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Districting modeling with exact contiguity constraints

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Abstract. A classic problem in planning is districting, which aims to partition a given area into a specified number of subareas according to required criteria. Size, compactness, and contiguity are among the most frequently used districting criteria. While size and compactness may be interpreted differently in different contexts, contiguity is an unambiguous topological property. A district is said to be contiguous if all locations in it are 'connected'—that is, one can travel between any two locations in the district without leaving it. This paper introduces a new integer-programming-based approach to districting modeling, which enforced contiguity constraints independently of any other criteria that might be additionally imposed. Three experimental models are presented, and tested with sample data on the forty-eight conterminous US states. A major implication of this paper is that the exact formulation of a contiguity requirement allows planners to address diverse sets of districting criteria.

1 Introduction

Districting is a process of partitioning an area into smaller regions, here referred to as districts, in response to some criteria. The term 'districting' may imply that boundaries of districts can lie anywhere in space, but districting is often cast as a problem of aggregating predefined discrete geographic units into larger clusters. Examples of such units include census tracts, zip-code areas, voting precincts, and land parcels.

Since the production of districting plans often requires a substantial amount of time and effort, computer methods are attractive if districting criteria are clear enough to be amenable to mathematical representation. Various techniques and algorithms have been developed for districting, and they tend to rely on a few basic models, such as transportation and location-allocation models. This is partly because most districting problems involve similar criteria, and also because efficient solution algorithms are available to those models. It is generally true that for any model to be useful in practice, it should ultimately be solvable (even at the cost of accuracy). Hence it seems even justifiable to relax essential criteria and/or add unnecessary criteria to make the resulting model tractable. Nevertheless, an alternative approach that respects the original problem statement more faithfully would have a theoretical contribution, and might eventually lead to good solution algorithms. Unfortunately, the development of such an approach has been hindered by the difficulty of explicitly modeling *contiguity*—ie the quality of being connected—which is inherent to districts.

The purpose of this paper is to propose a new strategy for districting modeling, which takes contiguity as an essential, unambiguous requirement, and allows varying interpretations of other criteria. To do so, I consider three fictitious districting problems and formulate them as mixed integer programming (MIP) models with exact contiguity constraints. These models do not cover all possible scenarios, but should suffice to demonstrate the flexibility of the contiguity-based districting modeling.

The rest of the paper is organized as follows. Section 2 reviews districting problems to identify typical districting criteria and modeling strategies. Section 3 reviews a series of contiguity constraint sets originally proposed for composing a single district.

These are integrated into prototype districting models in section 4. Section 5 tests the tractability of these models with varying numbers of districts. Section 6 concludes the paper.

2 Districting problems

Districting problems arise in various contexts ranging from political districting to sales districting to school districting. As discussed below, these problems share similar criteria and modeling strategies.

2.1 Districting criteria

For political districting there are three essential criteria: equal population, compactness, and contiguity. In fact, 'the' political districting problem often refers to one of creating compact and contiguous districts of equal population.

Equal population is based on the 'one-person one-vote' principle (Hess et al, 1965; Mehrotra et al, 1998; Williams, 1995). This requirement may not be achieved precisely, however, because it is virtually impossible to assign exactly the same number of voters to every district. Instead, it is more practical to set a tolerable limit on the deviation of each district from some target population. Still, the determination of such a limit is not an easy task, since even a slight deviation from strict population equality could be ruled unconstitutional (Mehrotra et al, 1998; Williams, 1995).

Compact and contiguous districts are called for to reduce the risk of 'gerrymandering'—that is, redrawing district boundaries in favor of a particular party or against its opposition (Garfinkel and Nemhauser, 1970; Harris, 1964; Mehrotra et al, 1998; Williams, 1995). The term 'compact' is intuitively understood as circle/square-like or consolidated rather than spread, but there is no universally accepted measure of compactness (Hess et al, 1965). Many different compactness measures have been proposed, and they tend to relate to distance. In political districting compactness is often related to moment of inertia—that is, the sum of the products of the area of each geographic unit and the squared distance from that unit to its district center. Some researchers (eg Hess et al, 1965; Hojati, 1996) emphasize a functional characteristic of compactness rather than a geometric one by substituting population for area in the calculation of moment of inertia. This physical quantity is obviously dependent on scale: the larger a district is, the greater moment of inertia it has and the less compact it is. To reduce this bias, Mehrotra et al (1998) replaced Euclidean distance with 'topological distance'—that is, the minimum number of geographic units to be traversed for traveling between two locations.

Contiguity, on the other hand, is an unambiguous property. It is easy to tell even by visual inspection whether or not a district is contiguous. Roughly, it can be said that "a district is contiguous if it is possible to walk from every point in the district to every other point without crossing the district" (Garfinkel and Nemhauser, 1970). Although contiguity is a major criterion in political districting, it tends to be excluded from a formal modeling process. Instead, this topological property is often implicitly addressed by pursuing compact districts, which tend to be contiguous. Other heuristics are also available that guarantee to make feasible, not necessarily optimal, contiguous districts (eg Brookes 1997; Church et al, 2003; Cova and Church, 2000; Mehrotra et al, 1998).

In addition to the three political districting principles, particular contexts may require other criteria, such as minority representation, community integrity, proportionality, presence of safe and/or swing districts, and similarity to the existing pattern (Williams, 1995). These secondary criteria are, in general, more political than demographic or geographic (Williams, 1995).

Sales districting much resembles political districting in that size, compactness, and contiguity are major criteria, although their interpretations and emphases may be different. The size of a sales district is not necessarily the actual number of people living there, but can be measured in terms of other attributes—for example, salesperson workload (Fleischmann and Paraschis, 1988; Hess and Samuels, 1971; Marlin, 1981; Segal and Weinberger, 1977) and sales potential (Shanker et al, 1975).

Geometric compactness seems less relevant in sales districting. Instead, companies tend to seek efficient districts in terms of travel cost. Because reduction of travel cost generally increases compactness, it is reasonable to regard travel efficiency as one type of compactness. Still, the economic interpretation of compactness requires a different measurement than the geometric counterpart; for example, as Marlin (1981) suggested, distance should not be squared if travel cost increases in proportion to distance. Also, the use of straight-line distance is sometimes discouraged, because it may underestimate the actual travel expense, particularly in urban areas. For a more realistic distance measurement, Segal and Weinberger (1977) placed a graph structure on a set of geographic units considering travel restrictions on the ground. Likewise, Zoltners and Sinha (1983) measured distance along an actual road network in the US.

Economic concerns also make contiguity less important, because the major purpose of sales districting is to make an efficient business environment, which may not need to look pretty.

School districting, too, is similar to political districting, except for two distinct features. First, a school district has a fixed center—that is, a school. The presence of district centers helps to evaluate compactness, as well as enforce contiguity (Shirabe, 2005). Second, a school has multiple grades, ranging from kindergarten to 12th grade, in a typical US school system. This imposes a capacity constraint not only on each school, but on each grade (Ferland and Guenette, 1990; Franklin and Koenigsberg, 1973). Time is also an important factor, since many students wish to continue to attend the same school as they move to higher grades (Ferland and Guenette, 1990; Holloway et al, 1975). Temporal complexity grows with other criteria, such as racial balance (Belford and Ratliff, 1972; Franklin and Koenigsberg, 1973; Knutson et al, 1980; Schoepfle and Church, 1989), community integrity, and travel safety (Holloway et al, 1975).

2.2 Districting models

Many of the districting criteria discussed above have been abstracted nicely to mathematical forms. Perhaps the most frequently used method is to cast a districting problem in the form of a transportation problem (see eg Marlin, 1981), which assigns geographic units to district centers in a way that minimizes the total travel distance/cost while balancing the supply of each unit and the demand of each district center. This approach is suitable for school districting, but could be problematic in most cases of political districting and some cases of sales districting because no apparent district centers may be given. Then a location-allocation model would be a good alternative, as it simultaneously decides the location of district centers and the allocation of geographic units. The location-allocation model is in general computationally more complicated than the transportation model. Yet an approximate solution can be obtained relatively easily by the following iterative procedure. At each iteration, district centers are fixed so that the model is solved as a transportation model; those geographic units which are split between two or more districts are then reassigned to only one district, according to some tie-breaking rule; finally, the centroids of the resulting districts are computed and designated as district centers for the next iteration. The process continues until identical solutions are generated successively. This heuristic dates back to Hess et al (1965).

There are at least three shortcomings in applying the models described above to districting problems. First, too much emphasis is placed on compactness. It is easy to cast the pursuit of compactness as the minimization of moment of inertia, but in practice compactness should be understood as "a loose constraint rather than an objective" (Mehrotra et al, 1998, page 1102). In political districting, for example, compactness need not be emphasized further when districts are compact enough to have no suspicion of gerrymandering. Second, district centers do not always exist a priori. As mentioned earlier, one can search for district centers through an iterative process starting with some arbitrarily chosen district centers. In this way, however, no solution proves to be optimal, nor is it known how good/poor obtained solutions are, compared with other possible solutions. Furthermore, as the number of districts increases, it becomes more difficult to guess correct initial centers. Third, contiguity is not guaranteed. It is true that compact districts are likely to result, but they are not necessarily compact enough to be contiguous. It is not advised to intensify compactness just to attain contiguity because it would distort the original purpose of districting.

I argue that, while the existing districting models undoubtedly remain valuable, it would be worthwhile to offer more alternatives to enhance our ability to deal with a wide variety of settings. Consider that we may want to minimize the maximum travel cost (Mehrotra et al, 1998), minimize the maximum population deviation (Garfinkel and Nemhauser, 1970), maximize the demand potential (Shanker et al, 1975), maximize the similarity to the current districting plan, and so on. In some cases more than one of these objective functions is expected to be optimized simultaneously. There are even cases in which only feasible districting patterns are sought without any objective function, because the characteristics of desirable districting patterns are sufficiently specified by constraints (which are to be either satisfied or not).

One obstacle to such flexible districting modeling has been the lack of exact formulation of contiguity constraints, although explicit consideration of contiguity (or fragmentation) in spatial optimization has been well investigated in the literature, particularly in the context of designing habitat conservation areas [see eg Cerdeira et al (2005) and Önal and Briers (2005) for recent development in this subject]. Zoltners and Sinha (1983), Mehrotra et al (1998), and Cova and Church (2000) are among the few to attempt to incorporate contiguity constraints into districting (or similar) models. In their models geographic units and their adjacency relationships are represented by a graph, and MIP constraints are designed so that any contiguous district corresponds to a subtree of a shortest-path tree rooted at a predetermined geographic unit. As a result, each district grows along shortest paths from its corresponding root, and contiguity is maintained. As they all pointed out, however, their contiguity constraints are not exact, so that some contiguous districts may be overlooked. To my best knowledge, Williams (2002) is the first to formulate exact contiguity constraints in MIP format. His experiment, however, suggests that his dual-graph-based formulation may suffer from computations difficulties in dealing with multiple-region contiguity problems with irregularly shaped geographic units—that is, districting problems. In this paper, to show the benefit of my modeling strategy, I employ the flow-based contiguity constraints of an earlier paper (Shirabe, 2005), which require fewer decision variables and constraints. The recent work by Xiao (2006) also found that the latter is more tractable, at least for a small-scale problem.

3 Contiguity constraints for a single district

I begin with the definition of contiguity, which was rather loosely described earlier, in terms of a graph. Let a vertex represent a geographic unit and an edge the adjacency relation between a pair of geographic units. We then say that a set of geographic units is contiguous if there is at least one path between every pair of geographic units. Considering the symmetry of every adjacency relation, the condition of contiguity can be redefined such that there is a geographic unit that can be reached from every other geographic unit. This is easy to prove because a path can be constructed between any two geographic units through such a 'hub'.

In an earlier paper (Shirabe 2005), I found the problem of selecting a contiguous district equivalent to one of selecting such a subnetwork that has one and only one vertex serving as a hub (or a sink), which receives one unit of flow from every other vertex (see figure 1).

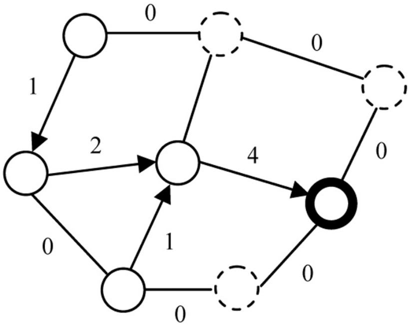

Figure 1. A flow representation of selection of a contiguous district. The solid circles and the dashed circles represent selected and nonselected geographic units, respectively. The bold circle is designated as a hub. The lines and the arrows represent adjacency relationships, and the latter have positive flow (originated from each selected geographic unit) as indicated by associated numbers.

This flow analogy led to the following set of MIP constraints that is satisfied by any contiguous district *and* guarantees that any selected district is contiguous.

$$\sum_{j|(i,j)\in A} y_{ij} - \sum_{j|(j,i)\in A} y_{ji} \geqslant x_i - mw_i, \quad \forall i \in I \ , \tag{1}$$

$$\sum_{i\in I} w_i = 1 \ , \tag{2}$$

$$\sum_{j|(j,i)\in A} y_{ji} \leqslant (m-1)x_i \, , \quad \forall i \in I \ , \tag{3}$$

$$x_i \in \{0,1\}, \quad \forall i \in I \ ,$$

$$w_i \in \{0,1\}, \quad \forall i \in I \ ,$$

$$y_{ij} \geqslant 0, \quad \forall (i,j) \in A \ ,$$

where I is the set of geographic units; A is the set of pairs of adjacent geographic units; m is the maximum allowable number of geographic units to constitute the district; x_i is a decision variable that equals 1 if the geographic unit i is assigned to the district and 0 otherwise; w_i is a decision variable that equals 1 if the geographic unit i is the hub, and 0 otherwise; and y_{ij} is a decision variable that indicates the amount of flow from geographic unit i to geographic unit j. Note that, if m is not specified, it should be equal to $|I|$.

If the number of geographic units (n) constituting the district is known, the above constraint set can be simplified by replacing constraints (1) and (3) with:

$$\sum_{j|(i,j)\in A} y_{ij} - \sum_{j|(j,i)\in A} y_{ji} = x_i - nw_i \, , \quad \forall i \in I \ , \tag{4}$$

and

$$\sum_{j|(j,\, i) \in A} y_{ji} \leqslant (n-1)x_i \,, \quad \forall i \in I \,. \tag{5}$$

Another variant is also useful when a hub unit, r, is selected in advance. It has the following form:

$$\sum_{j|(i,\, j) \in A} y_{ij} - \sum_{j|j,\, i) \in A} y_{ji} = x_i \,, \quad \forall i \in I | i \neq r \,, \tag{6}$$

$$\sum_{j|(j,\, i) \in A} y_{ji} \leqslant (m-2)x_i \,, \quad \forall i \in I | i \neq r \,, \tag{7}$$

$$\sum_{j|(j,\, r) \in A} y_{jr} \leqslant (m-1) \,, \tag{8}$$

$$x_i \in \{0{,}1\} \,, \quad \forall i \in I | i \neq r \,,$$

$$y_{ij} \geqslant 0 \,, \quad \forall (i, j) \in A | i \neq r \,,$$

$$x_r = 1 \,,$$

$$y_{rj} = 0 \,, \quad \forall j \in I | (r, j) \in A \,.$$

An advantage of all the three constraint sets is their modularity; that is, they can be embedded in any MIP districting regardless of any other requirement.

4 Contiguity-based districting models

To illustrate how a variety of districting models are based around the contiguity constraints presented in the preceding section, three hypothetical problems are considered here. They share a common dataset, which characterizes the conterminous forty-eight US states by population, state-to-state adjacency, and state-to-state distance. The problems are rather contrived to include the typical districting criteria discussed earlier with varying interpretation and emphasis. Still, it is fairly easy to adapt them to specific contexts that the reader might encounter. Each problem is formulated as an MIP model, which is then analyzed in terms of the numbers of binary (or $0-1$) variables, of continuous variables, and of main constraints. These indicate the model's theoretical complexity. Also reported is the CPU time spent in solving each model by an MIP solver, CPLEX 10 (ILOG Inc., Sunnyvale, CA), on a Pentium 4 CPU with 3.2 GHz clock speed and 2 GB of RAM. This gives some idea of the actual computational effort that might be required in practice.

The following notation is used throughout this section.

Sets:

I set of states;

A set of pairs of adjacent states;

K set of districts;

R set of district centers.

Parameters:

a_i population of state i;

d_{ij} distance between states i and j;

d_{ik} distance between state i and the center of district k;

m_k maximum allowable number of states to constitute district k;

n_k exact number of states to constitute district k;

p target population;

α tolerance on each district's population deviation from p.

Decision variables:
x_{ik} equals 1 if state i is assigned to district k, 0 otherwise;
y_{ijk} amount of flow from state i to state j for district k;
w_{ik} equals 1 if state i is chosen for district k's hub, 0 otherwise;
s_k^+ number of people by which district k exceeds the target population p;
s_k^- number of people by which district k falls short of the target population p.

Problem 1. Given four states designated as district centers—namely, California, Texas, New York, and Florida—partition the study area into four compact, contiguous regions of nearly equal population.

The problem consists of the three common districting criteria: (population) size, compactness, and contiguity. I have chosen to measure compactness in terms of the sum of the straight-line distance from each state to its district center. Contiguity here can be modeled by the third set of contiguity constraints, since each district center can play the role of a hub. The following is an MIP formulation of the problem.

$$\text{Minimize} \sum_{k \in K} \sum_{i \in I} d_{ik} x_{ik} \,, \tag{9}$$

subject to

$$\sum_{i \in I} a_i x_{ik} \geqslant (1-\alpha)p\,, \quad \forall k \in K\,, \tag{10}$$

$$\sum_{i \in I} a_i x_{ik} \leqslant (1+\alpha)p\,, \quad \forall k \in K\,, \tag{11}$$

$$\sum_{k \in K} x_{ik} = 1\,, \quad \forall i \in I - R\,, \tag{12}$$

$$\sum_{j|(i,j) \in A} y_{ijk} - \sum_{j|(j,i) \in A} y_{jik} = x_{ik}\,, \quad \forall k \in K, \forall i \in I - R\,, \tag{13}$$

$$\sum_{j|(j,i) \in A} y_{jik} \leqslant (m_k - 2)x_{ik}\,, \quad \forall k \in K, \forall i \in I - R\,, \tag{14}$$

$$\sum_{j|(j,i) \in A} y_{jik} \leqslant (m_k - 1)\,, \quad \forall k \in K, \forall i \in R | i = k\text{'s center} \tag{15}$$

$$x_{ik} \in \{0,1\}\,, \quad \forall k \in K, \forall i \in I - R\,,$$

$$y_{ijk} \geqslant 0\,, \quad \forall k \in K, \forall (i,j) \in A | i \notin R, \text{ and } j \notin R \text{ or } j = k\text{'s center}\,,$$

$$x_{ik} = 1\,, \quad \forall k \in K, \forall i \in R | i = k\text{'s center}\,,$$

$$x_{ik} = 0\,, \quad \forall k \in K, \forall i \in R | i \neq k\text{'s center},$$

$$y_{ijk} = 0\,, \quad \forall k \in K, \forall (i,j) \in A | i \in R, \text{ or } j \in R \text{ and } j \neq k\text{'s center}\,.$$

The objective of function (9) is to minimize the total distance between each state and its corresponding district center. Constraints (10) and (11) limit each district's population deviation. Constraint (12) requires each state to be allocated to one and only one district. Constraints (13)–(15) ensure that every district is contiguous. It is important to note that, although the present model regards compactness as a desired quality and population equality as a must-have condition, this is merely one possible interpretation of the problem.

The model contains $|K|(|I| - |K|)$ binary variables, approximately $|K||A|$ continuous variables, and $2|K|(|I| - |K|) + |I| + 2|K|$ main constraints. This particular example has 176 binary variables, 800 continuous variables, and 408 constraints. The model was solved with $\alpha = 0.1$, $p = 207\,597\,224/4$, and all $m_k = 30$ [it had been found that any combination of 31 states exceeds $(1+\alpha)p$]. An optimal solution (figure 2) was obtained in 0.31 CPU seconds. The model was tested with different sets of district centers, and was solved optimally within a second in most cases. It was also found that the solution time could be prolonged, depending not only on the configuration of district centers, but also on the number of districts. I have actually encountered some instances requiring a few seconds to be solved.

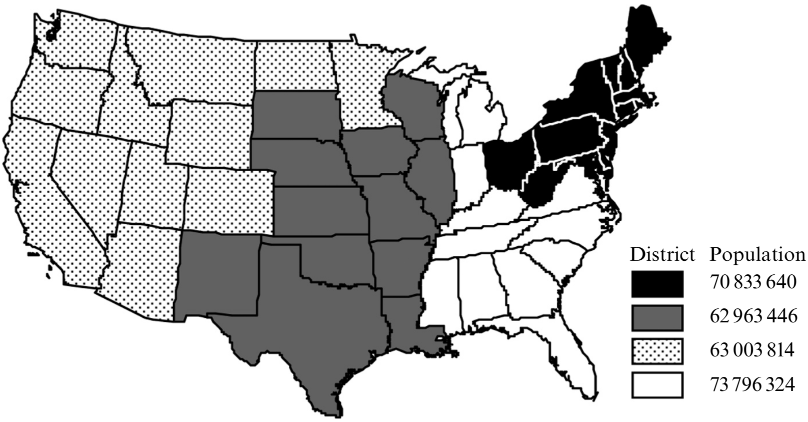

Figure 2. An optimal solution to problem 1.

Problem 2. Partition the study area into four contiguous districts of nearly equal population while making all the districts as equally compact as possible.

In this problem, it is preferred to distribute compactness evenly among all districts rather than to sacrifice some districts for the sake of the overall compactness. One approach to this end is to make the least compact district as compact as possible. As for contiguity, the original constraint set must be used because there are no fixed district centers. The problem is then formulated as follows.

$$\text{Minimize } z \tag{16}$$

subject to

$$d_{ij}(x_{ik} + x_{jk} - 1) \leqslant z, \quad \forall k \in K, \forall i, j \in I | i < j, \tag{17}$$

$$\sum_{k \in K} x_{ik} = 1, \quad \forall i \in I, \tag{18}$$

$$\sum_{i \in I} w_{ik} = 1, \quad \forall k \in K, \tag{19}$$

$$\sum_{j|(i,j) \in A} y_{ijk} - \sum_{j(j,i) \in A} y_{jik} \geqslant x_{ik} - m_k w_{ik} \quad \forall k \in K, \forall i \in I, \tag{20}$$

$$\sum_{j|(j,i) \in A} y_{jik} \leqslant (m_k - 1) x_{ik}, \quad \forall k \in K, \forall i \in I, \tag{21}$$

constraints (10) and (11), and

$$x_{ik}, w_{ik} \in \{0,1\}, \quad \forall k \in K, \forall i \in I,$$

$$y_{ijk} \geqslant 0, \quad \forall k \in K, \forall (i,j) \in A.$$

The objective of function (16), together with constraint (17), is to miminize the distance between the most distant pair of states that belong to the same district. Note that constraint (17) is defined only for ordered pairs of states in order to avoid redundant constraints. Constraint (18) ensures that each state is allocated to one and only one district. Constraints (19) – (21) enforce contiguity on every district.

The model contains $2|K||I|$ binary variables, $|K||A| + 1$ continuous variables, and $|K||I|(|I| + 3)/2 + |I| + 3|K|$ main constraints. This particular example involves 384 binary variables, 841 continuous variables, and 4956 constraints. The model (with $\alpha = 0.1$ and all $m_k = 30$) turned out to be a difficult large-scale MIP model. Still, a feasible solution is easier to find, since z always equals one of the 1128 $[= \binom{48}{2}]$ state-to-state distances. To see if a particular state-to-state distance, b, is feasible, the model is modified in a way that drops objective function (16) and replaces constraint (17) with:

$$d_{ij}(x_{ik} + x_{jk} - 1) \leqslant b\,, \quad \forall k \in K, \forall i, j \in I | i < j\,, \tag{22}$$

which is further equivalent to:

$$x_{ik} + x_{jk} \leqslant 1\,, \quad \forall k \in K, \forall i, j \in I | i < j, d_{ij} > b\;. \tag{23}$$

In principle, an optimal solution to the original problem can be found by solving this modified model 128 times, each for a different b. I have found the following iterative procedure more efficient. At each iteration, set b to the median of the current possible range for z. This range is initially bounded by the lowest and the highest of all d_{ij}. If the problem is then found to be infeasible, replace the current lower bound with the lowest d_{ij} greater than b; if a feasible solution is found, replace the current upper bound with the highest d_{ij} among all the state pairs assigned to the same district in that solution. The iteration terminates when it narrows the range down to one feasible d_{ij}; otherwise the original problem is infeasible (owing to the population deviation constraints). Every time a subproblem is solved, the z range shrinks to half or less, which makes finite the number of subproblems to be solved. For the present example, after solving ten such subproblems (of which seven were feasible) in a total of 1230.33 CPU seconds, an optimal solution (figure 3) was found.

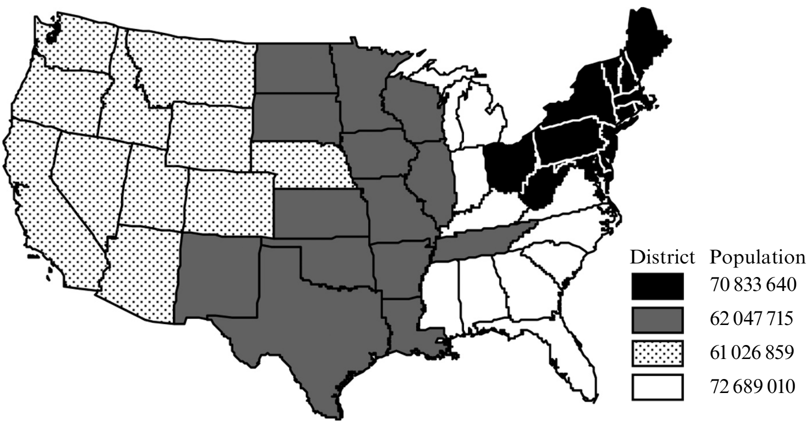

Figure 3. An optimal solution to problem 2.

Problem 3. Partition the study area into four contiguous districts of equal number of states, while making the overall population deviation as small as possible.

This is a good example of how important the explicit and independent treatment of contiguity is, because contiguity cannot be related to compactness or any other spatial property. I have formulated the problem in the following manner:

$$\text{minimize} \sum_{k \in K} (s_k^+ + s_k^-) \ , \tag{24}$$

subject to

$$\sum_{i \in I} a_i x_{ik} - p = s_k^+ - s_k^- , \quad \forall k \in K \ , \tag{25}$$

$$\sum_{j|(i,j) \in A} y_{ijk} - \sum_{j|(j,i) \in A} y_{jik} = x_{ik} - n_k w_{ik} , \quad \forall k \in K, \forall i \in I \ , \tag{26}$$

$$\sum_{j|(j,i) \in A} y_{jik} \leqslant (n_k - 1) x_{ik} , \quad \forall k \in K, \forall i \in I \ , \tag{27}$$

constraints (18) and (19), and

$$x_{ik}, w_{ik} \in \{0,1\} , \quad \forall k \in K, \forall i \in I \ ,$$

$$y_{ijk} \geqslant 0 , \quad \forall k \in K, \forall (i,j) \in A \ ,$$

$$s_k^+, s_k^+ \geqslant 0 , \quad \forall k \in K \ .$$

The objective of function (24) is to minimize the sum of the population deviations of all four districts. The population deviation of each district is expressed by constraint (25). Here notice that either s_k^+ or s_k^- is forced to be zero by the objective function. Constraints (26) and (27), together with constraint (19), guarantee that each district is contiguous.

The model contains $2|K||I|$ binary variables, $|K|(|A| + 2)$ continuous variables, and $2|K||I| + 2|K|$ main constraints. This particular example has 384 binary variables, 848 continuous variables, and 440 constraints. I failed to obtain an optimal solution to this model (with all $m_k = 12$), despite two days of computation, but found a feasible solution in about five CPU minutes. This solution is not too bad, as the most deviating district has about 10.85% (7 341 840 persons) fewer people than the target population (67 649 306 persons). A very good feasible solution (figure 4), in which the largest deviation from the target population was only 0.36%, was obtained in about 22.5 hours.

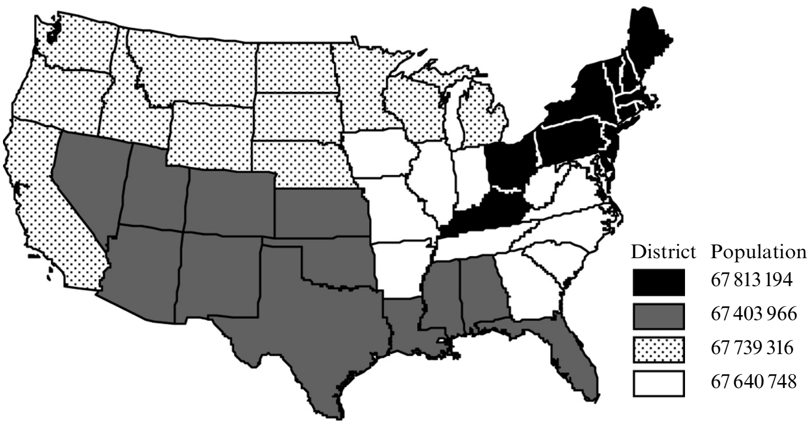

Figure 4. A good feasible solution to problem 3.

Unfortunately, these results do not indicate that the model is very tractable in a practical sense. Thus it might be worthwhile to try any standard MIP heuristic, such as Lagrangian relaxation, column generation, and cutting planes (Rardin, 1998). Otherwise, the problem might be dealt with using a known 'seeding' heuristic (see eg Hess and Samuels, 1971; Segal and Weinberger, 1977), which solves the problem with several promising sets of hubs and selects the best solution from among all generated solutions. This sort of technique involves another combinatorial problem of deciding an optimal initial set of hubs. Remember, however, that the model does not require a hub to be the center of a district, but just one of those units constituting that district. Thus the solution is not excessively affected by the initial choice of hubs. This should reduce the risk of combinatorial explosion. It is also worth noting that a solution obtained with this heuristic could be used as an upper bound for the original problem, which might reduce the MIP solver's scope of search for a true optimal solution, and hence its computation time.

A modification has been made to problem 3 by dropping the equal-state-number requirement. This version of the problem may be more reasonable in a practical sense, because as long as the resulting districts are equally populated, there seems no compelling reason for forcing them to have the same number of states. As seen in the next section, however, the price of this modification is greater computational difficulty, owing to the necessary replacement of the contiguity constraints (19), (26), and (27) with the original contiguity constraints (19)–(21), which are generally less tractable (Shirabe, 2005). Figure 5 shows the best feasible solution I have got in my limited computational experiment. In maximum deviation from the target population was 1.92% (1 231 855 persons).

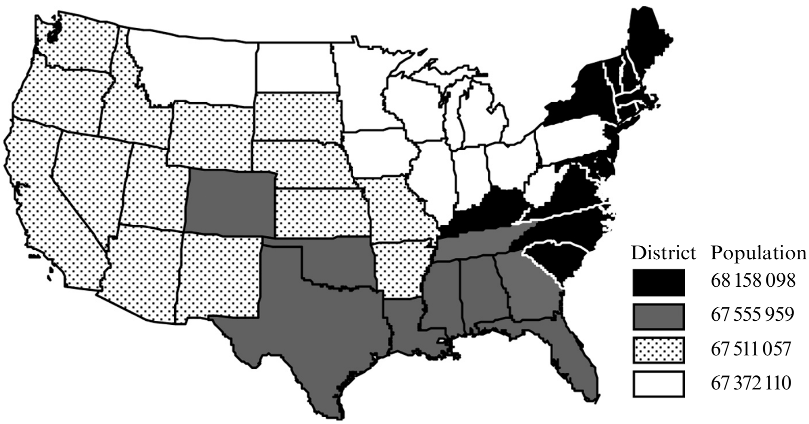

Figure 5. A feasible solution to problem 3 without the equal-state-number requirement.

5 Effects of the number of districts on tractability

With other conditions held constant, it is easy to see that the more geographic units are involved, the less tractable a districting model is, because more decision variables and constraints are required. The same seems true for the number of districts: the more districts, the less tractable the model. At the extreme case, in which the number of districts equals the number of geographic units, however, the model is in fact trivial, since each district contains only one geographic unit. This reasoning leads to a question of how the number of districts (or the size of each district) affects the tractability of a contiguity-based districting model.

To answer the question partially, I revisited problem 3 with varying numbers of districts. I chose this problem because it is always feasible, while problems 1 and 2 can be infeasible, depending on the selection of district centers and the specification of the limit on population deviation. Table 1(a) summarizes the results of an experiment with the condition that computation is terminated when the first feasible solution is found. Table 1(b) corresponds to an experiment with the condition that computation is terminated when 86 400 seconds (or 24 hours) elapse. These termination rules were due to the limited availability of my computational resource. The time of computation tended to increase and the quality of solution (in terms of the average population deviation) tended to decrease with the number of districts until the sixteen-district instance was reached. Despite the continuing increase of the number of decision variables and constraints, however, less computational time was spent and a better solution was found for the next instance. An intuitive interpretation of this is that the relative ease of composing smaller districts cancelled, and eventually overcame, the difficulty in finding more districts.

A similar experiment was conducted with the same problem, but without requiring that all districts have the same number of states. As indicated in table 2, the modified

Table 1. Computational results for problem 3.

Number of districts	Value of objective function	Time	Number of districts	Value of objective function	Average deviation
(a) First feasible solution			(b) 24-hour computation		
2	165 034.00	1.13	2	1 836.00	918.00
3	39 670 327.33	79.73	3	80 391.33	26 797.11
4	23 665 138.00	298.72	4	507 796.00	126 949.00
6	65 736 012.00	2 537.37	6	5 774 614.00	962 435.67
8	80 015 952.00	20.15	8	51 625 454.00	6 453 181.75
12	122 049 815.33	2 709.43	12	77 319 649.33	6 443 304.11
16	150 492 241.10	6 718.67	16	118 074 460.00	7 379 653.75
24	174 086 006.00	177.98	24	142 928 348.67	5 955 347.86

Note: The objective function is the sum of the population deviations of all four districts. The average deviation is the average of the population deviations of all four districts.

Table 2. Computational results for problem 3 without equal-state-number requirement.

Number of districts	Value of objective function	Time	Number of districts	Value of objective function	Average deviation
(a) First feasible solution			(b) 24-hour computation		
4	109 997 668.00	16.86	4	1 017 584.00	254 396.00
8	315 192 598.00	158.85	8	7 313 894.00	914 236.75
12	158 786 701.33	66 892.62	12	60 605 240.00	5 050 445.00
16	N	N	16	N	N
20	170 680 350.40	74 042.24	20	125 096 905.20	6 254 845.26
24	N	N	24	N	N
28	N	N	28	N	N
32	N	N	32	N	N
36	N	N	36	N	N
40	N	N	40	N	N
44	300 153 996.73	23 413.71	44	194 445 512.55	4 419 216.19

Note: The objective function is the sum of the population deviations of all four districts. The average deviation is the average of the population deviations of all four districts. 'N' indicates that no solution was obtained in the time limit of 24 hours.

problem was found to be much more difficult to solve. My computing environment failed to find even feasible solutions for six instances. With these limited results, I do not want to speculate excessively on how the tractability of the problem changes with the number of districts. Rather, the results should be taken as a confirmation of my finding (Shirabe 2005) that the original contiguity constraint set makes the districting problem harder.

6 Conclusion

This paper has introduced a new approach to districting modeling that incorporates exact contiguity constraints. I have not intended the proposed method to supersede existing ones (which are generally more tractable), but I have shown its capacity to deal with many different combinations of districting criteria.

The models presented in this paper can be solved by commercial MIP solvers if they involve a relatively small number of geographic units and districts. My experience suggests, however, that the models will suffer from computational difficulties as their scale grows. This relates to the combinatorial complexity of the problem under consideration. Although the validation of this should be left to future research, the most complex instances seem to be those in which a 'medium-sized' number of districts of mostly 'medium size' is to be selected, as opposed to a few districts of large size or many districts of small size. Thus, for many practical applications, it is necessary to rely on heuristic techniques that exploit the special features which individual problems might have. Even so, their exact formulation is essential for any objective evaluation of such techniques.

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