CHAPTER 3

FUZZY SETS AND FUZZY REASONING

This chapter introduces some fundamental concepts of fuzzy sets, fuzzy operations and approximate reasoning. It gives the definition of linguistic variable, membership function, fuzzy relation, fuzzy logic and approximate reasoning. Four fuzzy inference and two defuzzification methods are described. These concepts and definitions will be used in the subsequent chapter to present a neural fuzzy–logic modeling technique.

3.1 INTRODUCTION

Motivated by the conviction that traditional methods of systems analysis are unsuited for dealing with systems in which relations between variables do not lend themselves to representation in terms of differential or difference equations, Lotfi A. Zadeh proposed the first paper on fuzzy sets in 1965[3–1]. During the past several years, applications of fuzzy set theory have become productive and fruitful in industrial process control, because of the very active research in this area[3–2]. In short, fuzzy set theory is a body of concepts and techniques that give a form of mathematical precision to human reasoning which is imprecise and ambiguous. Contrary to traditional numerical techniques, much of human reasoning involves the use of linguistic variables whose values are fuzzy sets rather than exact numerical numbers.

Zadeh's principle of incompatibility states that as the complexity of a system to be modeled increases, our ability to make precise and yet significant statements about the system's behavior, using traditional system modeling techniques, diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics[3–3, 3–4].

3.2 FUZZY SETS

In crisp sets, the transition of an element between membership and nonmembership in a given set is abrupt. The membership value is either 1 or 0. In fuzzy sets, this transition can be gradual. The boundaries between fuzzy sets are vague and ambiguous. The membership value of an element in a fuzzy set is measured by a function that attempts to describe vagueness and ambiguity.

Let U be a collection of objects denoted by $\{u\}$. U is called the universe of discourse and u represents a generic element of U.

Definition 3.1: A fuzzy set $\underline{X}$ in a universe of discourse U is characterized by a membership function u_x which takes values in the interval [0,1], namely, $\mu_x : U \to [0,1]$.

Fuzzy sets are denoted by a set symbol with a understrike. A fuzzy set $\underline{X}$ in U may be represented as a set of ordered pairs of a generic element u and its grade of membership function, i.e.,

$$\underline{X} = \{(u, \mu_x(u)) | u \in U\} \tag{3.1}$$

where $\mu_x(u) \in [0, 1]$. When U is continuous and infinite, the fuzzy set $\underline{X}$ can be written as

$$\underline{X} = \int \frac{\mu_x}{u} \tag{3.2}$$

When U is discrete and finite, the fuzzy set $\underline{X}$ can be written as

$$\underline{X} = \frac{\sum_{i=1}^{n} \mu_x(u_i)}{u_i} \tag{3.3}$$

In both notations, the horizontal bar is not a quotient, but rather a delimiter. The numerator is the membership value associated with the element of the universe indicated in the denominator. The "∑" sign denotes a fuzzy union and the integral sign denotes a set union for continuous variables.

3.3 Fuzzy Set Operations

Let $\underline{A}$ and $\underline{B}$ be two fuzzy sets in U with membership functions μ_A and μ_B, respectively. The theoretic set operations of union, intersection and complement for fuzzy sets are defined via their membership functions.

Union: $$\mu_{A \cup B}(u) = \max\{\mu_A(u), \mu_B(u)\} \tag{3.4}$$

Intersection: $\mu_{A \cap B}(u) = \min\{\mu_A(u), \mu_B(u)\}$ (3.5)

Complement: $\mu_{\bar{A}}(u) = 1 - \mu_A(u)$ (3.6)

The diagrams in Figure 3.1 show these operations.

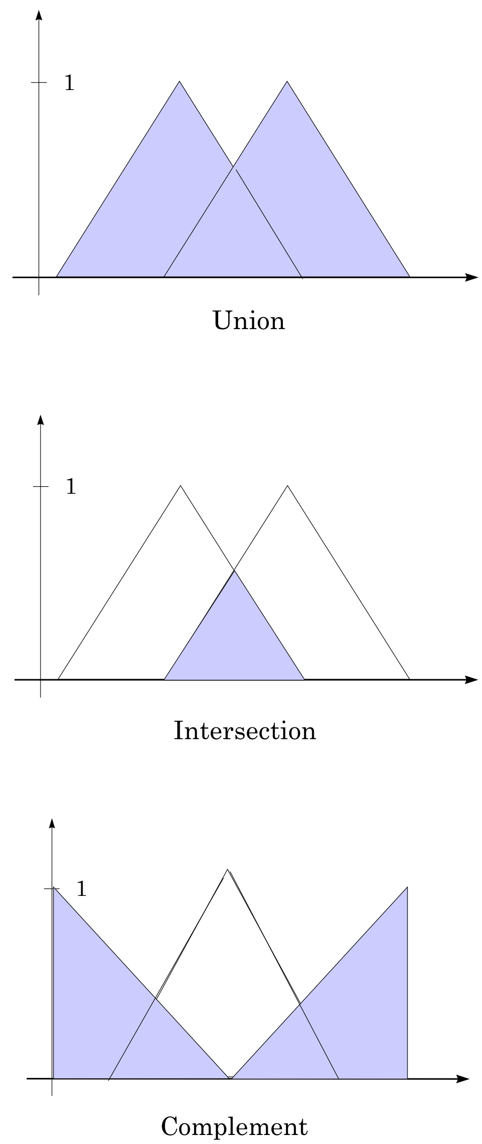

Fig. 3.1 Fuzzy Set Operations

If $A_1, A_2, \ldots, A_n$ are fuzzy sets(the understrike is omitted for simplicity), in $U_1, U_2, \ldots, U_n$, respectively, the Cartesian product of $A_1, A_2, \ldots, A_n$ is a fuzzy set in the product space $U_1 \times U_2 \times \cdots \times U_n$ with the membership function

$$\mu_{A_1 \times A_2 \times \cdots \times A_n}(u_1, u_2, \ldots, u_n) = \min\{\mu_{A_1}(u_1), \mu_{A_2}(u_2), \ldots, \mu_{A_n}(u_n)\}$$

$$= \mu_{A_1}(u_1) \times \mu_{A_2}(u_2) \times \ldots \times \mu_{A_n}(u_n) \qquad (3.7)$$

To express ambiguous relationships such as "you and your brother look very similar", and "*x* and *y* are almost equal", we use fuzzy relations instead of crisp relations.

Definition 3.2: The fuzzy relation R between V and U is a fuzzy set in the Cartesian Product expressed as

$$R_{V \times U} = \{((V, U), \mu_R(v, u)) | (v, u) \in V \times U\} \qquad (3.8)$$

The fuzzy relation in $U_1 \times U_2 \times \cdots \times U_n$ is given by

$$\begin{aligned} R_{U_1 \times U_2 \times \cdots \times U_n} = \{&((u_1, u_2, \ldots, u_n), \mu_R(u_1, u_2, \ldots, u_n)) \\ &| (u_1, u_2, \ldots, u_n) \in U_1 \times U_2 \times \cdots \times U_n\} \end{aligned} \qquad (3.9)$$

Other concepts used in this thesis are presented as follows.

Definition 3.3: Sup–star Composition: If R and S are fuzzy relations in $U \times V$ and $V \times W$, respectively, the composition of R and S is a fuzzy relation denoted by $R \bullet S$ and is defined by

$$R \bullet S = \left\{\left[(u, w), \sup_V(\mu_R(u, v) * \mu_S(v, w))\right], u \in U, v \in V, w \in W\right\} \qquad (3.10)$$

where $*$ could be any operator, namely, minimum, algebraic product, bounded product, or drastic product defined as follows.

Triangular norm: The triangular norm is a two–place function from [0,1] $[0,1]\times[0,1]\rightarrow[0,1]$, which includes the following operations defined for all $x,y\in[0,1]$:

Intersection: $x\wedge y=\min\{x,y\}$ (3.11)

Algebraic: $x\cdot y=xy$ (3.12)

Bounded: $x\otimes y=\max\{0,x+y-1\}$ (3.13)

Drastic Product: $$x\cap y=\begin{cases}x, y=1;\\ y, x=1;\\ 0, x,y<1.\end{cases}$$ (3.14)

The *support* of a fuzzy set $\underline{X}$ is the crisp set of all points u in U such that $\mu_x(u)>0$. The element u in U at which $\mu_x=0.5$, is called the *crossover* point. A fuzzy set whose support is a single point in U with $\mu_x=1$, is referred to as a fuzzy *singleton*. The *core* of a membership function for a fuzzy set $\underline{X}$ is defined as the region in U where $\mu_x(u)=1$. The boundaries of a membership function are the region in U where $0<\mu_x(u)<1$. Figure 3.2 illustrates the various regions as defined above. A normal fuzzy set $\underline{X}$ is one whose membership value is unity at least at one point in U. A convex fuzzy set $\underline{X}$ is one whose membership function satisfies

$$\mu_X\left(\lambda u_1+(1-\lambda)\mu_2\right)\geq\min\{\mu_X(u_1),\mu_X(u_2)\}$$ (3.15)

where $u_1,u_2\in U$ and $\lambda\in[0,1]$.

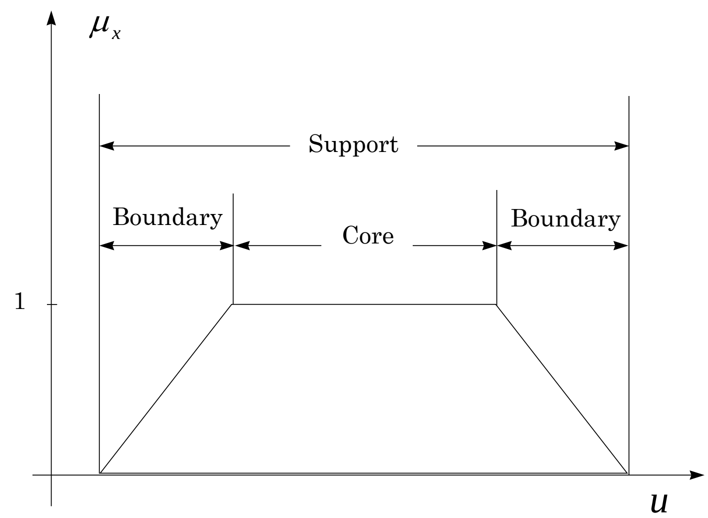

Fig. 3.2 Support, Core and Boundary

3.4 LINGUISTIC VARIABLES

A linguistic variable can be regarded as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms. A linguistic variable is characterized by a quintude $(u, T(u), U, G, M)$ in which u is the name of the variable; *T(u)* is the term set of u, that is, the set of names of linguistic values of u with each value being a fuzzy number defined on U; G is a syntactic rule for generating the names of values of u; and M is a semantic rule for associating with each value its meaning. For example, if temperature is interpreted as a linguistic variable, then its term set *T(temp)* could be

$$T(temp) = \{low, medium, high\} \tag{3.16}$$

where each term in *T(temp)* is characterized by a fuzzy set in a universe of discourse $U = [10,80]$. We might interpret '*low*' as 'a temperature below about 20', and '*medium*' as 'a temperature near 50', and '*high*' as 'a temperature above about 80'. These terms can be characterized as fuzzy sets whose membership functions are shown in Figure 3.3.

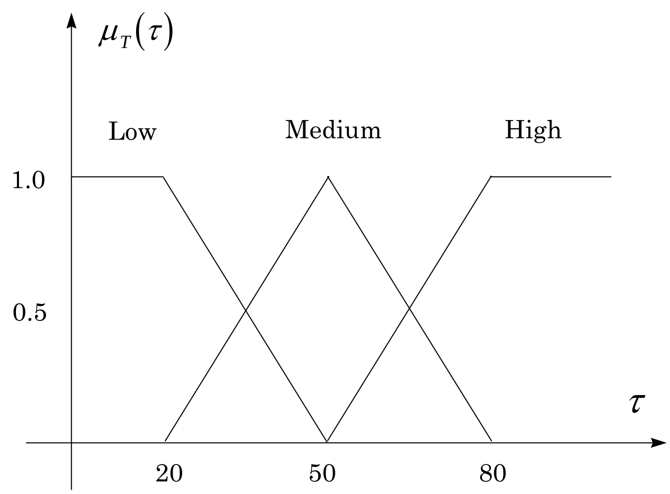

Fig. 3.3 Membership Functions of Temperature

3.5 Fuzzy Logic and Approximate Reasoning

Logic deals with propositions that may or may not be true. A proposition can be true on one occasion, but false on another. That 'A person of 5'7" is tall' is such an example. If this person is seen walking down an Edmonton street, this proposition may not hold true anymore. The truth value of the statement can be only one or zero, depending on whether it is true or false, respectively. By assuming only a true or false value to such a statement, we use crisp logic.

On the other hand, the proposition in the example can be said to be true to a certain degree, say 0.8, instead of 1. This is equivalent to saying that he is tall to the degree of 0.8. Here we are using fuzzy logic. Propositions that take truth values to a certain degree – from 0 to 1 are dealt with by fuzzy logic. A fuzzy logic proposition is a statement without clearly defined boundaries. The truth value assigned to a proposition can be any value on the interval [0,1]. The assignment of the truth value to a proposition is actually a mapping of the truth value from the interval [0,1] to the universe *U*. Propositions can be combined to form new propositions by means of logical operations.

Approximate reasoning or fuzzy inference plays a central role in all human thinking activities as well in the inquiry of the meaning of various soft or empirical sciences. Instead, certain form of approximate and vague reasoning is always involved apart from statistical inference in analysis of data. Unlike conventional expert systems where statements are executed consequential, the principal reasoning protocol behind fuzzy logic is a parallel processing paradigm where all the rules are fired. Each rule is a statement of relationship between model variables and fuzzy regions. When these rules are evaluated for their degrees of truth, they contribute to the output state of the solution variable set. The functional tie between the degree of truth in related fuzzy regions is called the method of implication. The functional tie between the fuzzy regions and the expected value of a set point is called the method of defuzzification.

3.6 INFERENCE METHODS

Fuzzy logic control describes algorithms for process control as a fuzzy relation between the state condition of the process to be controlled and the input to the process. A control algorithm comprises a set of fuzzy control rules expressed by "IF – THEN" expressions. The "IF" clause of the rules is called the *antecedent* and the "THEN" clause the *consequent*.

State variables have fuzzy values expressed by fuzzy sets. The output of a fuzzy control system is calculated by means of fuzzy inference given the state variables. The rule base of a fuzzy logic control(FLC) is usually defined from expert knowledge. For a multi–input single–output(MISO) system, the rule base has the form

$$R = \left\{ RB_1, RB_2, \ldots, RB_g \right\} \tag{3.17}$$

where RB_j represents the rule:

IF (x_1 is A_{1j} and x_2 is A_{2j} ... x_m is A_{mj}) THEN u is B_j

where

x_i = linguistic variables representing the process states,

u = control variable,

A_{ij}, B_j = linguistic labels $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$ of the above linguistic variables in the universe of discourse $V_1, V_2, \ldots, V_m$ and U, respectively.

Using the triangular norms, we can define conjunctions in approximate reasoning. A fuzzy control rule, "if x is $\underline{X}$ then y is $\underline{Y}$," is represented by a fuzzy implication function and is denoted by $\underline{X} \to \underline{Y}$, where $\underline{X}$ and $\underline{Y}$ are fuzzy sets in universes U and V with membership functions μ_x and μ_y, respectively.

The fuzzy conjunction is defined by

$$\underline{X} \to \underline{Y} = \underline{X} \times \underline{Y}$$

$$= \int_{U \times V} \mu_x(u) * \mu_y(v) / (u,v), \text{ all } u \in U \text{ and } v \in V \tag{3.18}$$

where $*$ is an operator representing a triangular norm.

To represent different types of fuzzy reasoning, we define two fuzzy implication functions. By using the definition of the fuzzy conjunction, Mamdani's mini–fuzzy implication, R_c, is obtained if the intersection operator is used, i.e.,

$$R_c = \underline{X} \times \underline{Y}$$

$$= \int_{U \times V} \mu_x(u) \wedge \mu_y(v) / (u,v) \tag{3.19}$$

This is the mini–operation rule of fuzzy implication by Mamdani[3–5]. Similarly, Larsen's product fuzzy implication, R_p, is obtained if the algebraic product is used, i.e.,

$$R_p = \underline{X} \times \underline{Y}$$

$$= \int_{U \times V} \mu_x(u) \cdot \mu_y(v) / (u,v) \tag{3.20}$$

This is the production operation rule of fuzzy implication by Larsen[3–6]. Now we are able to present the four types of fuzzy reasoning currently utilized in FLC applications.

(a) *Type 1 Fuzzy Reasoning*

This type of reasoning is associated with the use of Mamdani's minimum operation rule, R_c, as a fuzzy implication function. Suppose that the system is a multi–input single–output (MISO) system, the membership function of the control action produced by the *j*–th rule is given by

$$\mu_{B_j^{'}} = \alpha_j \wedge \mu_{B_j}(u) \tag{3.21}$$

where α_j is the firing strength of the *j*–th rule and is expressed as

$$\alpha_j = \mu_{A_{1j}}\left(x_1^0\right) \wedge \mu_{A_{2j}}\left(x_2^0\right) \wedge \ldots \wedge \mu_{A_{mj}}\left(x_m^0\right) \tag{3.22}$$

with $x_i^0 (i = 1,2,\ldots,m)$ being a fuzzy singleton and $\wedge$ denoting intersection. The membership function of the total control action of the *n*–rules can be obtained by

$$\mu_B(u) = \mu_{B_1^{'}} \vee \mu_{B_2^{'}} \vee \ldots \vee \mu_{B_n^{'}} \tag{3.23}$$

with $\vee$ denoting union. This fuzzy reasoning process is graphically illustrated in Figure 3.4. Assuming that $x_1 = x_1^0$ and $x_2 = x_2^0$. From fig. 3.4, it can be determined that

$$\alpha_1 = \mu_{A_{11}}\left(x_1^0\right) \wedge \mu_{A_{21}}\left(x_2^0\right) = \mu_{A_{21}}\left(x_2^0\right) \tag{3.24}$$

$$\alpha_2 = \mu_{A_{12}}\left(x_1^0\right) \wedge \mu_{A_{22}}\left(x_2^0\right) = \mu_{A_{12}}\left(x_1^0\right) \tag{3.25}$$

The value of the control variable produced by the first rule equals the trapezoid area in μ_{B_1}. The value of the control variable produced by the second rule equals the trapezoid area in μ_{B_2}. The final control action is produced by centroid defuzzification from the two trapezoids(see Section 3.7).

(b) *Type 2 Fuzzy Reasoning*

This type of reasoning is associated with the use of Larsen's product operation rule, R_p, as a fuzzy implication function. In this case, the membership function of the control action produced by the *j*–th rule is given by

$$\mu_{B_j^{'}} = \alpha_j \cdot \mu_{B_j}(u) \tag{3.26}$$

The membership function of the total control action of the n–rules can be obtained by the same formula as (3.22). Figure 3.5 shows this fuzzy reasoning process. The final control action is produced through defuzzification(see Section 3.7).

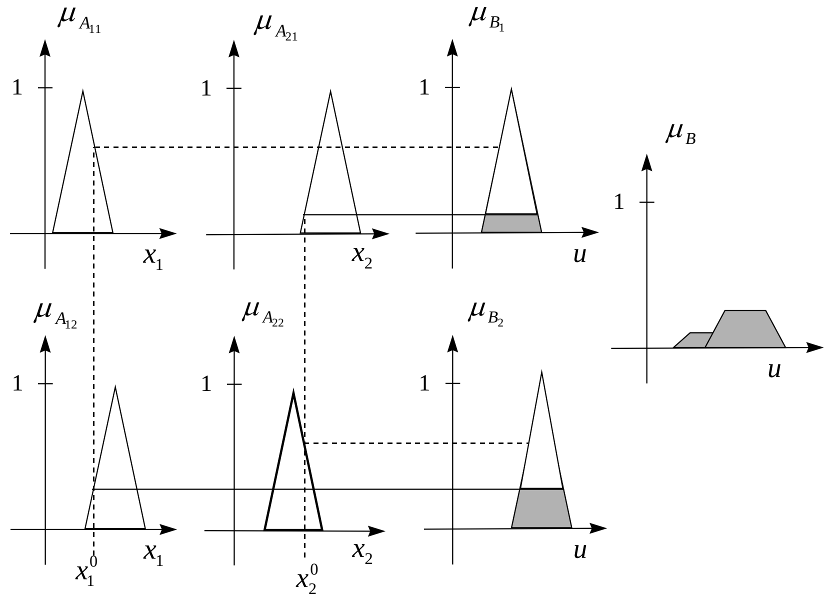

Fig. 3.4 Type 1 Fuzzy Reasoning

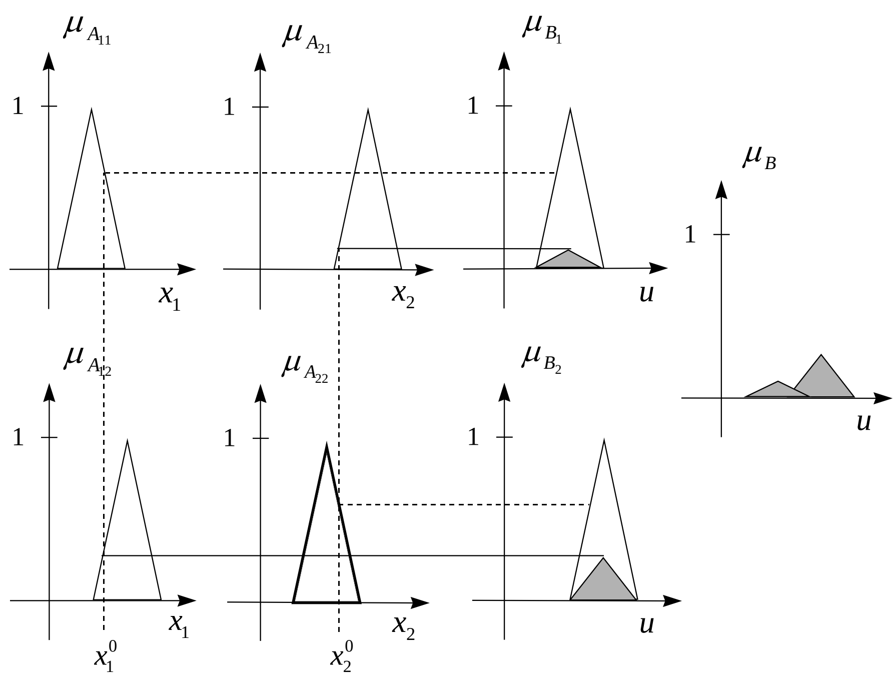

Fig. 3.5 Type 2 Fuzzy Reasoning

(c) *Type 3 Fuzzy Reasoning*

It was proposed by Tsukamoto[3–7]. In this mode of reasoning, the membership functions of fuzzy sets $A_{ij}\,(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$ are single valued, and $B_j\left(j=12,\ldots,n\right)$ are monotonic. The firing strength for the *j*–th rule is also given by (3.22). From this α_j, the output variable u_j can be determined by letting $\alpha_j=\mu_{B_j}\left(u_j\right)$ when μ_{B_j} is monotonic. A crisp control action may be expressed as the weighted combination of all the outputs, i.e.,

$$u_o=\frac{\sum_{j=1}^{n}\alpha_j u_j}{\sum_{j=1}^{n}\alpha_j} \tag{3.27}$$

(d) *Type 4 Fuzzy Reasoning*

It uses a function of the input linguistic variables as the consequence of a rule. The *j*–th fuzzy control rule is of the form

IF (x_1 is A_{1j} and x_2 is A_{2j} … x_m is A_{mj}) THEN $u=f_j\left(x_1,x_2,\ldots,x_m\right)$

where f_j is a function of the process variables $x_i\left(i=1,2,\ldots,m\right)$ defined in the input subspace. The inferred value of the control action from the *j*–th rule is $\alpha_j f_j\left(x_1^0,x_2^0,\ldots,x_m^0\right)$, and the total crisp control action is given by

$$u_o=\frac{\sum_{j=1}^{n}\alpha_j f_j\left(x_1^0,x_2^0,\ldots,x_m^0\right)}{\sum_{j=1}^{n}\alpha_j} \tag{3.28}$$

3.7 DEFUZZIFICATION

In the previous Section, we have outlined four fuzzy reasoning methods. In the first two methods, a space of fuzzy control actions is produced instead of a space of a crisp control actions. However, in practical applications, the latter is required. Therefore, to map from a fuzzy space to a crisp space, defuzzication is required.

Defuzzification is the final stage of fuzzy reasoning[3–8,3–9]. The evaluation of the fuzzy model propositions is done through an aggregation process such that a final fuzzy region for each solution variable is produced. This region is then decomposed using one of the defuzzification strategies.

(1) *Composite Maximum*

A maximum decomposition finds the domain point with the maximum truth. This technique applies to a narrower class of problems. Its attributes are that (i) the expected value is sensitive to a single rule that dominates the fuzzy rule set and (ii) the expected value tends to change from one point to another as the shape of the fuzzy region changes. In applications that need to assess the maximum fuzzy property, such as risk assessment, this may be the technique to use.

(2) *Composite Moments(Centroid)*

The centroid or center of gravity generates the center of gravity of the possibility distribution of a control action. Mathematically, in the case of a continuous universe, this method produces

$$u_o = \frac{\int \mu_{B'}(u) u du}{\int \mu_{B'}(u) du} \tag{3.29}$$

where $\int$ denotes conventional integral.

We have presented the two most commonly used methods of defuzzification. Unless we have reason to believe that a specialized method of defuzzification is required for our model, we will use one of these two techniques.