Spontaneous Synchronization of Oscillators

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1 Abstract

The purpose of this paper is to give an overview of the theory of identical coupled oscillators. Such oscillators are known to synchronize in phase after a time period, called spontaneous synchronization. The current model for describing this system is the Kuramoto model. This paper begins with a discussion of limit cycle oscillators. Then we analyze the Kuramoto model's dynamics as a limit cycle oscillator. The latter half of this paper is devoted to current research on spontaneous synchronization of networks of oscillators with different couplings.

2 Introduction

Spontaneous Synchronization was discovered by dutch physicist Christiaan Huygens. He invented the pendulum clock in the 18th century. Since it was the most accurate clock of its time, every Dutch ship was mandated to have two (in case one broke) on board in order to keep time over long voyages. On one of these ships, Huygens noticed that the pendulums on the two clocks would synchronize after a time, regardless of the initial positions.[6] Over centuries of research, this phenomenon has appeared in many unexpected places such as flashing groups of fireflies, laser arrays, and cardiac cells. A better understanding of spontaneous synchronization would allow people to apply it to electrical power-grids, parallel computing, and bridge engineering[1].
The main model of spontaneous synchronization is the Kuramoto model. This model described oscillators approaching a limit cycle through mean field coupling. This model has been used to explore this phenomenon as a many-body non-linear dynamical system [1]. The first part of this paper will discuss these types of systems and how they are modeled by Kuramoto.

The second part of this paper is devoted to the current research of Steven Strogatz. The original Kuramoto model assumes that every oscillator is coupled to every other oscillator. Strogatz complicates the model by investigating the dynamics of oscillators that are coupled in different 'networks'. His graph theory approach is intuitive and intriguing. His research is very applicable to physical systems because in real world examples such as fireflies and cardiac cells, there is a topology that allows some oscillators to couple while others do not, just as in his networks.

3 The Kuramoto Model

3.1 limit cycle oscillators

Spontaneous synchronization, as described in the introduction, is a self-limiting stable limit cycle. 'Self-limiting' means that once the system is set into motion, it will approach synchronization independent of the initial conditions. 'Stable' means that slight perturbations in the system will correct themselves and reinstate the synchrony. In other words, the system is bounded [4]. A helpful conceptual analogy is a metronome. Independent of the initial displacement, it soon converges to a characteristic frequency.

The way to achieve a dynamical system that fits this description is to have pumping and dampening. If the frequency/amplitude becomes to large then it must be dampened, and if it becomes too small then it must be pumped. This allows us to maintain the limit cycle. The paradigmatic equation for such a system is the Van der Pol equation:

$$\frac{d^2x}{dt^2} - (\gamma - x^2)\frac{dx}{dt} + \omega^2 x = 0 \tag{1}$$

Where the parameter γ is a real, positive constant. This means that the dx/dt term changes sign depending on whether x has magnitude that is larger or smaller than $\sqrt{\gamma}$. The system pumps displacements $|x| < \sqrt{\gamma}$ and damps displacements $|x| > \sqrt{\gamma}$. Therefore, the system asymptotically approaches a characteristic amplitude and frequency.

We can write Eq.1 as two coupled first-order equations:

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = (\gamma - x^2)v - \omega^2 x \tag{2}$$

And in a phase diagram (see Fig. 1) the trajectory given any initial point (x_0, v_0) approaches the stable periodic mapping. This periodic phase plot represents oscillations with a characteristic waveform, amplitude, and frequency and are limit-cycle oscillators.

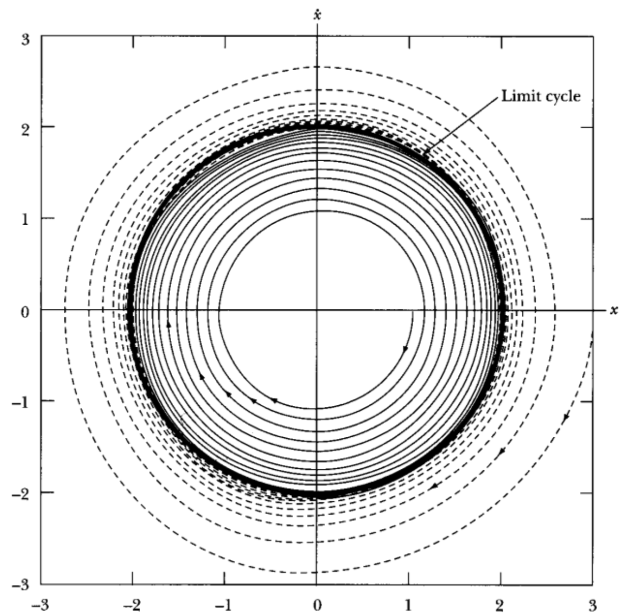

Figure 1: Phase diagram of the Van Der Pol Equation with $\mu = 0.05$. Solid line represents a system with initial conditions $(x, \dot{x}) = (1, 0)$ and the dotted line $(3, 0)$. Both systems converge to the limit cycle with radius 2.[4]

3.2 The Equation

Using math that is beyond the scope of this paper, Kuramoto showed that for any system of identical weakly coupled limit-cycle oscillators, the dynamics obeyed the phase equations:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \qquad i = 1, ..., N$$

Where θ_i is the phase of the ith oscillator and ω_i is its natural frequency. Every other oscillator, j, influences the ith oscillator based on the relative phase between them. Γ_{ij} is the 'interaction function'. This set of equations is too complicated, because the interaction function can involve complicated integrals and many Fourier harmonics. An analysis of a system is made more difficult because the topology is unspecified. Oscillators arranged in a chain, ring, lattice, etc. could change the dynamics of the system[2].

To make this model usable, Kuramoto assumed the mean field case, which meant assuming equally weighted, all-to-all, sinusoidal coupling:

$$\Gamma_{ij} = \frac{K}{N} \sin(\theta_j - \theta_i)$$

Where K is the coupling strength and the $1/N$ is to make it so the model doesn't blow up as $N \to \infty$. Substituting this relation above yields the full Kuramoto equations:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \frac{K}{N} \sin(\theta_j - \theta_i) \qquad i = 1, ..., N \tag{3}$$

Another assumption this model makes is that the natural frequencies ω_i are distributed unimodally about zero so that $g(\omega) = g(-\omega)$ where g is the distribution. This distribution must decrease monotonically and continuously to zero.[2][1]
The most popular way to visualize this model is to consider the oscillators' phasors as shown in Fig. 2. Then the system oscillators become a collection of particles moving on the unit circle. Each particle (oscillator) has a phasor $e^{i\theta_j(t)}$ that describes its position on the circle at time t. We can then define the 'order parameter' as:

$$r(t)e^{i\psi(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)} \tag{4}$$

This complex quantity is important because it represents the collective rhythm of the system. $\psi(t)$ is the average phase, and $r(t)$ measures the amount of phase coherence or synchrony in the system. If the oscillators are scattered equally around the circle, $r(t) = 0$, but if they are all in sync, $r(t) \to 1$. [2][1]

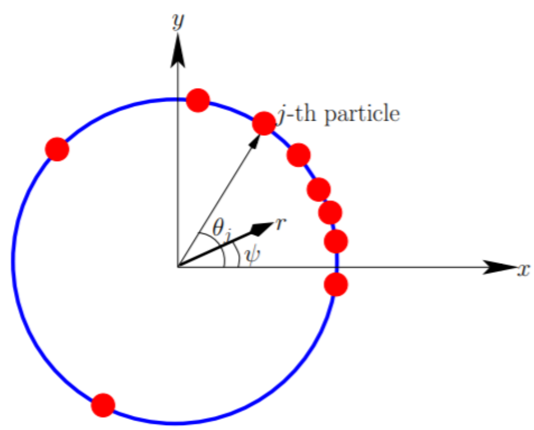

Figure 2: A system of Kuramoto oscillators represented by their phasors. Each particle has its own phasor, and r and ψ specify the order parameter of the system, quantifying how in synch the system is.[1]

If we multiply both sides of Eq(4) by $e^{-i\theta_i}$ and then take the imaginary parts of both sides, we get:

$$r \sin(\psi - \theta_i) = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

So we can substitute this back into Eq(3) to get

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i) \qquad i = 1, ..., N \tag{5}$$

This is a very helpful expression for understanding what the mean field theory assumption means. It is clear in the above expression that each oscillator is affected only by the mean field quantities r and ψ. The phase θ_i is pulled toward the mean phase, ψ, and the strength of this pull is dictated by r, this creates a feedback loop effect. As the oscillators become more synchronized, Kr becomes larger, and so the coupling strength increases.

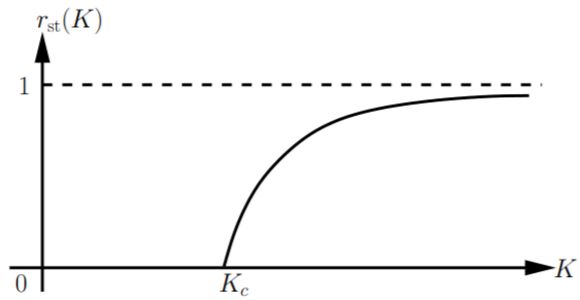

Figure 3: A diagram of the dependence of r_{st} on the coupling strength K (holding$g(\omega)$ constant). A phase transition occurs at the critical value K_c. This is also a bifurcation of the system[1]

Analysis of these equations (usually computationally) reveals that after a long time the system approaches a stationary state. There are two possible types of stationary states. The most obvious is a coherent stationary state, where all the oscillators become bunched and oscillate together. In coherent stationary states $r \to 1$ and ψ remains a function of time. The second stationary state is an incoherent state where the oscillators are equally spread out around the circle so that $r = 0$ and $\psi = const$. The oscillators will not synchronize without perturbation. Which stationary state the system falls into depends on the value of K, the initial phases, and the frequency distribution $g(\omega)$. Small values of K are more likely to lead to incoherent stationary states, and larger values of K are more likely to lead to coherent stationary states. This means for given initial conditions and frequency distribution, there is a critical value K_c and $K < K_c$ leads to incoherent states and $K > K_c$ leads to coherent states[1]. The stationary state of the order parameter r_{st} changes depending on the value of K in relation to K_c. Through a lot of mathematics, Kuramoto found that

$$K_c = \frac{2}{\pi g(0)}$$

The change that occurs at K_c is referred to as a phase transition[1] and can be visualized in Fig. 3.

4 Current Research

The Kuramoto model was first conjectured in 1975, and a lot of research has been done on it since then. I would like to skip ahead a few decades to look at the recent research of Steven Strogatz. The reason I want to focus on his research is because it is interesting and intuitive from a non-math background.
Remember that part of the mean field assumption made in the Kuramoto model was that each particle affects every other particle proportional to their relative phase. Strogatz has been researching different 'networks' and how they affect the dynamics of the system. He modifies which particles affect which other particles, creating unique networks with different dynamics. This research ties in very well with chaos theory.

Strogatz uses a slightly different formulation of the Kuramoto Equation

$$\frac{d\theta_i}{dt} = \sum_{k=0}^{n-1} A_{jk} \sin(\theta_k - \theta_j) \tag{6}$$

Which involves an adjacency matrix A which is composed of 1s and 0s. $A_{jk} = A_{kj} = 1$ if oscillator j affects oscillator k. This matrix defines the topology of the system. Strogatz defines globally synchronizing as asymptotically approaching in phase state from almost all initial conditions. Then, he wants to find the minimum connectivity that a system can have and be globally synchronizing. Connectivity, μ is defined as the percent of other oscillators that each oscillator is connected to. The question is, then, what is the lowest connectivity that can support the oscillators globally synchronizing? Call this the critical connectivity μ_c[5]. Strogatz uses 2D spherically symmetric structures called circulant networks to visualize the connectivity. Fig. 4 shows some of the networks he analyzed.

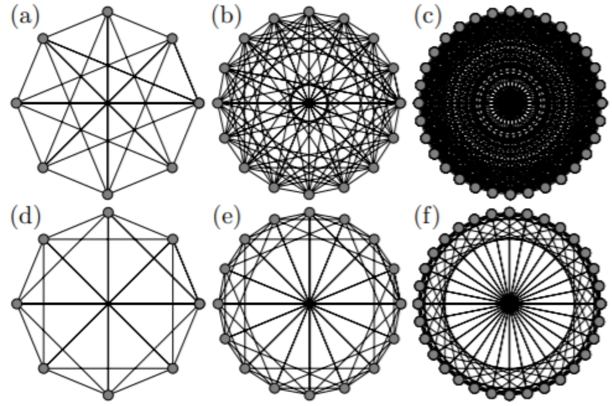

Figure 4: Circulant networks with 8, 16, and 32 oscillators. (a),(b), and (c) are networks that are globally syncrhonizing. This is because they are 'denser', meaning they have more connectivity. (d),(e), and (f) are networks that still globally synchronize, but are much sparser. Strogatz believes that these may be some of the sparsest networks that globally synchronize.[5]

Mathematically, the analysis of connectivity manifests as analysis of the adjacency matrix A. Circulent networks have an adjacency matrix where each row is a shifted version of the preceding row:

$$A = \begin{bmatrix} a_0 & a_1 & ... & a_2 & a_1 \\ a_1 & a_0 & a_1 & ... & a_2 \\ \vdots & & & & \vdots \\ a_1 & a_2 & ... & a_1 & a_0 \end{bmatrix}$$

So the main diagonal is all a_0 which is set to $a_0 = 0$ because no oscillator is coupled to itself. The important information from this matrix is its eigenvalues. Since this is a real symmetric matrix, its eigenvalues are integers. It has been shown [7] that the eigenvalues are given by:

$$\lambda_p = \sum_{s=1}^{n-1} a_s e^{2\pi i p s/n} \qquad 0 \leq p \leq n-1 \tag{7}$$

Which is an intriguing expression. Each of the n eigenvalues is the sum of the components in a row of the matrix with an attached phase factor. The eigenvectors are:

$$\vec{v}_p = \begin{bmatrix} 1 \\ \cos(\frac{2\pi p}{n}) \\ \vdots \\ \cos(\frac{2\pi p(n-1)}{n}) \end{bmatrix} \qquad \vec{w}_p = \begin{bmatrix} 0 \\ \sin(\frac{2\pi p}{n}) \\ \vdots \\ \sin(\frac{2\pi p(n-1)}{n}) \end{bmatrix}$$

To find equilibrium points of Eq. 6, arrange the θ_j elements into a vector $\vec{\theta} = [\theta_0, ..., \theta_{n-1}]^T$ and then create the vectors $\vec{c} = \cos(\vec{\theta})^T$ and $\vec{c} = \sin(\vec{\theta})^T$. (The superscript T denotes transpose). Then use $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$ (since we want $\dot{\theta} = 0$) to find that $\vec{\theta}$ is an equilibrium set of θ_j if and only iff:

$$(Diag(\vec{c}))A\vec{s} = (Diag(\vec{s}))A\vec{c} \tag{8}$$

Where $Diag(\vec{v})$ is a matrix with the elements of $\vec{v}$ as its diagonal. The above equation can be solved using the eigenvector relations to find that

$$\vec{\theta}_p = [\theta_0, \frac{2\pi p}{n} + \theta_0, ..., \frac{2\pi p(n-1)}{n} + \theta_0]^T$$

Shifting all of the angles by a constant angle will not change the result,so we set $\theta_0 = 0$

$$\vec{\theta}^{(p)} = [0, \frac{2\pi p}{n}, ..., \frac{2\pi p(n-1)}{n}]^T \tag{9}$$

The solutions with $p = 0$ are synchronous states, and the solutions with $p \neq 0$ are incoherent or 'twisted' states. The twisted states physically correspond to the oscillators' phases differing by a constant amount, so they are in equilibrium, but will never synchronize.[5]

Using the result above, Strogatz ran numerical simulations fore circulant networks $5 < n < 50$ to determine the connectivity needed for global spontaneous synchonization. In Fig. 5, it is clear that as n becomes large, the results converge to a value of $\mu_c \approx 0.75$. Strogatz continues by analyzing the stability of these equilibriums and the time taken to synchronize, but that is beyond the scope of this paper.

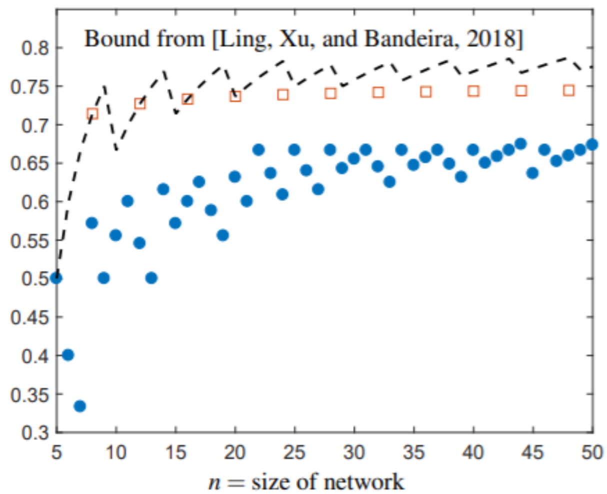

Figure 5: Connectivity of networks of different sizes. The blue dots indicate the densest networks that do not globally synchronize and the red squares represent the sparsest networks that do. The dashed line is the bound above which no all netwokrs globally synchronize. [5]

5 Summary

This paper began with classical mechanics, but the scope spanned a far vaster array of topics than anticipated. The content transferred from classical mechanics to nonlinear dynamics with the analysis of limit cycle oscillators. Then to graph theory with the circulant networks to spectral graph theory with analyzing the eigenvalues to find the connectivity of the graphs. I chose a classical mechanics topic that appears in the macroscopic world of physics[6], but diving into the research on the topic took me into more mathematical disciplines. It was quite a journey.

Even with all the mathematical complications, one can still see the similarities between this system and the systems studied in the non-linear dynamics and chaos sections of PHY235. The discussion of the Kuramoto model used phase diagrams, and the analysis can be described as the predictions of the limit cycle of a system. The transition at the critical coupling K_c is a bifurcation of the system. This paper took the basics of nonlinear dynamics and chaos that were discussed in class and focused in, using these techniques to analyze a particular phenomenon.

After spending a lot of time examining the literature on spontaneous synchronization, I think it is an exciting topic that still has some room to explore. During my research I found a

computer simulation that I played with a little: `https://www.complexity-explorables.org/explorables/ride-my-kuramotocycle/`. While watching this simulation, I noticed that the particles that have a faster angular frequency when $K < K_c$ will end up at the front of the pack when $K > K_c$. This intrigued me because it means that the synchronization translates the frequency to the phase. None of the literature mentioned this, so perhaps it is an area to explore.
Another area that I believe is ripe for research are the practical applications. Most of the literature is very mathematical. This paper skipped over a lot of derivations to avoid getting bogged down in pages of complicated equations. It is great that the mathematical analysis has come so far, but I think the physical part needs to catch up. It is known that there are many possible applications of this topic such as parallel computing and coupled lasers in arrays, but I couldn't find any examples of this model actually being implemented in those scenarios. My optics background leads me to question whether this could also be applied to modes in a fiber. In optical fibers there are issues of coupling or 'cross talk' between modes in fibers, and it seems like this model could be applied and reveal some things there. Overall, I think future research needs to focus more on physical applications.

Even though the this paper was more mathematically focused than intended, if I could go back in time I wouldn't change my topic because I enjoyed researching and writing about this. I have never been exposed to graph theory before and it is very interesting. I learned a lot more about nonlinear dynamics as well. This paper served to let me investigate a topic that I was passionate about, and I'm glad I had the opportunity.

References

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