Right Decisions from Wrong Predictions: A Mechanism Design Alternative to Individual Calibration

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Abstract

Decision makers often need to rely on imperfect probabilistic forecasts. While average performance metrics are typically available, it is difficult to assess the quality of individual forecasts and the corresponding utilities. To convey confidence about individual predictions to decision-makers, we propose a compensation mechanism ensuring that the forecasted utility matches the actually accrued utility. While a naive scheme to compensate decision-makers for prediction errors can be exploited and might be unsustainable in the long run, we propose a mechanism based on fair bets and online learning that provably cannot be exploited. We demonstrate an application showing how passengers could confidently optimize individual travel plans based on flight delay probabilities estimated by an airline.

1 Introduction

People and algorithms constantly rely on probabilistic forecasts (about medical treatments, weather, transportation times, etc.) and make potentially high-stake decisions based on them. In most cases, forecasts are not perfect, e.g., the forecasted chance that it will rain tomorrow does not match the true probability exactly. While average performance statistics might be available (accuracy, calibration, etc), it is generally impossible to tell whether any *individual* prediction is reliable (individually calibrated), e.g., about the medical condition of an *specific patient* or the delay of a *particular flight* (21; 1; 22). Intuitively, this is because multiple *identical* datapoints are needed to confidently estimate a probability from empirical frequencies, but identical datapoints are rare in real world applications (e.g. two patients are always different). Given these limitations, we study alternative mechanisms to convey confidence about *individual* predictions to decision-makers.

We consider settings where a single forecaster provides predictions to many decision makers, each facing a potentially different decision making problem. For example, a personalized medicine service could predict whether a product is effective for thousands of individual patients (19; 20; 2). If the prediction is accurate for 70% of patients, it could be accurate for Alice but not Bob, or vice-versa. Therefore, Alice might be hesitant to make decisions based on the 70% *average* accuracy. In this setting, we propose an insurance-like mechanism that 1) enables each decision maker to confidently make decisions as if the advertised probabilities were individually correct, and 2) is implementable by the forecaster with provably vanishing costs in the long run.

To achieve this, we turn to the classic idea (6; 14) that a probabilistic belief is equivalent to a willingness to take bets. We use the previous example to illustrate that if the forecaster is willing to take bets, a decision maker can bet with the forecaster as an "insurance" against mis-prediction. Suppose Alice is trying to decide whether or not to use a product. If she uses the product, she gains \$10 if the product is effective and loses \$2 otherwise. The personalized medicine service (forecaster) predicts that the product is effective with 50% chance for Alice. Under this probability Alice expects to gain \$4 if she decides to use the product, but she is worried the probability is incorrect. Alice proposes a bet: Alice pays the forecaster \$6 if the product is effective, and the forecaster pays Alice \$6 otherwise. The forecaster should accept the bet because under its own forecasted probability the bet is fair (i.e., the expectation is zero if the forecasted probabilities are true for Alice). Alice gets the guarantee that if she decides to use the product, *effective or not*, she gains \$4 — equal to her expected utility under the forecasted (and possibly incorrect) probability. In general, we show that *Alice has a way of choosing bets for any utility function and forecasted probability, such that her true gain equals her expected gain under the forecasted probability*.

From the forecaster's perspective, if the true probability that Alice's treatment is effective is actually 10%, then the forecaster will lose \$4.8 from this bet in expectation. However, in our setup, the forecaster makes probabilistic forecasts for many different decision makers, and each decision maker selects some bet based on their utility function and forecasted probability. The forecaster might gain or lose on *individual* bets, but it only needs to not lose on the entire set of bets *on average* for the approach to be sustainable. Intuitively, each decision maker's difference between forecasted gain and true gain can be averaged across the pool of decision makers. The difficult requirement that *each* difference should be negative has been reduced to an easier requirement that the *average* difference should be negative.

However, this protocol leaves the forecaster vulnerable to exploitation. For example, Alice already knows that the product will be ineffective; she could still bet with the forecaster for the malicious purpose of gaining \$6. Surprisingly we show that in the online setup (4), the forecaster has an algorithm to adapt its forecasts and guarantee vanishing loss in the long run, even in the presence of malicious decision makers. This is achieved by first using any existing online prediction algorithm to predict the probabilities, then applying a post processing algorithm to fine-tune these probabilities based on past gains/losses (similar to the idea of recalibration (16; 11)).

As a concrete application of our approach, we simulate the interaction between an airline and passengers with real flight delay data. Risk averse passengers might want to avoid a flight if there is possibility of delay and their loss in case of delay is high. We show if an airline offers to accept bets based on the predicted probability of delay, it can help risk-averse passengers make better decisions, and increase both the airline's revenue (due to increased demand for the flight) and the total utility (airline revenue plus passenger utility).

We further verify our theory with large scale simulations on several datasets and a diverse benchmark of decision tasks. We show that forecasters based on our post-processing algorithm consistently achieve close to zero betting loss (on average) within a small number of time steps. On the other hand, several seemingly reasonable alternative algorithms not only lack theoretical guarantees, but often suffer from positive average betting loss in practice.

2 Background

2.1 Decision Making with Forecasts

This section defines the basic setup of the paper. We represent the decision making process as a multi-player game between nature, a forecaster and a set of (decision making) agents. At every step t nature reveals an input observation x_t to the forecaster (e.g. patient medical records) and selects the hidden probability $\mu_t^* \in [0, 1]$ that $\Pr[y_t = 1] = \mu_t^*$ (e.g. probability treatment is successful), We only consider binary variables ($y_t \in \{0, 1\} = \mathcal{Y}$) and defer the general case to Appendix B.

The forecaster chooses a forecasted probability $\mu_t \in [0, 1]$ to approximate μ_t^*. We also allow the forecaster to represent the lack of knowledge about μ_t^*, i.e. the forecaster outputs a confidence $c_t \in [0, 1]$ where the hope is that $\mu_t^* \in [\mu_t - c_t, \mu_t + c_t]$.

At each time step, one or more agents can use the forecast μ_t and c_t to make decisions, i.e. to select an action $a_t \in \mathcal{A}$. However, for simplicity we assume that different agents make decisions at different time steps, so at each time step there is only a single agent, and we can uniquely index the agent by the time step t. The agent knows its own loss (negative utility) function $l_t : \mathcal{A} \times \mathcal{Y} \to [-M, M]$ (the forecaster does not have to know this) where $M \in \mathbb{R}_+$ is the maximum loss involved in each decision. This protocol is formalized below.

Protocol 1: Decision Making with Forecasts For $t = 1, \cdots, T$

1. Nature reveals $x_t \in \mathcal{X}$ to forecaster and chooses $\mu_t^* \in [0, 1]$ without revealing it
2. Forecaster reveals $\mu_t, c_t \in (0, 1)$ where $(\mu_t - c_t, \mu_t + c_t) \subset (0, 1)$
3. Agent t has loss function $l_t : \mathcal{A} \times \mathcal{Y} \to \mathbb{R}$ and reveals a_t selected according to μ_t, c_t and l_t
4. Nature samples $y_t \sim \text{Bernoulli}(\mu_t^*)$ and reveals y_t; Agent incurs loss $l_t(a_t, y_t)$

We make no assumptions on nature, forecaster, or the agents. They can choose any strategy to generate their actions, as long as they do not look into the future (i.e. their action only depends on variables that have already been revealed). In particular, we make no i.i.d. assumptions on how nature selects y_t and μ_t^*; for example, nature could even select them adversarially to maximize the agent's loss.

2.2 Individual Coverage

Ideally in Protocol 1 the forecaster's prediction μ_t, c_t should satisfy $\mu_t^* \in [\mu_t - c_t, \mu_t + c_t]$) for each individual t (this is often called individual coverage or individual calibration in the literature). However, many existing results show that learning individually calibrated probabilities from past data is often impossible (21; 1; 22) unless the forecast is trivial (i.e. $[\mu_t - c_t, \mu_t + c_t] = [0, 1]$).

One intuitive reason for this impossibility result is that in many practical scenarios for each x_t we only observe a single sample $y_t \sim \mu_t^*$. The forecaster cannot infer μ_t^* from a single sample y_t without relying on unverifiable assumptions.

2.3 Probability as Willingness to Bet

A major justification for probability theory has been that probability can represent willingness to bet (6; 12). For example, if you truly believe that a coin is fair, then it would be inconsistent if you are not willing to win \$1 for heads, and lose \$1 for tails (assuming you only care about average gain rather than risk). More specifically a forecaster that holds a probabilistic belief should be willing to accept any bet where it gains a non-negative amount in expectation.

For binary variables, we consider the case where a forecaster believes that a binary event $Y \in \{0, 1\}$ happens with some probability μ^* but does not know the exact value of μ^*. The forecaster only believes that $\mu^* \in [\mu - c, \mu + c] \subset [0, 1]$. The forecaster should be willing to accept any bet with non-negative expected return under *every* $\mu^* \in [\mu - c, \mu + c]$. For example, assume the forecaster believes that a coin comes up heads with at least 40% chance and at most 60% chance. The forecaster should be willing to win \$6 for heads, and lose \$4 for tails; similarly the forecaster should be willing to lose \$4 for heads, and win \$6 for tails.

More generally, according to Lemma 1 (proved in Appendix E), a forecaster believes that the probability of success $\Pr[Y = 1] = \mu^*$ of the binary event Y satisfies $\mu^* \in [\mu - c, \mu + c]$ if and only if she is willing to accept bets where she loses $b(Y - \mu) - |b|c, \forall b \in \mathbb{R}$.

Lemma 1. *Let $\mu, c \in (0, 1)$ such that $[\mu - c, \mu + c] \subset [0, 1]$, then a function $f : \mathcal{Y} \to \mathbb{R}$ satisfies $\forall \tilde{\mu} \in [\mu - c, \mu + c]$, $\mathbb{E}_{Y \sim \tilde{\mu}}[f(Y)] \leq 0$ if and only if for some $b \in \mathbb{R}$ and $\forall y \in \{0, 1\}$, $f(y) \leq b(y - \mu) - |b|c$.*

In words, a forecaster is willing to lose $f(Y)$ if f has non-positive expectation under every probability the forecaster considers possible. However, every such function f are smaller (i.e. forecaster loses less) than $b(Y - \mu) - |b|c$ for some $b \in \mathbb{R}$. Therefore, we only have to consider whether a forecaster is willing to accept bets of the form $b(Y - \mu) - |b|c$.

3 Decisions with Unreliable Forecasts

In Protocol 1, agents could make decisions based on the forecasted probability μ_t, c_t and the agent's loss l_t. For example, the agent could choose

$$a_t := \arg\min_{a \in \mathcal{A}} \mathbb{E}_{Y \sim \mu_t} l_t(a, Y) \tag{1}$$

to minimize the expected loss under the forecasted probability.

However, how can the agent know that this decision has low expected loss under the *true probability* μ_t^*? This can be achieved with two desiderata, which we formalize below:

We denote the agent's maximum / average / minimum expected loss under the forecasted probability as

$$L_t^{\max} = \max_{\tilde{\mu} \in \mu_t \pm c_t} \mathbb{E}_{Y \sim \tilde{\mu}}[l_t(a_t, Y)] \quad L_t^{\text{avg}} = \mathbb{E}_{Y \sim \mu_t}[l_t(a_t, Y)] \quad L_t^{\min} = \min_{\tilde{\mu} \in \mu_t \pm c_t} \mathbb{E}_{Y \sim \tilde{\mu}}[l_t(a_t, Y)]$$

and true expected loss as $L_t^* = \mathbb{E}_{Y \sim \mu_t^*}[l_t(a_t, Y)]$. If the agent knows that

Desideratum 1 $L_t^* \in [L_t^{\min}, L_t^{\max}]$
Desideratum 2 The interval size c_t is close to 0.

then the agent can infer that the true expected loss L_t^* is not too far off from the forecasted expected loss L_t^{avg}. This is because if c_t is small then $L_t^{\min}$ will be close to $L_t^{\max}$. Both L_t^* and L_t^{avg} will be sandwiched in the small interval $[L_t^{\min}, L_t^{\max}]$.

However, we show that desiderata 1 and 2 often cannot be achieved simultaneously. To guarantee $L_t^* \in [L_t^{\min}, L_t^{\max}]$ the forecaster in general must output individually correct probabilities (i.e. $\mu_t^* \in [\mu_t - c_t, \mu_t + c_t]$), as shown by the following proposition (proof in Appendix E).

Proposition 1. *For any $\mu_t, c_t, \mu_t^* \in (0,1)$ where $(\mu_t - c_t, \mu_t + c_t) \subset (0,1)$*

1. If $\mu_t^* \in [\mu_t - c_t, \mu_t + c_t]$ then $\forall l_t : \mathcal{Y} \times \mathcal{A} \to \mathbb{R}$ we have $L_t^* \in [L_t^{\min}, L_t^{\max}]$

2. If $\mu_t^* \notin [\mu_t - c_t, \mu_t + c_t]$ then $\exists l_t : \mathcal{Y} \times \mathcal{A} \to \mathbb{R}$ such that $L_t^* \notin [L_t^{\min}, L_t^{\max}]$

In words, unless $\mu_t^* \in [\mu_t - c_t, \mu_t + c_t]$ we cannot guarantee that $L_t^* \in [L_t^{\min}, L_t^{\max}]$ without assuming that the agent's loss function is special (e.g. it is a constant function). However, in Section 2.2 we argued that it is usually impossible to achieve $\mu_t^* \in [\mu_t - c_t, \mu_t + c_t]$ unless c_t is very large (i.e. $[\mu_t - c_t, \mu_t + c_t] = [0, 1]$). If c_t is too large, the interval $[L_t^{\min}, L_t^{\max}]$ will be large, and the guarantee that $L_t^* \in [L_t^{\min}, L_t^{\max}]$ would be practically useless even if it were true. This means the forecaster cannot convey confidence in individual predictions it makes, and as a result the agent can't be very confident about the expected loss it will incur.

3.1 Insuring against unreliable forecasts

Since it is difficult to satisfy desiderata 1 and 2 simultaneously, we consider relaxing desideratum 1. In particular, we study what guarantees are possible for each individual decision maker even when $\mu_t^* \notin [\mu_t - c_t, \mu_t + c_t]$, i.e., the prediction is wrong.

We consider the setup where each agent can receive some side payment (a form of "insurance" which could depend on the outcome Y, and could be positive or negative) from the forecaster, and we would like to guarantee

Desideratum 1' $\underbrace{L_t^* - \mathbb{E}_{Y \sim \mu_t^*}[\text{payment}(Y)]}_{\text{True expected loss w. side payment}} \in \underbrace{[L_t^{\min}, L_t^{\max}]}_{\text{Forecasted expected loss range}}$

In other words, we would like the expected loss under the true distribution to be predictable *once we incorporate the side payment.*

Note that desideratum 1' can be trivially satisfied if the forecaster is willing to pay any side payment to the decision agent. For example, an agent can choose $\text{payment}(Y) := \mathbb{E}_{Y \sim \mu_t}[l_t(a_t, Y)] - l_t(a_t, Y)$ to satisfy desideratum 1'. However, if the forecaster offers any side payment, it could be subject to exploitation. For example, decision agents could request the forecaster to pay \$1 under any outcome y_t. Such a mechanism cannot be sustainable for the forecaster.

3.2 Insuring with fair bets

Even though the forecaster cannot offer arbitrary payments to the decision agent, we show that the forecaster can offer a sufficiently large set of payments, such that [i] each decision agent can select a payment to satisfy Desideratum 1' and [ii] the forecaster has an algorithm to guarantee vanishing loss in the long run, even when the decision agents tries to exploit the forecaster.

In fact, the "fair bets" in Section 2.3 satisfy our requirement. Specifically, the forecaster can offer the set $\{\text{payment}(Y) := b(Y - \mu_t) - |b|c_t, \forall b \in [-M, M]\}$ as available side payment options. The constant $M \in \mathbb{R}_+$ caps the maximum payment each decision agent can request (in our setup l_t is also upper bounded by M). This set of payments satisfy both [i] (shown in this section) and [ii] shown in the next section).

Before we proceed to show [i] and [ii], for convenience, we formally write down the new protocol. Compared to Protocol 1, the decision agent selects some "stake" $b_t \in [-M, M]$, and receive side payment $b_t(Y - \mu_t) - |b_t|c_t$ from the forecaster.

Protocol 2: Decision Making with Bets For $t = 1, \cdots, T$

1. Nature reveals observation $x_t \in \mathcal{X}$ and chooses $\mu_t^* \in [0, 1]$ without revealing it
2. Forecaster reveals $\mu_t, c_t \in (0, 1)$ where $(\mu_t - c_t, \mu_t + c_t) \subset (0, 1)$

3. Agent t has loss function $l_t : \mathcal{A} \times \mathcal{Y} \to \mathbb{R}$ and reveals action $a_t \in \mathcal{A}$ and stake $b_t \in [-M, M]$ selected according to μ_t, c_t and l_t
4. Nature samples $y_t \sim \text{Bernoulli}(\mu_t^*)$ and reveals y_t
5. Agent incurs loss $l_t(a_t, y_t) - b_t(y_t - \mu_t) + |b_t|c_t$; forecaster incurs loss $b_t(y_t - \mu_t) - |b_t|c_t$

Denote the agent's true expected loss with side payment as (i.e. the LHS in Desideratum 1')

$$L_t^{\text{pay}} := \underbrace{L_t^*}_{\text{decision loss}} - \underbrace{\mathbb{E}_{Y \sim \mu_t^*}[b_t(Y - \mu_t) + |b_t|c_t]}_{\text{payment from forecaster}} \tag{2}$$

then we have the following guarantee: for any choice of μ_t, c_t, μ_t^*, a_t and l_t

Proposition 2. *If the stake $b_t = l_t(a_t, 1) - l_t(a_t, 0)$ then $L_t^{\text{pay}} \in [L_t^{\min}, L_t^{\max}]$*

For the more general version of the proposition in the multi-class setup, see Appendix B.1. In words, the agent has a choice of stake b_t that only depends on variables known to the agent (l_t and a_t) and does not depend on variables unknown to the agent (μ_t^*, y_t). If the agent chooses this b_t, she can be certain that desideratum 1' is satisfied, regardless of what the forecaster or nature does (they can choose any μ_t, c_t, μ_t^*).

This mechanism allows the agent to **make decisions as if the forecasted probability is correct**, i.e. as if $\mu_t^* \in [\mu_t - c_t, \mu_t + c_t]$. This is because Proposition 2 is true for any choice of action a_t (as long as the agent chooses b_t according to Proposition 2 after selecting a_t). Intuitively, for any action a_t the agent selects, she can guarantee to achieve a total loss close to $\mathbb{E}_{Y \sim \mu_t} l_t(a_t, Y)$ (assuming c_t is small). This is the same guarantee she would get as if $\mu_t^* \in [\mu_t - c_t, \mu_t + c_t]$.

In addition, if [ii] is satisfied (i.e. the forecaster has vanishing loss), the forecaster also doesn't lose anything, so should have no incentive to avoid offering these payments. We discuss this in the next section.

Algorithm 1: Post-Processing for Exactness

Invoke Algorithm 2 and 3 with $K = (T/\log T)^{1/4}$
for $t = 1, \cdots, T$ **do**
  Receive $\hat{\mu}_t$ and $\hat{c}_t$ from Algorithm 2. Receive λ_t from Algorithm 3
  Output $\mu_t = \hat{\mu}_t$, $c_t = \hat{c}_t + \lambda_t$. Input y_t and b_t
  Set $r_t = (b_t/\sqrt{|b_t|})(\mu_t - y_t) - \sqrt{|b_t|}\hat{c}_t$, $s_t = -\sqrt{|b_t|}$, Send (r_t, s_t) to Algorithm 3

Algorithm 2: Online Prediction

Choose any initial value for θ_1, ϕ_1
for $t = 1, \cdots, T$ **do**
  Input x_t and output $\hat{\mu}_t = \mu_{\theta_t}(x_t)$, $\hat{c}_t = c_{\phi_t}(x_t)$. Input y_t and b_t
  $\theta_{t+1} = \theta_t - \eta \frac{\partial}{\partial \theta}(\mu_{\theta_t}(x_t) - y_t)^2$, $\phi_{t+1} = \phi_t - \eta \frac{\partial}{\partial \phi}\left(b_t(\hat{\mu}_t - y_t) - |b_t|c_{\phi_t}(x_t)\right)^2$

4 Probability Forecaster Strategy

In this section we study the forecaster's strategy. As motivated in the previous section, the goal of the forecaster (in Protocol 2) is to:

1) have non-positive cumulative loss when T is large, so that the side payments are sustainable
2) output the smallest c_t compatible with 1), so that forecasts are as sharp as possible

Specifically, the forecaster's average cumulative loss (up to time T) in Protocol 2 is

$$\frac{1}{T} \sum_{t=1}^{T} b_t(\mu_t - y_t) - |b_t|c_t \tag{3}$$

Whether Eq.(3) is non-positive or not depends on the actions of all the players: forecaster μ_t, c_t, nature y_t and agent b_t. Our focus is on the forecaster, so we say that a sequence of forecasts $\mu_t, c_t, t = 1, 2, \cdots$ is **asymptotically sound** relative to $y_1, b_1, y_2, b_2, \cdots$ if the forecaster loss in Protocol 2 is non-positive, i.e.

$$\lim \sup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} b_t(\mu_t - y_t) - |b_t|c_t \leq 0 \tag{4}$$

In subsequent development we will use a stronger definition than Eq.(4). We say that a sequence of forecasts $\mu_t, c_t, t = 1, 2, \cdots$ is **asymptotically exact** relative to $y_1, b_1, y_2, b_2, \cdots$ if the forecaster loss in Protocol 2 is exactly zero, i.e.

$$\lim \sup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} b_t(\mu_t - y_t) - |b_t| c_t = 0 \tag{5}$$

Intuitively asymptotic soundness requires that the forecaster should not lose in the long run; asymptotic exactness requires that the forecaster should neither lose nor win in the long run — a stronger requirement.

The reason we focus on asymptotic exactness is because the forecaster should output the smallest possible c_t to achieve sharp forecasts. Observe that the left hand side of Eq.(4) is increasing if c_t decreases. Therefore, whenever the forecaster is asymptotically sound but not asymptotically exact (i.e. the left hand side in Eq.(4) is strictly negative), there is some room to decrease c_t without violating asymptotic soundness.

Algorithm 3: Swap Regret Minimization

Input: number of discrete interval K
Partition $[-1, 1]$ into equal intervals $[-1 = v_0, v_1), \cdots, [v_{K-1}, v_K = 1]$
For each interval init an empty set $\mathcal{D}_k$, set $v^0 = 0$
for $t = 1, \cdots, T$ **do**
    Initialize an empty ordered list $\mathcal{V}^t$
    Initialize $v^t = v^{t-1}$ and **while** $v^t \notin \mathcal{V}^t$ **do**
        $\lambda_t^{v^t} = \arg\inf_{\lambda \in [-1,1)} \frac{1}{|\mathcal{D}_{v^t}|} \sum_{r_t, s_t \in \mathcal{D}_{v^t}} (r_t + s_t \lambda)^2$. Append v^t to $\mathcal{V}^t$
        Set v^t as the k that satisfies $\lambda_t^{v^t} \in [v_k, v_{k+1})$
    Remove all elements before v^t from $\mathcal{V}^t$. Select v^t uniform randomly from $\mathcal{V}^t$
    Choose $\lambda_t = \lambda_t^{v^t}$ and send λ_t to Algorithm 1
    Receive (r_t, s_t) from Algorithm 1, add to $\mathcal{D}_{v^t}$

4.1 Online Forecasting Algorithm

We aim to achieve asymptotic exactness with minimal assumptions on $y_t, b_t, t = 1, 2, \cdots$ (we only assume boundedness). This is challenging for two reasons: an adversary could select $y_t, b_t, t = 1, 2, \cdots$ to violate asymptotic exactness as much as possible (e.g. decision agents could try to profit on the forecaster's loss); in Protocol 2 the agent's action b_t is selected *after* the forecaster's prediction μ_t, c_t are revealed, so the agent has last-move advantage.

Nevertheless asymptotic exactness can be achieved as shown in Theorem 1 (proof in Appendix D). In fact, we design a post-processing algorithm that modifies the prediction of a base algorithm (similar to recalibration (16; 11)). Algorithm 1 can modify any base algorithm (as long as the base algorithm outputs some μ_t, c_t at every time step) to achieve asymptotic exactness, even though the finite time performance could be hurt by a poor base prediction algorithm.

Theorem 1. *Suppose there is a constant $M > 0$ such that $\forall t$, $|b_t| \leq M$, there exists an algorithm to output μ_t, c_t in Protocol 2 that is asymptotically exact for μ_t^*, b_t generated by any strategy of nature and agent. In particular, Algorithm 1 satisfies*

$$\left(\frac{1}{T} \sum_{t=1}^{T} b_t(\mu_t - y_t) - |b_t| c_t \right)^2 = O\left(\sqrt{\frac{\log T}{T}} \right)$$

For this paper we use as our base algorithm a simple online gradient descent algorithm (23) shown in Algorithm 2. Specifically Algorithm 2 learns two regression models (such as neural networks with a single real number as output) μ_θ and c_ϕ. μ_θ is trained to predict μ_t^* by minimizing the standard L_2 loss $\min_\theta \sum_{\tau=1}^{t} (\mu_\theta(x_\tau) - y_\tau)^2$ while c_ϕ is trained to to minimize the squared payoff of each bet $\min_\phi \sum_{\tau=1}^{t} (b_\tau(\hat{\mu}_\tau - y_\tau) - |b_\tau| c_\phi(x_\tau))^2$

Based on Algorithm 2, Algorithm 1 learns an additional "correction" parameter $\lambda_t \in \mathbb{R}$ by invoking Algorithm 3. Intuitively, up to time t, if the forecaster has positive cumulative loss in Protocol 2, then the

c_ts have been too small in the past, Algorithm 1 will select a larger λ_t to increase c_t; conversely if the forecaster has negative cumulative loss, then the c_ts have been too large in the past, and Algorithm 1 will select a smaller λ_t to decrease c_t.

Despite the straight-forward intuition, the difficulty comes from ensuring Theorem 1 for *any* sequence of $y_t, b_t, t = 1, \cdots$. In fact, Algorithm 3 needs to be a swap regret minimization algorithm (3). For a detailed explanation and proof of why using Algorithm 3 can guarantee Theorem 1 see Appendix D.

4.2 Offline Forecasting

Our new definition of asymptotic soundness in Eq.(4) is related to existing notions of calibration. Asymptotic soundness depends on the set of bets $b_t, t = 1, \cdots, T$, which are further determined by the downstream decision tasks. In fact, we prove we can recover existing notions of calibration (5; 11; 15; 17) or multicalibration (13) for special decision tasks. For more details see Appendix B.2.

If a forecaster satisfies the existing notions of calibration, there is some set of functions $\mathcal{B}$, such that the forecaster is asymptotically sound if the agents choose $b_t = b(x_t)$ based on some $b \in \mathcal{B}$. The benefit is that once deployed, the forecaster does not have to be updated (compared to the online setup where the forecaster must continually update via Algorithm 1). However, the short-coming is that we must make strong assumptions on how the agents choose bets to insure themselves.

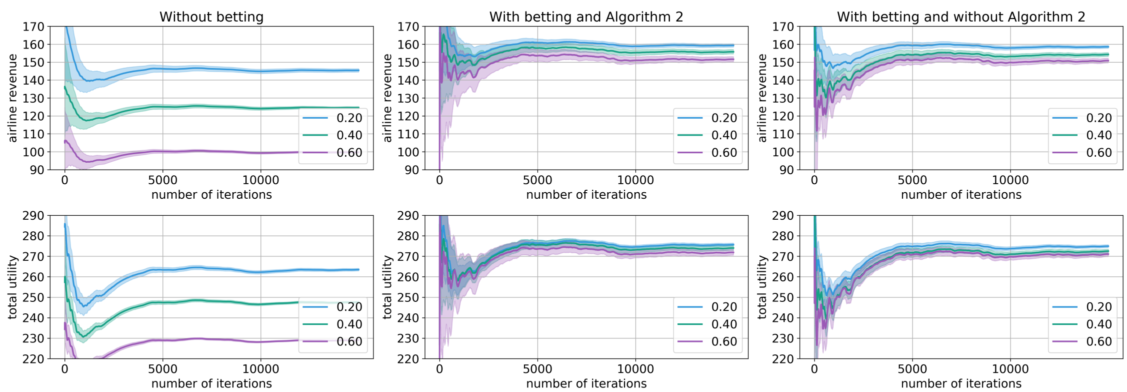

Figure 1: The airline's revenue (**Top**) and total utility (of both airline and passenger, **Bottom**) with and without the betting mechanism. Different colors represent the percentage of cautious passengers. The x-axis represents the number of flights that has happened, and the y-axis represents the average utility per passenger across all past flights. **Left**: Without the betting mechanism that insure passengers against delay **Middle and Right**: With the betting mechanism, the airline revenue increases (it is able to charge a higher ticket price due to increased demand) and the total utility increases. The middle panel is the utility with both Algorithm 1 and Algorithm 3, while the right panel only uses Algorithm 1 (i.e. it always sets $\lambda_t = 0$). In general the middle panel achieves faster convergence.

5 Case Study on Flight Delays

In this section we study a practical application that could benefit from our proposed mechanism. Compared to other means of transport, flights are often the fastest, but usually the least punctual. Different passengers may have different losses in case of delay. For example, if a passenger needs to attend an important event on-time, the loss from a delay can be very large, and the passenger might want to choose an alternative transportation method. The airline company could predict the probability of delay, and each passenger could use the probability to compute their expected loss before deciding to fly or not. However, as argued in Section 2.2, there is in general no good way to know that these probabilities are correct. Even worse, the airline may have the incentive to under-report the probability of delay to attract passengers.

Instead the airline can use Protocol 2 to convey confidence to the passengers that the delay probability is accurate. In this case, Protocol 2 has a simple form that can be easily explained to passengers as a "delay insurance". In particular, if a passenger buys a ticket, he can choose to insure himself against delay by specifying the amount b_t^1 he would like to get paid if the airplane is delayed. The airlines provides a quote on the cost b_t^0 (i.e. the passenger pays b_t^0 if the flight is not delayed). Note that this would be equivalent to Protocol 2 if the airline first predicts the probability of delay μ_t, c_t and then quotes $b_t^0 := \frac{b_t^1(\mu_t + c_t)}{1 - \mu_t - c_t}$.

If a passenger buys the right insurance according to Proposition 2, their expected utility (or negative loss) will be fixed — she does not need to worry that the predicted delay probability might be incorrect. In addition, if the airline follows Algorithm 1 the airline is also guaranteed to not lose money from the "delay insurance" in the long run (no matter what the passengers do), so the airline should be incentivized to implement the insurance mechanism to benefit its passengers "for free".

Passenger Model Since the passengers' utility functions are unknown, we model three types of passengers that differ by their assumptions on μ_t^* when they make their decision:

1. Naive passengers don't care about delays and assume the airline doesn't delay.
2. Trustful passengers assume the delay probability forecasted by the airline is correct.
3. Cautious passengers assume the worst (i.e. they choose actions that maximizes their worst case utility)

In this experiment we will vary the proportion of cautious passengers, and equally split the remaining passengers between naive and trustful. The naive and trustful passengers do not care about the risk of mis-prediction, so they do not buy the delay insurance (i.e. they always choose $b_t^1 = 0$), while cautious passengers always buy insurance that maximize their worst case utility.

5.1 Simulation Setup

Dataset We use the flight delay and cancellation dataset (7) from the year 2015, and use flight records of the single biggest airline (WN). As input feature, we convert the source airport, target airport, and scheduled time into one-hot vectors, and binarize the arrival delay into 1 (delay $>$ 20min) and 0 (delay $<$ 20min). We use a two layer neural network with the leaky ReLU activation for prediction.

Passenger Utility Let $y \in \{0, 1\}$ denote whether a delay happens, and $a \in \{0, 1\}$ denote whether the passenger chooses to ride the plane. We model the passenger utility (negative loss) as

$$-l(y, a) = \begin{cases} y = *, a = 0 & r^{\text{alt}} \\ y = 0, a = 1 & r^{\text{trip}} - c^{\text{ticket}} \\ y = 1, a = 1 & r^{\text{trip}} - c^{\text{ticket}} - c^{\text{delay}} \end{cases}$$

where r^{alt} is the utility of the alternative option (e.g. taking another transportation or cancelling the trip). r^{trip} is the reward of the trip, c^{ticket} is the cost of the ticket, and c^{delay} is the cost of a delayed flight. For each flight we sample potential passengers by randomly drawing r^{alt}, r^{trip} and c^{delay} (details in appendix).

Airline Pricing Based on the passenger type (naive, trustful, cautious) and passenger parameter r^{alt}, r^{trip} and c^{delay}, each passenger will have a maximum they are willing to pay for the flight. For simplicity we assume the airline will choose c^{ticket} at the highest price for which it can sell 300 tickets. The passengers who are willing to pay more than c^{ticket} will choose $a = 1$, and other passengers will choose $a = 0$.

5.2 Delay Insurance Improves Total Utility

The simulation results are shown in Figure 1. Using the betting mechanism is strictly better for both the airline's revenue (i.e. ticket price * number of tickets) and the total utility (airline revenue + passenger utility). This is because the cautious passengers always make decisions to maximize their worst case utility. With the betting mechanism, their worst case utility becomes closer to their actual true utility, so their decision ($a = 1$ or $a = 0$) will better maximize their true utility. The airline also benefits because it can charge a higher ticket price due to increased demand (more cautious passengers will choose $a = 1$).

We also consider several alternatives to Algorithm 3. The alternative algorithms do not provide theoretical guarantees; in practice, they also achieve worse convergence to the final utility. This is be a reason to prefer Algorithm 3 if the number of iterations T is small.

6 Conclusion

In this paper, we propose an alternative solution to address the impossibility of individual calibration based on an insurance between the forecaster and decision makers. Each decision maker can make decisions as if the forecasted probability is correct, while the forecaster can also guarantee not losing in the long run. Future work can explore other issues that arise from this protocol, such as honesty (10), fairness (9), and social/moral/legal implications.

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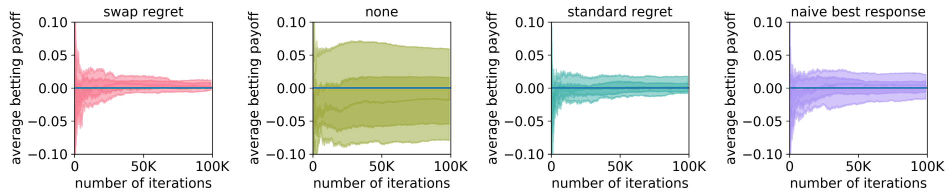

Figure 2: Comparing forecaster loss in Protocol 2 for different forecaster algorithms on MNIST (results for Adult dataset are in appendix C.2). Each plot is an average performance across 20 different decision tasks, where we plot the top 10%, 25%, 50%, 75%, 90% quantile in forecaster loss. If the forecaster achieves asymptotic exactness defined in Eq.(5), then the loss should be close to 0. **Left** panel is Algorithm 1, and the rest are other seemingly reasonable algorithms explained in Section A. The loss of a forecaster that use Algorithm 1 typically converges to 0 faster, while alternative algorithms often fail to converge.

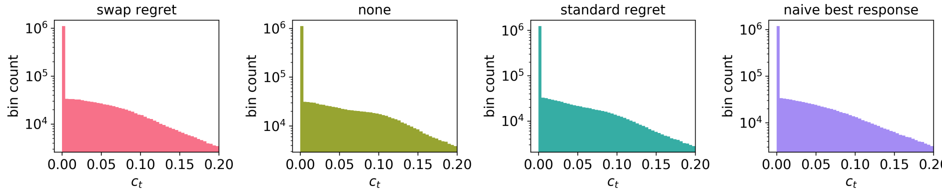

Figure 3: Histogram of the interval size c_t produced by the forecaster algorithm across all the tasks. There is no noticeable difference between the different algorithms. Notably the interval sizes are typically quite small, and big interval size is exponentially less common.

A Additional Experiments

We further verify our theory with simulations a diverse benchmark of decision tasks. We also do ablation study to show that Algorithm 3 is necessary. Several simpler alternatives often fail to achieve asymptotic exactness and have worse empirical performance.

Dataset and Decision Tasks We use the MNIST and UCI Adult (8) datasets. MNIST is a multi-class classification dataset; we convert it to binary classification by choosing $\Pr[Y = 1 \mid l(x) = i] = (i + 1)/11$ where the $l(x) \in \{0, 1, \cdots, 9\}$ is the digit category. We also generate a benchmark consisting of 20 different decision tasks. For details see Appendix C.2.

Comparison We compare several forecaster algorithms that differ in whether they use Algorithm 3 to adjust the parameter λ_t. In particular, **swap regret** refers to Algorithm 3; **none** does not use λ_t and simply set it to 0; **standard regret** minimizes the standard regret rather than the swap regret; **naive best response** chooses the λ_t that would have been optimal were it counter-factually applied to the past iterations.

Forecaster Model As in the previous experiment, we use a two layer neural network as the forecaster μ_θ and c_ϕ. For the results shown in Figure 6 we also use histogram binning (18) on the entire validation set to recalibrate μ_θ, such that μ_θ satisfies standard calibration (11).

Results The results are plotted in Figure 2,3 in the main paper and Figure 5,6 in Appendix C.2. There are three main observations: **1)** Even when a forecaster is calibrated, for individual decision makers, the expected loss under the forecaster probability is almost always incorrect. **2)** Algorithm 1 has good empirical performance. In particular, the guarantees of Theorem 1 can be achieved within a reasonable number of time steps, and the interval size c_t is usually small. **3)** Seemingly reasonable alternatives to Algorithm 1 often empirically fail to be asymptotically exact.

B Additional Results

B.1 Multiclass Prediction

For multiclass prediction, we suppose that Y can take K distinct values. We denote Δ^K as the K-dimensional probability simplex. For notational convenience we represent Y as a one-hot vector in $\mathbb{R}^K$, so $\mathcal{Y} = \{(1, 0, \cdots), (0, 1, \cdots), \cdots\}$.

Protocol 3: Decision Making with Bets, Multiclass At time $t = 1, \cdots, T$

1. Nature reveals $x_t \in \mathcal{X}$ and chooses $\mu_t^* \in \Delta^K$ without revealing it
2. Forecaster reveals $\mu_t \in \Delta^K$ and $c_t \in R_+^K$
3. Agent t has loss $l_t : \mathcal{Y} \times \mathcal{A} \to \mathbb{R}$ and chooses action a_t and $g_t \in \mathbb{R}^K$
4. Sample $y_t \sim \text{Categorical}(\mu_t^*)$ and reveal y_t
5. Agent total loss is $l_t(y_t, a_t) - \langle g_t, y_t - \mu_t \rangle + \langle |g_t|, c_t \rangle$, forecaster loss is $\langle g_t, y_t - \mu_t \rangle - \langle |g_t|, c_t \rangle$

As before we require the regularity condition that $\mu_c + c_t \in [0,1]^K$ and $\mu_t - c_t \in [0,1]^K$ (even though these are no longer on Δ^K, hence not probabilities.

Similar to Section 3 we can denote the agent's maximum / minimum expected loss under the forecasted probability as

$$L_t^{\max} = \max_{\tilde{\mu} \in \Delta^K, \tilde{\mu} \in \mu_t \pm c_t} \mathbb{E}_{Y \sim \tilde{\mu}}[l_t(a_t, Y)]$$
$$L_t^{\min} = \min_{\tilde{\mu} \in \Delta^K, \tilde{\mu} \in \mu_t \pm c_t} \mathbb{E}_{Y \sim \tilde{\mu}}[l_t(a_t, Y)]$$

and true expected loss as $L_t^* = \mathbb{E}_{Y \sim \mu_t^*}[l_t(a_t, Y)]$. As before denote

$$\mathrm{L}_t^{\text{pay}} = L_t^* + \mathbb{E}_{\mu^*}[\langle g_t, \mu_t - Y \rangle + \langle |g_t|, c_t \rangle]$$

Proposition 3. *If $g_t = l(\cdot, a_t) - \inf_{\gamma \in \mathbb{R}} \langle c_t, |l - \gamma 1| \rangle$ then $L_t^{\text{pay}} = L_t^{\max}$*

Proof of Proposition 3. As a notation shorthand we denote $l_t(a_t, Y)$ with the vector l, such that $l_i = l_t(a_t, Y = i)$. We first show a closed form solution for $L_t^{\max}$ which can be written as

$$\begin{aligned}
L_t^{\max} &= \sup_{\tilde{\mu} \in \Delta^K, \tilde{\mu} \in \mu_t \pm c_t} \mathbb{E}_{Y \sim \tilde{\mu}}[l_t(a_t, Y)] \\
&= \sup_{\tilde{\mu} \in \Delta^K, \tilde{\mu} \in \mu_t \pm c_t} \langle \tilde{\mu}, l \rangle && \text{Notation Change} \\
&= \langle \mu_t, l \rangle + \sup_{\delta\mu \in [-c_t, c_t], \langle \delta\mu, 1 \rangle = 0} \langle \delta\mu, l \rangle && \text{Algebric Manipulation} \\
&= \langle \mu_t, l \rangle + \sup_{\delta\mu \in [-c_t, c_t]} \inf_{\gamma \in \mathbb{R}} \langle \delta\mu, l \rangle - \gamma \langle \delta\mu, 1 \rangle && \text{Lagrangian} \\
&= \langle \mu_t, l \rangle + \inf_{\gamma \in \mathbb{R}} \sup_{\delta\mu \in [-c_t, c_t]} \langle \delta\mu, l \rangle - \gamma \langle \delta\mu, 1 \rangle && \text{Sion Minimax Theorem} \\
&= \langle \mu_t, l \rangle + \inf_{\gamma \in \mathbb{R}} \langle c_t, |l - \gamma 1| \rangle
\end{aligned}$$

Similarly we have

$$L_t^{\min} = \langle \mu_t, l \rangle - \inf_{\gamma \in \mathbb{R}} \langle c_t, |l - \gamma 1| \rangle$$

Denote the γ that achieves the infimum as γ^*. Comparing with L_t^{pay} we have

$$\begin{aligned}
L_t^{\text{pay}} &= L_t^* + \mathbb{E}_{\mu^*}[\langle g_t, \mu_t - Y \rangle + \langle |g_t|, c_t \rangle] \\
&= \langle \mu^*, l \rangle - \langle l - \gamma^* 1, \mu_t - \mu_t^* \rangle + \langle c_t, |l - \gamma^* 1| \rangle \\
&= \langle l, \mu_t \rangle + \langle c_t, |l - \gamma^* 1| \rangle && \langle \mu_t, 1 \rangle = 0, \langle \mu_t^*, 1 \rangle = 0 \\
&= L^{\max}
\end{aligned}$$

□

B.2 Offline Calibration

For this section we restrict to the i.i.d. setup, where we assume there are random variables X, Y with some distribution p_{XY}^* such that at each time step,

$$x_t \sim X \qquad \mu_t^* = \mathbb{E}[Y \mid x_t]$$

We also assume that the forecaster 's choice μ_t, c_t and the agent's choice b_t in Protocol 2 are computed by functions of x_t

$$\mu : x_t \mapsto \mu_t \quad c : x_t \mapsto c_t \quad b : x_t \mapsto b_t$$

The following definition is the specialization of asymptotic soundness in Eq.(4) to the i.i.d. setup

Definition 1. *We say that the functions μ, c are sound with respect to some set of functions $\mathcal{B} \subset \{\mathcal{X} \to [-M, M]\}$ if*

$$\sup_{g \in \mathcal{B}} \mathbb{E}[b(X)(\mu(X) - \mathbb{E}[Y \mid X]) - |b(X)|c(X)] \leq 0$$

If $c(x) \equiv 0$ we say μ is $\mathcal{B}$-calibrated.

B.2.1 Examples and Special Cases

Standard Calibration Standard calibration is defined as: for any $u \in [0, 1]$, among the X where $\mu(X) = u$ it is indeed true that Y is 1 with u probability. Formally this can be written as

$$\mathbb{E}[Y \mid \mu(X) = u] = u, \forall u \in [0, 1]$$

Deviation from this ideal situation is measured by the maximum calibration error (MCE).

$$\mathrm{MCE}(\mu) = \max_{u \in [0,1]} |\mathbb{E}[Y \mid \mu(X) = u] - u|$$

Note that the MCE may be ill-defined if there is an interval $(u_0, u_1) \subset [0, 1]$ such that $\mu(X) \in (u_0, u_1)$ with zero probability. We are going to avoid the technical subtlety by assuming that this does not happen, i.e. the distribution of $\mu(X)$ is supported on the entire set $[0, 1]$.

When $\mathcal{B}$ is the set of all possible functions $\mu(x) \to \mathbb{R}$ (i.e. it only depends on the probability forecast $\mu(x)$ but not x itself), we obtain the standard definition of calibration (5; 11), as shown by the following proposition

Proposition 4. *The forecaster function $\mu : \mathcal{X} \to [0, 1], c : x \mapsto c_0$ is sound with respect to $\mathcal{B} = \{b : \mu(x) \to \mathbb{R}\}$ if and only if the MCE error of μ is less than c_0.*

Proof. See Appendix E □

Multi-Calibration Multi-calibration achieves standard calibration for all subsets S in some collection of sets $\mathcal{S}$, and denote c_S as the indicator function of S (i.e. $c_S(x) = 1$ iff $x \in S$). Suppose $\mathcal{B}$ consists of all functions of the form $x, \mu \mapsto c_S(x)\tilde{b}(\mu)$ where $\tilde{b}$ is an arbitrary function. A forecaster that is sound with respect to this set of $\mathcal{B}$ is also multicalibrated.

B.2.2 Soundness and Calibration

This section aims to argue that if the forecaster achieves existing definitions of calibration, then it is sound under some assumptions on decision making agents.

Standard Calibration We assume that every decision agent select b_t according to Proposition 2 and the loss l_t does not depend on t (i.e. every decision maker has the same loss), in addition we assume that when given the same prediction μ_t the decision maker will always select the same a_t (for example, this would be true if every decision maker have the same loss, and they always choose the action that minimizes expected loss). Under these assumptions,a forecaster that satisfies standard calibration will also be asymptotically sound.

To see why this is, a_t only depends on μ_t, and l_t is independent of t; so $b_t := l_t(1, a_t) - l_t(0, a_t)$ will only depend on μ_t. This satisfies the condition of Appendix B.2.1; a forecaster that satisfies standard calibration will also be sound.

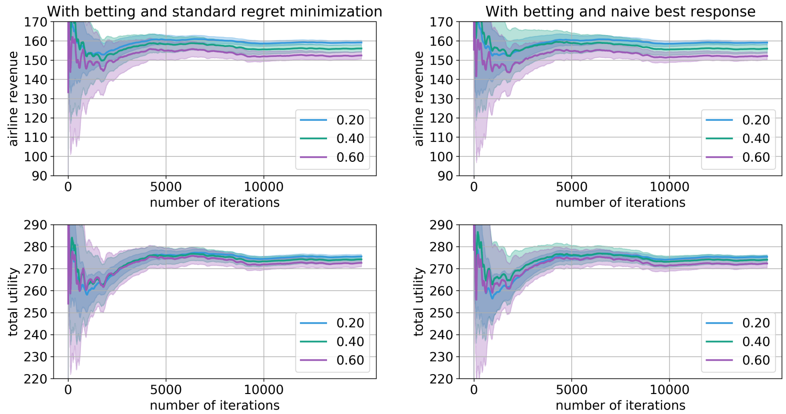

Figure 4: This plot extends Figure 1. We compare with additional Alternatives to Algorithm 3.

C Experiment Details and Additional Results

C.1 Airline Delay

Negative c_t In Protocol 2 c_t must be non-negative for its interpretation as a probability interval $[\mu_t - c_t, \mu_t + c_t]$. However if we only consider the flight delay insurance interpretation: airline pay passenger b_t^1 if flight delays and passenger pays airline $b_t^0 := \frac{b_t^1(\mu_t+c_t)}{1-\mu_t-c_t}$ if flight doesn't delay. These payments are meaningful for both positive and negative c_t; the passenger utility (with insurance) can be computed as $r^{\text{trip}} - c^{\text{ticket}} - (\mu_t + c_t)c^{\text{delay}}$, which is also meaningful for both positive and negative c_t. We find that allowing negative c_t improves the stability of the algorithm.

Passenger Model We sample r^{alt} as $\text{Uniform}(0, 200)$ and sample r^{trip} from $\text{Uniform}(0, 400)$. We assume the cost of delay can be more varied, so we sample it from the following process: $z \sim \text{Uniform}(4, 9)$ and $c^{\text{delay}} = 0.2e^z$. This gives us a cost of delay between $[10, 1600]$, but large values are less likely.

Additional Results We show additional comparison with other alternatives to Algorithm 3 in Figure 4. For details about these alternatives see Section A.

C.2 Additional Experiments

Decision Loss For each data point we associate an extra feature z used to define decision loss. For MNIST this is the digit label and for UCI Adult this is the age (binned by quantile into 10 bins). We simulate three kinds of decision losses; for each type of decision loss we randomly sample a few instantiations.

1. One-sided: we assume that $a \in [0, 1]$ and each decision loss $l(z, y, a)$ is large if $y \neq a$ and small if $y = a$. For different values of z there are different stakes (i.e. how much does the loss when $y = a$ differ from $y \neq a$).

2. Different Stakes: Each value of the decision loss $l(z, y, a)$ is a draw from $\mathcal{N}(0, z)$, which is used to capture the feature that certain groups of people have larger stakes

3. Random. Each value of the decision loss $l(z, y, a)$ is a draw from $\mathcal{N}(0, 10)$ but clipped to be within $[-10, 10]$.

Forecasted Loss vs. True Loss In Figure 6 we plot the relationship between the expected loss under the forecasted probability and the expected loss under the true probability (we can compute this for the MNIST dataset because the true probability is known as explained in Section A). Even if we apply histogram binning recalibration (explained in Section A), the individual probabilities are almost always incorrect.

Asymptotic Exactness In Figure 2 and Figure 5 we plot the average betting loss of the forecaster. Algorithm 1 consistently achieve better asymptotic exactness compared to alternatives.

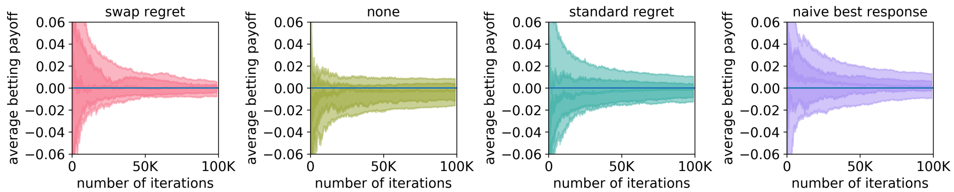

Figure 5: This plot is identical to Figure 2 but for the Adult dataset

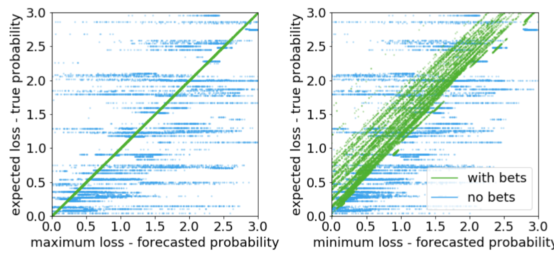

Figure 6: The expected loss under the forecaster utility vs. expected loss under the true probability. Each dot represents an individual probability forecast with a particular choice of loss function. We use histogram binning on the entire validation set to recalibrate the forecaster. Even though the forecaster is calibrated, the individual probabilities are often incorrect. Therefore, the expected loss under the forecasted probability often differs from the expected loss under the true probability (blue dots). On other hand, with additional payment from the bets, the expected total loss under true probability is always bounded between the minimum loss under the forecasted probability, and the maximum loss under the forecasted probability.

Average Interval Size In Figure 3 we plot the interval size c_t. A small c_t satisfies desideratum 2 in Section 3 and makes the guarantee in Proposition 2 useful for decision makers. We observe that most interval sizes are small, and larger intervals are exponentially unlikely.

D Proof of Online Prediction

The goal of Algorithm 3 is to select a sequence of λ_t to minimize the loss $\sum_{t=1}^T (r_t + s_t\lambda_t)^2$. However, instead of a standard online regret minimization, Algorithm 3 tries to minimize the swap regret. Let $L^1[-1,1]$ denote the set of 1-Lipshitz functions $\mathbb{R} \to [-1,1]$, define the swap regret as

$$R_T^{\text{swap}} = \sum_{t=1}^T (r_t + s_t\lambda_t)^2 - \inf_{\psi \in L^1[-1,1]} \sum_{t=1}^T (r_t + s_t\psi(\lambda_t))^2 \tag{6}$$

Intuitively, $\sum_{t=1}^T (r_t + s_t\psi(\lambda_t))^2$ is the loss of an alternative policy: whenever the algorithm selects λ_t, select $\psi(\lambda_t)$ instead. Swap regret measures the additional loss compared to the best alternative policy.

For Algorithm 3 we have the following Guarantee.

Theorem 2. *If there exists M_1, M_2 such that $\forall t, |s_t| \leq M_1, |r_t/s_t| \leq M_2$, then there exists constant $C(M_1, M_2) > 0$, for any choice of $K > 1$, the regret of Algorithm 3 is bounded by*

$$R_T^{\text{swap}} \leq C(M_1, M_2)K^2 \log T + \frac{1}{K^2}\sum_{t=1}^T s_t^2$$

In particular, if we choose $K^2 = \sqrt{T/\log T}$ then the swap regret is bounded by $O(\sqrt{T}\log T)$.

Before we prove Theorem 2 we show how to use it to prove Theorem 1

Proof of Theorem 1. Recall the definition of the swap regret

$$R_T^{\text{swap}} = \sum_{t=1}^T (r_t + s_t\lambda_t)^2 - \inf_{\psi \in L^1[-1,1]} \sum_{t=1}^T (r_t + s_t\psi(\lambda_t))^2$$

To prove this theorem we need the following Lemma, which is a inequality proved by simple algebra manipulations.

Lemma 2. *For any choice of* $r_t, s_t, \lambda_t, t = 1, \cdots, T$ *we have*

$$\left(\frac{1}{T}\sum_{t=1}^{T} s_t(r_t + s_t\lambda_t)\right)^2 \leq \frac{R_T^{\text{swap}}}{T^2}\sum_{t=1}^{T} s_t^2$$

Because at each iteration the algorithm selects $r_t = \frac{b_t}{\sqrt{|b_t|}}(\mu_t - y_t) - \sqrt{|b_t|}\hat{c}_t$ and $s_t = -\sqrt{|b_t|}$ we can plug this into Lemma 2 and conclude that for any sequence of λ_t (in particular, if the λ_t is chosen by Algorithm 3) must satisfy

$$\left(\frac{-1}{T}\sum_{t=1}^{T} b_t(\mu_t - y_t) - |b_t|\hat{c}_t - |b_t|\lambda_t\right)^2 \leq \frac{R_T^{\text{swap}}}{T}\frac{1}{T}\sum_{t=1}^{T}|s_t| \leq \frac{MR_T^{\text{swap}}}{T}$$

In addition we have

$$\left|\frac{r_t}{s_t}\right| = \left|\frac{b_t}{|b_t|}(\mu_t - y_t) + \hat{c}_t\right| \leq 2$$

So we can apply Theorem 2 to conclude $R_T^{\text{swap}} = O(\sqrt{T\log T})$. Combined we have

$$\left(\frac{1}{T}\sum_{t=1}^{T} b_t(\mu_t - y_t) - |b_t|(\hat{c}_t + \lambda_t)\right)^2 = O(M\sqrt{T\log T}/T) = O(\sqrt{\log T/T})$$

□

Now we proceed to prove Theorem 2

Proof of Theorem 2. To prove this theorem we first need the following Lemma

Lemma 3. *If there exists some* $M_1, M_2 > 0$ *such that* $\forall t$, $|\beta_t| \leq M_1$ *and* $|\alpha_t/\beta_t| \leq M_2$, *choosing* $\lambda_t = \arg\inf_{\lambda\in\mathbb{R}}\sum_{\tau=1}^{t-1}(\alpha_\tau + \beta_\tau\lambda)^2$ *satisfies for some constant* $C(M_1, M_2) > 0$

$$\sum_{t=1}^{T}(\alpha_t + \beta_t\lambda_t)^2 \leq \inf_{\lambda}\sum_{t=1}^{T}(\alpha_t + \beta_t\lambda)^2 + C(M_1, M_2)\log T$$

The proof strategy is to first bound the discretized swap regret, defined as follows

$$\tilde{R}_T^{\text{swap}} = \sum_{t=1}^{T}(r_t + s_t\lambda_t)^2 - \sum_{k=1}^{K}\inf_{\lambda}\sum_{t=1}^{T}\mathbb{I}(\lambda_t \in [v_k, v_{k+1}))(r_t + s_t\lambda)^2$$

As a notation shorthand we denote $c^t(\lambda) = (r_t + s_t\lambda)^2$ and $\mathcal{I}_k = [v_k, v_{k+1})$, for each k if we set $\alpha_t = r_t\mathbb{I}(\lambda_t \in \mathcal{I}_k)$ and $\beta_t = s_t\mathbb{I}(\lambda_t \in \mathcal{I}_k)$ with the convention that $0/0 = 0$. By the assumptions in Theorem 2 we know that $|\beta_t| \leq M_1$ and $|\alpha_t/\beta_t| \leq M_2$ so we can guarantee by Lemma 3 that

$$\begin{aligned}\sum_{t=1}^{T}\mathbb{I}(\lambda_t \in \mathcal{I}_k)c^t(\lambda_t^k) &= \sum_{t=1}^{T}(\alpha_t + \beta_t\lambda_t^k)^2 \\ &\leq \inf_{\lambda}\sum_{t=1}^{T}(\alpha_t + \beta_t\lambda)^2 + C(M_1, M_2)\log T = \inf_{\lambda}\sum_{t=1}^{T}\mathbb{I}(\lambda_t \in \mathcal{I}_k)c^t(\lambda) + C(M_1, M_2)\log T\end{aligned}$$

The total loss is given by

$$\sum_{t=1}^{T}c^t(\lambda_t) = \sum_{t=1}^{T}\sum_{k=1}^{K}\mathbb{I}(\lambda_t \in \mathcal{I}_k)c^t(\lambda_t^k) \leq \sum_{k=1}^{K}\inf_{\lambda}\sum_{t=1}^{T}\mathbb{I}(\lambda_t \in \mathcal{I}_k)c^t(\lambda) + C(M_1, M_2)K\log T$$

from which we can conclude that

$$\tilde{R}_T^{\text{swap}} := \sum_{t=1}^{T} c^t(\lambda_t) - \sum_{k=1}^{K} \inf_{\lambda} \sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_k) c^t(\lambda) \le C(M_1, M_2) K \log T$$

Finally we conclude the proof of the theorem with the following Lemma that bounds the difference between the discretized swap regret and the continuous swap regret.

Lemma 4. *In Algorithm 3,* $R_T^{\text{swap}} \le \tilde{R}_T^{\text{swap}} + \sum_{t=1}^{T} s_t^2 \frac{v_K - v_0}{K}$

□

Proof of Lemma 4. Denote $\mathcal{I}_k = [v_k, v_{k+1})$ and denote $\delta v = \max_k v_{k+1} - v_k$. In addition denote $\lambda_k^* = \arg\inf_\lambda \sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_k)(r_t + s_t\lambda)^2$

$$\begin{aligned}
& R_T^{\text{swap}} - \tilde{R}_T^{\text{swap}} \\
&= \sum_{k=1}^{K} \inf_{\lambda} \sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_k)(r_t + s_t\lambda)^2 - \inf_{\psi \in L^1} \sum_{t=1}^{T} (r_t + s_t\psi(\lambda_t))^2 && \text{Definition} \\
&= \sum_{k=1}^{K} \inf_{\lambda} \sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_k)(r_t + s_t\lambda)^2 - \inf_{\psi \in L^1} \sum_{k=1}^{K} \sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_K)(r_t + s_t\psi(\lambda_t))^2 && \text{Decompose 2nd term} \\
&\le \sum_{k=1}^{K} \left(\inf_{\lambda} \sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_k)(r_t + s_t\lambda)^2 - \inf_{\psi \in L^1} \sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_K)(r_t + s_t\psi(\lambda_t))^2 \right) && \text{Jensen} \\
&= \sum_{k=1}^{K} \left(\sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_k)(r_t + s_t\lambda_k^*)^2 - \inf_{\delta\psi \in L^1} \sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_K)(r_t + s_t(\lambda_k^* + \delta\psi(\lambda_t)))^2 \right) && \text{Change of variable} \\
&\le \sum_{k=1}^{K} \sum_{t=1}^{T} \mathbb{I}(\lambda_t \in \mathcal{I}_k) s_t^2 \delta_v^2 && \text{1-Lipschitzness} \\
&= \sum_{t=1}^{T} s_t^2 \delta_v^2
\end{aligned}$$

For Algorithm 3 we know that $\delta v = \frac{v_K - v_0}{K}$ because of the equal width partition. □

Lemma 3. *If there exists some $M_1, M_2 > 0$ such that $\forall t$, $|\beta_t| \le M_1$ and $|\alpha_t/\beta_t| \le M_2$, choosing $\lambda_t = \arg\inf_{\lambda \in \mathbb{R}} \sum_{\tau=1}^{t-1} (\alpha_\tau + \beta_\tau\lambda)^2$ satisfies for some constant $C(M_1, M_2) > 0$*

$$\sum_{t=1}^{T} (\alpha_t + \beta_t\lambda_t)^2 \le \inf_{\lambda} \sum_{t=1}^{T} (\alpha_t + \beta_t\lambda)^2 + C(M_1, M_2) \log T$$

Proof of Lemma 3. The proof strategy is similar to Chapter 4 of (4). Define $\lambda_t^* = \arg\inf_\lambda \sum_{\tau=1}^{t} (\alpha_\tau + \beta_\tau\lambda)^2$. In words the only difference between λ_t^* and λ_t is that λ_t^* can look one step into the future. Then by Lemma 3.1 of (4) we have

$$R_T := \sum_{t=1}^{T} (\alpha_t + \beta_t\lambda_t)^2 - \inf_{\lambda} \sum_{t=1}^{T} (\alpha_t + \beta_t\lambda)^2 \le \sum_{t=1}^{T} (\alpha_t + \beta_t\lambda_t)^2 - (\alpha_t + \beta_t\lambda_t^*)^2 \tag{7}$$

We introduce simplified notation $r_t(\lambda) = \sum_{\tau=1}^{t} (\alpha_\tau + \beta_\tau\lambda)^2$. So with the new notation $\lambda_t = \inf_\lambda r_{t-1}(\lambda)$ and $\lambda_t^* = \inf_\lambda r_t(\lambda)$. We can compute

$$r'_{t-1}(\lambda_t) = 0, \qquad r''_{t-1}(\lambda_t) = 2\sum_{\tau=1}^{t-1} \beta_\tau^2, \qquad r'''_{t-1}(\lambda) = 0 \tag{8}$$

Also denote $\delta\lambda_t = \lambda_t^* - \lambda_t$ we have

$$\begin{aligned}
\delta\lambda_t &= \arg\inf_{\delta\lambda} r_t(\lambda_t + \delta\lambda) \\
&= \arg\inf_{\delta\lambda} r_{t-1}(\lambda_t + \delta\lambda) + (\alpha_t + \beta_t\lambda_t + \beta_t\delta\lambda)^2 && \text{By definition} \\
&= \arg\inf_{\delta\lambda} r_{t-1}(\lambda_t) + r'_{t-1}(\lambda_t)\delta\lambda + \frac{1}{2}r''_{t-1}(\lambda_t)\delta\lambda^2 + \\
&\qquad (\alpha_t + \beta_t\lambda_t)^2 + 2(\alpha_t + \beta_t\lambda_t)\beta_t\delta\lambda + \beta_t^2\delta\lambda^2 && \text{Taylor expansion} \\
&= \arg\inf_{\delta\lambda} \sum_{\tau=1}^{t-1} \beta_\tau^2\delta\lambda^2 + 2(\alpha_t + \beta_t\lambda_t)\beta_t\delta\lambda + \beta_t^2\delta\lambda^2 && \text{Apply Eq.(8) and remove irrelevant terms} \\
&= -\frac{2(\alpha_t + \beta_t\lambda_t)\beta_t}{2\sum_{\tau=1}^{t-1}\beta_\tau^2 + 2\beta_t^2} = -\frac{(\alpha_t + \beta_t\lambda_t)\beta_t}{\sum_{\tau=1}^{t}\beta_\tau^2} && \text{Minimizer of quadratic}
\end{aligned}$$

Applying the new result to Eq.(7) we have

$$\begin{aligned}
R_T &\leq \sum_{t=1}^{T} (\alpha_t + \beta_t\lambda_t)^2 - (\alpha_t + \beta_t\lambda_t^*)^2 \\
&= \sum_{t=1}^{T} (2\alpha_t + \beta_t\lambda_t + \beta_t\lambda_t^*)(\beta_t\lambda_t - \beta_t\lambda_t^*) && \text{By } (a+b)(a-b) = a^2 - b^2 \\
&= \sum_{t=1}^{T} \left(|\alpha_t + \beta_t\lambda_t| + |\alpha_t + \beta_t\lambda_t^*|\right)|\beta_t\delta\lambda_t| && \text{Cauchy schwarz} \\
&= \sum_{t=1}^{T} 2|\alpha_t + \beta_t\lambda_t||\beta_t\delta\lambda_t| && \text{By } (\alpha_t + \beta_t\lambda_t^*)^2 \leq (\alpha_t + \beta_t\lambda_t)^2 \\
&= \sum_{t=1}^{T} 2|\alpha_t + \beta_t\lambda_t| \left|\beta_t \frac{(\alpha_t + \beta_t\lambda_t)\beta_t}{\sum_{\tau=1}^{t}\beta_\tau^2}\right| && \text{Insert expression for } \delta\lambda_t \\
&\leq \sum_{t=1}^{T} 2(\alpha_t + \beta_t\lambda_t)^2 \frac{\beta_t^2}{\sum_{\tau=1}^{t}\beta_\tau^2} && \text{Cauchy schwarz} \\
&= \sum_{t=1}^{T} 2(\alpha_t/\beta_t + \lambda_t)^2 \frac{\beta_t^4}{\sum_{\tau=1}^{t}\beta_\tau^2} \\
&\leq \left(\max_t 2(\alpha_t/\beta_t + \lambda_t)^2\right) \sum_{t=1}^{T} \frac{\beta_t^4}{\sum_{\tau=1}^{t}\beta_\tau^2} && \text{Holder inequality} \\
&\leq 8M_2^2 \sum_{t=1}^{T} \frac{\beta_t^4}{\sum_{\tau=1}^{t}\beta_\tau^2} && |\lambda_t| \leq M_2
\end{aligned}$$

Finally we apply the Lemma 5 to conclude that

$$R_T \leq 8M_2^2 M_1^2 \log(T+1)$$

Lemma 5. *For any sequence* $\beta_t, t = 1, \cdots, T$ *such that* $|\beta_t| \leq M, \forall t$ *we have* $\sum_{t=1}^{T} \frac{\beta_t^4}{\sum_{\tau=1}^{t}\beta_\tau^2} \leq M^2 \log(T+1)$

□

Finally we prove the remaining unproved Lemmas

Lemma 2. *For any choice of* $r_t, s_t, \lambda_t, t = 1, \cdots, T$ *we have*

$$\left(\frac{1}{T}\sum_{t=1}^{T} s_t(r_t + s_t\lambda_t)\right)^2 \leq \frac{R_T^{\text{swap}}}{T^2}\sum_{t=1}^{T} s_t^2$$

Proof of Lemma 2. Without loss of generality assume $\frac{1}{T}\sum_{t=1}^T s_t(r_t + s_t\lambda_t) > 0$, find some $\epsilon > 0$ such that

$$\sum_{t=1}^T s_t(r_t + s_t\lambda_t) = \sum_{t=1}^T s_t^2 \epsilon$$

Such an ϵ can always be found because the range of the RHS is $[0, +\infty)$ as $\epsilon \in [0, +\infty)$ (unless all the s_t are zero, in which case the Lemma is trivially true). Therefore, there must be a solution to the equality. Because the function $\lambda_t \mapsto \lambda_t + \lambda$ is 1-Lipshitz, we have

$$\begin{aligned}
R_T^{\text{swap}} &\geq \sum_{t=1}^T (r_t + s_t\lambda_t)^2 - \inf_\lambda \sum_{t=1}^T (r_t + s_t(\lambda_t + \lambda))^2 && \text{Choose a particular } \psi \\
&\geq \sum_{t=1}^T (r_t + s_t\lambda_t)^2 - \sum_{t=1}^T (r_t + s_t(\lambda_t - \epsilon))^2 && \text{Choose a particular } \lambda \\
&= \sum_{t=1}^T (2r_t + 2s_t\lambda_t - s_t\epsilon)s_t\epsilon = 2\left(\sum_{t=1}^T s_t(r_t + s_t\lambda_t)\right)\epsilon - \sum_t s_t^2\epsilon^2 = \sum_t s_t^2\epsilon^2
\end{aligned}$$

Therefore we have

$$\left(\frac{1}{T}\sum_{t=1}^T s_t(r_t + s_t\lambda_t)\right)^2 = \frac{1}{T^2}\left(\sum_{t=1}^T s_t^2\right)^2 \epsilon^2 \leq \frac{R_T^{\text{swap}}}{T^2}\sum_{t=1}^T s_t^2$$

□

Lemma 5. *For any sequence $\beta_t, t = 1, \cdots, T$ such that $|\beta_t| \leq M, \forall t$ we have $\sum_{t=1}^T \frac{\beta_t^4}{\sum_{\tau=1}^t \beta_\tau^2} \leq M^2 \log(T+1)$*

Proof of Lemma 5. First observe that for any j if we fix the values of $\beta_t, t \neq j$, then choosing $\beta_j = M$ always maximizes $\sum_{t=1}^T \frac{\beta_t^4}{\sum_{\tau=1}^t \beta_\tau^2}$. Therefore, we have

$$\sum_{t=1}^T \frac{\beta_t^4}{\sum_{\tau=1}^t \beta_\tau^2} \leq \sum_{t=1}^T \frac{M^4}{\sum_{\tau=1}^t M^2} = M^2 \sum_{t=1}^T \frac{1}{t} \leq M^2 \int_{t=1}^{T+1} \frac{1}{t} = M^2 \log(T+1)$$

□

E Additional Proofs

Proposition 1. *For any $\mu_t, c_t, \mu_t^* \in (0,1)$ where $(\mu_t - c_t, \mu_t + c_t) \subset (0,1)$*

1. If $\mu_t^* \in [\mu_t - c_t, \mu_t + c_t]$ then $\forall l_t : \mathcal{Y} \times \mathcal{A} \to \mathbb{R}$ we have $L_t^* \in [L_t^{\min}, L_t^{\max}]$

2. If $\mu_t^* \notin [\mu_t - c_t, \mu_t + c_t]$ then $\exists l_t : \mathcal{Y} \times \mathcal{A} \to \mathbb{R}$ such that $L_t^* \notin [L_t^{\min}, L_t^{\max}]$

Proof of Proposition 1. Part I: without loss of generality assume $l_t(a_t, 1) > l_t(a_t, 0)$, denote $L_t = \mathbb{E}_\mu[l_t(a_t, Y)]$ and we also use the notation shorthand $l_t(y)$ to denote $l_t(a_t, y)$. Since $\mu^* \in [\mu_t - c_t, \mu_t + c_t]$ we have

$$\begin{aligned}
|L_t - L_t^*| &\leq \sup_{\mu^* \in \mu_t \pm c_t} |\mathbb{E}_{Y\sim\mu_t}[l_t(Y)] - \mathbb{E}_{Y\sim\mu^*}[l_t(Y)]| \\
&= \sup_{\mu^* \in \mu_t \pm c_t} |\mu_t l_t(1) + (1-\mu_t)l_t(0) - \mu_t^* l_t(1) - (1-\mu_t^*)l_t(0)| \\
&= \sup_{\mu^* \in \mu_t \pm c_t} |(\mu_t - \mu_t^*)(l_t(1) - l_t(0))| \\
&\leq c_t(l_t(1) - l_t(0))
\end{aligned}$$

by similar algebra as above we also have

$$L_t - L_t^{\min} = c_t(l_t(1) - l_t(0))$$

$$L_t^{\max} - L_t = c_t(l_t(1) - l_t(0))$$

therefore it must be that $L_t^* \geq L_t^{\min}$ and $L_t^* \leq L_t^{\max}$.

Part II: Choose $l_t(a_t, y) = y$, suppose $\mu_t^* < \mu_t - c_t$

$$L_t^* = \mathbb{E}_{Y\sim\mu^*}[Y] = \mu_t^*$$
$$L_t^{\min} = \min_{\tilde{\mu}\in\mu_t\pm c_t} \mathbb{E}_{Y\sim\tilde{\mu}}[Y] = \mathbb{E}_{Y\sim\mu_t - c_t}[Y] = \mu_t - c_t$$

but this would imply that $L_t^* < L_t^{\min}$.

Suppose $\mu_t^* > \mu_t + c_t$

$$L_t^* = \mathbb{E}_{\mu^*}[Y] = \mu_t^*$$
$$L_t^{\max} = \max_{\tilde{\mu}\in\mu_t\pm c_t} \mathbb{E}_{Y\sim\tilde{\mu}}[Y] = \mathbb{E}_{Y\sim\mu_t + c_t}[Y] = \mu_t + c_t$$

but this would imply that $L_t^* > L_t^{\max}$. □

Lemma 1. *Let $\mu, c \in (0,1)$ such that $[\mu - c, \mu + c] \subset [0,1]$, then a function $f : \mathcal{Y} \to \mathbb{R}$ satisfies $\forall\tilde{\mu} \in [\mu - c, \mu + c]$, $\mathbb{E}_{Y\sim\tilde{\mu}}[f(Y)] \leq 0$ if and only if for some $b \in \mathbb{R}$ and $\forall y \in \{0,1\}$, $f(y) \leq b(y-\mu) - |b|c$.*

Proof of Lemma 1. If: if for some $b \in \mathbb{R}$ we have $f(y) \leq b(y - \mu) - |b|c$ then for any $\tilde{\mu}$ such that $\tilde{\mu} \in [\mu - c, \mu + c]$ or equivalently $|\tilde{\mu} - \mu| \leq c$ we have

$$\mathbb{E}_{Y\sim\tilde{\mu}}[f(Y)] \leq \mathbb{E}_{Y\sim\tilde{\mu}}[b(Y - \mu) - |b|c] = b(\tilde{\mu} - \mu) - |b|c \leq |b||\tilde{\mu} - \mu| - |b|c \leq 0$$

Only if: If $\mu = 1$ or $\mu = 0$ then the proof is trivial; we consider the case where $\mu \in (0,1)$. Suppose for any $\tilde{\mu} \in [\mu - c, \mu + c]$ we have $\mathbb{E}_{Y\sim\tilde{\mu}}[f(Y)] \leq 0$ we have (by instantiating a few concrete values for $\tilde{\mu}$)

$$f(1)(\mu - c) + f(0)(1 - \mu + c) \leq 0 \tag{9}$$
$$f(1)(\mu + c) + f(0)(1 - \mu - c) \leq 0 \tag{10}$$

Choose some b such that $f(1) = b(1-\mu) - |b|c$. Such a b must exist because the range of $b \mapsto b(1-\mu) - |b|c$ is $\mathbb{R}$. If $b < 0$ then by Eq.(9) we have

$$b(1 - \mu + c)(\mu - c) + f(0)(1 - \mu + c) \leq 0, \qquad f(0) \leq -b(\mu - c) = b(0 - \mu) - |b|c$$

Conversely if $b \geq 0$ by Eq.(10) we have

$$b(1 - \mu - c)(\mu - c) + f(0)(1 - \mu + c) \leq 0, \qquad f(0) \leq -b(\mu + c) = b(0 - \mu) - |b|c$$

In either cases this is equivalent to $\forall y \in \{0,1\}$, $f(y) \leq b(y - \mu) - |b|c$. □

Proposition 2. *If the stake $b_t = l_t(a_t, 1) - l_t(a_t, 0)$ then $L_t^{\text{pay}} \in [L_t^{\min}, L_t^{\max}]$*

Proof of Proposition 2. For convenience denote $l(Y) := l_t(Y, a_t)$. Without loss of generality assume $l(1) > l(0)$

$$\begin{aligned}
L_t^{\min} &= \min_{\tilde{\mu}\in\mu_t\pm c_t} \mathbb{E}_{\tilde{\mu}}[l(Y)] = (\mu_t - c_t)l(1) + (1 - \mu_t + c_t)l(0) = \mu_t l(1) + (1 - \mu_t)l(0) - (l(1) - l(0))c_t \\
&= \mu_t^* l(1) + (\mu_t - \mu_t^*)l(1) + (1 - \mu_t^*)l(0) - (\mu_t - \mu_t^*)l(0) - (l(1) - l(0))c_t \\
&= \mathbb{E}_{\mu_t^*}[l(Y)] - (l(1) - l(0))\mathbb{E}_{\mu_t^*}[Y - \mu] - (l(1) - l(0))c_t] \leq L_t^{\text{pay}}
\end{aligned}$$

and

$$\begin{aligned}
L_t^{\max} &= \min_{\tilde{\mu}\in\mu_t\pm c_t} \mathbb{E}_{\tilde{\mu}}[l(Y)] = (\mu_t + c_t)l(1) + (1 - \mu_t - c_t)l(0) = \mu_t l(1) + (1 - \mu_t)l(0) + (l(1) - l(0))c_t \\
&= \mu_t^* l(1) + (\mu_t - \mu_t^*)l(1) + (1 - \mu_t^*)l(0) - (\mu_t - \mu_t^*)l(0) + (l(1) - l(0))c_t \\
&= \mathbb{E}_{\mu_t^*}[l(Y)] - (l(1) - l(0))\mathbb{E}_{\mu_t^*}[Y - \mu] + (l(1) - l(0))c_t = L_t^{\text{pay}}
\end{aligned}$$

□

Proposition 4. *The forecaster function $\mu : \mathcal{X} \to [0,1], c : x \mapsto c_0$ is sound with respect to $\mathcal{B} = \{b : \mu(x) \to \mathbb{R}\}$ if and only if the MCE error of μ is less than c_0.*

Proof. If the MCE error of μ is less than c_0, denote $U = \mu(X)$ by definition we have, for every $U \in [0, 1]$

$$|U - \mathbb{E}[Y \mid U]| \leq c_0 \tag{11}$$

For any $b \in \mathcal{B}$, denote $b(X) := \tilde{b}(\mu(X)) = \tilde{b}(U)$ we have

$$\begin{aligned}
\mathbb{E}[b(X)(\mu(X) - Y) - |b(X)|c(X)] &= \mathbb{E}\left[\mathbb{E}[\tilde{b}(U)(\mu(X) - Y) - |\tilde{b}(U)|c_0 \mid U]\right] && \text{Iterated Expectation} \\
&= \mathbb{E}[\tilde{b}(U)\mathbb{E}[\mu(X) - Y \mid U] - |\tilde{b}(U)|c_0] && \mathbb{E}[UZ \mid U] = U\mathbb{E}[Z \mid U] \\
&= \mathbb{E}[\tilde{b}(U)(U - \mathbb{E}[Y \mid U]) - |\tilde{b}(U)|c_0] && \text{Linearity} \\
&\leq \mathbb{E}[|\tilde{b}(U)||U - \mathbb{E}[Y \mid U]| - |\tilde{b}(U)|c_0] && \text{Cauchy Schwarz} \\
&= \mathbb{E}[|\tilde{b}(U)| \left(|U - \mathbb{E}[Y \mid U]| - c_0\right)] \leq 0 && \text{By Eq.(11)}
\end{aligned}$$

which shows that μ, c is sound.

Conversely suppose there is some interval (u_0, u_1) such that whenever $U \in (u_0, u_1)$

$$U - \mathbb{E}[Y \mid U] > c_0$$

we can choose $b(X) := \tilde{b}(U) = \mathbb{I}(U \in [u_0, u_1])$ we have

$$\mathbb{E}[b(X)(\mu(X) - Y) - |b(X)|c(X)] = \mathbb{E}[|\tilde{b}(U)| \left(|U - \mathbb{E}[Y \mid U]| - c_0\right)] > 0$$

so the forecaster is not sound. We can show a similar proof when

$$U - \mathbb{E}[Y \mid U] < c_0$$

□