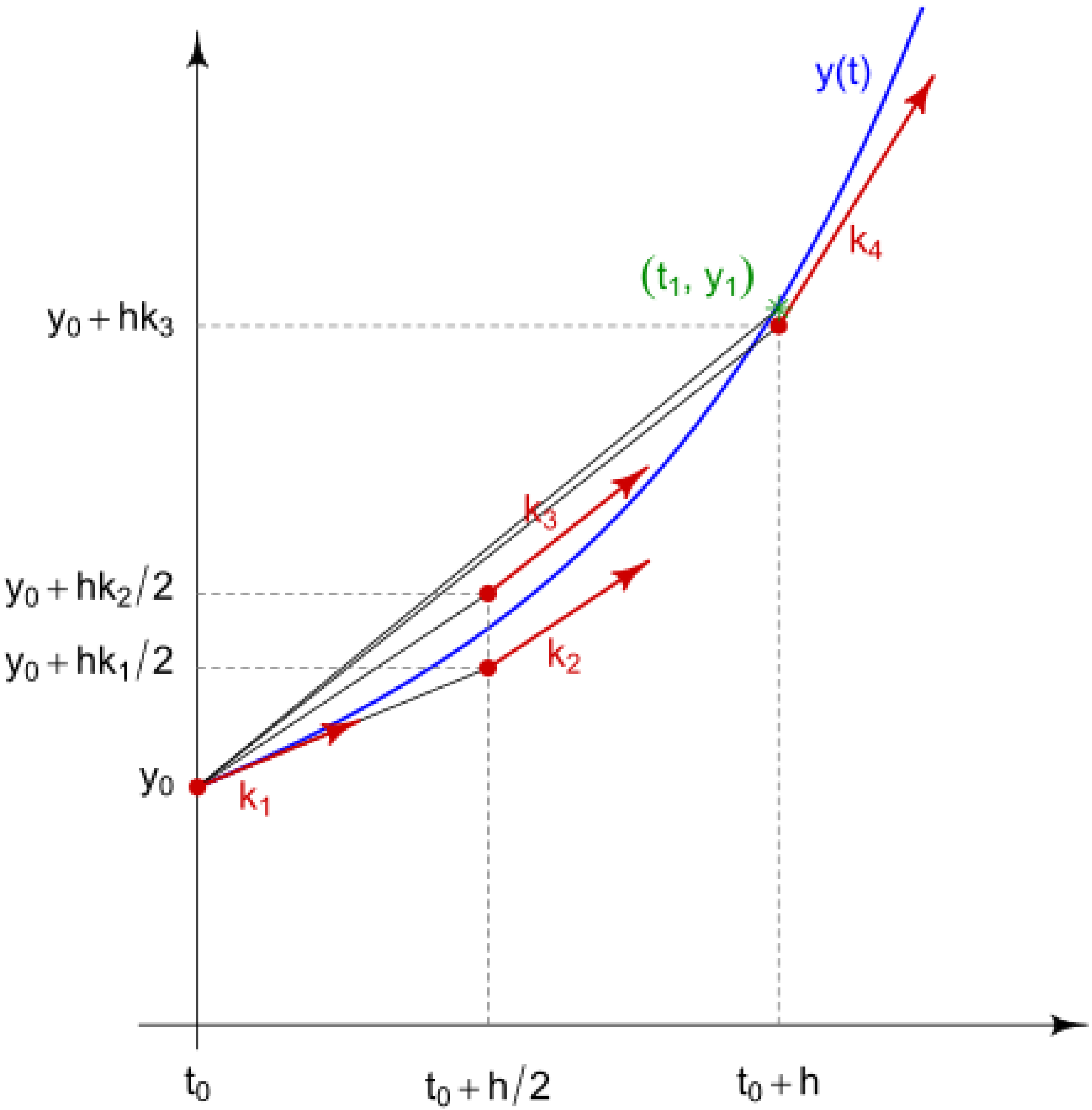
Definition 2 (Runge-Kutta Method of Order 4)**.**

$$w_0 = y_0$$

$$w_{i+1} = w_i + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

where

$$\begin{aligned}
s_1 &= f(t_i, w_i) \\
s_2 &= f(t_i + \frac{h}{2}, w_i + \frac{h}{2} s_1) \\
s_3 &= f(t_i + \frac{h}{2}, w_i + \frac{h}{2} s_2) \\
s_4 &= f(t_i + h, w_i + h s_3)
\end{aligned}$$



y(t)
k_4
(t_1, y_1)
$y_0 + hk_3$
k_3
$y_0 + hk_2/2$
$y_0 + hk_1/2$
k_2
y_0
k_1
t_0
$t_0 + h/2$
$t_0 + h$

Example 2. Apply RK4 with $h = 0.5$ to approximate $y(2)$

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

Table 2: h=0.5

t	**ye**	**yt**	**ytay**	**ym**	**yrk4**	**y**
0	1	1	1	1	1	1
0.5	1	1.125	1.125	1.125	1.13313802	1.13314845
1	1.25	1.6171875	1.58203125	1.59960938	1.6485277	1.64872127
1.5	1.88	2.93115234	2.76855469	2.8493042	3.07797616	3.08021685
2	3.28	6.59509277	5.96969604	6.27737331	7.36680329	7.3890561

Table 3: h = 0.1

t	**ye**	**yt**	**ytay**	**ym**	**yrk4**	**y**
0.0	1	1	1	1	1	1
0.1	1	1.005	1.005	1.005	1.00501252	1.00501252
0.2	1.01	1.0201755	1.02012525	1.02015038	1.02020134	1.02020134
0.3	1.0302	1.04598594	1.04583241	1.04590917	1.04602786	1.04602786
0.4	1.061106	1.08322304	1.08290717	1.0830651	1.08328706	1.08328707
0.5	1.10355024	1.1330513	1.13250431	1.13277778	1.13314845	1.13314845
0.6	1.15872775	1.1970687	1.19620768	1.19663813	1.19721735	1.19721736
0.7	1.22825142	1.27739201	1.27611435	1.27675305	1.27762128	1.27762131
0.8	1.31422902	1.37677311	1.37494941	1.37586101	1.37712769	1.37712776
0.9	1.41936734	1.4987552	1.49621995	1.49748712	1.49930236	1.4993025
1.0	1.5471104	1.64788135	1.64442053	1.64615016	1.64872101	1.64872127
1.1	1.70182144	1.82997223	1.82530679	1.82763821	1.83125172	1.83125221
1.2	1.8890218	2.05249686	2.04626018	2.04937642	2.05443232	2.05443321
1.3	2.11570441	2.32506844	2.31677578	2.32091879	2.32797624	2.32797781
1.4	2.39074598	2.6601108	2.64911726	2.65460889	2.66445351	2.66445624
1.5	2.72545042	3.07375803	3.05920061	3.06647146	3.08021217	3.08021685
1.6	3.13426799	3.58707562	3.56779272	3.57742227	3.59663183	3.59663973
1.7	3.63575086	4.22772733	4.20214626	4.21491892	4.24183896	4.24185214
1.8	4.25382851	5.03226384	4.99824287	5.01522665	5.0530685	5.05309032
1.9	5.01951764	6.04928436	6.00388934	6.0265471	6.0799356	6.07997145
2.0	5.973226	7.34383122	7.28301796	7.31336557	7.38899753	7.3890561