

A computational approach to unbiased districting[☆]

Clemens Puppe^{a,*}, Attila Tasnádi^b

^a Department of Economics, University of Karlsruhe, D–76128 Karlsruhe, Germany

^b Department of Mathematics, Corvinus University of Budapest, H–1093 Budapest, Fővám tér 8, Hungary

ARTICLE INFO

Article history:

Received 19 May 2008

Accepted 22 May 2008

Keywords:

Redistricting

Gerrymandering

NP-complete problems

ABSTRACT

In the context of discrete districting problems with geographical constraints, we demonstrate that determining an (ex post) unbiased districting, which requires that the number of representatives of a party should be proportional to its share of votes, turns out to be a computationally intractable (NP-complete) problem. This raises doubts as to whether an independent jury will be able to come up with a “fair” redistricting plan in case of a large population; that is, there is no guarantee for finding an unbiased districting (even if such exists). We also show that, in the absence of geographical constraints, an unbiased districting can be implemented by a simple alternating-move game among the two parties.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

Districting is well known to be a critical determinant of the representation of population in legislatures. The process of how the shape and structure of districts is brought about has therefore received considerable interest both from political scientists and economists, see e.g. Besley and Preston [1], Coate and Knight [2], Friedman and Holden [3], Gelman and King [4], Gilligan and Matsusaka [5], Gul and Pesendorfer [6], Sherstyuk [7] and Shotts [8] for recent contributions. From an economic perspective, the districting problem becomes particularly salient when the political parties themselves can actively and strategically influence the shape and structure of districts. This is the case in most US states in which the legislature has primary responsibility for creating a redistricting plan, often subject to approval by the state governor. By consequence, the literature has focused on strategically optimal *gerrymandering*, i.e. the optimal manipulation of districts from the viewpoint of the involved parties in order to maximize their number of representatives.

In five states, however, congressional redistricting is carried out by an independent bipartisan commission (in Arizona, Hawaii, Idaho, New Jersey and Washington). Moreover, Iowa and Maine give independent bodies the authority to propose redistricting plans, which have to be approved by legislature.¹ Accordingly, the present paper asks whether and how a “fair” districting can be achieved by an impartial and independent arbiter or jury. Specifically, a districting will be called (ex post) *unbiased* if the number of seats won by a party is proportional to its share of votes in the entire population. While this problem has an easy solution in the absence of further constraints (Proposition 1), our main result shows that finding an unbiased districting represents an NP-complete problem in the case of geographical constraints. In states with a large population and many districts it can be thus very difficult to find an unbiased districting plan. In case of congressional elections in the United States, striking examples are California with 53 representatives (districts) and Texas with 32 representatives.²

[☆] We are very grateful to three anonymous referees for their helpful comments and suggestions. The second author gratefully acknowledges financial support from the Hungarian Academy of Sciences (MTA) through the Bolyai János research fellowship.

* Corresponding author.

E-mail addresses: puppe@wior.uni-karlsruhe.de (C. Puppe), attila.tasnadi@uni-corvinus.hu (A. Tasnádi).

¹ For more details on redistricting practice, also outside the US, see e.g. <http://en.wikipedia.org/wiki/Redistricting> (accessed: 01/15/2008).

² The axiomatic theory of determining the number of representatives in proportion to the states respective populations has been developed by Balinski and Young [9].

We also investigate whether an unbiased districting can be implemented by the parties themselves without the involvement of neutral bodies through an appropriate mechanism. To this behalf, we introduce a simple alternating-move game in which two parties sequentially determine the districts. Compared to the setting with an independent jury, the informational requirements are different, and arguably much weaker in this case since only the involved parties have to know the distribution of their respective supporters. We show that an unbiased districting results as the (essentially unique) backwards induction outcome of the game in the absence of geographical constraints.

Algorithms determining districtings that satisfy near population equality, geographical compactness and contiguity have been already given by Garfinkel and Nemhauser [10] and Hess et al. [11] among others. An algorithm meeting the listed three criteria is ex ante unbiased since it does not favor a particular party in advance. However, an actual districting obtained by such an algorithm can definitely favor a party, and therefore, it can be ex post biased. In addition, Altman [12] also points out that achieving any of these ex ante unbiasedness criteria results in NP-hard districting problems. To cope with the computational complexity of determining ex ante unbiased districtings, for instance, Bação et al. [13] employ genetic algorithms, Chou and Li [14] apply the q -state Potts model of statistical physics and Mehrotra et al. [15] use heuristics.

The remainder of the paper is organized as follows. The next section presents our framework with basic definitions and our notation. Section 3 contains our main result, the NP-completeness of determining an unbiased districting plan with geographical constraints. In fact, we show by example that an unbiased districting may not even exist in some cases. In Section 4, we investigate the simple alternating move game described above. Section 5 offers some concluding remarks.

2. The framework

We assume that a set of voters has to be partitioned into a given number of equal districts in each of which candidates of two parties, say parties A and B , compete for winning a seat. A district is “won” by a candidate if he/she receives the majority of votes. We shall denote the number of voters by n and the set of voters by $N := \{1, \dots, n\}$. Similarly, the number of districts is denoted by d and the set of districts by $D := \{1, \dots, d\}$. We assume that d divides n .³

We assume that the voters have deterministic and known party preferences. This is clearly a simplification of reality which, however, allows us to obtain several insightful results. Relaxing these assumptions could be the aim of further research. The voters’ party preferences are summarized by the mapping $v : N \rightarrow \{A, B\}$ with $v(j) = A$ interpreted as “voter j votes for (prefers) party A ”. The number of supporters of parties A and B are denoted by n_A and n_B , respectively. Let us assume for simplicity that there exists a positive integer k such that $n = d(2k + 1)$. Thus, each district must consist of $2k + 1$ voters and, assuming full participation, each district is won by either party A or party B . In particular, we exclude in all districts the possibility of a draw.

Most of the literature investigates districting problems without geographical constraints (an exception is Sherstyuk [7]). We introduce the following simple but quite general framework that allows us to incorporate geographical constraints.

Definition (Geography). A non-empty family $\mathcal{S} \subset 2^N$ of subsets of N is called a *geography* if (i) for all $S \in \mathcal{S}$, $\#S = 2k + 1$, and (ii) there exist $S_1, \dots, S_d \in \mathcal{S}$ such that $\{S_1, \dots, S_d\}$ forms a partition of N .

Definition (Districting). For a given geography $\mathcal{S} \subset 2^N$ a mapping $f : N \rightarrow D$ is called a *districting* if $f^{-1}(i) \in \mathcal{S}$ for all $i \in D$ and $\bigcup_{i \in D} f^{-1}(i) = N$.

Observe that if \mathcal{S} consists of all $2k + 1$ sized subsets of N , then we obtain as a special case *districting without geographical constraints*.

A districting f and voters’ preferences v jointly determine the number of districts won by parties A and B , which we denote by $F(f, v, A)$ and $F(f, v, B)$, respectively. Party A wins the (congressional) election if $F(f, v, A) > F(f, v, B)$ and loses the election if $F(f, v, A) < F(f, v, B)$. The following definition is central to our approach to “fair” districting. In what follows $\lfloor x \rfloor$ stands for the largest integer not greater than x and $\lceil x \rceil$ stands for the smallest integer not less than x , for any real number x .

Definition (Biasedness). For given voters’ preferences $v : N \rightarrow \{A, B\}$ a districting $f : N \rightarrow D$ is *unbiased* if $F(f, v, A) = \lfloor d \frac{n_A}{n} \rfloor$ or $F(f, v, A) = \lceil d \frac{n_A}{n} \rceil$. A districting is *biased* if it is not unbiased.

Thus, a districting plan is unbiased if the number of districts won by each party respects their relative strength in the population as close as possible. Without geographical constraints, an unbiased districting can be found quite easily.

Proposition 1. *An unbiased districting without geographical constraints can be determined in polynomial time, and more specifically, even in linear time.*

Proof. Fill $\lfloor d \frac{n_A}{n} \rfloor$ districts with voters of party A , $\lfloor d \frac{n_B}{n} \rfloor$ districts with voters of party B and the remaining district (whenever $d \frac{n_A}{n}$ is not an integer) with the remaining $2k + 1$ voters.

The simple algorithm given in the proof of Proposition 1 in particular shows that without geographical constraints an unbiased districting is always feasible. However, this is not always the case in the presence of geographical constraints. We verify this based on the “rectangular country” shown in Fig. 1. Party A ’s supporters are indicated by empty circles and party

³ This is without much loss of generality, since otherwise we can introduce dummy voters in proportion of the supporters of each party to overcome indivisibilities.

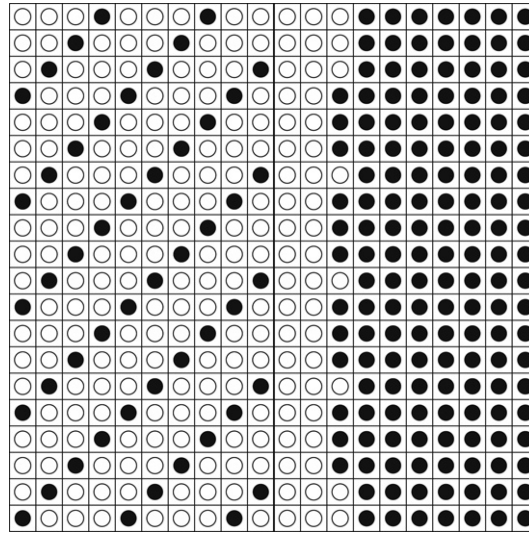


Fig. 1. Rectangular country.

B 's supporters are indicated by solid circles; it can be verified that $n_A = n_B = 200$. We assume that $k = 2$, i.e. district size is 5, and that therefore $d = 80$ districts have to be formed. Two voters are considered *adjacent* if they have a common boundary (edge), and a district is *connected* if two voters living in the same district are “reachable” through a sequence of adjacent voters. We impose the simple restriction on the districting that only connected districts can be formed, which defines a geography \mathcal{G} for the rectangular country. Under the distribution of voters' preferences shown in Fig. 1 and under the given geographical constraint, party B loses the election (since it cannot win more than $39 = d/2 - 1$ districts) although it has exactly the same number of supporters as party A . To verify this, observe that if a district contains one voter from the left hand side (first ten columns) of the country, then it cannot be won by Party B . Therefore, winning districts for party B must consist only of voters from the right hand side (last ten columns) of the country. Since, for instance, the voter in the top row and 11th column (a party A voter) can only be put in a winning district for party A , it is impossible to create 40 winning districts for party B .

3. Districting is NP-complete

Our main concern is whether an impartial arbiter or judge can determine an unbiased districting for a given geography \mathcal{G} on N from a computational perspective. We establish that even the associated decision problem, i.e. deciding the *existence* of an unbiased districting, is a computationally intractable NP-complete problem. We call this problem UNBIASED DISTRICTING.

To prove the NP-completeness of UNBIASED DISTRICTING, we shall reduce EXACT COVER BY m -SETS ($m \geq 3$), a well-known NP-complete problem,⁴ to UNBIASED DISTRICTING. EXACT COVER BY m -SETS asks if a given set X with cardinality m possesses an exact cover from a given set system \mathcal{C} of m -element subsets (henceforth m -sets) of X (i.e. $C_1, \dots, C_q \in \mathcal{C}$ and $\bigcup_{i=1}^q C_i = X$), where we can assume that $\#\mathcal{C} \geq q$.

Theorem. UNBIASED DISTRICTING is NP-complete.

Proof. First, we verify that the unbiasedness of a districting f can be checked in polynomial time, and therefore UNBIASED DISTRICTING \in NP. Assume that the set of party A voters is represented by $\{1, 2, \dots, n_A\}$ and the set of party B voters by $\{n_A + 1, \dots, n\}$. A district of size $2k + 1$ is encoded by a sequence of distinct positive integers not greater than n , a districting f by a sequence of d districts, and a geography by $2k + 1, n_A, n_B, s = \#\mathcal{G}$ and the sequence S_1, \dots, S_s of possible districts. The unbiasedness of a given districting f can be checked by counting the number of winning districts for party A while reading the encoding of f .

Second, we reduce EXACT COVER BY $2k + 1$ -SETS to UNBIASED DISTRICTING. We start with the motivating example shown in Fig. 2 to illustrate our construction of a districting problem associated with a given instance of EXACT COVER BY $2k + 1$ -SETS. The empty circles stand for the elements to be covered by disjoint 5-sets ($k = 2$), which we regard as the party A voters in the districting problem. The given instance of EXACT COVER BY 5-SETS, i.e. the set system \mathcal{C} of 5-sets of party A

⁴ See Garey and Johnson [16, p. 53] for EXACT COVER BY 3-SETS and EXACT COVER BY 4-SETS. The NP-completeness of the EXACT COVER BY m -SETS for $m \geq 5$ can be shown in an analogous way.

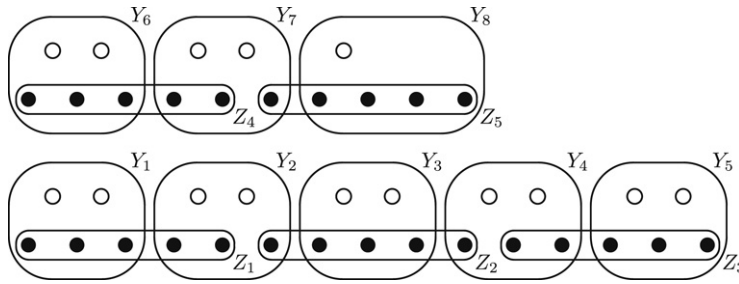


Fig. 2. $k = 2$, $d = 8$ and $n = 40$.

supporters, specifies possible districts that are not shown in Fig. 2 since we allow for arbitrary systems of such sets. The solid circles indicate the voters of party B. We obtain the desired geography \mathcal{G} (on the set of all voters) by adding the sets Y_1, \dots, Y_8 and Z_1, \dots, Z_5 to \mathcal{C} as shown in Fig. 2. In the figure, we also see that $n_A = 15$ and $n_B = 25$, thus an unbiased districting requires exactly 3 winning districts for party A. The crucial observation is that an unbiased districting cannot contain any of the districts Y_1, \dots, Y_8 . Indeed, among all admissible districts (i.e. those in \mathcal{G}) only the districts in \mathcal{C} are winning districts for party A. Moreover, \mathcal{C} contains at most 3 mutually disjoint districts. Therefore, a districting containing a set Y_i cannot contain at the same time 3 districts from \mathcal{C} . This shows that an unbiased districting exists if and only if the given instance of EXACT COVER BY 5-SETS has a solution.

Now let us turn to the general case and take an instance \mathcal{C} on X of EXACT COVER BY $2k+1$ -SETS, where $\#X = (2k+1)c$ for some integer c . The elements of X will be the voters of party A, and thus $n_A = \#X$. Let $a = \lfloor \frac{(2k+1)c}{k} \rfloor$ and $r = (2k+1)c \bmod k$ (the remainder of the division of $(2k+1)c$ by k). Party B will (by construction) have either $y = a(k+1) + 2k+1-r$ voters if $r > 0$ or $y = a(k+1)$ voters if $r = 0$, and we shall denote the associated set of voters by Y .

We claim that y is divisible by $2k+1$. First, we consider the case of $r > 0$. Since $(2k+1)c = ak + r$,

$$\begin{aligned} y &= a(k+1) + 2k+1-r \\ &= a(k+1) + 2k+1 - ((2k+1)c - ak) \\ &= (2k+1)(a+1-c), \end{aligned}$$

which proves our claim for $r > 0$. Second, assume that $r = 0$. Since then $(2k+1)c = ak$, we have

$$\begin{aligned} y &= a(k+1) \\ &= (2k+1)c + a \\ &= (2k+1)c + \frac{c}{k}(2k+1). \end{aligned}$$

Now y is divisible by $2k+1$ because $\gcd(2k+1, k) = \gcd(k, 1) = 1$ by the Euclidean algorithm and therefore c is divisible by k , since all the terms are integers, and hence y is clearly divisible by $2k+1$.⁵

Next we construct a geography \mathcal{G} on $N = X \cup Y$. First, pick a partition Z_1, \dots, Z_u of Y into $2k+1$ -sets. Second, we partition X into k -sets X_1, \dots, X_a and in addition into an r -set X_{a+1} if $r > 0$. Third, we partition Y into $k+1$ -sets Y'_1, \dots, Y'_a and in addition into a $2k+1-r$ -set Y'_{a+1} if $r > 0$. Fourth, match sets X_1, \dots, X_a with sets Y'_1, \dots, Y'_a , respectively, to obtain $2k+1$ -sets Y_1, \dots, Y_a consisting of k voters of party A and $k+1$ voters of party B. Moreover, match set X_{a+1} with set Y'_{a+1} if $r > 0$, to obtain $2k+1$ -set Y_{a+1} with more voters of party B than party A. Let $t = a$ if $r = 0$ and $t = a+1$ if $r > 0$. Finally, let $\mathcal{G} = \mathcal{C} \cup \{Y_1, \dots, Y_t, Z_1, \dots, Z_u\}$ completing the construction of geography \mathcal{G} .

Since sets Y_1, \dots, Y_t determine a districting with district size $2k+1$ on N , we have associated an instance of UNBIASED DISTRICTING with an arbitrary instance of EXACT COVER BY $2k+1$ -SETS.

Because $n_B = y = c'(2k+1)$ for some positive integer c' , party A receives exactly $d \frac{n_A}{n} = c$ winning districts by an unbiased districting. Remember that the set of winning districts for party A equals \mathcal{C} and one can select at most c disjoint districts from \mathcal{C} . Hence, a districting for geography \mathcal{G} is unbiased if and only if it does not contain a set from Y_1, \dots, Y_t , since otherwise party A wins fewer than c districts. Therefore, the necessary and sufficient condition for the existence of an unbiased districting is the existence of an exact cover of X by $2k+1$ -sets from the given set system \mathcal{C} . Thus, we have reduced EXACT COVER BY $2k+1$ -SETS to UNBIASED DISTRICTING.

Finally, we show that our reduction can be done in polynomial time. Assume that the given instance of EXACT COVER BY $2k+1$ -SETS is given by a sequence $C_1, \dots, C_v \subseteq X$ of $2k+1$ -sets, where the elements of X are encoded by integers $\{1, 2, \dots, n_A\}$ and $v \geq c$. Clearly, the input length in integers equals $v(2k+1)$. Since the reduction produces $t+u$ new $2k+1$ -sets, $2c \leq t \leq 3c$ and $u \leq t \leq 3c$, the required number of computations is linear in c and at most linear in the size of input, which completes the proof.

⁵ gcd stands for the greatest common divisor.

The theorem says that at the current state of computer science (i.e. unless $P = NP$) we cannot give an efficient (polynomial time) algorithm to determine whether a given geography allows an unbiased districting. This also implies the non-existence of an efficient algorithm for determining an unbiased districting if it exists. Thus, for a given district size an increase in the number of districts increases the required number of computation steps radically (again assuming $P \neq NP$). Clearly, one can easily come up with exponential time procedures; however, these can work well only for “small” problems.

The theorem does, of course, not exclude the existence of a polynomial time algorithm for a set of reasonably restricted geographies. A natural step would be to consider geographies satisfying a kind of planarity condition. This issue could be addressed in future research, however, we conjecture that considering only “planar geographies” will not turn UNBIASED DISTRICTING into a polynomial time problem.

Since an unbiased districting can thus not be determined in polynomial time, we may ask whether an approximation version of our problem, i.e. determining the *least-biased* districting, could be approximated in polynomial time. Though this is an important question and should be investigated in future research, we focus here on the pure decision problem. Indeed, even as the pure decision problem, UNBIASED DISTRICTING seems to be of particular importance, especially in the case of two almost equally strong parties in which even a small bias of one seat induced by a particular districting can decide the outcome of the election. We conjecture that, based on a recent inapproximability result by Hazan, Safra and Schwartz [17] for the so-called m -set packing problem, least-biased districting cannot be approximated in polynomial time with a factor of $\Omega\left(\frac{m}{\log m}\right)$ unless $P = NP$.

4. A districting game

Our theorem suggests that an unbiased districting cannot be easily worked out by an independent jury unless the population is very small. An alternative way to prevent partisan gerrymandering is to consider appropriate institutions according to which the parties can determine the redistricting themselves.

Maybe the simplest institution serving this purpose is an alternating-move game in which the parties determine districts sequentially. In the case of geographical constraints, one has to make sure that, at any stage, the party at move is only allowed to select from those set of possible districts that do not prevent the continuation of the game. The goal of the parties is to maximize the number of their own winning districts.

Assume without loss of generality that $n_A \leq n_B$ and, for simplicity, that party B is the first mover. Thus, first party B selects a district, then party A selects a district, then party B again selects a district, and so on.

Proposition 2. *In the absence of geographical constraints, the backwards induction outcome of the above specified game determines an unbiased districting.*

Proof. We start by considering the first $m = \lfloor d \frac{n_A}{n} \rfloor$ moves of both parties, for which we claim that both parties can ensure winning at least m districts already in this part of the game.

First, the second mover (party A) can ensure m winning districts by copying party B 's strategy in the following sense: If party B selected at its i th move a district D_{2i-1} consisting of m_A voters of party A and m_B voters of party B , then party A can always construct in its i th move a district D_{2i} consisting of m_B supporters of party A and m_A supporters of party B .⁶

Second, the first mover (party B) can also ensure m winning districts in a similar way. At its first move party B creates a winning district by a margin of one voter and waits for party A 's first move. Now party B can copy party A 's move in the following sense: If party A selected at its i th move a district D_{2i} consisting of m_A voters of party A and m_B voters of party B , then party B can always construct at its $i + 1$ th move a district D_{2i+1} consisting of m_B supporters of party A and m_A supporters of party B , which proves our claim.

Finally, if party A could still form a winning district, party B continues in its $m + 1$ th move by creating a winning district by a margin of one voter as in its first move, and thereafter there are too few supporters of party A left to form a winning district for party A , which completes the proof.

Let us remark that taking party A as a first mover of the alternating-move game would also result in an unbiased districting; however, in this case party A might even achieve $\lceil d \frac{n_A}{n} \rceil$ winning districts if the fractional part of $d \frac{n_A}{n}$ is large enough. Hence, this game can have a first mover advantage.

The above game cannot deliver an efficient solution for the general case of unbiased districting with geographical constraints due to our above theorem. However, implementing a districting through the simple alternating-move game can nevertheless be regarded as a (more or less) satisfactory “solution” to the districting problem since it effectively limits the possibilities of manipulating the outcome through strategic gerrymandering for each party.

5. Conclusion

In this paper, we have studied the problem of determining an unbiased (“fair”) redistricting plan. We have shown, by example, that even under simple geographical constraints such as connectedness of each district, an unbiased districting

⁶ Observe that this can be done since $i \leq m$.

may not exist. As our main result, we proved that determining whether a given geography admits an unbiased districting is an NP-complete problem. Applied to congressional elections in the US, one may therefore expect that in states with a large population (and thus also a large number of districts), the problem of finding a fair solution to the districting problem becomes very difficult and computationally intractable. While the problem is completely absent in the states of Alaska, Delaware, Montana, North Dakota, South Dakota, Vermont and Wyoming because each of them has just one representative, it can become extremely complex in states such as Texas with its 32 representatives, or even California with 53 representatives.

As a possible solution to the problem of avoiding partisan gerrymandering more generally, we have considered a simple alternating-move game in which parties sequentially choose districts. In the absence of geographical constraints, this game indeed delivers an unbiased districting as the equilibrium outcome. A study of the equilibria in the general case is left to future research.

References

- [1] T. Besley, I. Preston, Electoral bias and public choice: Theory and evidence, *Quart. J. Econ.* 122 (4) (2007) 1473–1510.
- [2] S. Coate, B. Knight, Socially optimal districting: A theoretical and empirical Exploration, *Quart. J. Econ.* 122 (4) (2007) 1409–1471.
- [3] J.N. Friedman, R.T. Holden, Optimal gerrymandering: Sometimes pack, but never crack, *Amer. Econ. Rev.* 98 (1) (2008) 113–144.
- [4] A. Gelman, G. King, Enhancing democracy through legislative redistricting, *Amer. Polit. Sci. Rev.* 88 (3) (1994) 541–559.
- [5] T.W. Gilligan, J.G. Matsusaka, Public choice principles of redistricting, *Public Choice* 129 (3–4) (2006) 381–398.
- [6] R. Gul, W. Pesendorfer, Strategic redistricting. Mimeographed (2007).
- [7] K. Sherstyuk, How to gerrymander: A formal analysis, *Public Choice* 95 (1–2) (1998) 27–49.
- [8] K.W. Shotts, Gerrymandering, legislative composition and national policy outcomes, *Amer. J. Polit. Sci.* 46 (2) (2002) 398–414.
- [9] M.L. Balinski, H.P. Young, *Fair Representation: Meeting the Ideal of One Man, One Vote*, second edition, Brookings Institution Press, Washington D.C., 2001.
- [10] R.S. Garfinkel, G.L. Nemhauser, Optimal political districting by implicit enumeration techniques, *Manage. Sci.* 16 (8) (1970) 495–508.
- [11] S.W. Hess, J.B. Weaver, H.J. Siegfeldt, J.N. Whelan, P.A. Zitlau, Nonpartisan political redistricting by computer, *Oper. Res.* 13 (6) (1965) 998–1006.
- [12] M. Altman, Is automation the answer? The computational complexity of automated redistricting, *Rutgers Comput. Technol. Law J.* 23 (1) (1997) 81–142.
- [13] F. Bação, V. Lobo, M. Painho, Applying genetic algorithms to zone design, *Soft Comput.* 9 (5) (2005) 341–348.
- [14] C.-I. Chou, S.P. Li, Taming the gerrymander—statistical physics approach to political districting problem, *Phys. A* 369 (2) (2006) 799–808.
- [15] A. Mehrotra, E.L. Johnson, G.L. Nemhauser, An optimization based heuristic for political districting, *Manage. Sci.* 44 (8) (1998) 1100–1114.
- [16] M.R. Garey, D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman and Company, San Francisco, 1979.
- [17] E. Hazan, S. Safra, O. Schwartz, On the complexity of approximating k-set packing, *Comput. Complexity* 15 (1) (2006) 20–39.