Graphs' decompositions
and
resolutions of combinatorial problems

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Objectives

Objectives (of my project)

- *The first goal of this project is, starting from a graph whose tree representation is known, to solve corresponding combinatorial problems.*
  The k-coloring problem, the max clique problem or the Hamilton path problem can be explored.
- *The second purpose is to implement a graph decomposition calculator.*
- *Finally, it can be considered extensions by working on the efficiency of implementations on large-size structures, or improving the shape of displayed results.*

Objectives (of my presentation)

Now, I am going to present :

- *important concepts to understand the goals of my project ;*
- *an example where these concepts help to solve a problem of coloration.*

Contents

Tree decomposition of a graph

Definition (Tree decomposition of a graph)

A tree T is a tree decomposition of a graph G where its nodes are arranged satisfying the following properties :

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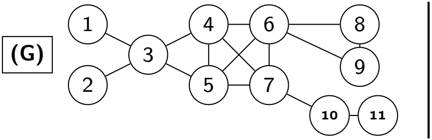

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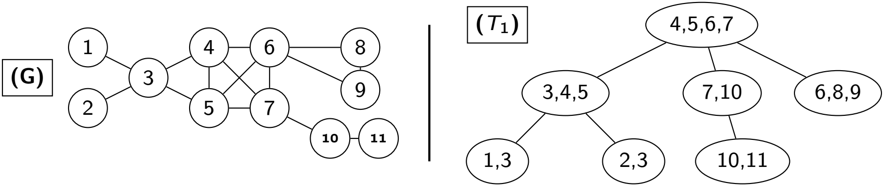

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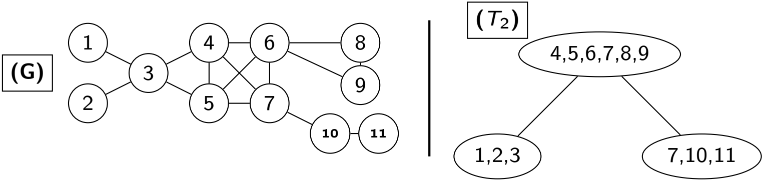

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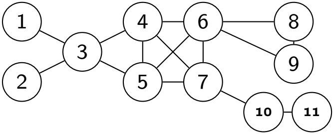
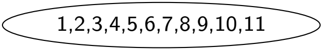


(T_3)



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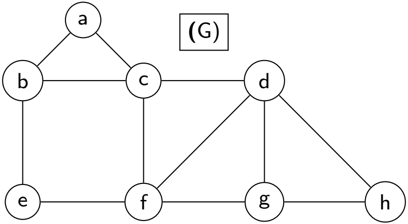

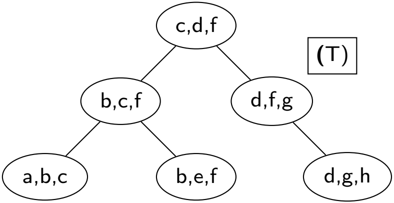

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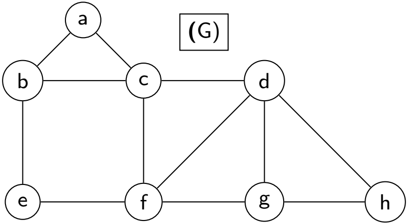
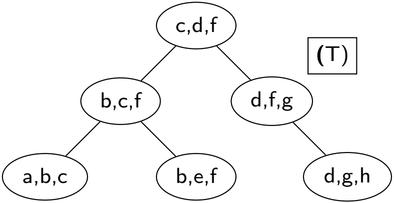




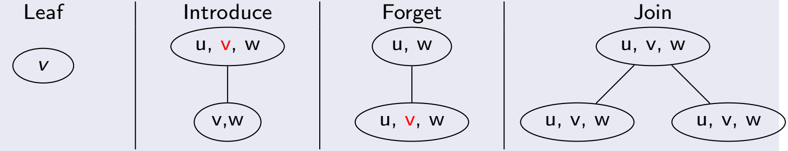
Treewidth(G)=3-1=2

Nice tree

Definition (nice tree)

A tree decomposition is *nice* if every node x is one of the following 4 types :

- **Leaf :** no children, $|B_x| = 1$ (B_x names a bag of the tree containing x)
- **Introduce :** 1 child y, $B_x = B_y \cup \{v\}$ for some vertex v
- **Forget :** 1 child y, $B_x = B_y \backslash \{v\}$ for some vertex v.
- **Join :** 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$

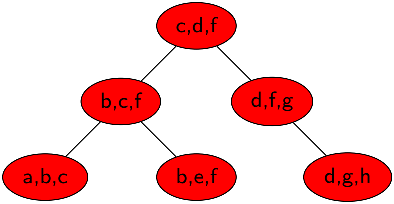


Remark

- *A tree decomposition can be turned into a nice tree decomposition*
- *A nice tree can be used to simplify a proof, or to find an easy program to solve a problem (as we will see later on).*

Example

This is a tree decomposition of the graph seen previously. How can we get a nice tree from it ?



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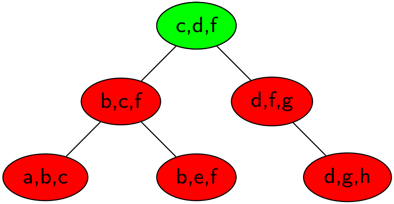

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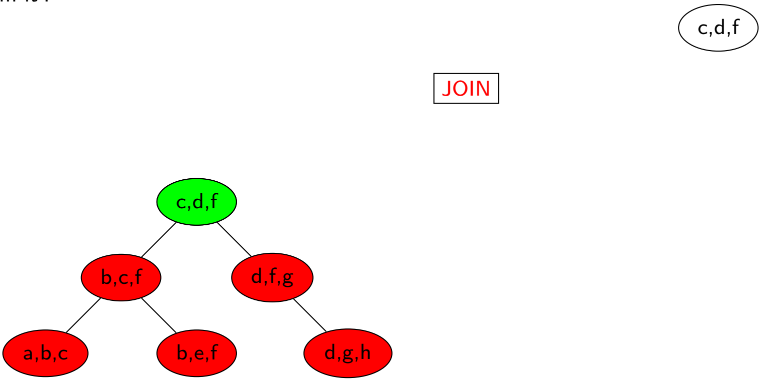

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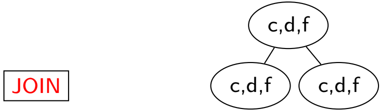

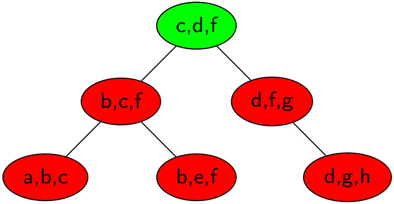

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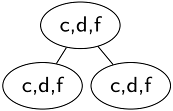
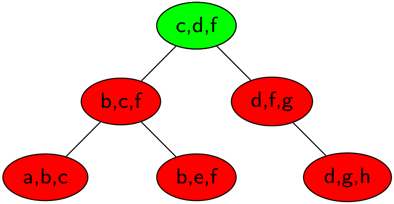


JOIN

INTRODUCE



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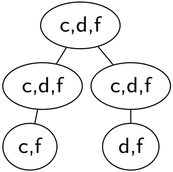

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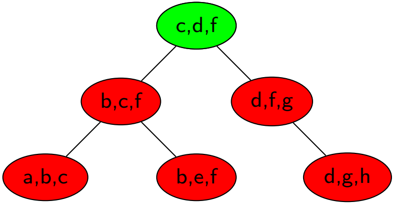

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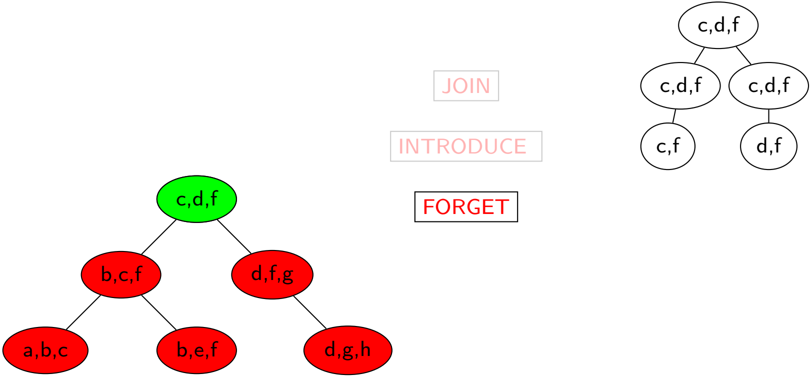

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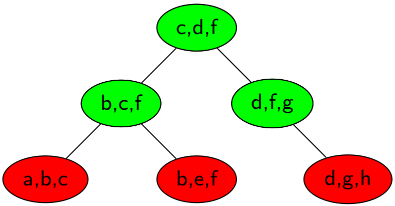

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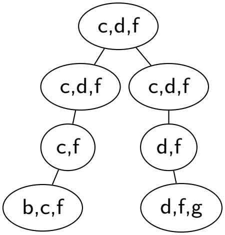

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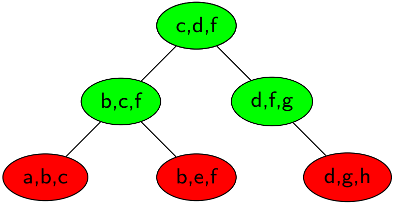

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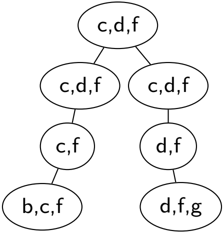

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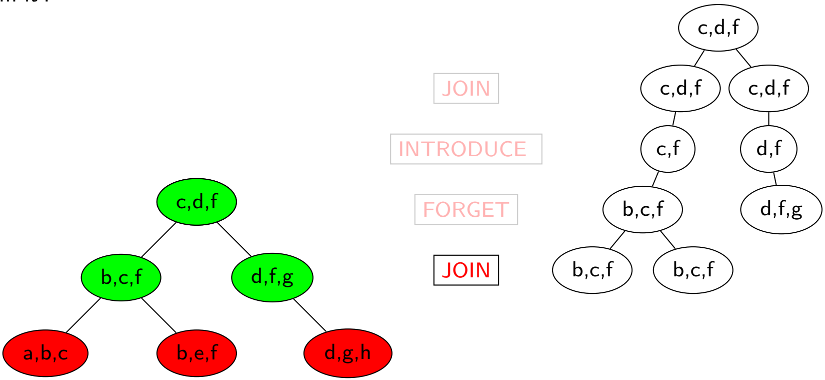

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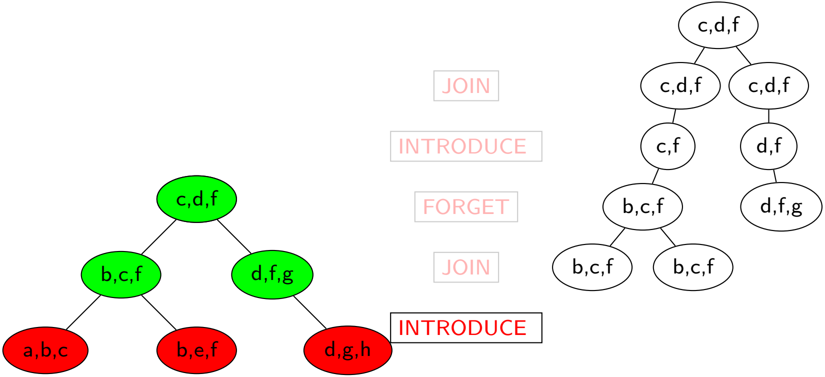

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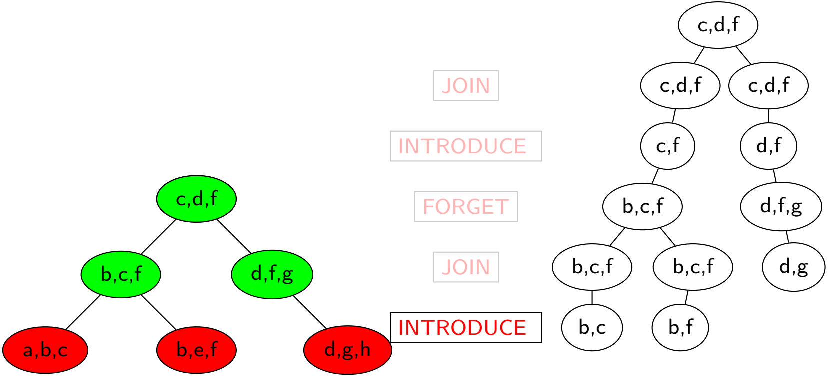

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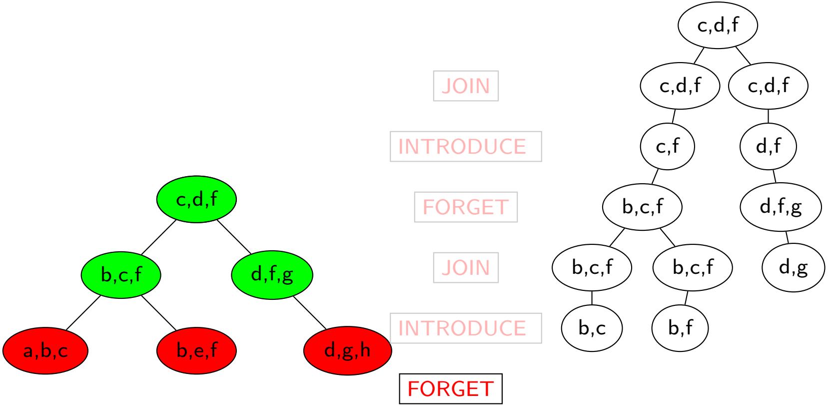

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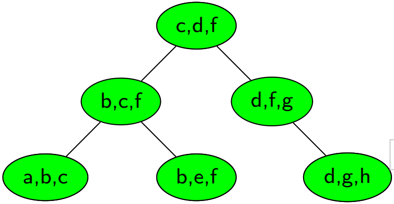

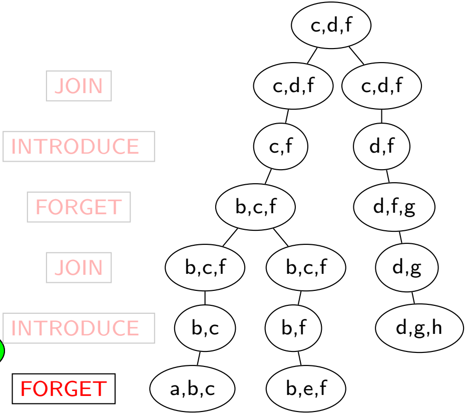

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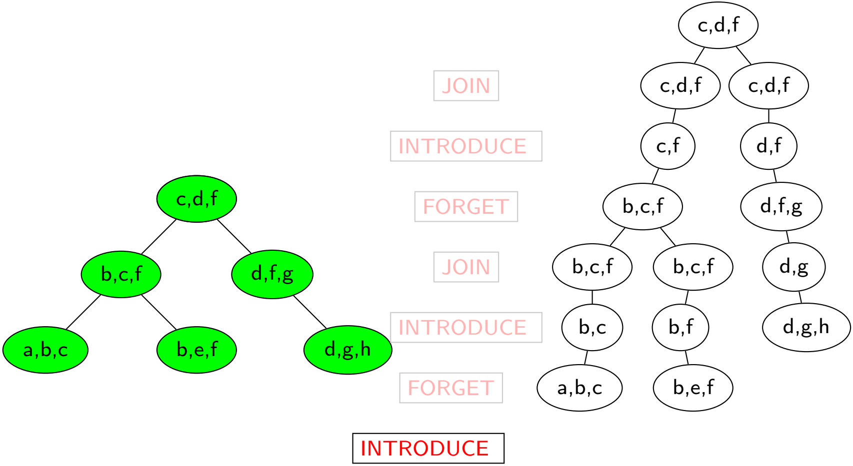

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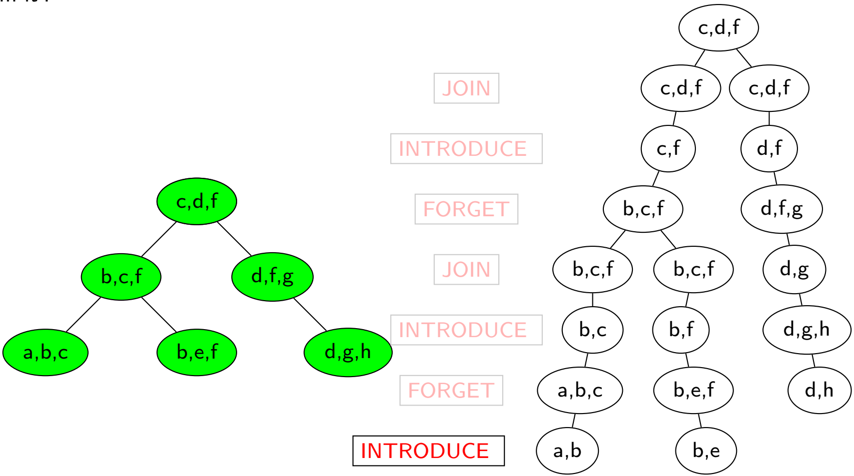

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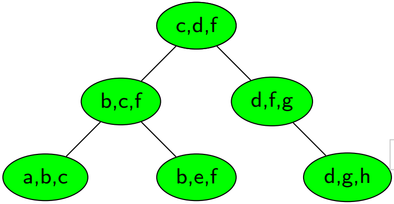

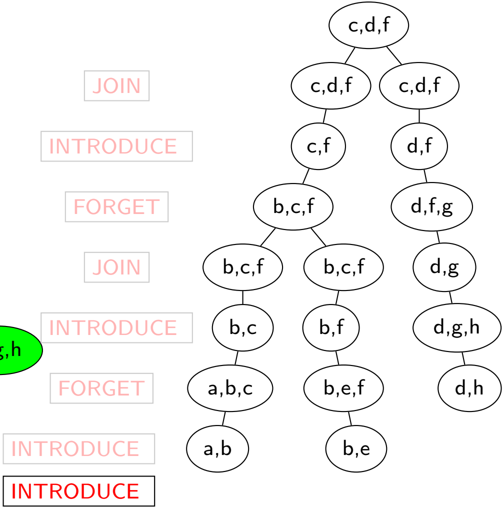

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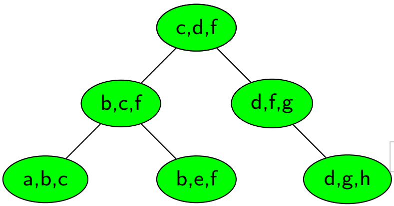

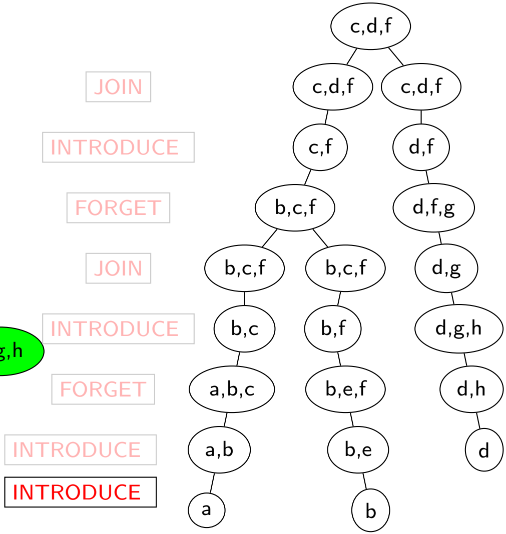

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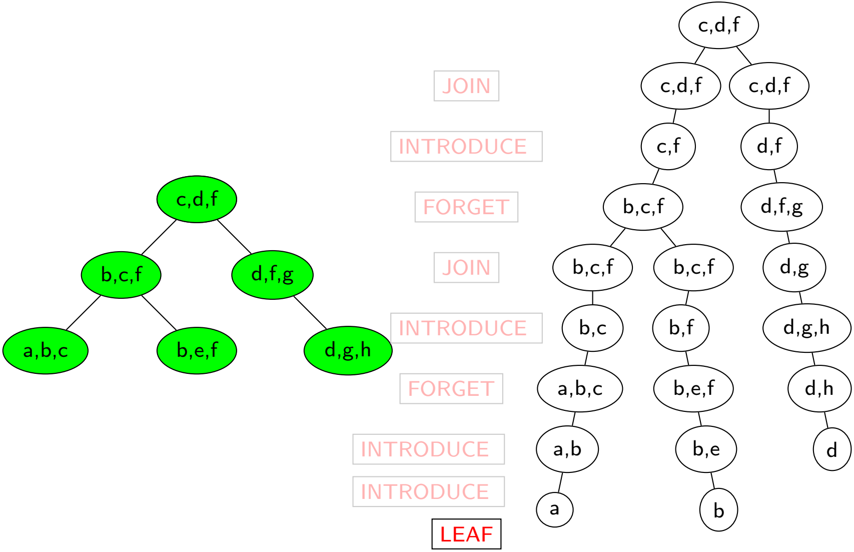

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Let (G) be a graph and k an integer. We want to know if it is possible to draw each vertice of the graph so that two neighbors have never the same color and with only k colors.
This problem is a problem of decisions problem which is NP-complet.

Solution

tree-width : *k-color is possible for a graph (G) if and only if* $k > treewidth(G)$*.*

nice tree : *a nice tree of (G) gives a way to find a k-coloration of (G) (if* $k > treewidth(G)$*).*

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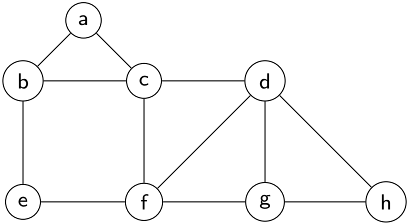
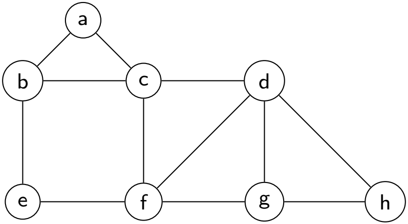
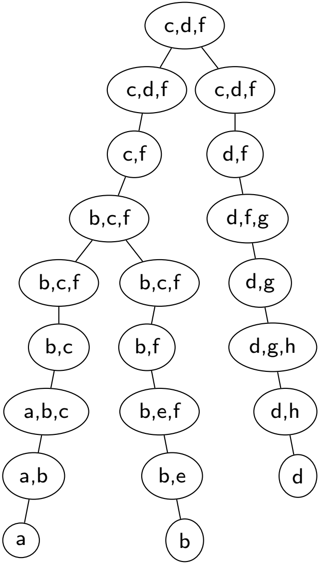
Illustration : we will use the previously trees to solve the problem with this graph :



nice tree of the graph

- treewith(G)=2 : we can solve the problem with 3 colors
- we can fix a color for a "Leaf"
- when we meet an "Introduce" we add a color
- when we meet a "Forget" we can state that the vertex which has disappeared won't come back (property of the decomposition) and so we can reuse its color.





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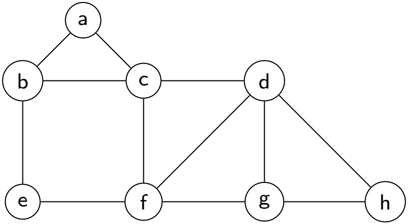

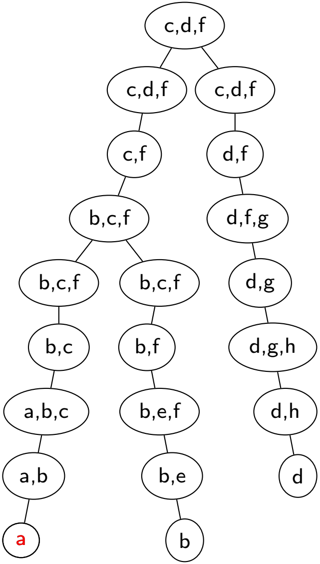

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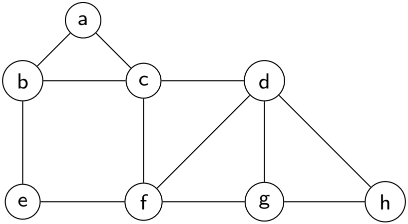

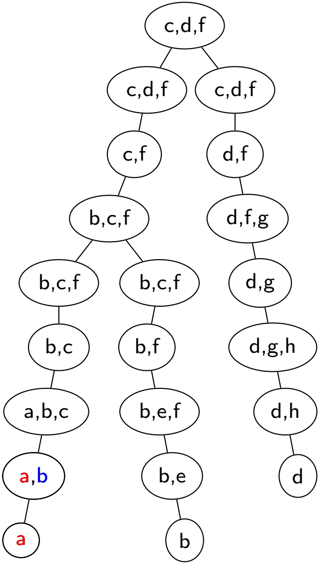

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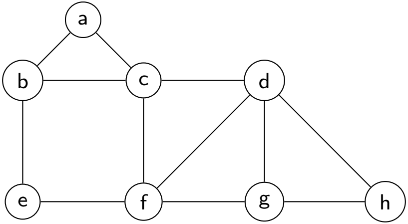

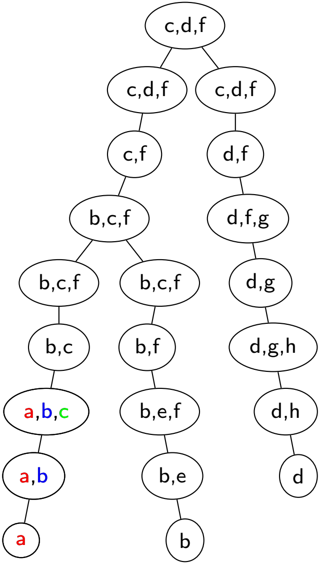

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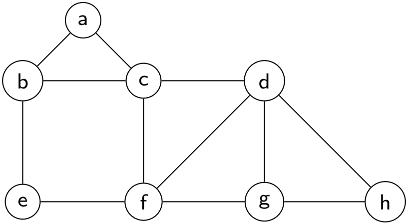

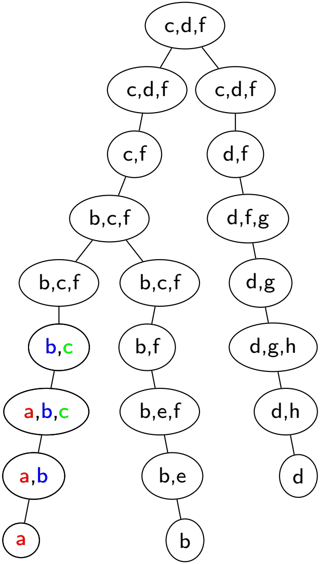

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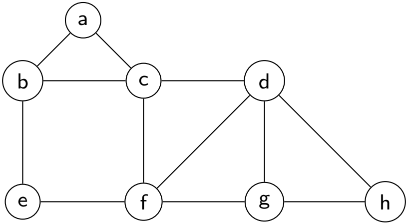

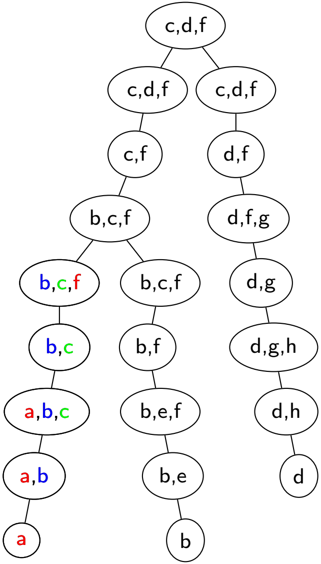

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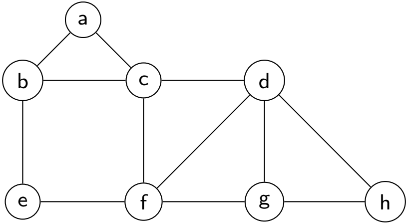

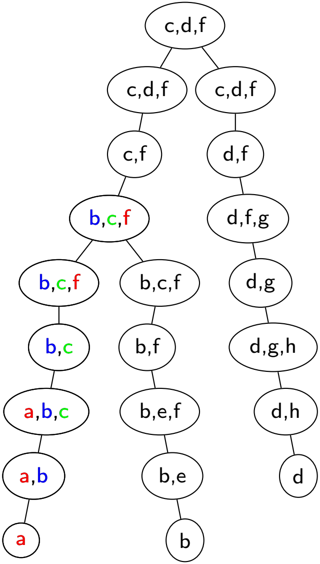

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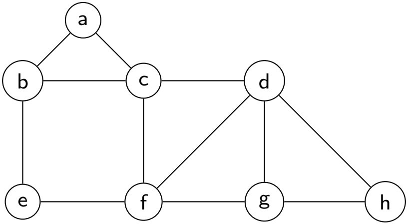

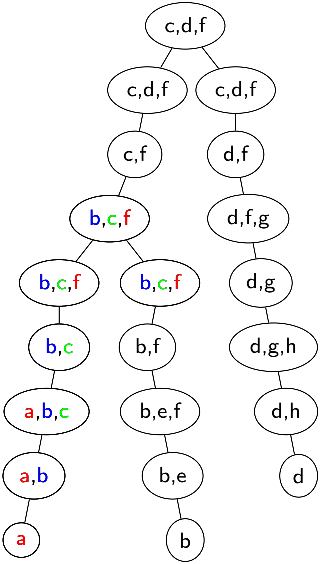

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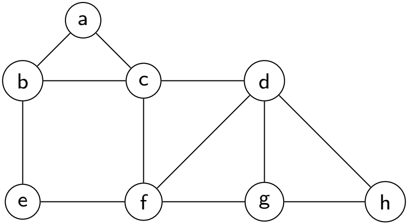

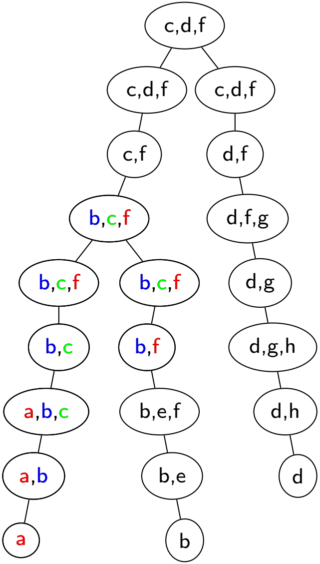

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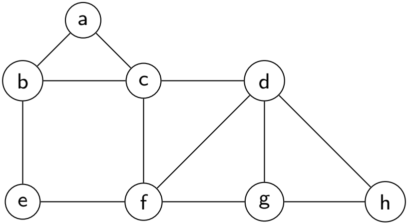

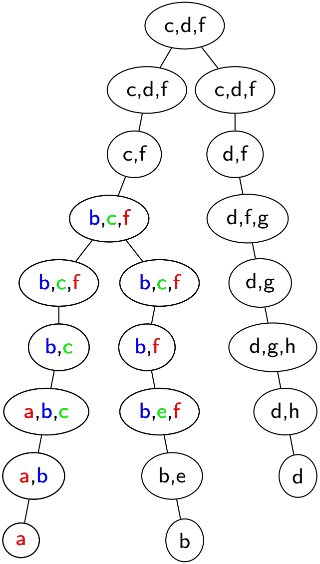

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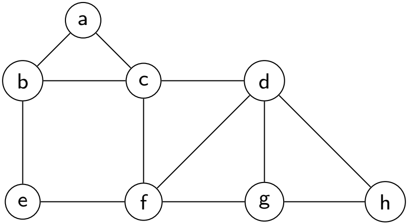

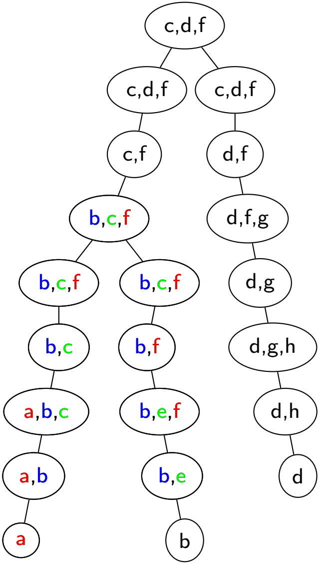

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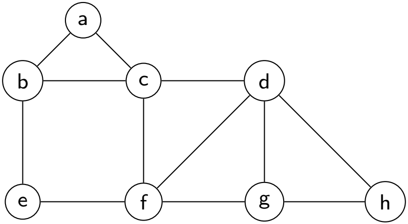

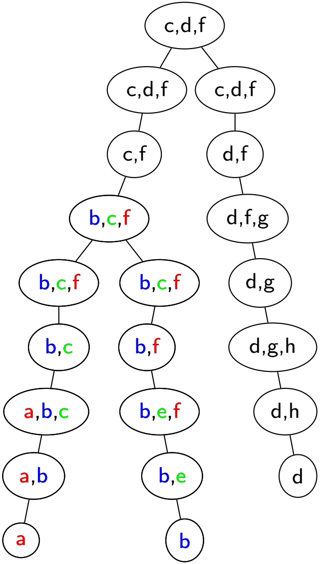

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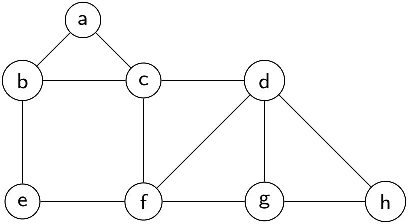

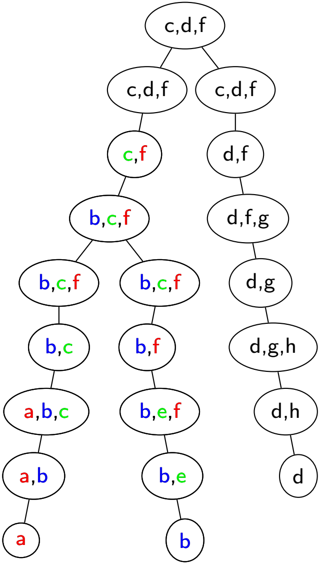

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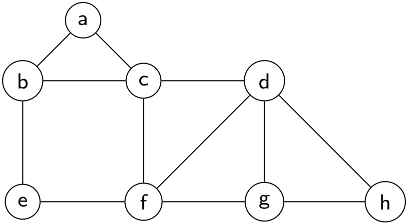

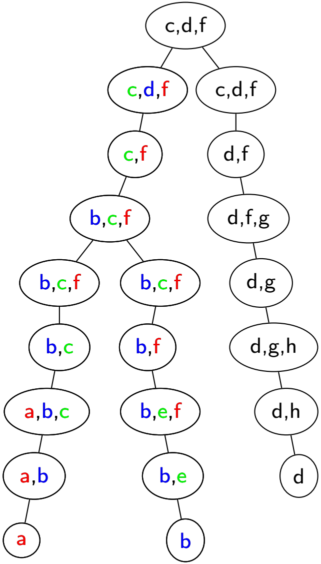

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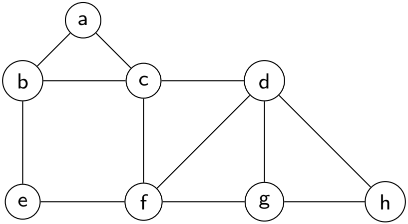

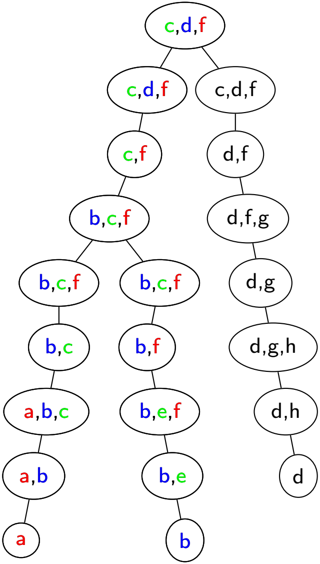

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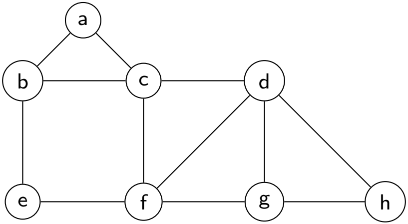

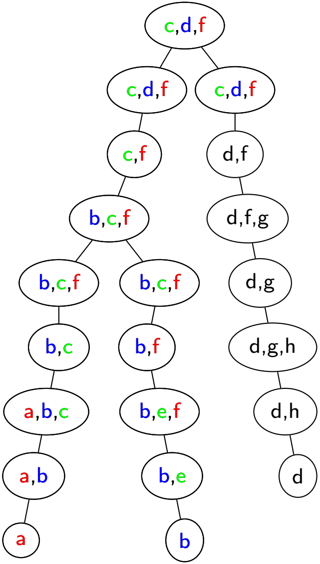

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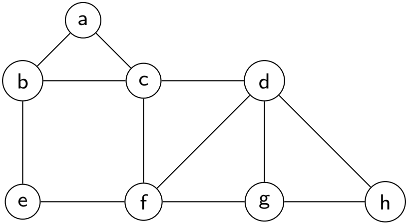

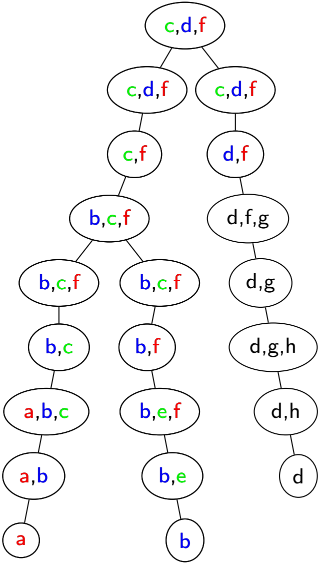

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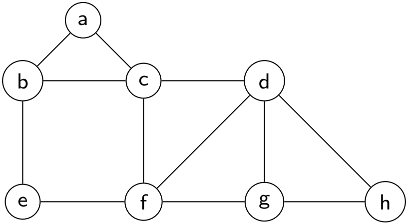

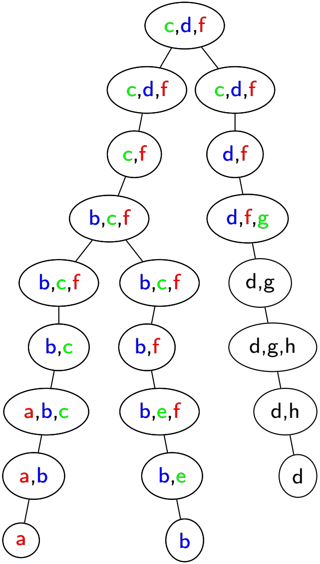

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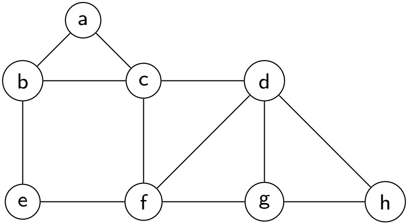

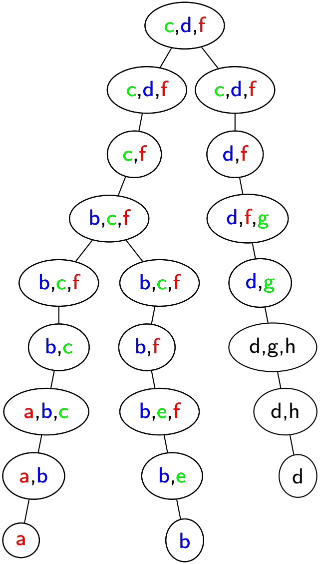

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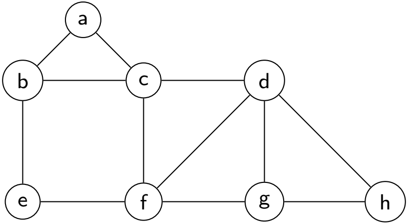

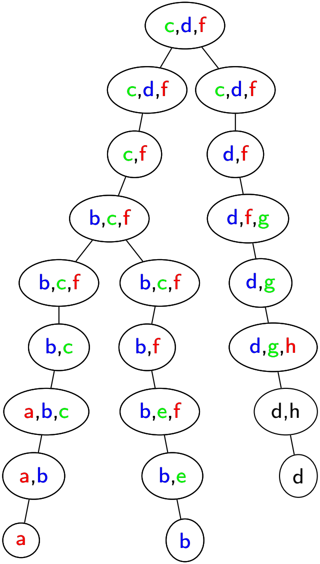

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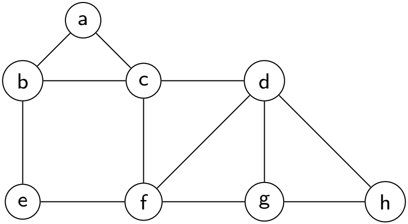

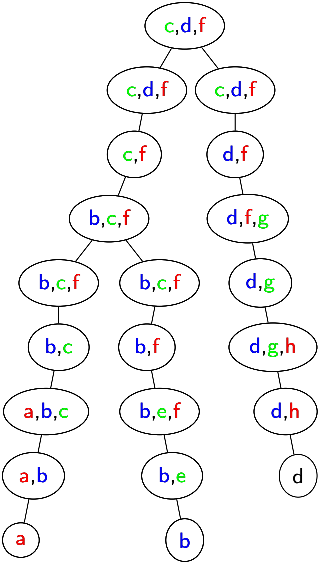

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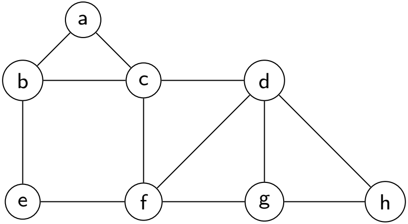

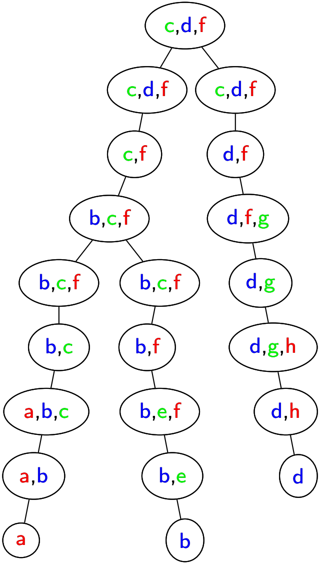

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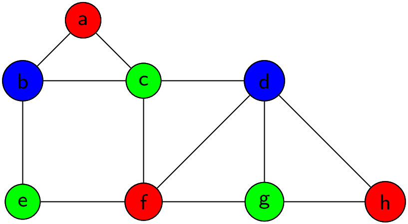

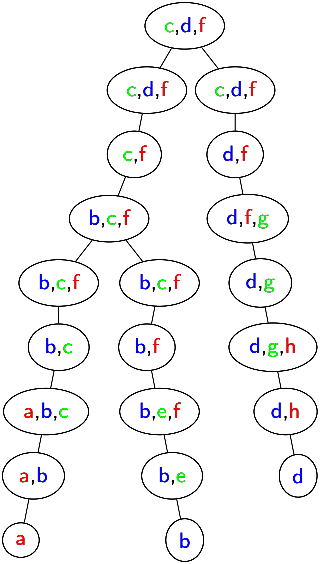

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