SET-1

MID-2

1. write the Maclaurin's series expansion.

If $f: [0,x] \to R$ is such that

i). f^{n-1} is continuous on $[0,x]$

ii). f^{n-1} is derivable on $(0,x)$ and $p \in Z^+$

then $\exists$ a real number $\theta \in (0,1)$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x).$$

2. If $u = x\sin y$, $v = y\sin x$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

Sol^n: Given

$u = x\sin y$, $v = y\sin x$

$\frac{\partial u}{\partial x} = (1)\sin y$ $\qquad \frac{\partial v}{\partial x} = y\cos x$

$\frac{\partial u}{\partial y} = x\cos y$ $\qquad \frac{\partial v}{\partial y} = (1)\sin x$

now,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \sin y & x\cos y \\ y\cos x & \sin x \end{vmatrix} = \sin x.\sin y - xy\cos x\cos y //$$

3. write the taylor's series expansion of two variables.

Ans: the Taylor series expansion for a function of two variables, $f(x,y)$ around a points $(x-a)$ & $(y-b)$ is

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots //$$

4. find $\Gamma\left(\frac{9}{2}\right)$.

Sol^n: To find $\Gamma\left(\frac{9}{2}\right)$

we knoco that $\Gamma(n) = (n-1)\Gamma(n-1)$

put $n = 9/2$

$$\therefore \Gamma\left(\frac{9}{2}\right) = \left(\frac{9}{2}-1\right)\Gamma\left(\frac{9}{2}-1\right)$$

$$= \frac{7}{2}\Gamma\left(\frac{7}{2}\right)$$

$$= \frac{7}{2}\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$$

$$= \frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right) = \frac{35}{4}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{105}{8}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{105}{16}\sqrt{\pi}.//$$

6.
(a). Expand $\log(1+x)$ by Maclaurin's series expansion.

Sol^n: Let $f(x) = \log(1+x)$ $\qquad$ $f(0) = \log 1 = 0$

$f'(x) = \dfrac{1}{1+x}$ $\qquad$ $f'(0) = \dfrac{1}{1+0} = 1$

$f''(x) = \dfrac{-1}{(1+x)^2}$ $\qquad$ $f''(0) = \dfrac{-1}{1} = -1$

$f'''(x) = (-1)(-2)(1+x)^{-3}$ $\qquad$ $f'''(0) = \dfrac{2}{(1+0)^3} = 2$

$= \dfrac{2}{(1+x)^3}$

$f^{IV}(x) = 2(-3)(1+x)^{-4}$ $\qquad$ $f^{IV}(0) = \dfrac{-6}{(1+0)^4} = -6$

$= \dfrac{-6}{(1+x)^4}$

by Maclaurin's series

$f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0) + \dfrac{x^3}{3!} f'''(0) + \cdots$

$\therefore f(x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$

$= x - \frac{x^2}{2!} + \frac{x^3}{3} - \cdots$ //

(b). obtain Taylor series expansion of e^x about $x = -1$ (or) $x = +1$.

Sol: Let $f(x) = e^x$

about $x = -1$

put $x + 1 = t \Rightarrow x = t - 1$

$\therefore f(x) = e^x$

$= e^{t-1}$

$= e^t \cdot e^{-1} = \frac{e^t}{e} = \frac{1}{e}\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]$ $\because [t = x+1]$

$= \frac{1}{e}\left[1 + \frac{(x+1)}{1!} + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \cdots\right]$ //

7. (a). find 'c' for cauchy Mean value theorem for $f(x) = e^x$, $g(x) = e^{-x}$ in $[a, b]$.

Sol: Let $f(x) = e^x$, $g(x) = e^{-x}$ on $[a, b]$

$f(x)$ & $g(x)$ are continuous on $[a, b]$ &

$f(x)$, $g(x)$ are derivable on (a, b)

also $g'(x) = -e^{-x} \neq 0$

by cauchy Mean value theorem

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \longrightarrow (1)$$

here $f(x) = e^x$ $\quad$ $g(x) = e^{-x}$

$f(a) = e^a$ $\quad$ $g(a) = e^{-a}$

$f(b) = e^b$ $\quad$ $g(b) = e^{-b}$

& $f'(x) = e^x$ $\quad$ & $g'(x) = -e^{-x}$

from eq^n (1)

$$\therefore \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\Rightarrow \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -\frac{e^c}{\frac{1}{e^c}}$$

$$\Rightarrow \frac{\cancel{e^b - e^a}}{\frac{\cancel{e^a - e^b}}{e^a \cdot e^b}} = -e^c \cdot e^c$$

$$\Rightarrow -e^{a+b} = -e^{c+c}$$

$$\Rightarrow a+b = 2c \Rightarrow \boxed{c = \frac{a+b}{2}}$$ //

(b). obtain Taylor series expansion of $\sin x$ in powers of $x - \frac{\pi}{4}$.

Sol: Let $f(x) = \sin x$ at $a = \pi/4$

here

$f'(x) = \cos x$ $\quad$ $f(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$

$f''(x) = -\sin x$ $\quad$ $f'(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$

$f'''(x) = -\cos x$ $\quad$ $f''(\pi/4) = -\sin \pi/4 = -1/\sqrt{2}$

$f^{IV}(x) = \sin x$ $\quad$ $f'''(\pi/4) = -\cos \pi/4 = -1/\sqrt{2}$

$f^{IV}(\pi/4) = \sin \pi/4 = 1/2$

by Taylor series

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$= \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2\left(\frac{-1}{\sqrt{2}}\right) + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3\left(\frac{-1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \cdots\right]$$ //

8. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ then show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Given

$x = r\sin\theta$ co

now,

∂x

Given

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

now,

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi\,(1) \qquad \frac{\partial y}{\partial r} = \sin\theta\sin\phi\,(1) \qquad \frac{\partial z}{\partial r} = \cos\theta\,(1)$

$\frac{\partial x}{\partial \theta} = r\cos\phi\cdot\cos\theta \qquad \frac{\partial y}{\partial \theta} = r\sin\phi\cdot\cos\theta \qquad \frac{\partial z}{\partial \theta} = r.(-\sin\theta)$

$= -r\sin\theta$

$\frac{\partial x}{\partial \phi} = r\sin\theta.(-\sin\phi) \qquad \frac{\partial y}{\partial \phi} = r\sin\theta.\cos\phi \qquad \frac{\partial z}{\partial \phi} = 0$

consider $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\phi\cos\theta & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$

$= \sin\theta\cos\phi\,[0 + r^2\sin^2\theta\cos\phi] - r\cos\phi\cos\theta\,[0 - r\sin\theta\cos\theta\cos\phi]$

$- r\sin\theta\sin\phi\,[-r\sin^2\theta\sin\phi - r\sin\phi\cos^2\theta]$

$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta.\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\sin^2\phi\cos^2\theta$

$= r^2\sin\theta\cos^2\phi\,[\sin^2\theta + \cos^2\theta] + r^2\sin\theta\sin^2\phi\,[\sin^2\theta + \cos^2\theta]$

$= r^2\sin\theta\cos^2\phi\,(1) + r^2\sin\theta\sin^2\phi\,(1)$

$= r^2\sin\theta\,[\sin^2\phi + \cos^2\phi]$

$= r^2\sin\theta\,(1) = r^2\sin\theta$ //

9). find the dimension of the rectangular box, when top is opened volume is 108 cubic

9) A)

Let x, y, z be the dimensions of a rectangular parallelopiped opened at top

$V = xyz$

given $V = 108$ cubic feet.

$xyz = 108$

$\Rightarrow xyz - 108 = 0 \longrightarrow (1)$

here $\phi(x,y,z) = xyz - 108$

$S = xy + 2yz + 2zx$

Let $f(x,y,z) = xy + 2yz + 2zx \longrightarrow (2)$

$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$

$= xy + 2yz + 2zx + \lambda(xyz - 108)$

$= xy + 2yz + 2zx + \lambda xyz - \lambda 108$

now, $\frac{\partial F}{\partial x} = y + 2z + \lambda yz$ $\quad \frac{\partial F}{\partial y} = x + 2z + \lambda xz$ $\quad \frac{\partial F}{\partial z} = 2y + 2x + \lambda xy$

$\therefore \frac{\partial F}{\partial x} = 0$

$\Rightarrow y + 2z + \lambda yz = 0$

$\Rightarrow \lambda yz = -(y + 2z)$

$\Rightarrow \lambda = \frac{-(y+2z)}{yz}$

$\frac{\partial F}{\partial y} = 0$

$\Rightarrow x + 2z + \lambda xz = 0$

$\Rightarrow \lambda xz = -(x + 2z)$

$\Rightarrow \lambda = \frac{-(x+2z)}{xz}$

$\frac{\partial F}{\partial z} = 0$

$\Rightarrow 2y + 2x + \lambda xy = 0$

$\Rightarrow \lambda xy = -(2y + 2x)$

$\Rightarrow \lambda = \frac{-(2y+2x)}{xy}$

consider

$\frac{-(y+2z)}{yz} = \frac{-(x+2z)}{xz}$

$\Rightarrow xy + 2xz = xy + 2yz$

$\Rightarrow 2xz = 2yz$

$\Rightarrow \boxed{x = y}$

&

$\frac{-(x+2z)}{xz} = \frac{-(2y+2x)}{xy}$

$y(x+2z) = z(2y+2x)$

$xy + 2yz = 2yz + 2xz$

$xy = 2xz$

$\Rightarrow \boxed{y = 2z}$

$\therefore x = y = 2z$

now, $xyz = 108$ $\quad \because \left[\begin{array}{l} y = x \\ z = x/2 \end{array}\right]$

$\Rightarrow x(x)\left(\frac{x}{2}\right) = 108$

$\Rightarrow x^3 = (108)2$

$\Rightarrow x^3 = 216$

$\Rightarrow x^3 = 6^3$

$\Rightarrow \boxed{x = 6}$

$\therefore y = 6$ & $z = \frac{x}{2} = \frac{6}{2} = 3$

dimensions are $x = 6, y = 6, z = 3$.//

10). Relation between Beta & Gamma function

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Proof: consider

$$\Gamma(m) = \int_0^\infty e^{-x} . x^{m-1} dx$$

put $x = yt$

$dx = y\,dt$

$$\therefore \int_0^\infty e^{-yt} . (yt)^{m-1} y\,dt$$

$$\Rightarrow \Gamma(m) = \int_0^\infty e^{-yt} . y^{m-1} . t^{m-1} y\,dt$$

$$= \int_0^\infty e^{-yt} . y^m t^{m-1} dt$$

$$\Rightarrow \frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yt} . t^{m-1} dt$$

multiply $\int_0^\infty e^{-y} . y^{m+n-1} dy$ on both sides

$$\Rightarrow \int_0^\infty \frac{\Gamma(m)}{y^m} . e^{-y} . y^{m+n-1} dy = \int_0^\infty \int_0^\infty e^{-yx} . e^{-y} . y^{m+n-1} dy . x^{m-1} dx$$

$$\Rightarrow \Gamma(m) \int_0^\infty e^{-y} . y^{n-1} dy = \int_0^\infty \int_0^\infty e^{-y(x+1)} . y^{m+n-1} dy . x^{m-1} dx$$

$$\Rightarrow \Gamma(m)\cdot\Gamma(n) = \int_0^\infty \frac{\Gamma(m+n)}{(x+1)^{m+n}}\cdot x^{m-1}dx \quad \because \left[\Gamma(n) = \int_0^\infty e^{-x}\cdot x^{n-1} \right. \quad -(9)$$

$$\left. \frac{\Gamma(n)}{K^n} = \int_0^\infty e^{-kx}\cdot x^{n-1}dx \right]$$

$$= \Gamma(m+n)\int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}}dx$$

$$= \Gamma(m+n)\,B(m,n).$$

$$\Rightarrow \boxed{B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}}$$

11. (a). Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$.

Sol: Given integral

$$I = \int_{x=0}^1 \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right]dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{\underbrace{[\sqrt{1+x^2}]^2}_{a^2}+y^2}dx \qquad \left[\because \int\frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_{y=0}^{\sqrt{1+x^2}}dx$$

$$= \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \tan^{-1}0\right]dx$$

$$= \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}(1) - 0\right]dx$$

$$= \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\tan\frac{\pi}{4}\right)dx = \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}dx$$

$$= \frac{\pi}{4}\log\left[x+\sqrt{x^2+1}\right]_0^1$$

$$= \frac{\pi}{4}\left[\log(1+\sqrt{1+1}) - \log(0+\sqrt{0+1})\right] = \frac{\pi}{4}\cdot\log(1+\sqrt{2}) - 0$$

$$= \frac{\pi}{4}\log(1+\sqrt{2}).$$

evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz.$

sd: Given Integral

$$I=\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[(x+z)(y)+\frac{y^2}{2}\right]_{x-z}^{x+z}dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[(x+z)\left[x+z-x+z\right]+\frac{1}{2}\left[(x+z)^2-(x-z)^2\right]\right]dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[(x+z)(2z)+\frac{1}{2}(4xz)\right]dx\,dz$$

$$=2\int_{-1}^{1}\int_{0}^{z}(xz+z^2+xz)\,dx\,dz$$

$$=2\int_{-1}^{1}\int_{0}^{z}(2xz+z^2)\,dx\,dz=2\int_{-1}^{1}\left(2z\frac{x^2}{2}+xz^2\right)_0^z dz$$

$$=2\int_{-1}^{1}\left[z(z^2)+\cancel{z}z^2\right]-0\;dz$$

$$=2\int_{-1}^{1}2z^3\,dz=4.\left(\frac{z^4}{4}\right)_{-1}^{1}$$

$$=1-(-1)^4$$

$$=0.$$

6). Evaluate : $\int_0^1\int_0^x x\,dy\,dx$

$$=\int_0^1 x(y)_0^x\,dx$$

$$=\int_0^1 x(x-0)\,dx=\int_0^1 x^2\,dx=\left(\frac{x^3}{3}\right)_0^1=\frac{1}{3}-0=\frac{1}{3}.$$