SET - 2

1. Maclaurin's series expansion

statement $f : [0, x] \in R$ if $\exists$.

(i) f^{n-1} is continuous on $[0, x]$

(ii) f^{n-1} is differentiable on $[0, x]$ on $R \in Z^{+}$ then $\exists$ a real number $\theta \in (0, 1)$ $\exists$

$$f^{n} = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \frac{x^{n-1}}{(n-1)!} f^{n-1}(0)$$

2. Properties of Jacobian transformation.

* $$\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1$$

* $$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

* Implicit functions $\dfrac{dy}{dx} = -\dfrac{\partial f / \partial x}{\partial f / \partial y}$

3. Taylor's series expansion of two variables

$$f(x, y) = f(a, b) + (x-a) f_x(a, b) + (y-b) f_y(a, b)$$
$$+ \frac{1}{2!} \left[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b) \right] + \cdots$$

4. $\Gamma 7/2 = (7/2 - 1)(7/2 - 2)(7/2 - 3)\Gamma(7/2 - 3)$

$= (5/2)(3/2)(1/2)\Gamma(1/2)$

$= \frac{15}{8}\sqrt{\pi}$

5. relation between beta and gamma function

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{where } m>0,\ n>0$$

6 (a) $f(x) = e^x,\ g(x) = e^{-x}$ in $[a,b]$

let $f(x) = e^x$ and $g(x) = e^{-x}$ on $[a,b]$

i) $f(x), g(x)$ are Continuous on $[a,b]$ and

ii) $f(x), g(x)$ are differentiable on (a,b)

iii) Cauchy's mean value theorem

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$f(x) = e^x$	$g(x) = e^{-x}$.
$f(a) = e^a$	$g(a) = e^{-a}$
$f(b) = e^b$	$g(b) = e^{-b}$
$f'(x) = e^x$	$g'(x) = -e^{-x}$
$f'(c) = e^c$	$g'(c) = -e^{-c}$

$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -\frac{e^c}{\frac{1}{e^c}}$$

$$\frac{e^b - e^a}{\frac{e^a - e^b}{e^a \cdot e^b}} = -e^c \cdot \frac{e^c}{1}$$

$$e^b - e^a \times \frac{e^a \cdot e^b}{e^a - e^b} = -e^{2c}$$

$$\frac{+(e^a - \cancel{e^b})}{(e^a - \cancel{e^b})} \cdot e^{a+b} = +e^{2c}$$

$$e^{a+b} = e^{2c}$$

$$a+b = 2c$$

$$c = \frac{a+b}{2}$$

$c \in (a,b)$ which verifies the Cauchy's mean value theorem.

b) $\log(1+x)$ – maclaurins series.

$f(x) = \log(1+x)$ | $f(0) = \log(1+0) = \log 1 = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ | $f'(0) = \frac{1}{1+0} = 1$

$f''(x) = -1(1+x)^{-1-1} = \frac{-1}{(1+x)^2}$ | $f''(0) = \frac{-1}{(1+0)^2} = \frac{-1}{1} = -1$

$$f'''(x) = -1(-2)(x+1)^{-2-1} \qquad\qquad -1(x+1)^{-2}$$

$$= 2(x+1)^{-3}$$

$$= \frac{2}{(x+1)^3}$$

$$f'''(0) = \frac{2}{(0+1)^3} = \frac{2}{1^3} = 2$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$\log(1+x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \ldots$$

$$= \frac{x}{1!} - \frac{x^2}{2!} + \frac{2x^3}{3!} + \ldots$$

$$= x - \frac{x^2}{2} - \frac{\cancel{2}x^3}{3 \times \cancel{2} \times 1} + \ldots$$

$$= x - \frac{x^2}{2} - \frac{x^3}{3} + \ldots$$

7(a) Obtain taylor's series expansion of sinx in power of $x-\frac{\pi}{4}$

Sol: Let $f(x) = \sin x$.

The taylor's series expansion of f(x) in powers of $x-\frac{\pi}{4}$ is given by [In terms of x-a].

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Here $f(x) = \sin x$ $\quad a = \frac{\pi}{4}$.

$$\sin x = f\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)}{1!} f'\left(\frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right) \ldots$$

$f(x) = \sin x$. $\quad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \sin 45° = 1/\sqrt{2}$

$f'(x) = \cos x$. $\quad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \cos 45° = 1/\sqrt{2}$.

$f''(x) = -\sin x$. $\quad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -1/\sqrt{2}$

$f'''(x) = -\cos x$, $\quad f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -1/\sqrt{2}$

$= 1/\sqrt{2} + \left(x-\frac{\pi}{4}\right) 1/\sqrt{2} + \frac{\left(x-\frac{\pi}{4}\right)}{2!}(-1/\sqrt{2}) + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!}(-1/\sqrt{2}) \ldots$

$= 1/\sqrt{2}\left[1 + \left(x-\frac{\pi}{4}\right) - \frac{(x-\pi/4)^2}{2!} - \frac{\left(x-\frac{\pi}{4}\right)^3}{3!} + \ldots\right].$

(7)(5) Verify taylor's theorem for $f(x) = (1-x)^{5/2}$ with lagrages form remainder upto two forms in the interval $[0,1]$.

sol. consider $f(x) = (1-x)^{5/2}$ in $[0,1]$.

i) $f(x)$, $f'(x)$ are continuos in $[0,1]$.

$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$

$f(x) = (1-x)^{1/2}.$

$f'(x) = \frac{5}{2}(1-x)^{5/2} \quad \frac{d}{dx}(1-x).$

$= \frac{5}{2}(1-x)^{\frac{5-2}{2}}\left[\frac{d}{dx}(1) - \frac{d}{dx}(x)\right]$

$= \frac{5}{2}(1-x)^{3/2}[0-1].$

$f'(x) = \frac{-5}{2}(1-x)^{3/2}.$

$f''(x) = \frac{-5}{2}\left[\frac{3}{2}(1-x)^{3/2-1/1} \cdot \frac{d}{dx}(1-x\right].$

$= \frac{-15}{4}[1-x]^{\frac{3-2}{2}}(-1)].$

$= \frac{15}{4}[1-x]^{1/2}.$

$f(x)$ is, differentiable in $(0,1)$ we know that taylor series expression with lagranges form remainder has.

$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(\theta x).$

$a = 0; \quad x = 1.$

$f(x) = (1-x)^{5/2}.$

$= (1-1)^{5/2}$

$= 0^{5/2} = 0. \quad$ Here $f(x) = 0.$

$f(a)$.

$f(x) = (1-x)^{5/2}$.

$f(a) = (1-a)^{5/2}$

$= (1-0)^{5/2}$

$= 1^{5/2} = 1$.

$\boxed{f(a) = 1}$.

$f'(a)$.

$f'(x) = -5/2\,(1-x)^{3/2}$.

$f'(a) = -5/2\,(1-a)^{3/2}$.

$= -5/2\,(1)^{3/2} = -5/2\,(1)$.

$f''(\theta x) = f''(\theta)$.

$f''(x) = \frac{15}{4}(1-x)^{1/2}$

$f''(\theta) = \frac{15}{4}(1-\theta)^{1/2}$.

$0 = 1 + \frac{(1-0)}{1!}(-5/2) + \frac{(1-0)^2}{2!}\,\frac{15}{4}(1-\theta)^{1/2}$.

$0 = \left(\frac{1}{1} - \frac{5}{2}\right) + \frac{15}{8}(1-\theta)^{1/2}$.

$0 = \frac{2-5}{2} + \frac{15}{8}(1-\theta)^{1/2}$.

$0 = \frac{-3}{2} + \frac{15}{8}(1-\theta)^{1/2}$,

$3/2 = \frac{15}{8}(1-\theta)^{1/2}$.

$\frac{3}{\cancel{2}} \times \frac{\cancel{8}\,4}{\cancel{15}\,5} = (1-\theta)^{1/2}$.

$\frac{4}{5} = \sqrt{1-\theta}$.

$\sqrt{1-\theta} = 4/5$.

S.O.B.S.

$(\sqrt{1-\theta})^2 = (4/5)^2$

$1-\theta = 16/25 \Rightarrow$

$\frac{1}{1} - \frac{16}{25} = \theta$.

$\theta = \frac{25-16}{25}$.

$\theta = 9/25$.

$\boxed{\theta = 0.35}$

Q) If $u = \log(x^3+y^3+z^3-3xyz)$ then ST $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

Sol:

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right).$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log(x^3+y^3+z^3-3xyz).$$

$$= \frac{1}{x^3+y^3+z^3-3xyz}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz).$$

$$= \frac{1}{x^3+y^3+z^3-3xyz}\left(3x^2+0+0-3yz\frac{\partial}{\partial x}(x)\right).$$

$$= \frac{1}{x^3+y^3+z^3-3xyz}[3x^2-3yz].$$

$$= \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}.$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}.$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}.$$

$$\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right]$$

$$= \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}.$$

$$= \frac{3(x^2-yz+y^2-xz+z^2-xy)}{x^3+y^3+z^3-3xyz}.$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{3}{x+y+z}.$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right).$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right).$$

$$= 3\frac{\partial}{\partial x}(x+y+z)^{-1} + 3\frac{\partial}{\partial y}(x+y+z)^{-1} + 3\frac{\partial}{\partial z}(x+y+z)^{-1}.$$

$$= 3\left[-1(x+y+z)^{-1-1}\right. + 3\left[-1(x+y+z)^{-1-1}\right] + 3(x+y+z)^{-1-1}.$$

$$= -3\left[\frac{1}{(x+y+z)^2}\right] - 3\left[\frac{1}{(x+y+z)^2}\right] - 3\left[\frac{1}{(x+y+z)^2}\right].$$

9. If $f(x,y) = e^x \log(1+y)$ interms of x & y upto 3rd degree.

Sol: Given

$$f(x,y) = e^x \log(1+y) \qquad f(0,0) = 0$$

$$f_x(x,y) = e^x \log(1+y) \qquad f_x(0,0) = 0$$

$$f_{xx}(x,y) = e^x \log(1+y) \qquad f_{xx}(0,0) = 0$$

$$f_y(x,y) = \frac{e^x}{1+y} \qquad f_y(0,0) = 1$$

$$f_{yy}(x,y) = \frac{-e^x}{(1+y)^2} \qquad f_{yy}(0,0) = -1$$

$$f_{xy}(x,y) = e^x \cdot \frac{1}{1+y} \qquad f_{xy}(0,0) = 1$$

$$f_{xxy}(x,y) = \frac{e^x}{1+y} \qquad f_{xxy}(0,0) = 1$$

$$f_{xyy}(x,y) = \frac{-e^x}{(1+y)^2} \qquad f_{xyy}(0,0) = -1$$

$$f_{xxx}(x,y) = e^x \log(1+y) \qquad f_{xxx}(0,0) = 0$$

$$f_{yyy}(x,y) = \frac{2e^x}{(1+y)^3} \qquad f_{yyy}(0,0) = 2$$

By Maclaurin's series

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$$

$$+ \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)]$$

$$= 0 + 0 + y + \frac{1}{2!}[0 + 2xy - y^2] + \frac{1}{3!}[0 + 3x^2y - 3xy^2 + 2y^3]$$

$$= y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} \text{ //}$$

(a). Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol: Given

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2+y^2}\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\, dy\, dx \qquad \because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \int_{x=0}^{1} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_{y=0}^{\sqrt{1+x^2}} \frac{1}{\sqrt{1+x^2}} dx$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \tan^{-1}\frac{0}{\sqrt{1+x^2}}\right] dx$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}}\left[\frac{\pi}{4}\right] dx$$

$$= \frac{\pi}{4} \log\left[x+\sqrt{x^2+1}\right]_0^1$$

$$= \frac{\pi}{4}\left[\log(1+\sqrt{1+1}) - \log(0+\sqrt{0+1})\right]$$

$$= \frac{\pi}{4}\log(1+\sqrt{2}). //$$

(b). Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

Sol: Given

$$I = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[(x+z)(y)+\frac{y^2}{2}\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[(x+z)(x+z)-(x+z)(x-z)\right]+\frac{1}{2}\left[(x+z)^2-(x-z)^2\right] dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left[x^2 + xz + xz + z^2 - x^2 + \cancel{x}z - x\cancel{z} + z^2\right] + \frac{1}{2}\left[(x+z)^2 - (x-z)^2\right] dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left[2xz + 2z^2 + \frac{1}{2}(4xz)\right] dx\,dz$$

$$= 2\int_{-1}^{1}\int_{0}^{z} (2xz + z^2)\, dx\,dz$$

$$= 2.\int_{-1}^{1} \left[2z.\left(\frac{x^2}{2}\right)_0^z + z^2 (x)_0^z\right] dz$$

$$= 2\int_{-1}^{1} \left[z(z^2) + z^2(z) - (0+0)\right] dz$$

$$= 2\int_{-1}^{1} 2z^3\, dz \quad = \quad 4\left(\frac{z^4}{4}\right)_{-1}^{1}$$

$$= 1 - (-1)^4 = 0 //$$

10(a) S.T. $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$

$-\log x = t$

$\log x = -t$

$e^{\log x} = e^{-t}$

$x = e^{-t}$

$dx = -e^{-t}\,dt$

U.L. $x = 1$ then $t = -\log 1 = 0$

L.L $x = 0$ then $t = -\log 0 = \infty$

$$\int_0^1 \frac{1}{\sqrt{-\log x}}\,dx = \int_\infty^0 \frac{-e^{-t}\,dt}{\sqrt{t}} = \int_0^\infty e^{-t} \cdot t^{-1/2}\,dt$$

$$= \int_0^\infty e^{-t}\, t^{1/2-1}\,dt$$

$$= \Gamma(1/2) = \sqrt{\pi}$$

BETA

10(b) Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx$ in terms of Beta function

Sol

put $x^5 = t$

$\left((x)^5\right)^{1/5} = t^{1/5}$

$x = t^{1/5}$

$dx = \frac{1}{5} t^{1/5-1}\,dt$

$dx = \frac{1}{5} t^{-4/5}\,dt$

U.L. $x = 1$ then $t = 1^5 = 1$

L.L. $x = 0$ then $t = 0^5 = 0$

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{(t^{1/5})^2}{\sqrt{1-t}} \cdot \frac{1}{5} \cdot t^{-4/5}\,dt$$

$$= \frac{1}{5}\int_0^1 t^{2/5} \cdot t^{-4/5} \cdot \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{5}\int_0^1 t^{2/5-4/5} \cdot (1-t)^{-1/2}\,dt$$

$$= \frac{1}{5}\int_0^1 t^{-2/5} \cdot (1-t)^{-1/2}\,dt$$

$$= \frac{1}{5}\int_0^1 t^{3/5-1}\,(1-t)^{\frac{1}{2}-1}\,dt$$

$$= \frac{1}{5}\,\beta\left(\frac{3}{5}, \frac{1}{2}\right)$$