Newton-Cotes Formulas

In Calculus, we introduce the ideas of numerical integration in a geometric manner. We first recall the Trapezoid Rule.

Trapezoid Rule

To approximate the integral $\int_a^b f(x)dx$ for a continuous function $f(x)$,

1. Subdivide the interval into n subintervals, producing $n+1$ grid points $x_0 = 1, \ldots, x_n = b$ (as opposed to n internal points for the mesh grid $t_1, \ldots, t_n$ for BVP's last handout).
2. Compute the width of each interval $\Delta x = \frac{b-a}{n}$.
3. Compute $f(x_i)$ for $i = 0, \ldots, n$.
4. The Trapezoid approximation is given by $T_n = \frac{\Delta x}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)]$.

Theorem 1. *If $f''(x)$ is continuous and $|f''(x)| \leq M$ for x in $[a, b]$, the error associated with the approximation T_n satisfies $|E_T| \leq \frac{M(b-a)^3}{12n^2}$.*

Example 1. Compute the Trapezoid approximation for $\int_{-1}^{3} 4x^3 dx$ and $n = 4$. Then compute a bound on the error.

We can now pose the problem in numerical analysis terms.

Simpson's Rule

For the Simpson's approximation, compute $S_n = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+4f(x_{n-1})+f(x_n)]$. The coefficients alternate between 4 and 2, and should end with 4. So n has to be even.

Theorem 2. *If $f^{(4)}(x)$ is continuous and $|f^{(4)}x)| \leq M_4$ for x in $[a,b]$, the error associated with the approximation S_n satisfies $|E_S| \leq \frac{M_4(b-a)^5}{180n^4}$.*

Example 2. Recompute $\int_{-1}^{3} 4x^3 dx$ using Simpson's Rule with $n = 4$. Compute the error bound.

The Simpson's Rule formula is found by replacing the degree 1 interpolation with a degree 2. We derivate the formula by starting with the first three points, x_0, x_1, x_2.

Composite Newton-Cotes Formulas

As described, these "Trapezoid" and "Simpson's" Rules are actually *composite* versions of the rules, as they can be applied to many *panels*, not just a single formula using the endpoints. They are part of a larger class of approximation methods called Newton-Cotes Methods, which integrate an interpolating polynomial for a set of points from the desired function. Because they are methods that use the endpoints, they are called *closed* methods. There also exist *open* Newton-Cotes Methods, such as the midpoint rule (Riemann sums where the function value at the middle of the interval).

Definition 3. The *degree of precision* of a numerical integration method is the greatest integer k for which all degree k or less polynomials are integrated exactly by the method.

Question 4. What are the degrees of precision of the Trapezoid and Simpson's Rules?

Question 5. Find the degree of precision of the degree 3 Newton-Cotes formula, called the *Simpson's 3/8 Rule*,

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)).$$