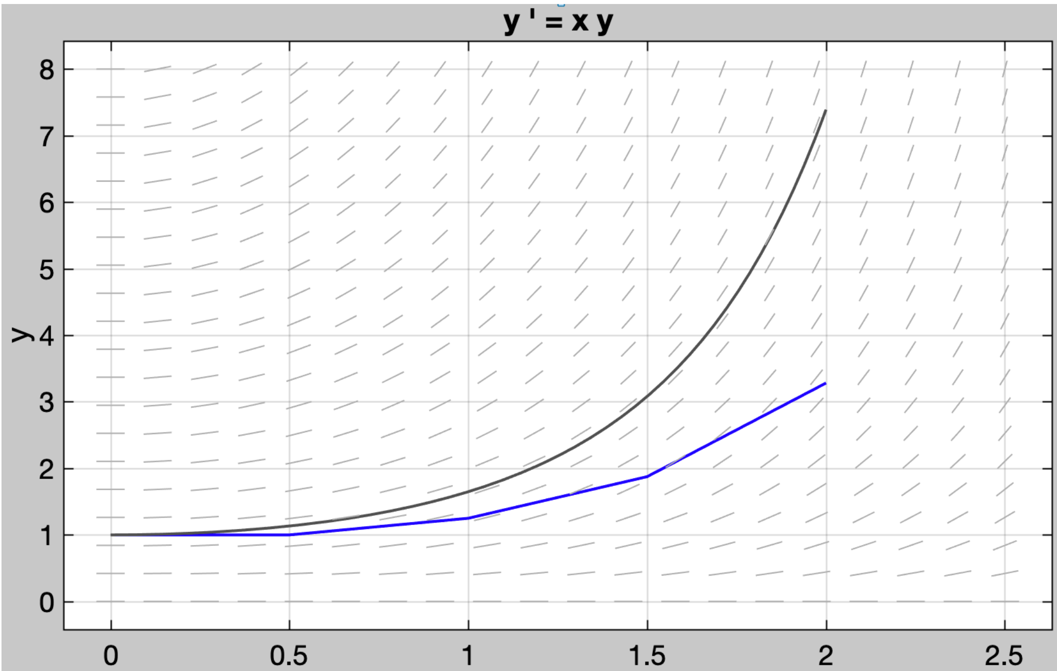
Initial Value Problems

We motivate our first improvement on Euler's method by again returning to the slope field.



So average together the slope at the current approximation, and the slope at what would be the next approximation from Euler's method.

Definition 1 (Explicit Trapezoid Method)**.**

$$w_0 = y_0$$

$$w_{i+1} = w_i + h\left(\frac{f(t_i, w_i) + f(t_i + h, w_i + hf(t_i, w_i))}{2}\right)$$

Example 1. Apply the explicit trapezoid method with $h = \frac{1}{2}$ to approximate $y(2)$ for the IVP.

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

Table 1: h = 0.5

t	**ye**	**yt**	**y**
0	1	1	1
0.5	1	1.125	1.13314845
1	1.25	1.6171875	1.64872127
1.5	1.88	2.93115234	3.08021685
2	3.28	6.59509277	7.3890561

Table 2: h = 0.1

t	ye	yt	y
0	1	1	1
0.1	1	1.005	1.00501252
0.2	1.01	1.0201755	1.02020134
0.3	1.0302	1.04598594	1.04602786
0.4	1.061106	1.08322304	1.08328707
0.5	1.10355024	1.1330513	1.13314845
0.6	1.15872775	1.1970687	1.19721736
0.7	1.22825142	1.27739201	1.27762131
0.8	1.31422902	1.37677311	1.37712776
0.9	1.41936734	1.4987552	1.4993025
1	1.5471104	1.64788135	1.64872127
1.1	1.70182144	1.82997223	1.83125221
1.2	1.8890218	2.05249686	2.05443321
1.3	2.11570441	2.32506844	2.32797781
1.4	2.39074598	2.6601108	2.66445624
1.5	2.72545042	3.07375803	3.08021685
1.6	3.13426799	3.58707562	3.59663973
1.7	3.63575086	4.22772733	4.24185214
1.8	4.25382851	5.03226384	5.05309032
1.9	5.01951764	6.04928436	6.07997145
2	5.973226	7.34383122	7.3890561

Why is it called a Trapezoid Method?

Note that if $f(t, y) = f(t)$, we have

$$w_{i+1} = w_i + \frac{h}{2}(f(t_i) + f(t_i + h)).$$

Now it's a trapezoid formula!

So what's the local truncation error this time?

The Taylor expansion of the error (regardless of method) is

$$y_{i+1} = y_i + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(c)$$

assuming that y''' is continuous and as always, c is some number between t_i and t_{i+1}.

We want to make this look more like the Trapezoid Method. Note that if

$$y' = f(t, y),$$

then

$$y''(t) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y)y'(t) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y)f(t, y).$$

Substitution gives a new version of the Taylor expansion,

$$y_{i+1} = y_i + hy'(t_i) + \frac{h^2}{2}(\frac{\partial f}{\partial t}(t_i, y_i) + \frac{\partial f}{\partial y}(t_i, y_i)f(t_i, y_i)) + \frac{h^3}{6}y'''(c).$$

We also have a Taylor expansion:

$$f(t_i + h, y_i + hf(t_i, y_i)) = f(t_i, y_i) + h\frac{\partial f}{\partial t}(t_i, y_i) + hf(t_i, y_i)\frac{\partial f}{\partial y}(t_i, y_i) + \mathcal{O}(h^2).$$

So the Trapezoid Method can be written

$$w_{i+1} = y_i + \frac{h}{2}(f(t_i, y_i) + h(\frac{\partial f}{\partial t}(t_i, y_i) + f(t_i, y_i)\frac{\partial f}{\partial y}(t_i, y_i)) + \mathcal{O}(h^2))$$

$$= y_i + hf(t_i, y_i) + \frac{h^2}{2}(\frac{\partial f}{\partial t}(t_i, y_i) + f(t_i, y_i)\frac{\partial f}{\partial y}(t_i, y_i)) + \mathcal{O}(h^3)$$

Therefore the local truncation error is

$$|y_{i+1} - w_{i+1}| = \mathcal{O}(h^3) = Ch^3$$

Then revisiting this theorem from the last lecture:

Theorem 2. *Assume that $f(t, y)$ has a Lipschitz constant L for the variable y and that the value y_i of the solution of the initial value problem at t_i is approximated by w_i from a one-step ODE solver with local truncation error $e_i \leq Ch^{k+1}$ for some constant C and $k \geq 0$. Then for each $a < t_i < b$, the solver has global truncation error*

$$g_i = |w_i - y_i| \leq \frac{Ch^k}{L}(e^{L(t_i - a)} - 1).$$

Definition 3. If an ODE solver satisfies Theorem 2 as $h \to 0$, then we say the solver has *order k*.

What is the order of the Explicit Trapezoid Method?

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Methods of Higher Orders

In this section, we show that methods of all orders exist. These methods are called Taylor Methods - because, of course, they continue using the Taylor expansions.

The Taylor expansion of $y(t)$, assuming it is at least $k + 1$ times continuously differentiable, is

$$y(t + h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \cdots + \frac{h^k}{k!}y^{(k)}(t) + \frac{h^{k+1}}{(k+1)!}y^{(k+1)}(c)$$

Definition 4 (Taylor Method of Order k)**.**

$$w_0 = y_0$$

$$w_{i+1} = w_i + hf(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i) + \cdots + \frac{h^k}{k!} f^{(k-1)}(t_i, w_i)$$

Here the prime notations refer to the total derivative of f. For example,

$$f'(t, y) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y)y'(t) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y)f(t, y).$$

By design, the local truncation error is the error term $\frac{h^{k+1}}{(k+1)!} y^{(k+1)}(c)$ and so the method is order k.

Example 2. Write out the first order and second order Taylor Method formulas.

Order one:

$$w_{i+1} = w_i + hf(t_i, w_i)$$

is Euler's Method.

Order two is:

$$\begin{aligned} w_{i+1} &= w_i + hf(t_i, w_i) + \frac{h^2}{2}(f'(t_i, w_i)) \\ &= w_i hf(t_i, w_i) + \frac{h^2}{2}(\frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial t}(t, y)f(t, y)) \end{aligned}$$

Example 3. Apply the second-order Taylor Method with $h = \frac{1}{2}$ to

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}$$

$w_{i+1} = w_i + ht_i w_i + \frac{h^2}{2}(w_i + t_i^2 w_i)$

Table 3: $h = 0.5$

t	ye	yt	ytay	y
0	1	1	1	1
0.5	1	1.125	1.125	1.13314845
1	1.25	1.6171875	1.58203125	1.64872127
1.5	1.88	2.93115234	2.83007813	3.08021685
2	3.28	6.59509277	6.32029724	7.3890561

Example 4. Find the order three Taylor method formula for the differential equation $y' = y - t^2 + 1$.

$f'(t, y) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = (y - t^2 + 1) - 2t$

and $f''(t, y) = \frac{d}{dt}(y - t^2 - 2t + 1) = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1$

and the formula is

$w_{i+1} = w_i + h(w_i + t_i^2 + 1) + \frac{h^2}{2}(w_i - t_i^2 + 1 - 2t_i) + \frac{h^3}{6}(w_i - t_i^2 - 2t_i - 1)$

These terms could be reordered for more efficient evaluation, but doing so is not required.

Question 5. What's the biggest drawback of a Taylor Method verses the previous two?

We had to find the partial derivatives, by hand, before we could use it. You might not be able to find them. Formulas of the same orders can be developed that do not require these partial derivatives, so they are rarely used in practice. The popular ones are Runge-Kutta Methods, up next.