Partitioned Matrices

When we think of a matrix as a list of column vectors, we are essentially turning each column of the matrix into its own *part* of a *partition* of A.

Example 1. The four columns of A are a partition of A into four parts.

$$A = \left[\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & -1 \end{array}\right]$$

□

Any use of vertical and horizontal lines will partition a matrix into *submatrices.* A submatrix is labeled in the same way as an element of a matrix, meaning, the subscripts denote the row and column number. Remember: lower case letters a are scalars, bold letters $\mathbf{a}$ or with a line over it $\bar{a}$ are vectors, and capital letters A are matrices.

Example 2. The matrix A is partitioned into 6 submatrices.

$$A = \left[\begin{array}{ccc|cc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ -1 & -2 & 1 & 2 & -1 & -2 \\ \hline 0 & 1 & 3 & 5 & 8 & 9 \end{array}\right]$$

They are labeled:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Submatrix A_{12} is the 2×2 matrix $\begin{bmatrix} 4 & 5 \\ 2 & -1 \end{bmatrix}$

□

1. Partition the following matrix into 9 parts where A_{22} is 3×3 and A_{32} is 2×3.

$$\begin{bmatrix} 8 & 4 & 2 & 1 & 0 \\ -1 & 3 & -7 & 9 & e \\ 0 & 0 & 1 & 1 & \pi \\ \ln(2) & 0 & -1,000 & 0 & 4 \\ 1 & 2 & 4 & 8 & 16 \\ \sqrt{2} & \sqrt{3} & \sqrt{5} & \sqrt{7} & \sqrt{11} \end{bmatrix}$$

2. Suppose I_6 is partitioned into four square submatrices. Fill in the missing names of the submatrices:

$$I_6 = \left[\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right]$$

Block addition and block multiplication work very much the same way as regular matrix addition and multiplication. For instance, assuming that the matrices A and B are partitioned in the same manner (same size submatrices), we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

3. Let $A = \begin{bmatrix} I & 0 \\ A_{21} & I \end{bmatrix}$ and $B = \begin{bmatrix} I & 0 \\ B_{21} & I \end{bmatrix}$.
   (a) Find $A + B$ and AB.

   (b) Suppose $AB = I$. What is B's relationship with A? What must B_{21} be in terms of the submatrices of A?

LU factorizations

The LU factorization is primarily a way of storing the information of a row reduction. You may recall the airplane example of the start of Chapter 2, in which the same system (same matrix A) is solved thousands of times for different right hand sides ($\mathbf{b}$). This could be done by computing A^{-1} once, and then $A^{-1}\mathbf{b}$ for each of the right-hand sides. However, in practice, the LU factorization is more stable. For more information on why LU is preferable to A^{-1} on a computer, see the notes of page 129.

L and U are two matrices so named because they are Lower and Upper triangular. A lower triangular matrix is one that has all zeros above the main diagonal. An upper triangular matrix is one that has all zeros below the main diagonal. Formally, an LU factorization of an $m \times n$ matrix A is

$$A = LU$$

where L is an $m \times m$ unit lower triangular matrix and U is an $m \times n$ upper triangular matrix. *Unit* means that L has only ones on its diagonal.

1. The matrix $A = \begin{bmatrix} 6 & 4 & 2 & 4 \\ -3 & -12 & 1 & 1 \\ 0 & 5 & 1 & 1 \end{bmatrix}$ has LU factorization:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 6 & 4 & 2 & 4 \\ 0 & -10 & 2 & 3 \\ 0 & 0 & 2 & 2.5 \end{bmatrix}$$

Confirm that L times U equals A.

The LU factorization lets us write a system $A\mathbf{x} = \mathbf{b}$ as $LU\mathbf{x} = \mathbf{b}$, and equivalently, $L(U\mathbf{x}) = \mathbf{b}$. This can be solved in two stages, where the row reduction steps are relatively simple due to the triangular structure of L and U. Let $\mathbf{y}$ represent $U\mathbf{x}$.

- First solve $L(\mathbf{y}) = \mathbf{b}$
- Then solve $U\mathbf{x} = \mathbf{y}$

2. Let $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 6 & 4 & 2 \\ 0 & -10 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$.

   (a) First, write $L\mathbf{y} = \mathbf{b}$ as an augmented matrix. Then row reduce to reduced echelon form.

   (b) Now, write $U\mathbf{x} = \mathbf{y}$ as an augmented matrix, and row reduce to reduced echelon form.

Computing an LU factorization by hand requires limiting our options on row operations. You may not:

1. Swap rows. Yes, this is a big limitation.
2. Scale the row to which the zero is being introduced.
3. Choose any number for scaling that does not introduce a 0.

This makes the process less nice for numbers (you won't be able to avoid fractions), but also more straightforward. For each column, the top number, excluding those in a row that already contains a pivot, must be chosen as your pivot. All row operations should be written as $R_m - (c)R_p$.

Example 3. For $A = \begin{bmatrix} 6 & 4 & 2 & 4 \\ -3 & -12 & 1 & 1 \\ 0 & 5 & 1 & 1 \end{bmatrix}$, the factorization goes like this.

1. Begin with $L = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$ and $U = \begin{bmatrix} & & & \\ 0 & & & \\ 0 & 0 & & \end{bmatrix}$.
2. The first pivot must be the first element of the matrix, which is 6.
3. This pivot must be used to introduce a 0 to the first element of the second row. This is row operation $R_2 - (-\frac{1}{2})R_1$. The scalar in front of the pivot row is the element that goes into L at that location.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ & & 1 \end{bmatrix} \qquad U = \begin{bmatrix} & & & \\ 0 & & & \\ 0 & 0 & & \end{bmatrix}$$

4. Repeat for all rows below the pivot row. Since we do not need to introduce a 0 to the last row, we are effectively performing $R_3 - 0R_1$. We now have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & & 1 \end{bmatrix} \qquad U = \begin{bmatrix} & & & \\ 0 & & & \\ 0 & 0 & & \end{bmatrix}$$

5. The row that remains unchanged is copied as-is into the same row of U.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & 4 & 2 & 4 \\ 0 & & & \\ 0 & 0 & & \end{bmatrix}$$

6. A is now row equivalent to $\begin{bmatrix} 6 & 4 & 2 & 4 \\ 0 & -10 & 2 & 3 \\ 0 & 5 & 1 & 1 \end{bmatrix}$. The process begins again; -10 must be the pivot.
7. The next row operation is $R_3 - (-\frac{1}{2})R_2$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & 4 & 2 & 4 \\ 0 & & & \\ 0 & 0 & & \end{bmatrix}$$

8. The second row of the row reduced A is remaining unchanged, so copy it into U.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & 4 & 2 & 4 \\ 0 & -10 & 2 & 3 \\ 0 & 0 & & \end{bmatrix}$$

9. Update A. $A = \begin{bmatrix} 6 & 4 & 2 & 4 \\ 0 & -10 & 2 & 3 \\ 0 & 0 & 2 & 2.5 \end{bmatrix}$
10. Copy the last unchanged row into U, completing the factorization.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & 4 & 2 & 4 \\ 0 & -10 & 2 & 3 \\ 0 & 0 & 2 & 2.5 \end{bmatrix}$$

$\square$

Because of these limitations, not every matrix even has an LU factorization.

Example 4. $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ does not have an LU factorization. The process immediately fails since the pivot is 0, and we have a nonzero below it.

$\square$

3. Compute the LU factorization of $A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & -6 & 6 \\ 2 & 9 & 8 \\ 1 & 3 & 0 \end{bmatrix}$