# Material Summary: Calculus

## Limits

**1.1 Limits**

* Natural definition
  + Given a function , "nudge" the input around a given value
    - As a result, the function value changes
  + Limit of at the point : what approaches as approaches
* A black background with a black square

  Description automatically generated with medium confidenceNotation:
* Mathematical definition
  + Gives us a nice way to define "approaching a value"
  + For any positive and
    - A black background with a black square

      Description automatically generated with medium confidenceIf
    - A black background with a black square

      Description automatically generated with medium confidenceThen
  + Also called "epsilon-delta" definition
  + What are these numbers? Arbitrary, they only  
    need to be positive
    - A diagram of a function

      Description automatically generatedIt's very useful to **make them really small**

**1.2 Limits in Python**

* To find the limit of a function at a point, just apply the definition
  + Generate several values of x around a
    - Don't forget to include positive and negative "nudges"
  + A screenshot of a computer program

    Description automatically generatedPrint the function values at those points
* Some functions don't have a value at certain points
  + A black background with a black square

    Description automatically generated with medium confidenceBut they are defined "around" these points
  + **The limit exists** even though the function value doesn't
* A black background with a black square

  Description automatically generated with medium confidenceSome limits can be infinite:
* Some functions "jump"
  + The limits "from the left" and "from the right" are different
    - Therefore, the limit is not defined
    - We say the function is not continuous at that point
  + Example:
    - In this case, but the limit does not exist

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## A diagram of a circle with arrows Description automatically generatedDerivatives

**2.1 Calculus Motivation**

* Say you want to compute the area of a circle
  + It is but why?
  + Remember how you can divide a shape into simpler  
    shapes and sum their areas to get the total area
    - One way: cut it like cake: see [this video](https://www.youtube.com/watch?v=YokKp3pwVFc)
    - Another way: concentric rings
  + If you "cut" and "straighten" each ring, you'll get a trapezoid
    - If your ring is very, very thin; it will actually be close to a rectangle



* + - Example:
    - **Set the difference to be very, very, veeeeeeery small:**
    - … and you get calculus :)
* Even in this simple example, there are the notions about derivativesand integrals; even the fundamental theorem of calculus

**2.2 Derivatives and Velocity**

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  Description automatically generated with medium confidenceWe all know that
  + But that's mostly useless
  + Travelling is not done at a uniform velocity, it's not a fixed number  
    but a function of time:
* Instantaneous velocity: A black background with a black square

  Description automatically generated with medium confidence
* Computing instantaneous velocity from travelled distance
  + Say, ; say we start at and finish at
    - Final distance:
  + Average speed:
  + But we cover different distances for the same time
    - From :
    - From :
    - From :
    - And neither of these is even close to the average speed
* Let's calculate the instantaneous velocity
  + Fix time at
  + But… how can we move if time is fixed?
* Let's apply our previous idea
  + Nudge time a tiny bit and see how the distance changes
  + :
  + :
    - More generally, if we nudge time from to , we'll get   
      an approximation of the instantaneous velocity: A black background with a black square

      Description automatically generated with medium confidence
    - This approximation will get increasingly more accurate as becomes smaller
    - Smaller better approximation of
* How does the velocity behave as ?
  + Note that we **cannot** set , this will freeze time
  + Math notation: if , we write it as A black background with a black square

    Description automatically generated with medium confidence
    - We now have a nice definition of velocity
  + But what does it mean mathematically?
    - Velocity = rate of change of travelled distance over time
  + The rate of change of a function as its argument changes, is called the **first derivative** of with respect to
  + Math notation: or
    - Note that is only notation, it is not equal to
  + A black background with a black square

    Description automatically generated with medium confidenceDefinition:

A black background with a black square

Description automatically generated with medium confidence**2.3 Geometric Interpretation**

* Look at the chord and the triangle
* As , approaches
  + The chord becomes the same as  
    the tangent line at point
  + The angle : slope of the tangent lineA black background with a black square

    Description automatically generated with medium confidence
  + Geometrically, the derivative at a given point is **equal** to the slope of the tangent line to the function at this point
* This is what calculus is all about
  + A screen shot of a graph

    Description automatically generated**Zooming in really close** until everything appears as a straight line

**2.4 Calculating Derivatives**

* Note that we have two definitions
  + Derivative of at a fixed point (e.g. ): this is a number
  + Derivative of at any point: this is another function
* Calculate the derivative of at
  + We're doing a numerical approximation
  + A white background with black and green text

    Description automatically generatedWe can't work with infinitesimally small but we can get away with something quite small
  + We can also do this **analytically**
  + A fancy term for "with pen and paper"

**2.5 Calculating Derivatives Analytically**

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  Description automatically generated with medium confidenceLet's take a relatively simple function like
* We're looking for approximation and is small, so let's ignore
  + Ignoring higher-order terms is completely valid (and is done often) A black background with a black square

    Description automatically generated with medium confidence
  + Note that the derivative **does not depend** on the tiny shift
* We can do this for every function
  + We have precomputed [tables of derivatives](http://www.math.com/tables/derivatives/tableof.htm)

**2.6 Properties of Derivatives**

* The derivative of a constant () is 0
* Derivatives are linear
* Product rule
* Derivative of a function composition
  + Also called **chain rule**
  + Looks better in the other notation:
* We can prove these using the geometric intuition or the definition
  + This is left as an exercise for the reader :)

**2.7 Higher-Order Derivatives**

* The second derivative of a function is the first derivative of its first derivative
  + Interpretation: "rate of change of the rate of change"
  + … a.k.a. acceleration
  + Notation:
* This can be applied arbitrary many times
  + E.g., rate of change of acceleration: third derivative
    - a.k.a. "jerk"… don't ask me why
  + Third, fourth, etc. derivatives; -th derivative notation:
    - E.g.,

**2.8 Function Extrema**

* A red and blue curves on a black background

  Description automatically generatedEven if we don't know the function,  
  its derivatives give us useful information
* Consider the drawn function
  + The smallest value of is calleda **global minimum**
  + Conversely, largest value:   
    **global maximum**
* These are collectively called extrema (plural of extremum)
* Smallest / largest value of in a tiny range: local min / max
* More formally, we say has a maximum at, say, if the function value is bigger than the function values immediately to the left and right
  + The complete definition involves limits
  + The points of min / max (e.g., ) are called **critical points**
* Notice how the tangent line behaves
  + At max / min,
  + Around max / min, changes its sign
* Also notice that if in a given interval, the function increases
  + If , the function decreases
* Therefore, if behaves like this
  + Increasing; stop; decreasing local maximum
  + Decreasing; stop; increasing local minimum
* The second derivative gives us more informationabout whether the function is "concave up" or "concave down"
  + More specifically, its sign
  + A red and black line

    Description automatically generatedA graph of function and function of function

    Description automatically generated with medium confidenceThese are sometimes called convex and concave functions

## A graph of a function Description automatically generatedIntegrals

**3.1 Area under a Function**

* Look back to the motivating example
* How can we find the area "under" a curve given by a function?
  + What is the shaded area ( if )?
* **Approach:** approximate and zoom in
* Divide the x-axis into equal intervals
* Approximate the area with trapezoids

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Description automatically generated with medium confidence

* If the intervals in are really small, the trapezoids will look like rectanglesA black background with a black square

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* Smaller better approximation

**3.2 Integral of a Function**

* At the limit, , so we write
* A black background with a black square

  Description automatically generated with medium confidenceThe sum is denoted differently:
  + This is called the **definite integral** of
  + **Note:** don't forget the after the function!
* **Indefinite integral**: the same, without the end points
  + Like derivatives, the definite integral is a number
  + The indefinite integral is a function of
* Calculating integrals
  + Analytically – very difficult (unlike derivatives)
  + Numerically – apply the trapezoidal rule
    - Use a small number , like before

## Fundamental Theorem of Calculus

**4.1 Antiderivatives**

* The antiderivative of a function is such a function that
  + It's also called the primitive function of
  + Note that since the derivative of a constant is zero, there are many antiderivatives:
  + Therefore, we can know the antiderivative only up to an arbitrary additive constant
* If we do definite integrals, the does not apply – we know the area exactly
* If we do indefinite integrals, we must always add the constant
* The indefinite integral of a function is related to its antiderivative and can be reversed via differentiation

**4.2 Fundamental Theorem of Calculus**

* The definite integral of a function can be computed using one of its infinitely many antiderivatives
* Simply, differentiation and integration are inverse functions
* Proof: [Khan Academy](https://www.youtube.com/watch?v=C7ducZoLKgw)
* Intuition
  + The sum of infinitesimal changes in a quantity over time adds up to the net change in quantity
  + Think about distance and velocity again

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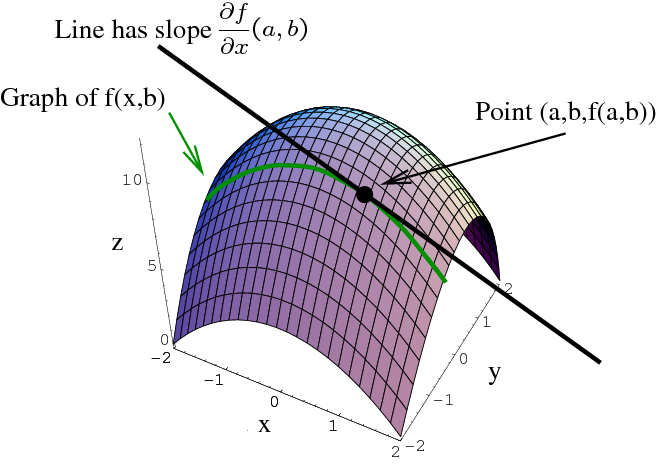
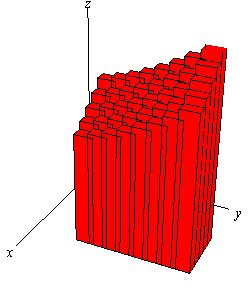
## Calculus in Many Dimensions

**5.1 Generalizations**

* The notions of derivatives and integrals generalize to more dimensions
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  Description automatically generated with medium confidenceDerivatives: take the derivative w.r.t. one variable, treat the other variables as "parameters" **partial derivatives**
* Yet more confusing notation: is the same as , it's just used for many dimensions
* Integrals: 1D intervals can become curves or planes
* A black background with a black square

  Description automatically generated with medium confidenceApply the same "zooming in" technique



**5.2 Gradient Descent**

* Optimization method
  + Used for finding local extrema
* Gradient: or
  + A black background with a black square

    Description automatically generated with medium confidenceA combination of vector and derivative:
    - "Multi-dimensional derivative"
    - A vector whose components are the partial derivatives w.r.t. every variable
  + Shows where the **steepest rise in slope** is
* If we follow the gradient, we'll arrive at a maximum
  + Conversely, negative gradient takes us to a minimum
* Iterative procedure
  + Continue to apply until close enough
* Not guaranteed to find global extrema
  + May get "stuck" in a local extremum

**5.3 Example: Gradient Descent**

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  Description automatically generated with medium confidenceFind a local minimum of the function
  + A screenshot of a computer

    Description automatically generatedStart at