Lemma: Let x R. Then there is an integer n st n – 1 x < n, i.e., x [n-1, n]

Proof: By the Archimedean Principle there is a natural number m1 st x < m1 and there is a natural number m2 st –x < m2 i.e. –m2 < x

So –m2 < x < m1, i.e. x (-m2, m1)

But [-m2, m1) = [-m2, -m2 +1) U [-m2 + 1, -m2 + 2) U … U [0, 1) U [1, 2) U … … U [m1 – 2, m1 – 1) U [m1 – 1, m1)

And this is a disjoint union. So x belongs to exactly one of the intervals on the right hand side i.e. x [n-1, n) some n {-m2 + 1, …, m1}

Theorum: Between any two real numbers, there is a rational number.

Proof: Let x and y be real numbers with x < y. Need a rational with m Z and n N, st x < < y

Roughwork: nx < m < ny (want gap between nx and ny to be bigger than 1, guarantees an integer in that interval) => ny – nx > 1 => n(y – x) > 1

By Archimedean Principle there is n N with n(y – x) > 1 (as y – x > 0). Then nx – ny > 1, i.e., ny > nx + 1

Now by previous lemma there is an integer m with m – 1 nx < m (Note m - 1 nx => m nx + 1)

Then nx < m nx + 1 < ny => nx < m < ny => x < < y as required

A real number that is not rational is called irrational e.g., is irrational

Theorum: Between any two real numbers there is an irrational number

Proof: Let x and y be real numbers with x < y. Then and are real numbers and < By the previous result rational m/n with < < But then x < < y and (?) r = is irrational