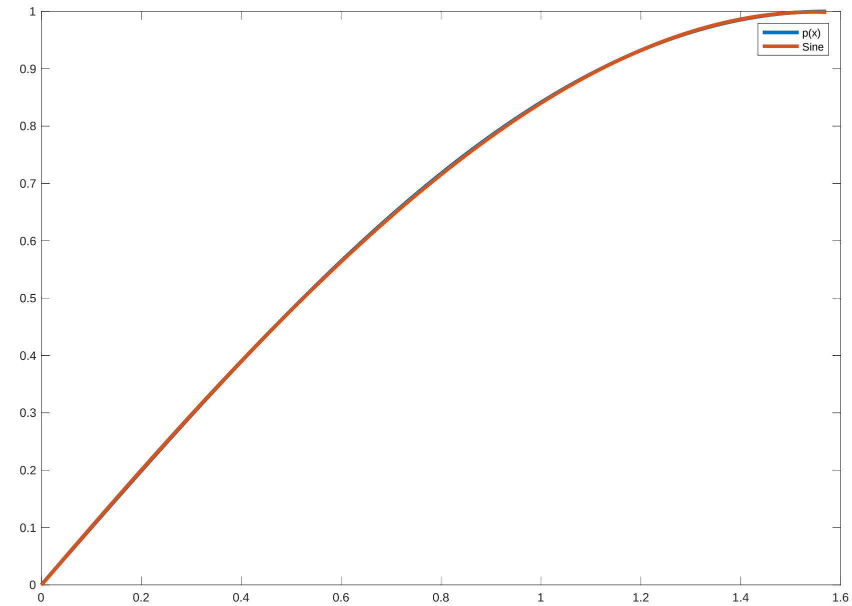
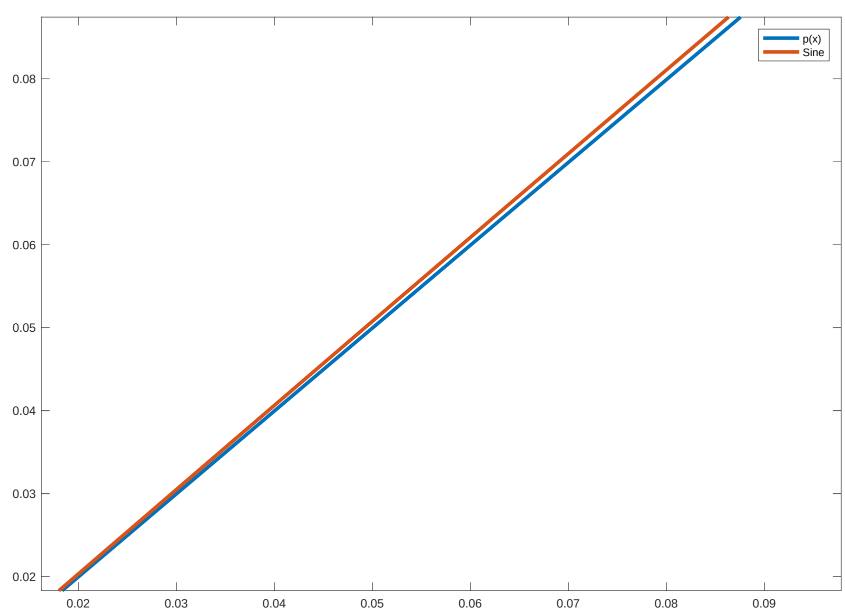
Lagrangian Interpolation

Example 1. Interpolate the function $f(x) = \sin(x)$ at 4 equally spaced points on $[0, \frac{\pi}{2}]$. Use 4 digits of accuracy.

Example 2. How well would this polynomial serve as a sine key for a calculator?


p(x)
Sine


p(x)
Sine

x	$\sin(x)$	$P_3(x)$	error
1	0.8415	0.8411	0.0004
2	0.9093	0.9102	0.0009
4	-0.7568	-0.7557	0.0011
1000	0.8269	0.8263	0.0006

So we're getting about 3 digits of accuracy. We'd like better, but that does seem like a good start!

Formally, the interpolation error is the difference between the true function and the interpolating function.

Definition 1. The *interpolation error* at x is $f(x) - P(x)$.

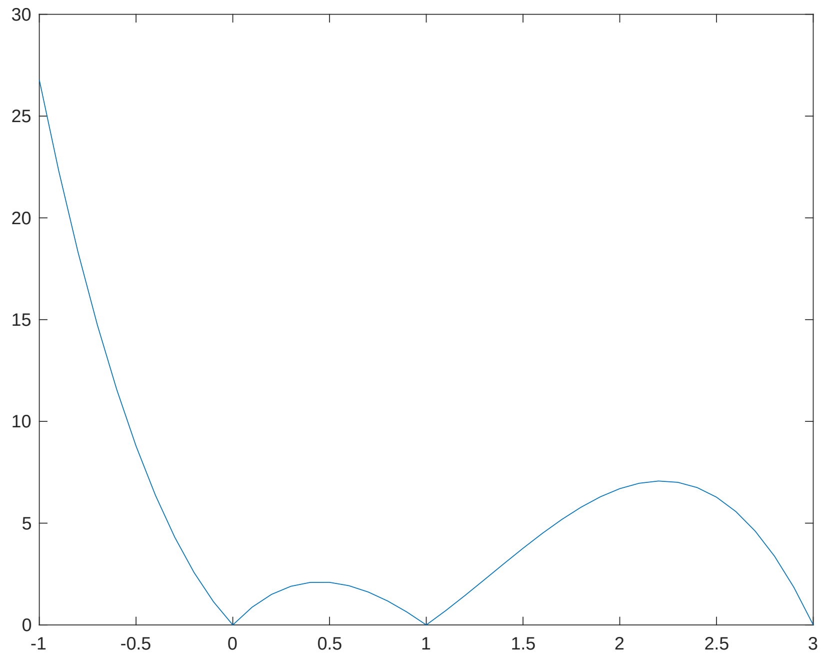
Theorem 2. *Assume that $P(x)$ is the degree at most $n-1$ interpolating polynomial fitting the n points $(x_1, y_1), \ldots, (x_n, y_n)$. The interpolation error is*

$$f(x) - P(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{n!} f^{(n)}(c)$$

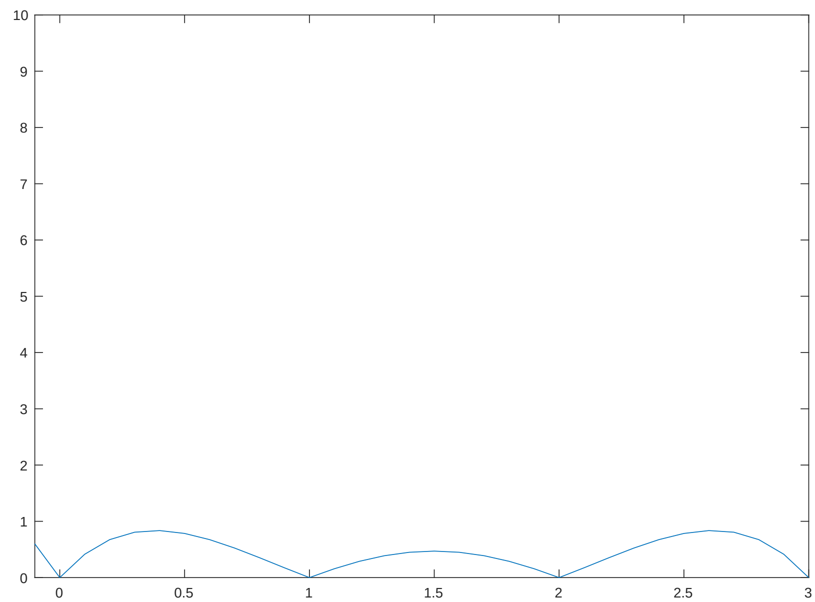
where c lies between the smallest and largest of the numbers $x, x_1, \ldots, x_n$.

Example 3. Use Theorem 2 to compute a bound on the error at $x = 1$ using P_3 above.

Example 4. Consider the function $f(x) = e^x$ and its interpolating polynomial using the values $x = 0, 1$ and 3. Find an expression for the error on the interval $[0, 3]$. At what x is it largest/smallest?



30
25
20
15
10
5
0
-1
-0.5
0
0.5
1
1.5
2
2.5
3



When using an even grid, it is almost always the case that the error is lowest in the middle and bigger at the ends of the interval.

However, there's a real danger to using regularly spaced points for polynomial interpolation.

Example 5. Suppose you have a function like a trig function whose values are between -1 and 1. Also suppose that at your evenly spaced interpolating values, the function evaluates to 0 except at one point in the middle, where it is 1.

Look at $r(x) = \dfrac{1}{1+12x^2}$ and $f(x) = \dfrac{\cos(\frac{\pi}{2}x)}{\max|x|, 1}$

Introduce smaller step size

This interpolation behavior is called the *Runge phenomenon*. Basically, the Runge phenomenon describes the extreme amount of wiggle we observe with high-degree polynomial interpolation at evenly spaced points. The function $r(x)$ above is called the Runge example.

Question 3. What should we do to get better accuracy near the ends of the interval?