Discovering Opportunities in New York City's Discovery Program: an Analysis of Affirmative Action Mechanisms

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Discovery program (`DISC`) is an affirmative action policy used by the New York City Department of Education (NYC DOE) to increase the number of admissions for disadvantaged students at specialized high schools. For instance, in the 2016-2017 academic year, this program resulted in about $1,350$ more disadvantaged students being admitted, almost tripling the number of disadvantaged students prior to the introduction of the program. However, our empirical analysis of the student-school matches from the 12 recent academic years (2005-06 to 2016-17) shows that about 950 in-group blocking pairs were created each year amongst disadvantaged group of students, impacting about 650 disadvantaged students every year. Moreover, we find that this program usually benefits *lower-performing* disadvantaged students more than *top-performing* disadvantaged students (in terms of the ranking of their assigned schools), thus unintentionally creating an incentive to under-perform.

In this work, we explore two affirmative action policies that can be used to minimally modify and improve the discovery program: the minority reserve (`MR`) and the joint-seat allocation (`JSA`) mechanism. Theoretically, by employing choice functions, we show that (i) both `MR` and `JSA` result in no in-group blocking pairs, and (ii) `JSA` is weakly group strategy-proof, ensures that at least one disadvantaged is not worse off, and when reservation quotas are carefully chosen then no disadvantaged student is worse-off. We show that each of these properties is not satisfied by `DISC`. In the general setting, we show that there is no clear winner in terms of the matchings provided by `DISC`, `JSA` and `MR`, from the perspective of disadvantaged students. We however characterize a condition for markets, that we term *high competitiveness*, where `JSA` dominates `MR` for disadvantaged students. This condition is verified, in particular, in certain markets when there is a higher demand for seats than supply, and the performances of disadvantaged students are significantly lower than that of advantaged students. Data from NYC DOE satisfy the high competitiveness condition, and for this dataset our empirical results corroborate our theoretical predictions, showing the superiority of `JSA`. We believe that the discovery program, and more generally affirmative action mechanisms, can be changed for the better by implementing the `JSA` mechanism, leading to incentives for the top-performing disadvantaged students while providing many benefits of the affirmative action program.

Key words: stable matching, school choice, affirmative action, discovery program, choice functions, fairness

1. Introduction

There is a pervasive problem in the way students are evaluated and given access to higher education (Ashkenas et al. 2017, Boschma and Brownstein 2016, Capers IV et al. 2017). Promising students are often unable to get admission at the top schools because the path to getting admitted to these schools requires extensive training at various levels, starting as early as when students are 3 years old (Shapiro 2019b). It is no surprise then that underrepresented minorities, especially those with lower household income and lower family education, are systematically screened-out of the education pipeline. In fact, in many cities, schools remain highly segregated (Shapiro 2021, 2019a). Disparate opportunities in accessing high-quality education is one of the main causes of income imbalance and social immobility in the United States (Orfield and Lee 2005). It is expected that this disparity will only become more acute due to COVID-induced loss of jobs and strain on low-income families. Now more than ever, affirmative action policies, such as quota-based mechanisms and training programs, are critical and offer practical remedies for increasing representation of under-represented minorities and disadvantaged groups in public schools in the U.S. (Hafalir et al. 2013).

In this work, we study theoretically and empirically the characteristics of the *Discovery Program*[1], which is an affirmative action program used by the New York City Department of Education (NYC DOE) in an effort to increase the number of disadvantaged students at specialized high schools (SHS) (NYCDOE 2019). SHSs span the five boroughs of NYC (Table 3), and are among the most competitive ones in NYC. For admission, these high schools consider *only* students' score on the Specialized High School Admissions Test (SHSAT). Around 5000 students are admitted every year to SHSs. The discovery program reserves some seats for disadvantaged students that are assigned at the end of the regular admission process, after student's participation in a 3-week enrichment program during the summer.

The discovery program has been instrumental in creating opportunities for disadvantaged students (classified with respect to socio-economic factors), increasing the number of admitted students to these extremely competitive public high schools in NYC. In 2020, for example, Mayor Bill de Blasio called for an expansion of discovery program, with 20% seats at SHSs reserved for the program. This expansion resulted in 1,350 more disadvantaged students being admitted to these specialized schools (NYCDOE 2019, Veiga 2020).

In this work, we dive deep into the student-school matching produced by the discovery program. Our empirical analysis shows that under a reasonable assumption on students' preferences over

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[1] https://www.schools.nyc.gov/enrollment/enrollment-help/meeting-student-needs/diversity-in-admissions
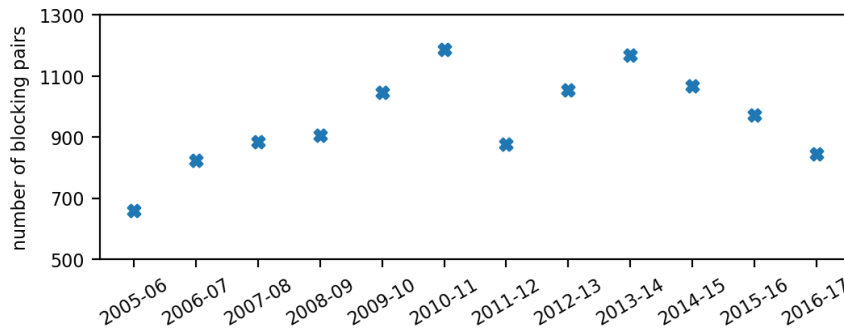
Figure 1   Number of blocking pairs amongst disadvantaged students under the discovery program mechanism across the last 12 years, which impacted around 650 students each year.

schools which we term *school-over-seat*[2], the matchings from 12 recent academic years (2005-06 to 2016-17) created about 950 in-group blocking pairs each year amongst disadvantaged students, impacting about 650 disadvantaged students every year (see Figure 1). A blocking pair is a pair of student s_1 and school c_1 that prefer each other to their matches, thus violating the priority of student s_1 at school c_1 and creating dissatisfaction among students and schools. Moreover, we find that this program benefits *lower-performing* disadvantaged students more than *top-performing* disadvantaged students (in terms of their rankings of their assigned schools), thus unintentionally creating an incentive to under-perform. See Figure 2 for a depiction of our empirical analysis, where top-performing students (with ranks $0 \sim 500$) attend less preferred schools under the discovery program, unlike low-performing students (rank 500-1000) who get matched to better ranked schools (lower numeric rank is better). These drawbacks of the discovery program are not simply an artifact of the data from NYC DOE, but are, as we show theoretically, properties about the current implementation of the discovery program. Therefore, our goal in this paper is to explore other affirmative action mechanisms, so that we can propose practical modifications to how the discovery program is implemented, while alleviating the above-mentioned drawbacks.

In particular, we compare the discovery program (`DISC`) together with two other affirmative action mechanism: *minority reserve* (`MR`) and *joint seat allocation* (`JSA`). These latter mechanisms are also quota-based, where schools reserve a certain proportion of their seats for disadvantaged students. Minority reserve, in contrast to the discovery program, allocates the reserved seats to disadvantaged students before the general admission. This mechanism has been well studied in the literature (see, e.g., Hafalir et al. (2013)). The joint seat allocation, on the other hand, allocates reserved and general (i.e., non-reserved) seats at the same time, while allowing disadvantaged students to take general seats (if they are able to compete) and otherwise revert to reserved seats.

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[2] This hypothesis assumes that students' preference lists over schools are not affected by whether they are required to participate in the three-week summer enrichment program. See Section 1.1 for details.
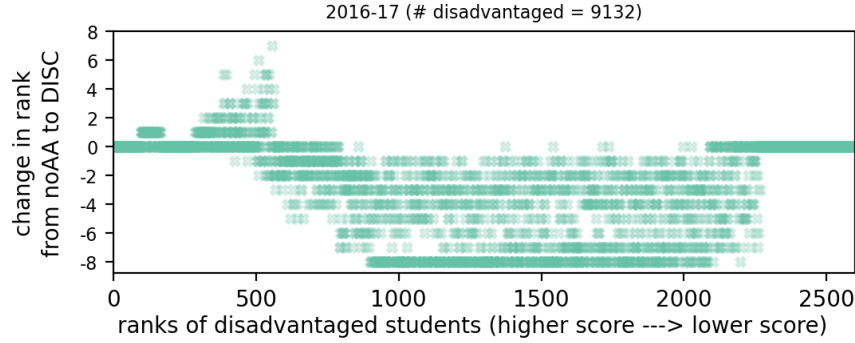
2016-17 (# disadvantaged = 9132)

Figure 2   Change in rank (where a negative change means getting to a more preferred school) of assigned schools for all disadvantaged students from `noAA` to `DISC` (we plot `DISC - noAA`), ranked by the quality of students. Top students (ranked 0-500) are matched to worse schools under `DISC`, whereas the lower performing disadvantaged students are matched to better schools.

This mechanism is inspired by the joint seat allocation process for admission to Indian Institutes of Technology[3] (JoSAA 2020) and this is the first work to study this to the best of our knowledge. We compare these three affirmative action policies with respect to the baseline stable matching mechanism, `noAA`, which does not incorporate affirmative action policies (Gale and Shapley 1962). We discuss our key contributions next.

1.1. Main results

We first show properties of affirmative action mechanisms under the school-over-seat hypothesis, i.e., students' preferences over schools are not influenced by whether they are admitted via general seats or reserved seats (in the case of NYC SHSs, reserved seats additionally require a 3-week summer program). We next discuss weak dominance amongst the three affirmative action mechanisms, showing that `JSA` outperforms `MR` under a condition that we term *high competitiveness* of markets. Finally, we empirically validate our theoretical results using data from NYC DOE, and make a policy recommendation for the discovery program.

1.1.1. Properties of Affirmative Action Mechanisms.

Question 1. Which affirmative action mechanisms considered in the paper satisfy reasonable notions of fairness such as absence of in-group blocking pairs and strategy-proofness? What is the impact of these affirmative action policies on the disadvantaged group of students?

We explore four useful properties for affirmative action mechanisms for each of `noAA` (the mechanism that does not reserve seats for disadvantaged students), `DISC`, `JSA` and `MR` mechanisms and

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[3] The actual mechanism used by the Joint Seat Allocation Authority is more complicated than the version we study here in the paper. In particular, in our setting, we assume that there are two disjoint types of students: disadvantaged and non-disadvantaged. However, in the actual implementation (see, e.g., Baswana et al. (2019)), students are categorized through multiple dimensions (e.g., caste, gender).

briefly explain these properties here (see Sections 2 and 3 for formal definitions): (i) *strategy-proofness*: this property means that the best strategy of students is to honestly report their preferences; (ii) *absence of in-group blocking pairs*: this is a fairness condition which ensures there is no priority violation for students; (iii) the third property asks for the mechanism not to worsen (with respect to the mechanism with no affirmative action) the assignment of *at least* one disadvantaged student[4]; and (iv) the fourth property asks *all* disadvantaged students not to be worse-off in a restricted scenario called *smart-reserve*. Reservation quotas are a smart reserve[5] if the number of seats reserved for disadvantaged students is no less than the number of disadvantaged students admitted without affirmation actions.

We summarize our results in Table 1. As one can immediately see from the table, the current implementation of the discovery program does not satisfy any of the attractive features we investigate, yet the other two affirmative action mechanisms, MR and JSA, satisfy all these properties. This is even true when *all the schools rank students in the same order*, as in the NYC SHS admission market where students are ranked based on their SHSAT scores. We additionally demonstrate these findings empirically by computational experiments using the admission data on NYC SHSs (the details can be found in Section 5). These results suggest that the discovery program could benefit by replacing the current implementation with either minority reserve or joint seat allocation. This result calls for a direct comparison of those mechanisms.

	noAA	DISC	MR	JSA
weakly group strategy-proof	✓ [DF]	✗ (Ex 2)	✓ [HYY]	✓ (Prop 6)
no in-group blocking pairs	✓ [GS]	✗ (Ex 2)	✓ (Prop 4)	✓ (Prop 7)
at least one disadvantaged student not worse off	NA	✗ (Ex 1)	✓ [HYY]	✓ (Thm 4)
no disadvantaged student worse off if smart reserve	NA	✗ (Ex 2)	✓ [HYY]	✓ (Thm 5)

Table 1   Summary of properties of affirmative action mechanisms under the school-over-seat assumption. NA means not applicable. Previously known results and their corresponding citations are given in square brackets, with: [DF] Dubins and Freedman (1981); [HYY] Hafalir et al. (2013); and [GS] Gale and Shapley (1962); other results are accompanied by the labels of examples, propositions, or theorems used to answer the questions.

1.1.2. Dominance across Affirmative Action Mechanisms.

[4] Intuitively, one might expect property (iii) to be so weak that it is trivially satisfied. However, the discovery program does not satisfy it in general.

[5] This requirement was first proposed and studied by Hafalir et al. (2013), and they showed that such a condition is achievable either in an ad-hoc fashion or by using historical data on school admissions.

Question 2. Considering a fixed reservation quota, does one of the affirmative action mechanisms (DISC, JSA or MR) (weakly) dominate another one for disadvantaged students, i.e., do all disadvantaged students weakly prefer the schools they are matched to under one mechanism compared to the other?

We say that a mechanism A *(weakly) dominates* another mechanism B for disadvantaged students if A places all disadvantaged students in schools they like at least as much as the schools they are placed in by B. Our results from Table 1 seem to suggest that the discovery program mechanism could be dominated by either minority reserve or joint seat allocation. However, this is *not* the case, as shown by the results we summarize in Table 2. All three mechanisms are incomparable, even under some pretty restrictive hypothesis: (1) schools rank students in the same order; and/or (2) reservation quotas being a smart reserve. The first hypothesis is common in markets where students' ranking is based on an entrance exam, such as the one for NYC SHSs. The only exception to the incomparability results is that the mechanism noAA without affirmative action, under the second hypothesis, is dominated by minority reserve and joint seat allocation[6].

	noAA			MR	DISC	JSA
noAA				(✗) (✗) (✗ Ex 4)	(✗) (✗) (✗ Ex 4)	(✗) (✗) (✗ Ex 4)
MR	(✗ [HYY])	(✓ [HYY])	(✓)		(✗) (✗) (✗ Ex 5)	(✗) (✗) (✗ Ex 6)
DISC	(✗)	(✗)	(✗ Ex 2)	(✗) (✗) (✗ Ex 5)		(✗) (✗) (✗ Ex 5)
JSA	(✗ 3)	(✓ Thm 5)	(✓)	(✗) (✗) (✗ Ex 6)	(✗) (✗) (✗ Ex 5)	

Table 2   The table answer the following question under the school-over-seat assumption: does the "row" mechanism dominates the "column" mechanism for disadvantaged students? We answer the question for three restricted domains: (1) schools share a common ranking of the students, (2) the reservation quotas is a smart reserve, and (3) both. The answers are given in the exact order. All answers are accompanied by the citations with [HYY] Hafalir et al. (2013) or the labels of the examples or theorems used to answer the questions, except for cases when the answer for one domain can be inferred from that of another domain.

To be able to identify crucial interventions for the discovery program, we study the behavior of the JSA and MR mechanisms in markets that satisfy a condition which we call *high competitiveness*. This is a novel ex-post condition which guarantees that JSA weakly dominates MR for disadvantaged students. This condition is verified by our data from NYC DOE, where in fact JSA outperforms MR for disadvantaged students. We also show reasonable conditions on the primitives of the market that imply high competitiveness. Roughly speaking, the high competitiveness condition is satisfied when the demand for seats (i.e., number of students) is much larger than the supply, and when

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[6] We would like to point out that this exception is simply another way of expressing the same results related to the third property in Table 1.

disadvantaged students are performing systematically worse than advantaged students[7]. See Theorem 6 for the formal statement. We discuss next how our experiments validate our theoretical result and provide a practical policy recommendation for changes to the discovery program.

1.1.3. Case Study based on Data from New York City's Department of Education. We validate our theoretical results with extensive computational experiments using data we obtained from NYC DOE for the 2005-2006 to 2016-2017 academic years, where we label students as advantaged or disadvantaged based on the criteria given by the discovery program. First, we show that, in practice as well, the discovery program suffers from all the theoretical drawbacks we presented in Table 1 (see Section 5.1 for details), except the third property (as it requires the construction of an extreme case). We find that for reservation quotas set to 20%, on average there are 950 blocking pairs for disadvantaged students which impact around 650 students each year. Considering the changes in rank to matched schools, DISC mechanisms is the only one under which disadvantaged students can be worse-off (i.e., which hurts some disadvantaged students). In particular, this hurts the top-performing disadvantaged students much more, and helps the low-performing disadvantaged students (see Figure 2). The discovery program is also not strategy proof: some of the aforementioned top-performing students may truncate their preference lists (i.e., remove some less preferred schools from their honestly submitted preference lists), so that they skip the competition for general seats at these less preferred schools and aim directly for reserved seats at more preferred schools.

In addition, by observing the distribution of SHSAT scores for both the advantaged and disadvantaged groups of students, we notice that disadvantaged students are performing systematically worse than advantaged students (see Figure 3b), which would undoubtedly lead to underrepresentation of disadvantaged students at these SHSs without affirmative actions. Because of this observation and of the very limited number of seats when compared to the students applying to SHSs, we expect the market to the highly competitive and thus all disadvantaged students would weakly prefer their assignment under JSA than under MR. We indeed observe these characteristics for the NYC SHS admission market across all academic years we have data for (see Figure 3a and Figure 4b). This leads to the policy recommendation we present in this work.

1.1.4. Policy Recommendation. Overall, our work paves the way to make the discovery program fairer for disadvantaged students. In particular, we provide an answer to how the existing practice of the discovery program can be changed minimally to improve the outcome for the disadvantaged group of students, so that the program aligns with the incentives to perform better.

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[7] We hereafter refer to non-disadvantaged students as advantaged students for the sake of nomenclature, and it only implies that advantaged students do not suffer from the disadvantages that affect disadvantaged students.

Our Proposal: We propose that the program takes into account the preferences of students in terms of the schools v/s seats. Is attending a particular school more important than the type of seat they are assigned to or vice versa? We believe that most students should be willing to take a one-time 3-week summer program to attend a school they prefer, rather than not taking the program and attending, for 4 years, a school they prefer less (e.g., we find that this hypothesis is supported by the fact that preferences appear often to be strongly polarized for certain schools due to, e.g., geographical considerations, details are reported in the Appendix, Section E). Although this seems reasonable, unfortunately such preferences are currently not collected in the data provided by the NYC DOE.

Under the school-over-seat assumption, we find that the many drawbacks of the current implementation of the discovery program can be corrected by following the *joint seat allocation* mechanism. For the NYC Specialized High School market – and, more generally, for highly competitive markets – joint seat allocation gives a matching that is weakly better for disadvantaged students, when compared to matching output by the other replacement mechanism studied in this paper, both in theory and in practice.

Although powerful, the modification we propose requires *minimal* modification: there is essentially no change in terms of what students and schools should report to the DOE (preference lists for both and admission capacity for schools), and there is no change in terms of the algorithm (the deferred acceptance algorithm (Gale and Shapley 1962), which is currently in implementation). Given this information, to implement the `JSA` mechanism, one only needs to compute an equivalent instance where students' preference lists are expanded to be over reserved and general seats at schools, so that the matching we desire to obtain can be easily recovered from the matching obtained under the classical stable matching model on this equivalent instance. See Section 4.3 for details.

Before we delve deeper into our model and results, we would like to highlight a trade-off that any constrained resource allocation problem faces. Diverting some resources to the disadvantaged groups implies taking some resources that are currently assigned to the advantaged groups. In this work as well, we find from our empirical analysis, that advantaged students always weakly prefer their assignment under `MR` compared to `JSA`. For all the academic years we analyze, we find that about 3% of the advantaged students are worse off under `JSA` than under `MR` (i.e., about 97% of them are matched to the same school under the two mechanisms); and among the 3%, most of them experience a drop in the rank of assigned schools that is at most two. See Figure 4b for details of one academic year. We consider this impact to be minimal compared to the ill-treatment faced by the disadvantaged students.

1.2. The techniques

The affirmative action mechanisms introduced in this paper seem to entail different algorithms applied to the same preferences lists of students and schools. However, it turns out that an equivalent, yet mathematically more convenient way is to view their assignment outputs as obtained from the same algorithm applied, however, to different input instances. There are two approaches by which we can obtain such a reformulation.

This first approach is to employ *choice functions*, which are a general and powerful way to model the preference lists of agents in matching markets. In particular, all choice functions needed to model the mechanisms in this paper satisfy the *substitutability*, *consistency*, and q_c-*acceptance* properties (see Section 2.2). Under such properties, stable matchings are known to exist and satisfy strong structural and algorithmic properties (see, e.g., Alkan (2002), Faenza and Zhang (2021), Roth (1984)). This reformulation[8] allows us to analyze the assignments under different mechanisms as the outputs of one or more rounds of Roth's generalization (Roth 1984) of the classical deferred acceptance algorithm by Gale and Shapley (1962). As a result, to show properties of the assignment obtained from an affirmative action mechanism, we can directly use properties of its choice functions, of stable matchings, as well as the properties of the generalized deferred acceptance algorithm.

The second approach is to expand students' original preferences over schools to preferences over reserved and general seats at schools. Under this reformulation, assignments under different affirmative action mechanisms can be obtained simply by applying the classical deferred acceptance algorithm over the equivalent instances. This allows us to deduce interesting properties of the mechanisms (e.g., strategy-proofness), by leveraging on classical results on stable matchings.

1.3. Related literature

The problem of assigning students to schools (without affirmative action) was first studied by Gale and Shapley in their seminal work (Gale and Shapley 1962). Abdulkadiroğlu and Sönmez (2003) then analyzed the algorithm in the context of school choice and recommended school districts to replace their current mechanisms with either this algorithm or another algorithm, called the *top trading cycle algorithm*. Since then, these mechanisms have been widely adopted by many cities in the United States, such as New York City and Boston.

The first attempt of incorporating affirmative action with the stable mechanism occurred in this pioneering work (Abdulkadiroğlu and Sönmez 2003), where they extended their analysis to a simple

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[8] We note in passing, that, this reformulation allows a central planner to access many stable matchings, using recent results by Faenza and Zhang (2021), which provide alternatives to the matchings output by the mechanisms considered in this paper.

affirmative action policy, using majority quotas. However, Kojima (2012) then analyze the effects of these proposed affirmative action policies, as well as priority-based policies, and showed that in some cases, the mechanisms might hurt disadvantaged students, the very group these policies are trying to help. Hafalir et al. (2013) further analyze the effect empirically through simulated data and suggested that this phenomenon might be quite common, and does not just happen in theory due to special edge cases. In addition, to overcome the efficiency loss, they propose the minority reserve mechanism.

Since then, there has been an abundance of literature, studying and proposing solutions for the efficiency loss due to affirmative action, such as Afacan and Salman (2016), Doğan (2016), Echenique and Yenmez (2015), Ehlers et al. (2014), Fragiadakis and Troyan (2017), Jiao and Shen (2021), Kominers and Sönmez (2016), Nguyen and Vohra (2019).

Another popular form of affirmative action is what is called *priority-based* (see, e.g., Hafalir et al. (2013), Jiao and Shen (2021), Kojima (2012)), which creates a higher priority for disadvantaged students by, e.g., boosting their scores. Though this mechanism satisfies important properties such as strategy-proofness and absence of in-group blocking pairs, its practical use is being largely debated. For example, in 2019, the college board proposed adding an adversity score to SAT scores to account for socio-economic differences, however, this was met with severe pushback (Jaschik 2019). In another lawsuit at the University of Michigan challenging a priority-based mechanism that assigned 20 points extra to disadvantaged students, the system was declared unconstitutional by the Supreme Court (Gratz_vs_Bollinger 2003). Faenza et al. (2020) investigates the effects of policies where scores for minority students are boosted before the admission process by extra training, additional resources, etc. Since the goal of this work is to focus on operational suggestions to the discovery program, we do not explore priority-based mechanisms.

1.4. Outline

The rest of the paper is organized as follows. In Section 2, we introduce the basic model and related concepts for stable matchings and stable matching mechanisms. In Section 3, we formally introduce the affirmative action mechanisms considered in this paper and investigate their properties (i.e., answering Question 1). We then compare these mechanisms in Section 4 and provide the answer to Question 2. In Section 4.3, we show that the three affirmative action mechanisms considered in the paper differ in terms of how students' preference over reserved seats and general seats are interpreted. Lastly, in Section 5, we dive into the data on NYC SHS admission, demonstrate our theoretical findings empirically.

2. Model and Notations

2.1. Matchings and mechanisms

Let S and C denote a finite set of students and schools respectively. Let $G = (S \cup C, E)$ be a bipartite graph, where two sides of nodes are students and schools, and the edge set E represents the schools which students find *acceptable* (i.e., would like to attend). Every student $s \in S$ has a strict preference relation over the schools they find acceptable and the option of being unassigned (denoted by $\emptyset$), which we call the *preference list* of student s, and we denote it by $>_s$. Formally, for two options $c_1, c_2 \in C \cup \{\emptyset\}$, $c_1 >_s c_2$ means that student s strictly prefers c_1 to c_2. Note that for every student-school pair (s, c), if $(s, c) \in E$, we have $c >_s \emptyset$; otherwise, $\emptyset >_s c$. There are two types of students, *advantaged* (or *majority*) and *disadvantaged* (or *minority*), denote by S^M and S^m respectively. That is, $S = S^M \dot\cup S^m$ where $\dot\cup$ is the disjoint union operator. On the other hand, every school c has a quota $q_c \in \mathbb{N} \cup \{0\}$, which represents the maximum number of students it can admit, and a strict *priority order* $>_c$ over the students: for any two students $s_1, s_2 \in S$, $s_1 >_c s_2$ means that student s_1 has a higher *priority* (e.g., higher test score) than student s_2 at school c.

Let $>_S \equiv \{>_s : s \in S\}$, $>_C \equiv \{>_c : c \in C\}$, and $\mathbf{q} \equiv \{q_c : c \in C\}$ denote the collection of students' preference lists, schools' priority orders, and schools' quotas, respectively. Moreover, we write $> \equiv \{>_S, >_C\}$. An *instance (or market)* is thus denoted by $(G, >_S, >_C, \mathbf{q})$ or $(G, >, \mathbf{q})$.

A *matching* μ (of an instance) is a collection of student-school pairs such that every student is incident to at most one edge in μ and every school c is incident to at most q_c edges in μ. For student $s \in S$ and school $c \in C$, we denote by $\mu(s)$ the school student s is matched (or assigned) to, and by $\mu(c)$ the set of students school c is matched (or assigned) to, under matching μ.

For every school $c \in C$, let $q_c^R \in \{0, 1, \cdots, q_c\}$ denote the number of seats reserved to disadvantaged students at school c, and let $q_c^G := q_c - q_c^R$ denote the number of general seats at school c. We call $\mathbf{q}^R := \{q_c^R : c \in C\}$ the *reservation quotas*.

A *(quota-based matching) mechanism* is a function that maps every instance, together with reservation quotas, to a *matching*. Given an instance $I = (G, >, \mathbf{q})$, a mechanism ϕ, and reservation quotas $\mathbf{q}^R$, let $\phi(I, \mathbf{q}^R)$ denote the matching obtained under the mechanism ϕ with reservation quotas $\mathbf{q}^R$. Sometimes, when the reservation quotas are clear from context, we simply denote the matching as $\phi(I)$.

Let μ_1, μ_2 be two matchings. We say μ_1 *(weakly) dominates μ_2 for disadvantaged students* if $\mu_1(s) \geq_s \mu_2(s)$ for all disadvantaged students $s \in S^m$. If moreover $\mu_1 \neq \mu_2$ (i.e., there is at least one disadvantaged student $s \in S^m$ such that $\mu_1(s) >_s \mu_2(s)$), then we say μ_1 *Pareto dominates μ_2 for disadvantaged students*. Consider a student-school pair $(s, c) \in E$, it is a blocking pair of matching μ *for disadvantaged students* if $s \in S^m$, $c >_s \mu(s)$, and there exists a disadvantaged student

$s' \in \mu(c) \cap S^m$ such that $s >_c s'$; and it is a blocking pair of matching μ *for advantaged students* if $s \in S^M$, $c >_s \mu(s)$, and there exists an advantaged student $s' \in \mu(c) \cap S^M$ such that $s >_c s'$. Collectively, a blocking pair is called an *in-group* blocking pair if it is a blocking pair for either disadvantaged or advantaged students.

Fix reservation quotas $\mathbf{q}^R$. A mechanism ϕ is *strategy-proof* if for any instance I and for any student $s \in S$, there is no preference list $\widetilde{>}_s$ such that $\phi(\widetilde{I}, \mathbf{q}^R)(s) >_s \phi(I, \mathbf{q}^R)(s)$, where $\widetilde{I}$ is obtained from I by replacing $>_s$ with $\widetilde{>}_s$. In other words, a mechanism is strategy-proof if no student has the incentive to misreport their preference list. As a stronger concept, a mechanism is *weakly group strategy-proof* if for any instance I and for any *group* of students $S_1 \subseteq S$, there are no preference lists $\{\widetilde{>}_s : s \in S_1\}$ such that for every student $s \in S_1$, $\phi(\widetilde{I}, \mathbf{q}^R)(s) >_s \phi(I, \mathbf{q}^R)(s)$, where $\widetilde{I}$ is obtained from I by replacing $>_s$ with $\widetilde{>}_s$ for every $s \in S_1$. That is, a mechanism is weakly group strategy-proof if no group of students can jointly misreport their preference lists so that everyone in the group is strictly better off. Note that if a mechanism is weakly group strategy-proof, it is strategy-proof.

Consider two mechanisms ϕ_1 and ϕ_2. If $\phi_1(I, \mathbf{q}^R)$ (weakly) dominates $\phi_2(I, \mathbf{q}^R)$ for disadvantaged students for all instances I, we say that mechanism ϕ_1 *(weakly) dominates* mechanism ϕ_2 for disadvantaged students. If neither ϕ_1 nor ϕ_2 dominates the other mechanism, we say they are *not comparable* or *incomparable*.

2.2. Choice functions

To unify the treatment of different affirmative action mechanisms, we next introduce the concept of choice functions. Under each mechanism, every school $c \in C$ is endowed with a *choice function* $\mathcal{C}_c : 2^S \to 2^S$: for every subset of students $S_1 \subseteq S$, $\mathcal{C}_c(S_1)$ represents the students whom school c would like to admit among those in S_1. In particular, for every $S_1 \subseteq S$, we have $\mathcal{C}_c(S_1) \subseteq S_1$ and $|\mathcal{C}_c(S_1)| \leq q_c$. Choice function $\mathcal{C}_c$ is a function of the priority order $>_c$ and quotas q_c^R and q_c^G, and its exact definition depends on the specific mechanism (see Section 4). Students' preferences are still described by a strict order over a subset of schools.

For all the affirmative action mechanisms studied in this paper, every school c's choice function $\mathcal{C}_c$ satisfies the following classical (see, e.g., Alkan (2002)) properties: *substitutability*, *consistency*, and q_c*-acceptance*[9]. Thus, for the rest of the paper, unless otherwise specified, these properties are always assumed. For some mechanisms, $\mathcal{C}_c$ is additionally q_c*-responsive*. Substitutability states that whenever a student is selected from a pool of candidates, they will also be selected from a smaller subset of the candidates; consistency is also called "irrelevance of rejected contracts", which means removing rejected candidates from the input will not change the output; q_c-acceptance means that

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[9] q_c-acceptance is also referred to as *quota-filling* by some authors. However, we prefer to use q_c-acceptance since it highlights the quota.

the choice function tries to fill the q_c positions as much as possible; and q_c-responsiveness means that there is an underlying priority order over the students and the choice function simply selects q_c students with the highest priorities whenever available. The precise definition of these concepts is included in Appendix A.

For any nonnegative integer q, a priority order over the students $>$, and a subset of students $S_1 \subseteq S$, let $\max(S_1, >, q)$ denote the $\min(q, |S_1|)$ highest ranked students (i.e., students with the highest priorities) of S_1 according to the priority order $>$. We further note that q-responsiveness implies substitutability, consistency, and q-acceptance. Indeed, q-responsive choice functions are the "simplest" choice functions and are mostly studied in the matching literature, including the seminal work by Gale and Shapley (1962) and in practical school choice (Abdulkadiroğlu et al. 2005a,b).

2.3. Stable matchings

Consider an arbitrary collection of schools' choice functions $\mathcal{C} := \{\mathcal{C}_c : c \in C\}$. Note that the q_c-acceptant property implies that for every school c, we must have $\mathcal{C}_c(\mu(c)) = \mu(c)$ by any matching μ by the definition of matchings. A matching μ is *stable* (in instance I under choice functions $\mathcal{C}$) if there is no student-school pair $(s, c) \in E$ such that $c >_s \mu(s)$ and $s \in \mathcal{C}_c(\mu(c) \cup \{s\})$. When such a student-school pair exists, we call it a *blocking pair* of μ, or we say that the edge (or pair) *blocks* μ. Note that the definition of matchings only depends on the instance, not on the choice functions; whereas the definition of stability depends on both.

When the choice function is q_c-responsive (i.e., induced by a priority order and a quota), the definition of stability with respect to choice functions is equivalent to the standard definition in the classical model without choice functions. In particular, the condition $s \in \mathcal{C}_c(\mu(c) \cup \{s\})$ can then be stated as: either school c's seats are not fully assigned (i.e., $|\mu(c)| < q_c$) or s has a higher priority over some students that are assigned to c (i.e., $\exists s' \in \mu(c)$ such that $s >_c s'$).

Among all stable matchings of a given instance and choice functions, there is one that *dominates* every stable matching, where matching μ_1 is said to *dominate* matching μ_2 if $\mu_1(s) \geq_s \mu_2(s)$ for all students $s \in S$. This stable matching is called the *student-optimal* stable matching, and it can be obtained by the *student-proposing deferred acceptance algorithm* (Gale and Shapley 1962, Roth 1984), which we describe next. The algorithm runs in *rounds*. At each round k, every student applies to their most preferred school that has not rejected them; and every school c, with $S_c^{(k)}$ denoting the set of students who applied to it in the current round, *temporarily* accepts students in $\mathcal{C}_c(S_c^{(k)})$ and rejects the rest. The algorithm terminates when there is no rejection. For any instance I and choice functions $\mathcal{C}$, we denote by $\texttt{SDA}(I, \mathcal{C})$ the matching output by the student-proposing deferred acceptance algorithm.

3. Affirmative Action Mechanisms

For the rest of the section, we fix an instance $I = (G, >, \mathbf{q})$ and reservation quotas $\mathbf{q}^R$. The choice functions of schools depend on the mechanisms, and we introduce them in details in each subsection. We also discuss the features of the mechanisms in their corresponding subsections.

3.1. No affirmative action

The simplest mechanism is the one without affirmative action. That is, schools do not distinguish students of different types. The choice function of school c under the no affirmative action mechanism is q_c-responsive, simply induced from its priority order: for all subset of students $S_1 \subseteq S$,

$$\mathcal{C}_c^{\text{noAA}}(S_1) := \max(S_1, >_c, q_c).$$

We denote by $\mu^{\text{noAA}} := \text{SDA}(I, \mathcal{C}^{\text{noAA}})$ the matching under the no affirmative action mechanism. Although this matching can be obtained from the original and simpler deferred acceptance algorithm proposed by Gale and Shapley (1962), we present the mechanism from a choice function point of view so that it is consistent with later sections. The no affirmative action mechanism has the following two properties. Theorem 1 was shown by Dubins and Freedman (1981), and Proposition 1 is immediate from the fact that μ^{noAA} admits no blocking pairs under $\mathcal{C}^{\text{noAA}}$ and the definition of choice functions $\mathcal{C}^{\text{noAA}}$.

THEOREM 1 (**Dubins and Freedman (1981)**). *The no affirmative action mechanism is weakly group strategy-proof.*

PROPOSITION 1. *μ^{noAA} does not admit in-group blocking pairs.*

3.2. Minority reserve

Under minority reserve, the choice function of every school $c \in C$, denoted by $\mathcal{C}_c^{\text{MR}}$, is defined as follows (Hafalir et al. 2013): for every subset of students $S_1 \subseteq S$,

$$\mathcal{C}_c^{\text{MR}}(S_1) = \underbrace{\max(S_1 \cap S^m, >_c, q_c^R)}_{=:S_1^R; \text{ reserved seats}} \,\dot\cup\, \underbrace{\max\left(S_1 \setminus S_1^R, >_c, q_c - |S_1^R|\right)}_{\text{remaining seats}}.$$

That is, every school first accepts disadvantaged students from its pool of candidates up to its reservation quota, and then fills up the remaining seats from the remaining candidates. Note that if there is a shortage of disadvantage students (i.e., $|S_1 \cap C^m| < q_c^R$), then the remaining reserved seats become open to advantaged students. We defer all proofs in this subsection to Appendix C.1.

PROPOSITION 2. *Choice function $\mathcal{C}_c^{\text{MR}}$ is substitutable, consistent, and q_c-acceptant.*

Since substitutability and consistency guarantee the existence of stable matchings (Aygün and Sönmez 2013, Hatfield and Milgrom 2005, Roth 1984), stable matchings exist under choice functions $\mathcal{C}^{\mathtt{MR}}$ and we denote by $\mu^{\mathtt{MR}} \coloneqq \mathtt{SDA}(I, \mathcal{C}^{\mathtt{MR}})$ the matching under minority reserve with reservation quotas $\mathbf{q}^R$. Minority reserve has several desirable properties, which we formally state below. Theorem 2 states that at least one disadvantaged student is not worse off when compared to the no affirmative action mechanism, which is not necessarily true for other affirmative action mechanisms (see, e.g., Kojima (2012)); and Theorem 3 states that when the reservation quotas are "carefully" chosen, no disadvantaged student is worse off. Reservation quotas $\mathbf{q}^R$ are said to be a *smart reserve* if $q_c^R \geq |\mu^{\mathtt{noAA}}(c)|$ for all $c \in C$.

PROPOSITION 3 (**Hafalir et al. (2013)**). *Minority reserve is weakly group strategy-proof.*

THEOREM 2 (**Hafalir et al. (2013)**). *For any reservation quota $\mathbf{q}^R$, there exists a disadvantaged student $s \in S^m$ such that $\mu^{\mathtt{MR}}(s) \geq_s \mu^{\mathtt{noAA}}(s)$.*

THEOREM 3 (**Hafalir et al. (2013)**). *If the reservation quotas $\mathbf{q}^R$ is a smart reserve, then $\mu^{\mathtt{MR}}$ dominates $\mu^{\mathtt{noAA}}$ for disadvantaged students.*

The following claim follows directly from the fact that $\mu^{\mathtt{MR}}$ is stable under choice functions $\mathcal{C}^{\mathtt{MR}}$ and the definition of $\mathcal{C}^{\mathtt{MR}}$.

PROPOSITION 4. *$\mu^{\mathtt{MR}}$ does not admit in-group blocking pairs.*

3.3. Discovery program

This mechanism is adapted from the mechanism used by NYC DOE for increasing the number of disadvantaged students at the city's eight specialized schools, which are considered to be the best public schools. Instead of distributing reserved seats to disadvantaged students at the beginning as the minority reserve (i.e., to top ranked disadvantaged students), the discovery program mechanism[10] distributes reserved seats to disadvantaged student at the end of seat-assignment procedure. One of the reasons for allocating reserved seats to lower ranked disadvantages students is that disadvantaged students who are admitted via reserved seats are required to participate in a 3-weeks *summer enrichment program* as a preparation for the specialized high schools.

However, for the sake of comparison (with other mechanisms), we assume that students' preference for schools are not affected by whether they are required to participate in the summer program – that is, students are indifferent between general and reserved seats at each school. We assume this *school-over-seat* hypothesis for the rest of the paper, and we discuss its validity in the Appendix, Section E.

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[10] https://www.schools.nyc.gov/enrollment/enrollment-help/meeting-student-needs/diversity-in-admissions

When there is a shortage of disadvantaged students, reserved seats could go unassigned under the discovery program mechanism. Although this is usually not of concern in real-world applications, since there are usually more students than available seats, we nevertheless present the discovery program mechanism in a more general case where vacant reserved seats are de-reserved.

The algorithm for the discovery program mechanism has three stages. Schools' choice functions at all stages are the simple q-responsive choice function $\mathcal{C}^{\mathtt{noAA}}$. The mechanism starts by running the deferred acceptance algorithm on instance $(G, >, \mathbf{q}^G)$ to obtain matching $\mu_1^{\mathtt{DISC}}$ for the general seats; it then runs the deferred acceptance algorithm on the instance restricted to the disadvantaged students that are not yet assigned $(G[C \cup \{s \in S^m : \mu_1^{\mathtt{DISC}}(s) = \emptyset\}], >, \mathbf{q}^R)$ to obtain matching $\mu_2^{\mathtt{DISC}}$ for reserved seats; and it lastly runs the deferred acceptance algorithm on the instance restricted to the advantaged students that are not yet assigned $(G[C \cup \{s \in S^M : \mu_1^{\mathtt{DISC}}(s) = \emptyset\}, >, \mathbf{q}^E)$ with $q_c^E = q_c^R - |\mu_2^{\mathtt{DISC}}(c)| \; \forall c \in C$ to obtain matching $\mu_3^{\mathtt{DISC}}$ for vacant reserved seats. The final matching combines the matchings obtained at these three stages: $\mu^{\mathtt{DISC}} := \mu_1^{\mathtt{DISC}} \dot\cup \mu_2^{\mathtt{DISC}} \dot\cup \mu_3^{\mathtt{DISC}}$.

Although the mechanism intends to help disadvantaged students, it could actually hurt them. As we show through Example 1 below, under the discovery program mechanism, it is possible that all disadvantaged students are worse off. Moreover, the discovery program mechanism could create blocking pairs for disadvantaged students, incentivize disadvantaged students to misrepresent their preference lists, and might hurt disadvantaged students even when the reservation quotas are a smart reserve. See Example 2.

EXAMPLE 1. Consider the instance with students $S^M = \{s_1^M, s_2^M\}$, $S^m = \{s_1^m\}$ and schools $C = \{c_1, c_2\}$. The quotas of schools are $q_{c_1} = 2$ and $q_{c_2} = 1$, and both schools have priority order $s_1^M > s_2^M > s_1^m$. Both advantaged students prefer c_1 to c_2, whereas the disadvantaged student prefers c_2 to c_1. It is easy to see that under the no affirmative action mechanism,

$$\mu^{\mathtt{noAA}} = \{(s_1^M, c_1), (s_2^M, c_1), (s_1^m, c_2)\}.$$

Now consider the discovery program mechanism with reservation quotas $q_{c_1}^R = 1$ and $q_{c_2}^R = 0$. Then,

$$\mu^{\mathtt{DISC}} = \{(s_1^M, c_1), (s_2^M, c_2), (s_1^m, c_1)\}.$$

Under the discovery program mechanism, the disadvantaged student s_1^m is not only assigned to a school less preferred less, but is also now required to participate in the summer program. $\triangle$

EXAMPLE 2. Consider the instance with students $S^M = \{s_1^M, s_2^M, s_3^M\}$, $S^m = \{s_1^m, s_2^m, s_3^m\}$ and schools $C = \{c_1, c_2\}$. The quotas of schools are $q_{c_1} = 3$ and $q_{c_2} = 2$, and both schools have priority order $s_1^M > s_2^M > s_1^m > s_3^M > s_2^m > s_3^m$. All students prefer c_1 to c_2. Without affirmative action, we have

$$\mu^{\mathtt{noAA}}(c_1) = \{s_1^M, s_2^M, s_1^m\}, \quad \mu^{\mathtt{noAA}}(c_2) = \{s_3^M, s_2^m\}.$$

Now assume that the reservation quotas are $q_{c_1}^R = q_{c_2}^R = 1$, which in particular is a smart reserve. Under the discovery program mechanism with these reservation quotas, we have

$$\mu^{\text{DISC}}(c_1) = \{s_1^M, s_2^M, s_2^m\}, \quad \mu^{\text{DISC}}(c_2) = \{s_1^m, s_3^m\}.$$

Disadvantaged student s_1^m is worse off under μ^{DISC} than under μ^{noAA}. In addition, μ^{DISC} admits a blocking pair (s_1^m, c_1) for disadvantages students as s_1^m prefers c_1 to c_2 and s_1^m has a higher priority than s_2^m at c_1. Moreover, s_1^m has the incentive to misreport the preference list: if s_1^m were to report the preference list as $c_1 > \emptyset$, the matching under the discovery program mechanism would have been the same as μ^{noAA}. $\triangle$

Example 1 and Example 2 together show that none of the desirable properties we investigate in this paper (see Table 1) holds for the discovery program mechanism.

3.4. Joint seat allocation

The mechanism of joint seat allocation we introduce here is inspired by the mechanism used for admission to Indian Institutes of Technology (JoSAA 2020). It allocates the general and reserved seats at the same time, while only allowing disadvantaged students to take the reserved seats when they cannot get admitted via the general seats. Under this mechanism, the choice function of every school $c \in C$, denoted by $\mathcal{C}_c^{\text{JSA}}$, is defined as follows. For every subset of students $S_1 \subseteq S$,

$$\mathcal{C}_c^{\text{JSA}}(S_1) = \underbrace{\max(S_1, >_c, q_c^G)}_{=:S_1^G; \text{ general seats}} \dot{\cup} \underbrace{\max\left(S_1 \cap S^m \setminus S_1^G, >_c, q_c^R\right)}_{=:S_1^R; \text{ reserved seats}} \dot{\cup} \underbrace{\max(S_1 \setminus (S_1^G \cup S_1^R), >_c, q_c - |S_1^G \cup S_1^R|)}_{\text{remainning seats}}.$$

A prominent distinction between joint seat allocation and minority reserve is that in the former, "highly ranked" disadvantaged students are admitted via general seats and do not take up the quotas for reserved seats. Intuitively, this opens up more opportunities for disadvantaged students and one would expect all disadvantaged students to be weakly better off under joint seat allocation than under minority reserve. This is true for instances where the competition for seats is high, but is not true for general instances. See Section 4 and Theorem 6 for more discussions on the comparison between these two mechanisms. We defer all proofs in this subsection to Appendix C.2.

PROPOSITION 5. *Choice function $\mathcal{C}_c^{\text{JSA}}$ is substitutable, consistent, and q_c-acceptant.*

Proposition 5 implies that stable matchings exist under joint seat allocation, and we denote the student-optimal stable matching by $\mu^{\text{JSA}} := \text{SDA}(I, \mathcal{C}^{\text{JSA}})$.

All positive results of minority reserve extend to joint seat allocation. We formalize the statements below. The proof of Proposition 6 and Theorem 4 follow by constructing an equivalent instance where, in particular, students have preference lists over general and reserved seats at different

schools. This idea is similar to that given in Hafalir et al. (2013), but the equivalent instances are different under minority reserve and joint seat allocation (see Section 4.3 for details, where we additionally construct similar equivalent instances for the discovery program mechanism). The main reason for establishing such equivalent instances is that it allows us to directly use the strategy-proofness result for the classical stable matching model (i.e., no affirmative action).

PROPOSITION 6. *Joint seat allocation is weakly group strategy-proof.*

THEOREM 4. *For any reservation quota $\mathbf{q}^R$, there exists a disadvantaged student $s \in S^m$ such that $\mu^{\text{JSA}}(s) \geq_s \mu^{\text{noAA}}(s)$.*

For the following theorem, we give a novel proof that directly follow the procedure of the deferred acceptance algorithm and use the properties of choice functions $\mathcal{C}^{\text{JSA}}$. Our approach is different from the one given in Hafalir et al. (2013) for the similar property of minority reserve.

THEOREM 5. *If the reservation quotas are a smart reserve, then μ^{JSA} dominates μ^{noAA} for disadvantaged students.*

When the reservation quota is not a smart reserve, it is possible that μ^{noAA} Pareto dominates μ^{JSA} for disadvantaged students, which can be readily seen from the same example for minority reserve presented in Hafalir et al. (2013). See Example 3 in Appendix B.1.

As Proposition 4, the following claim follows directly from the fact that μ^{JSA} is stable under choice functions $\mathcal{C}^{\text{JSA}}$ and the definition of $\mathcal{C}^{\text{JSA}}$.

PROPOSITION 7. *μ^{JSA} does not admit in-group blocking pairs.*

4. Comparison of Affirmative Action Mechanisms

In this section, we investigate how different mechanisms introduced in the previous section compare with each other.

4.1. Is there a winning mechanism for disadvantaged students?

To begin with, we would like to answer the following question regarding any two mechanisms: does one mechanism dominate the other mechanism for disadvantaged students? We consider three domains which impose restrictions on the instance or the reservation quotas. They are: (1) schools share a common priority order over the students (i.e., *universal* priority order), (2) the reservation quotas are a smart reserve, and (3) both smart reserve and universal priority order. We summarized the results in Table 2. Note that for a pair of mechanisms, a positive answer for (1) or (2) implies a positive answer for (3) and a negative answer for (3) implies negative answers for both (1) and (2). These allow us to simplify the presentations given in Table 2.

From Table 2, we can see that no two mechanisms are comparable in the general domain (i.e., all instances included). In addition, even in the restricted domains, most of the mechanisms are not comparable, with the exception that minority reserve and joint seat allocation dominate the no affirmative action mechanism when the reservation quotas are a smart reserve.

These results are show as follows. We first observe that the no affirmative action mechanism does not dominate the other mechanisms, through a rather trivial example included in Appendix B.2 (see Example 4). We then compare the three mechanisms with affirmative action in Example 5 and Example 6 in Appendix B.2.

4.2. Joint seat allocation vs minority reserve: the high competitiveness hypothesis

To further compare minority reserve and joint seat allocation, we consider a special condition on the market, that we term *high competitiveness of the market:*

$$|\mu^{\text{MR}}(c) \cap S^m| \leq q_c^R \text{ for every school } c \in C.$$

Note that this is an ex-post condition that is based on the outcome μ^{MR} of a specific mechanism – namely, minority reserve. In particular, this condition asks that minority students not occupy general seats in the matching μ^{MR}.

Under the high competitiveness hypothesis, joint seat allocation dominates minority reserve for disadvantaged students. We formalize the statement in Theorem 6.

THEOREM 6. *If μ^{MR} satisfies that for every school $c \in C$, $|\mu^{\text{MR}}(c) \cap S^m| \leq q_c^R$ (*high competitiveness hypothesis)*, then μ^{JSA} dominates μ^{MR} for disadvantaged students.*

High competitiveness can be connected to primitives of the market. Intuitively, it is satisfied when disadvantaged students are systematically performing worse than advantaged students and when there is a shortage of seats at all schools. In other words, this condition is satisfied if after the initial allocation of reserve seats to top ranked disadvantaged students, the remaining disadvantaged students are not able to compete with the advantaged students for general seats[11]. This condition is not uncommon in markets with limited resources.

In Section 5 we show empirically that, in particular, the market of NYC SHS is highly competitive using their admission data. Below we state a rigorous statement connecting primitives of the market and high competitiveness.

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[11] High competitiveness is also satisfied in the trivial case when there are so many reserved seats, that all disadvantaged students get one, but this is rarely seen in the real world – and does not happen in our data from NYC SHSs.

THEOREM 7. *Consider a family of markets with an increasing number of students and schools, where the preference lists of students are i.i.d. such that the probabilities of any two schools ranking first in a student's preference list coincide. Assume that schools have the same (reservation) quota and they share the same ranking of students, and that $q - 1 > q^R > n \log n$, where n is the number of schools. If, for some $\epsilon \in (0,1)$ the $(n \log n + (q - q^R)n \log \log n)$-ranked advantaged student exists and is ranked above the $(1-\epsilon)q^R n$-ranked disadvantaged student (where rankings of students are within their respective groups), then the market is highly competitive with probability $1 - o(1)$.*

The proofs of Theorem 6 and Theorem 7 can be found in Appendix C.3, together with a discussion on Theorem 7.

4.3. Equivalent Interpretation

In this subsection, we take a different approach and instead of comparing the outputs. We compare how mechanisms interpret the inputs, and particularly how students' original preferences over schools are translated to their preferences over reserved and general seats at all schools.

We present alternative representations of the inputs under three mechanisms. That is, for each of the three matchings – $\mu^{\mathtt{MR}}$, $\mu^{\mathtt{DISC}}$, and $\mu^{\mathtt{JSA}}$ – we show how to construct an auxiliary instance such that the matching corresponds to the student-optimal stable matching of the auxiliary instance without affirmative action.

The reason for developing these auxiliary instances is three-fold. First, it allows us to prove many of the properties (e.g., weakly group strategy-proofness) of the joint seat allocation mechanism, since we can now apply results developed for the classical stable matching model. Second, it completely removes the cost of implementing a new mechanism for the DOE. That is, the DOE does not need to develop a new algorithm incorporating choice functions, and can use the same algorithm as in their current system. Lastly, these auxiliary instances elucidate a simple difference of the three mechanisms: they differ in how students' preferences over general and reserved seats at all schools are extracted from their original preferences over schools.

We start by describing the common components of these auxiliary instances, which are the set of schools, their quotas, and their priority orders over the students. Every school $c \in C$ is divided into two schools c' and c'', where c' represents the part with general seats and has quota $q_{c'}^{\mathtt{aux}} := q_c - q_c^R$, and c'' is the part with reserved seats and has quota $q_{c''}^{\mathtt{aux}} := q_c^R$. Let $C^{\mathtt{aux}} = \{c' : c \in C\} \cup \{c'' : c \in C\}$ be the new set of schools after the division, and for every $c \in C^{\mathtt{aux}}$, let $\omega(c)$ denote its corresponding school in the original instance. Then, graph $G^{\mathtt{aux}}$ has vertices and edges:

$$V(G^{\mathtt{aux}}) = C^{\mathtt{aux}} \cup S, \text{ and } E(G^{\mathtt{aux}}) = \{(s,c) : s \in S, c \in C^{\mathtt{aux}}, (s, \omega(c)) \in E\}.$$

The priority order over the students by school c' is the same as that of school c (i.e., $>_{c'}^{\text{aux}}=>_c$); and that by school c'' is defined as follows: for two students $s_1, s_2 \in S$,

$$
s_1 >_{c''}^{\text{aux}} s_2 \Leftrightarrow
\begin{cases}
s_1 \in S^m \text{ and } s_2 \in S^M; \text{ or} \\
s_1, s_2 \in S^m \text{ and } s_1 >_c s_2; \text{ or} \\
s_1, s_2 \in S^M \text{ and } s_1 >_c s_2.
\end{cases}
$$

The choice function $\mathcal{C}_c^{\text{aux}}$ of every school $c \in C^{\text{aux}}$ is q_c^{aux}-responsive and is simply induced from priority order $>_c^{\text{aux}}$. We state the choice functions here to be consistent with our approach in previous sections. However, they are not necessary to obtain the student-optimal stable matching as the classical deferred acceptance algorithm suffice.

The only component remaining is the preference lists of students, which depends on the specific affirmative action mechanism, and we describe those next. The proofs of their correctness are included in Appendix C.4.

Minority reserve. The original preference list $c_1 >_s c_2 >_s \cdots >_s c_k$ of student s is modified as:

$$
c_1'' >_s^{\text{MR-a}} c_1' >_s^{\text{MR-a}} c_2'' >_s^{\text{MR-a}} c_2' >_s^{\text{MR-a}} \cdots >_s^{\text{MR-a}} c_k'' >_s^{\text{MR-a}} c_k'.
$$

Although the relative ranking of the schools remains the same, students prefer reserved seats to general seats. Let $I^{\text{MR-a}} := (G^{\text{aux}}, >_S^{\text{MR-a}}, >_C^{\text{aux}}, \mathbf{q}^{\text{aux}})$ denote the auxiliary instance, and let $\mu^{\text{MR-a}} := \text{SDA}(I^{\text{MR-a}}, \mathcal{C}^{\text{aux}})$ denote the student-optimal stable matching of the auxiliary instance.

PROPOSITION 8 (Hafalir et al. (2013)). *For every student $s \in S$, $\mu^{\text{MR}}(s) = \omega(\mu^{\text{MR-a}}(s))$.*

Discovery program. The original preference list $c_1 >_s c_2 >_s \cdots >_s c_k$ of student s becomes:

$$
c_1' >_s^{\text{DISC-a}} c_2' >_s^{\text{DISC-a}} \cdots >_s^{\text{DISC-a}} c_k' >_s^{\text{DISC-a}} c_1'' >_s^{\text{DISC-a}} \cdots >_s^{\text{DISC-a}} c_k''.
$$

Students prefer general seats over reserved seats; and within each type of seats, the ranking of the schools is the same as that of the original instance. Similarly, we denote the auxiliary instance by $I^{\text{DISC-a}} := (G^{\text{aux}}, >_S^{\text{DISC-a}}, >_C^{\text{aux}}, \mathbf{q}^{\text{aux}})$, and let $\mu^{\text{DISC-a}} := \text{SDA}(I^{\text{DISC-a}}, \mathcal{C}^{\text{aux}})$ denote the student-optimal stable matching of the auxiliary instance.

PROPOSITION 9. *For every student $s \in S$, $\mu^{\text{DISC}}(s) = \omega(\mu^{\text{DISC-a}}(s))$.*

Joint seat allocation. The original preference list $c_1 >_s c_2 >_s \cdots >_s c_k$ of student s becomes:

$$
c_1' >_s^{\text{JSA-a}} c_1'' >_s^{\text{JSA-a}} c_2' >_s^{\text{JSA-a}} c_2'' >_s^{\text{JSA-a}} \cdots >_s^{\text{JSA-a}} c_k' >_s^{\text{JSA-a}} c_k''.
$$

Similar to minority reserve, the relative ranking of the schools remains the same as that of the original instance; but different from minority reserve, students prefer general seats to reserved seats. Again, we let $I^{\text{JSA-a}} := (G^{\text{aux}}, >_S^{\text{JSA-a}}, >_C^{\text{aux}}, \mathbf{q}^{\text{aux}})$ denote the auxiliary instance, and let $\mu^{\text{JSA-a}} := \text{SDA}(I^{\text{JSA-a}}, \mathcal{C}^{\text{aux}})$ denote the student-optimal stable matching of the auxiliary instance.

PROPOSITION 10. *For every student $s \in S$, $\mu^{\text{JSA}}(s) = \omega(\mu^{\text{JSA-a}}(s))$.*

5. Data on NYC Specialized High Schools

In this section, we analyze and compare the mechanisms on real-world datasets[12]. There is a total of 12 anonymized datasets, each for one of the 12 consecutive academic years from 2005-06 to 2016-17. Entries of each dataset include (1) students' IDs, (2) their scores for the Specialized High School Admissions Test (see Table 3 for a list of specialized high schools), (3) their (possibly, non-complete) preference lists of these eight schools, (4) their middle schools, (5) which school they are admitted to (which could be empty), and other information that are not relevant for our analysis.

B	Bronx High School of Science
T	Brooklyn Technical High School
R	Staten Island Technical High School
L	Brooklyn Latin
Q	Queens High School for the Sciences at York
M	High School of Mathematics, Science and Engineering at City College
S	Stuyvesant High School
A	High School of American Studies at Lehman College

Table 3   School code and school name of NYC specialized high schools.

Immediately from the dataset, we can extract the number of students applying for these specialized high schools and the capacities of each schools (i.e., the number of students admitted). On average, about $27,000$ students take the SHSAT exam every year, and among them, about $8,000$ (which is about 30%) are disadvantaged students. In terms of admission, about $5,100$ students receive an offer, out of whom about 820 (which is about 16%) are disadvantaged students.

To label each student as advantaged or disadvantaged, we follow the definition currently used by NYC DOE for the discovery program:

To be eligible for the Discovery program, a Specialized High Schools applicant must

1. *Be one or more of the following: a student from a low-income household, a student in temporary housing, or an English Language Learner who moved to NYC within the past four years; and*

2. *Have scored within a certain range below the cutoff score on the SHSAT; and*

3. *Attend a high-poverty school. A school is defined as high-poverty if it has an Economic Need Index (ENI) of at least 60%.*

The second condition is related to eligibility, and not specifically to whether a student is disadvantaged, so we do not incorporate that when labeling the students. For the first set of conditions, we use an accompanying dataset which contains students' demographic information. However, since

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[12] The dataset is under a non-disclosure agreement with NYC DOE.

the information given in the dataset are not exactly the same as those specified in the definition, we slightly modify the first condition: "be one or more of the following: (1) eligible for free or reduced price lunch or has been identified by the Human Resources Administration (HRA) as receiving certain types of public assistance; or (2) an English Language Learner". For the last condition, we obtain the ENIs of NYC middle schools from a school quality report of academic year 2017-2018, which can be downloaded from the NYC Open Data website[13].

To obtain schools' universal priority order $>_C$ over the students, we assign to every student a unique *lottery* number, denoted as ℓ_s, for tie-breaking. For any two students $s_1, s_2 \in S$, s_1 has a higher priority than s_2 (i.e., $s_1 >_C s_2$) only when s_1 has a higher score than s_2 or when they have the same score but $\ell_{s_1} < \ell_{s_2}$. This idea of using lottery numbers for tie breaking has been used in practice (see, e.g., Abdulkadiroğlu et al. (2009)).

Combining all components, the final dataset for analysis contains the following information for each student: unique identification number, test score, preference list, indicator for whether they are disadvantaged students, and lottery number.

First in Section 5.1, we analyze the outcome of the discovery program mechanism under the current guideline, and we provide some additional observations besides the theoretical results in Section 3.3. We then compare, in Section 5.2, the outcomes from all three mechanisms. For most of the experiments, we only include results of the latest academic year, since they are qualitatively similar for all academic years. Full results of all academic years can be found in Appendix D.

We also investigate and discuss the school-over-seat hypothesis by analyzing the patterns of students' preference lists, which can be found in Appendix E.
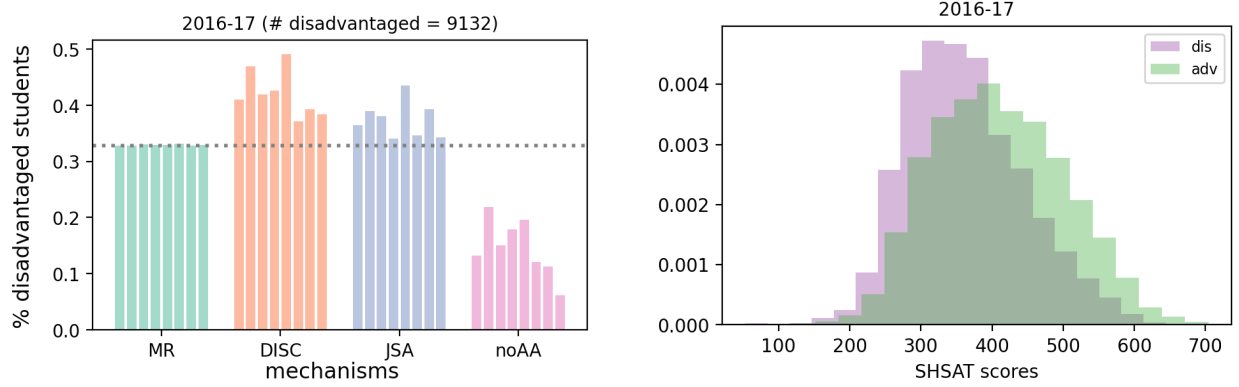
5.1. Results: the discovery program

We start by analyzing the performance of the discovery program mechanism, where the reservation quota of every school c is set to be $q_c^R := \lceil q_c \times 20\% \rceil$, since 20% is the number recommended in a proposal by NYCDOE (2019). We show two negative results of the discovery program mechanism, one of which has been discussed theoretically in Section 3.

Recall that the discovery program is the only mechanism that admits in-group blocking pairs (see the summary in Table 1). In Figure 1, we show the number of blocking pairs for disadvantaged students across all academic years. On average, there are about 950 blocking pairs for disadvantaged students every academic year involving about 650 disadvantaged students.

We also conducted a simple experiment to show that the discovery program is not strategy-proof. In this experiment, we first identify the top ranked disadvantaged student s who is not admitted

[13] https://data.cityofnewyork.us/Education/2017-2018-School-Quality-Reports-Elem-Middle-K-8/g6v2-wcvk

(a) Proportions of disadvantaged students admitted, with bars from left to right corresponding to schools: B, T, R, L, Q, M, S, A. The dotted line represents the proportion of disadvantaged students among all applicants.

(b) The distribution of the SHSAT scores of advantaged students (labeled "adv") and disadvantaged students (labeled "dis").

Figure 3 Affirmative action increases the number of disadvantaged students admitted.

to his most preferred school, and we then modify the preference list of s so that this most preferred school is the only school on the preference list (i.e., removing all other schools and considering them as unacceptable). We notice that with the modified preference list, disadvantaged student s is then able to go to the most preferred school. Hence, under the discovery program, students could lie about their preferences in order to go to more preferred schools.

5.2. Results: comparison of three mechanisms

For experiments in this section, we choose the reservation quotas so that they are consistent with the proportion of disadvantaged students in the market: $q_c^R = \lceil q_c \times \frac{|S^m|}{|S^M|} \rceil$, $\forall c \in C$. We choose these reservation quotas simply because they are a reasonable choice and are a smart reserve, and we would like to point out that one could slightly increase or decrease these numbers without affecting the findings in this section qualitatively.

Proportion of disadvantaged students admitted. In Figure 3a, we show that all mechanisms with affirmative action can increase the proportion of disadvantaged students admitted to these schools. More specifically, under joint seat allocation and the discovery program mechanism, the numbers of disadvantaged students admitted exceeds the reservation quotas. This is because disadvantaged students with high scores can take up general seats under these two mechanisms. On the other hand, for minority reserve, the numbers of disadvantaged students admitted match exactly the reservation quota. This is because after disadvantaged students take up the reserved seats, the remaining disadvantaged students cannot compete against advantaged students for the general

seats and are thus not admitted. The phenomenon is exactly the high competitiveness condition we discussed in Section 4.2 and is particularly true for our dataset since the number of students are much higher than the number of available seats, and disadvantaged students are performing systematically worse than advantaged students, as one can see in Figure 3b.

The figure seems to suggest that, for a fixed quota, the discovery program mechanism is better for disadvantaged students, as the number of disadvantaged students admitted to any school is the largest. However, this is not true when we examine the matching more closely.

Effects of affirmative actions to individual students. As opposed to Figure 3a which shows the effects of affirmative action mechanisms on disadvantaged students as a whole group, we show in Figure 4a these effects on individual levels. In particular, we examine the change in rank of the schools assigned to students under these mechanisms as compared to under the no affirmative action mechanism. For instance, if a student is matched to their third choice (i.e., rank of assigned school is 3) under the no affirmative action mechanism, but is matched to their first choice (i.e., rank of assigned school is 1) under minority reserve, then their change in rank of assigned school is -2 under minority reserve.

The main takeaway of Figure 4a is that when the reservation quotas are a smart reserve, the discovery program mechanism is the only one under which disadvantaged students can be worse off, as it is the only mechanism with markers on the positive axis. This is consistent with our discussion in Section 3 (see Table 1). We further investigate who are the disadvantaged students that are worse off under the discovery program, and we show the results in Figure 2. Interestingly, the disadvantaged students who are performing relatively well are the ones who are being admitted to schools they prefer less (dots on the upper left side of Figure 2). These are essentially the disadvantaged students who are assigned to general seats during the first stage of the discovery program mechanism. Because there are fewer seats during the first stage of the discovery program mechanism (as compared to the no affirmative action mechanism), the competition is fiercer and thus, these disadvantaged students got assigned to worse schools. Not only does this phenomenon imply that the discovery program mechanism is unfair to these well-performing disadvantaged students, but it also hints at a situation where students have the incentive to under-perform in the admission exams. This certainly is in sharp contrast to the purpose of education and should not be a consequence of any applicable mechanism.

Joint seat allocation dominates minority reserve. In Figure 4a, we see that for each negative change in rank of assigned schools, the markers of joint seat allocation are in general higher than those of minority reserve. It seems to suggest that matching $\mu^{\texttt{JSA}}$ dominates matching $\mu^{\texttt{MR}}$ for disadvantaged students. To understand if this is true, we directly compare these two matchings

(a) Change from `noAA` to an affirmative action mechanism, for disadvantaged students

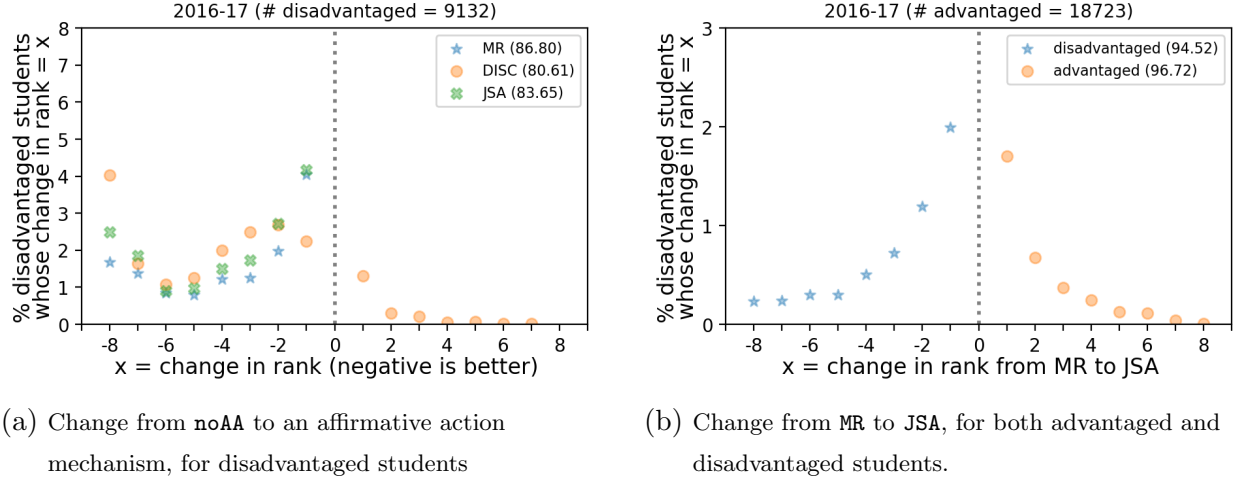(b) Change from `MR` to `JSA`, for both advantaged and disadvantaged students.

Figure 4   Percentage of (dis)advantaged students (w.r.t. the total number of (dis)advantaged students) whose change in rank of assigned schools is a certain value. The number in each legend label is for when $x = 0$.

and confirm the hypothesis (see Figure 4b). In fact, we observe the same dominance relation for all academic years. This prompts us to investigate the reason behind it, especially given that this dominance relation is not true in general as we discussed in Section 4. This dominance is a consequence of the data satisfying the high competitiveness hypothesis defined in Section 4.2 (see Figure 3a): the number of disadvantaged students admitted under minority reserve should not exceed the reservation quotas.

6. Conclusion and Discussion

In this paper, we study three quota-based affirmative action mechanisms, and compare their outcomes for disadvantaged students under the school-over-seat hypothesis. We show that although the discovery program is instrumental in providing opportunities for disadvantaged students, the current implementation suffers from some drawbacks both theoretically and empirically. In addition, we show that to improve the discovery program, although there is no clear winner between joint seat allocation and minority reserve in general settings, the former is better for the NYC specialized high school market.

One caveat of our results is that they are based on the school-over-seat hypothesis, for which current data do not offer a definitive validation. Our experiments on the polarization of the preference data (see Appendix E) and the fact that the length of the summer program (3 weeks) is minimal when compared to the length of a high-school cycle (4 years) seem to suggest that this

hypothesis is reasonable. However, other factors may come into play, such as the social stigma attached to being admitted via reserved seats[14].

This leads to two interesting directions for future work. As a first step, we believe it would be beneficial to explicitly collect students' expanded preference. Not only will these data confirm or invalidate the school-over-seat hypothesis, but they will also provide insights on the similarity or heterogeneity of the structure of students' expanded preference lists. In the case where the school-over-seat hypothesis fails, then a valuable next step would be to design a matching mechanism that account for individual students' expanded preference lists, while maintaining a number of desirable features such as strategy-proof and absence of in-group blocking pairs.

Acknowledgments

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[14] We are not aware of this stigma being present in NYC SHSs, but it is definitely present in other markets employing some form of affirmative action mechanisms (Aygun and Turhan 2020).

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Appendix A: Missing Definition for Choice Functions

DEFINITION 1 (SUBSTITUTABILITY). Choice function $\mathcal{C}_c$ is substitutable if for any set of students S_1, $s \in \mathcal{C}_c(S_1)$ implies that for all $S_2 \subseteq S_1$, $s \in \mathcal{C}_c(S_2 \cup \{s\})$.

DEFINITION 2 (CONSISTENCY). Choice function $\mathcal{C}_c$ is consistent if for any sets of students S_1 and S_2, $\mathcal{C}_c(S_1) \subseteq S_2 \subseteq S_1$ implies $\mathcal{C}_c(S_1) = \mathcal{C}_c(S_2)$.

DEFINITION 3 (q_c-ACCEPTANCE). Choice function $\mathcal{C}_c$ is q_c-acceptant if for any set of students S_1, $|\mathcal{C}_c(S_1)| = \min(q_c, |S_1|)$.

DEFINITION 4 (q_c-RESPONSIVE). Choice function $\mathcal{C}_c$ is q_c-responsive if there exists a priority order $>$ over the students such that for any set of students S_1, $\mathcal{C}_c(S_1) = \max(S_1, >, q_c)$. In such case, we say $\mathcal{C}_c$ is *induced* by priority order $>$ (and quota q_c).

Appendix B: Missing Examples

B.1. From Section 3.4

EXAMPLE 3. Consider the instance with students $S^M = \{s_1^M\}$, $S^m = \{s_1^m, s_2^m\}$ and schools $C = \{c_1, c_2, c_3\}$, each with a quota of 1. All schools have priority order $s_1^M > s_1^m > s_2^m$. Students' preference lists are given below:

s_1^M	s_1^m	s_2^m
c_1	c_3	c_1
c_3	c_1	c_2

Without affirmative action, the resulting matching is

$$\mu^{\texttt{noAA}} = \{(s_1^M, c_1), (s_2^m, c_2), (s_1^m, c_3)\}.$$

Consider the reservation quotas $q_{c_1}^R = 1$ and $q_{c_2}^R = q_{c_3}^R = 0$. Then,

$$\mu^{\texttt{MR}} = \mu^{\texttt{JSA}} = \{(s_1^m, c_1), (s_2^m, c_2), (s_1^M, c_3)\}.$$

Disadvantaged student s_2^m is indifferent between the two matchings, but disadvantaged student s_1^m strictly prefers $\mu^{\texttt{noAA}}$ to $\mu^{\texttt{JSA}}$. That is, $\mu^{\texttt{noAA}}$ Pareto dominates $\mu^{\texttt{JSA}}$ for disadvantaged students.   △

B.2. From Section 4.1

EXAMPLE 4. Consider the instance with students $S^M = \{s_1^M\}$, $S^m = \{s_1^m, s_2^m\}$ and schools $C = \{c_1, c_2\}$. Both schools have a quota of 1, and a reservation quota of 1. All students prefer school c_1 to c_2. Both schools have priority order $s_1^M > s_1^m > s_2^m$. Then,

$$\mu^{\texttt{noAA}} = \{s_1^M, c_1\}, \{s_1^m, c_2\}, \text{ and } \mu^{\texttt{MR}} = \mu^{\texttt{DISC}} = \mu^{\texttt{JSA}} = \{s_1^m, c_1\}, \{s_2^m c_2\}.$$

That is, the matching under any of the mechanisms with affirmative action Pareto dominates the matching obtained without affirmative action for disadvantaged students.   △

EXAMPLE 5. Consider the instance with students $S^M = \{s_1^M, s_2^M\}$, $S^m = \{s_1^m, s_2^m\}$ and schools $C = \{c_1, c_2\}$. Both schools have a quota of 2 and a reservation quota of 1. All students prefer school c_1 to c_2, and all schools have priority order $s_1^M > s_1^m > s_2^M > s_2^m$. Then,

$$\mu^{\texttt{noAA}} = \mu^{\texttt{MR}} = \mu^{\texttt{JSA}} = \{s_1^M, c_1\}, \{s_1^m, c_1\}, \{s_2^M, c_2\}, \{s_2^m, c_2\},$$

and

$$\mu^{\texttt{DISC}} = \{s_1^M, c_1\}, \{s_2^m, c_1\}, \{s_1^m, c_2\}, \{s_2^M, c_2\}.$$

Note that the reservation quotas is a smart reserve. Disadvantaged student s_2^m strictly prefers $\mu^{\texttt{DISC}}$ to the other matching, while s_1^m strictly prefers the other matching to $\mu^{\texttt{DISC}}$. △

EXAMPLE 6. Consider the instance with students $S^M = \{s_1^M, s_2^M, s_3^M\}$, $S^m = \{s_1^m, s_2^m, s_3^m, s_4^m\}$ and schools $C = \{c_1, c_2, c_3, c_4\}$. The quotas and reservation quotas of schools, and the preference lists of students are given below.

c	c_1	c_2	c_3	c_4
q_c	1	1	1	2
q_c^R	1	1	0	1

s_1^M	s_2^M	s_3^M	s_1^m	s_2^m	s_3^m	s_4^m
c_2	c_1	c_4	c_2	c_4	c_3	c_4
c_3		c_3	c_3		c_1	

All schools have priority order $s_1^M > s_1^m > s_2^M > s_2^m > s_3^M > s_3^m > s_4^m$. To see that the reservation quotas is a smart reserve, the matching under the no affirmative action mechanism is

$$\mu^{\texttt{noAA}} = \{s_1^m, c_1\}, \{s_1^M, c_2\}, \{s_2^M, c_3\}, \{s_2^m, c_4\}, \{s_3^M, c_4\}.$$

The matchings under minority reserve and joint seat allocation are:

$$\mu^{\texttt{MR}} = \{s_2^M, c_1\}, \{s_1^m, c_2\}, \{s_3^m, c_3\}, \{s_2^m, c_4\}, \{s_3^M, c_4\};$$
$$\mu^{\texttt{JSA}} = \{s_2^M, c_1\}, \{s_1^m, c_2\}, \{s_3^M, c_3\}, \{s_2^m, c_4\}, \{s_4^m, c_4\}.$$

Disadvantaged student s_1^m and s_2^m are indifferent between $\mu^{\texttt{MR}}$ and $\mu^{\texttt{JSA}}$, s_3^m strictly prefers $\mu^{\texttt{MR}}$ to $\mu^{\texttt{JSA}}$, but s_4^m strictly prefers $\mu^{\texttt{JSA}}$ to $\mu^{\texttt{MR}}$. △

Appendix C: Missing Proofs

C.1. From Section 3.2

Proof of Proposition 2. The substitutability property was shown in Hafalir et al. (2013), but we include the proof here for completeness. Let $S_1 \subseteq S$ be a subset of students, $s \in \mathcal{C}_c^{\texttt{MR}}(S_1)$ be a student selected by the choice function, and S_2 be a subset of students such that $s \in S_2 \subseteq S_1$. We want to show that $s \in \mathcal{C}_c^{\texttt{MR}}(S_2)$. Consider the following two cases. The first case is when $s \in S_1^R$. Here, it is immediate that $s \in S_2^R := \max(S_2 \cap S^m, >_c, q_c^R)$ since $S_2 \cap S^m \subseteq S_1 \cap S^m$ and thus, $s \in \mathcal{C}_c^{\texttt{MR}}(S_2)$.

The other case is when $s \in \mathcal{C}_c^{\texttt{MR}}(S_1) \setminus S_1^R$. Our argument for the first case implies that $S_1^R \cap S_2 \subseteq S_2^R$ and thus, we have $S_2 \setminus S_2^R \subseteq S_2 \setminus S_1^R \subseteq S_1 \setminus S_1^R$. Hence, we also have $s \in \mathcal{C}_c^{\texttt{MR}}(S_2)$.

Next, for consistency, let S_2 be a subset of students with $\mathcal{C}_c^{\texttt{MR}}(S_1) \subseteq S_2 \subseteq S_1$, and we want to show that $\mathcal{C}_c^{\texttt{MR}}(S_1) = \mathcal{C}_c^{\texttt{MR}}(S_2)$. By the definition of the choice function, it is clear that $S_1^R = S_2^R$ since $S_1^R \subseteq S_2$. With the same reasoning, we additionally have $\max(S_1 \setminus S_1^R, >_c, q_c - |S_1^R|)) = \max(S_2 \setminus S_1^R, >_c, q_c - |S_1^R|)) = \max(S_2 \setminus S_2^R, >_c, q_c - |S_2^R|))$. Therefore, the claim follows.

Lastly, for q_c-acceptance, we first have that $|\mathcal{C}_c^{\texttt{MR}}(S_1)| \leq |S_1^R| + q_c - |S_1^R| = q_c$, where the inequality follows directly from the definition. It remains to show that when $|S_1| < q_c$, we have $\mathcal{C}_c^{\texttt{MR}}(S_1) = S_1$. This is immediate from the definition of the choice function. $\square$

Proof of Propsoition 4. Assume by contradiction that (s, c) is an in-group blocking pair for $\mu^{\texttt{MR}}$. Let s' be the student in the same group as s such that $s' \in \mu^{\texttt{MR}}(c)$ and $s >_c s'$. Then, by definition of $\mathcal{C}_c^{\texttt{MR}}$, we have $s \in \mathcal{C}_c^{\texttt{MR}}(\mu^{\texttt{MR}}(c) \cup \{s\})$, which means (s, c) is a blocking pair for $\mu^{\texttt{MR}}$. However, this contradicts stability of $\mu^{\texttt{MR}}$. $\square$

C.2. From Section 3.4

Proof of Proposition 5. The proof steps are similar to that of Proposition 2 for minority reserve. Let $S_1 \subseteq S$ be a subset of students. First, for substitutability, let $s \in \mathcal{C}_c^{\texttt{JSA}}(S_1)$ and let S_2 be a subset of students such that $s \in S_2 \subseteq S_1$. We want to show that $s \in \mathcal{C}_c^{\texttt{JSA}}(S_2)$ and we consider the following three cases. The first case is when $s \in S_1^G$. In this case, it is immediate that $s \in S_2^G := \max(S_2, >_c, q_c^G)$ since $S_2 \subseteq S_1$. This first case in particular implies that $S_1^G \cap S_2 \subseteq S_2^G$ and thus, $S_2 \setminus S_2^G \subseteq S_2 \setminus S_1^G \subseteq S_1 \setminus S_1^G$. Hence, in the second case where $s \in S_1^R$, we similarly have $s \in S_2^R := \max(S_2 \cap S^m \setminus S_2^G, >_c, q_c^R)$. Note that this argument for the second case also implies that $S_2 \setminus (S_2^G \cup S_2^R) \subseteq S_1 \setminus (S_1^G \cup S_1^R)$. Hence, for the last case where $s \in \max(S_1 \setminus (S_1^G \cup S_1^R), >_c, q_c - |S_1^G \cup S_1^R|)$, we also have $s \in \max(S_2 \setminus (S_2^G \cup S_2^R), >_c, q_c - |S_2^G \cup S_2^R|)$. Therefore, in all these three cases, we have $s \in \mathcal{C}_c^{\texttt{JSA}}(S_2)$ and thus $\mathcal{C}_c^{\texttt{JSA}}$ is substitutable.

Next, for consistency, let S_2 be a subset of students with $\mathcal{C}_c^{\texttt{JSA}}(S_1) \subseteq S_2 \subseteq S_1$, and we want to show that $\mathcal{C}_c^{\texttt{JSA}}(S_1) = \mathcal{C}_c^{\texttt{JSA}}(S_2)$. By the definition of the choice function, it is clear that $S_1^G = S_2^G$ since $S_1^G \subseteq S_2$. Moreover, we have $S_1^R = S_2^R$ since $S_1^R \subseteq S_2 \cap S^m \setminus S_2^G$. With the same reasoning, we additionally have that $\max(S_1 \setminus (S_1^G \cup S_1^R), >_c, q_c - |S_1^G \cup S_1^R|) = \max(S_2 \setminus (S_2^G \cup S_2^R), >_c, q_c - |S_2^G \cup S_2^R|)$. Therefore, the choice function is consistent.

Lastly, for q_c-acceptant, we first have that $|\mathcal{C}_c^{\texttt{JSA}}(S_1)| \leq |S_1^G| + |S_1^R| + q_c - |S_1^G| - |S_1^R| = q_c$, where the inequality follows directly from the definition. It remains to show that when $|S_1| < q_c$, we have $\mathcal{C}_c^{\texttt{JSA}}(S_1) = S_1$. This is immediate from the definition of the choice function. $\square$

Proof of Proposition 6. Assume by contradiction that there exists a group of students $S_1 \subseteq S$ who can jointly misreport their preference lists so that every one in S_1 is strictly better off. Now

consider the auxiliary instance introduced in Section 4.3, where the relative ranking of schools by each student remains the same as that of the original instance. As a result, this strategic behavior by S_1 can be translated to a strategic behavior in the auxiliary instance due to Proposition 10. That is, $S_1 \subseteq S$ can accordingly misreport their preferences lists in the auxiliary instance so that every one in S_1 is better off. However, this contradicts Theorem 1, which states that strategic behaviors are not possible in the auxiliary instance. This concludes the proof. □

Proof of Theorem 4. Assume by contradiction that there is reservation quotas $\mathbf{q}^R$ such that $\mu^{\texttt{noAA}}(s) >_s \mu^{\texttt{JSA}}(s)$ for every disadvantaged student $s \in S^m$. Then, consider an alternative instance where every disadvantaged student s misreports his or her preference list where $\mu^{\texttt{noAA}}(s)$ is the only acceptable school. Let $\widetilde{G}$ and $\widetilde{>}_S$ be the resulting graph and preference lists of the students. In the following, we consider the alternative instance $\widetilde{I} = (\widetilde{G}, \widetilde{>}_S, >_C, \mathbf{q})$ and we claim that $\mu^{\texttt{noAA}}$ is stable in instance $\widetilde{I}$ under choice functions $\mathcal{C}^{\texttt{JSA}}$. Assume by contradiction that $\mu^{\texttt{noAA}}$ admits a blocking pair (s, c). Since all disadvantaged students are matched to their first choice, it must be that $s \in S^M$. Then, $s \in \mathcal{C}_c^{\texttt{JSA}}(\mu^{\texttt{noAA}}(c) \cup \{s\})$ implies that there is a student $s' \in \mu^{\texttt{noAA}}(c)$ such that $s >_c s'$. However, this means $s \in \mathcal{C}^{\texttt{noAA}}(\mu^{\texttt{noAA}}(c) \cup \{s\})$, which contradicts stability of $\mu^{\texttt{noAA}}$ in the original instance I under choice functions $\mathcal{C}^{\texttt{noAA}}$. Hence, $\mu^{\texttt{noAA}}$ is stable in instance $\widetilde{I}$ with choice functions $\mathcal{C}^{\texttt{JSA}}$. Since $\texttt{SDA}(\widetilde{I}, \mathcal{C}^{\texttt{JSA}})$ is the student-optimal stable matching, it dominates $\mu^{\texttt{noAA}}$ and thus, every disadvantaged student is strictly better off under $\texttt{SDA}(\widetilde{I}, \mathcal{C}^{\texttt{JSA}})$ as compared to $\mu^{\texttt{JSA}}$. However, this contradicts Proposition 6 which states that the joint seat allocation mechanism is weakly group strategy-proof. □

Proof of Theorem 5. Assume by contradiction that there exists disadvantaged students s with $\mu^{\texttt{noAA}}(s) >_s \mu^{\texttt{JSA}}(s)$. Let s_1 be the *first* disadvantaged student that is rejected by $c_1 := \mu^{\texttt{noAA}}(s_1)$ during the deferred acceptance algorithm on instance I with choice functions $\mathcal{C}^{\texttt{JSA}}$. Assume this rejection happens at round k. Let $S_k^{\texttt{JSA}}$ denote the set of students who apply to school c_1 during round k. In addition, let $S^{\texttt{noAA}}$ denote the set of students who have ever applied to c_1 throughout the deferred acceptance on instance I with choice functions $\mathcal{C}^{\texttt{noAA}}$. It has been shown in Roth (1984) that $\mathcal{C}_{c_1}^{\texttt{noAA}}(S^{\texttt{noAA}}) = \mu^{\texttt{noAA}}(c_1)$. Thus, $s_1 \in \max(S^{\texttt{noAA}} \cap S^m, >_{c_1}, q_{c_1}^R)$ by definition of choice function $\mathcal{C}_{c_1}^{\texttt{noAA}}$ and the assumption that the reservation quotas are a smart reserve (i.e., $q_{c_1}^R \geq |\mu^{\texttt{noAA}}(c_1)|$). Moreover, by our choice of s_1, we have $S_k^{\texttt{JSA}} \cap S^m \subseteq S^{\texttt{noAA}} \cap S^m$. Therefore, $s_1 \in \max(S_k^{\texttt{JSA}} \cap S^m, >_{c_1}, q_{c_1}^R)$, which then implies $s_1 \in \mathcal{C}_{c_1}^{\texttt{JSA}}(S_k^{\texttt{JSA}})$ by definition of choice function $\mathcal{C}_{c_1}^{\texttt{JSA}}$. However, this contradicts our assumption that s_1 is rejected by c_1 at round k, concluding the proof. □

Proof of Proposition 7. Assume by contradiction that (s, c) is an in-group blocking pair. Let s' be the student in the same group as s such that $s' \in \mu^{\texttt{JSA}}(c)$ and $s >_c s'$. Then, by definition of $\mathcal{C}_c^{\texttt{JSA}}$, we have $s \in \mathcal{C}_c^{\texttt{JSA}}(\mu^{\texttt{JSA}}(c) \cup \{s\})$, which means (s, c) is a blocking pair of $\mu^{\texttt{JSA}}$. However, this contradicts stability of $\mu^{\texttt{JSA}}$. □

C.3. From Section 4.2, with a discussion

Proof of Theorem 6. Assume by contradiction there exists disadvantaged students s such that $\mu^{\mathtt{MR}}(s) >_s \mu^{\mathtt{JSA}}(s)$. Consider the execution of the deferred acceptance algorithm with choice functions $\mathcal{C}^{\mathtt{JSA}}$, and let s_1 be the first disadvantaged student who is rejected by $\mu^{\mathtt{MR}}(s_1) \coloneqq c_1$. Assume this rejection happens at round k of the deferred acceptance algorithm. Let $S_k^{\mathtt{JSA}}$ denote the set of students who apply to school c_1 during round k. In addition, let $S^{\mathtt{MR}}$ denote the set of students who have ever applied to school c_1 during the execution of the deferred acceptance algorithm with choice functions $\mathcal{C}^{\mathtt{MR}}$. It has been shown in Roth (1984) that $\mathcal{C}_{c_1}^{\mathtt{MR}}(S^{\mathtt{MR}}) = \mu^{\mathtt{MR}}(c_1)$, which then implies that $s_1 \in \max(S^{\mathtt{MR}} \cap S^m, >_{c_1}, q_{c_1}^R)$ by definition of choice function $\mathcal{C}_{c_1}^{\mathtt{MR}}$ and our assumption that $|\mu^{\mathtt{MR}}(c_1)| \leq q_{c_1}^R$. Moreover, our choice of student s_1 implies that $S_k^{\mathtt{JSA}} \cap S^m \subseteq S^{\mathtt{MR}} \cap S^m$ and thus, we also have $s_1 \in \max(S_k^{\mathtt{JSA}} \cap S^m, >_{c_1}, q_{c_1}^R)$. Therefore, $s_1 \in \mathcal{C}_{c_1}^{\mathtt{JSA}}(S_k^{\mathtt{JSA}})$ by definition of choice function $\mathcal{C}_{c_1}^{\mathtt{JSA}}$. However, this contradicts our assumption that s_1 is rejected by c_1 at round k, concluding the proof. $\square$

Proof of Theorem 7. Recall that, under $\mathtt{MR}$, a student applies to her favorite school's reserved seats, and, if rejected, to the same school's non-reserved seat (see Section 4). We want to estimate the ranking, among disadvantaged students, of the *bottleneck* student – that is, the first disadvantaged student that is not admitted through a reserved seat at her most preferred school (hence, the student may either be admitted to her most preferred school via a general seat, or be admitted to another school, or not be admitted to any school).

We reformulate this problem in the classical balls in bins setting: given n bins and a series of balls, each inserted in exactly one bin chosen uniformly at random, which is the first ball k that is inserted in a bin with already q^R balls? Classical bounds (see, e.g., Raab and Steger (1998)) imply that, in the $q^R > n \log n$ regimen, $k \geq (1 - \epsilon) q^R n$ with probability $1 - o(1)$ for any $\epsilon \in (0, 1)$ – in particular, for the ϵ from the hypothesis of the theorem. Interpreting schools as bins, disadvantaged students as balls, and assigning students to their most preferred schools as inserting balls to bins, we obtain that, with probability $1 - o(1)$, the bottleneck student is ranked at least $(1 - \epsilon) q^R n$ among disadvantaged students.

The market is highly competitive if and only if any disadvantaged student ranked at par or worse than the bottleneck student does not get a general seat in any school. For this to happen, the bottleneck student must be ranked worse than an advantaged student that we call *lucky applicant*. This is the worst-ranked advantaged student that would get a non-reserved seat in the market obtained from the original market with the number of seats being $q - q^R$, no reservation quota, and no disadvantaged student (call such a market *restricted*). So we want to compute the ranking, among advantaged students, of the lucky applicant. We can use again the balls and bins analogy

from above. Denote by $b(q - q^R, n)$ the random variable denoting the smallest p such that, when ball p is extracted, all bins already have at least $(q - q^R)$ balls inserted. From Erdős and Rényi (1961), we know that for any real x, we have

$$\lim_{n \to \infty} \mathbb{P}(b(q - q^R, n) - 1 < n \log n + n(q - q^R - 1) \log \log n + nx) = e^{-\frac{e^{-x}}{(q - q^R - 1)!}}.$$

Taking $x = \log \log \log n$, we have

$$
\begin{aligned}
\lim_{n \to \infty} \mathbb{P}(b(q - q^R, n) - 1 < n \log n + n(q - q^R - 1) \log \log n + n \log \log \log n) &= \lim_{n \to \infty} e^{-\frac{e^{-\log \log \log n}}{(q - q^R - 1)!}} \\
&\geq \lim_{n \to \infty} e^{-e^{-\log \log \log n}} \\
&= 1.
\end{aligned}
$$

Hence, with probability $1 - o(1)$, each school is ranked first at least $(q - q^R - 1)$ times when we look at the preference lists of the best $n \log n + (q - q^R)n \log \log n$ advantaged students. Thus, with high probability, all the advantaged students that are admitted to a seat in the restricted market – in particular, the lucky applicant – are contained in the $(n \log n + (q - q^R)n \log \log n)$-best ranked advantaged students. It suffices therefore that the worst of them is ranked above the bottleneck student – as it is required by the hypothesis – to conclude that the market is highly competitive.

□

Let us discuss the hypothesis from Theorem 7. We restricted to markets where schools share a unique preference list of students. This condition applies, for instance, to the way universities rank incoming student across China and in Indian IITs, as well as in NYC SHSs. The condition on the the equal probability of each school appearing first in preference lists apply, for instance, in classical random markets, such as Knuth et al. (1990), Pittel (1989, 1992). $q^R > n \log n$ applies when there are few schools compared to the number of seats, while the condition on the relative rankings of students applies when disadvantaged students perform systematically worse than advantaged students. For a comparison, in the data from NYC DOE, we have that the average reservation quota is $q^r = 208 > n = 8$, the average number of seats at each school is $q = 635$, $n + n(q - q^R) = 3424$, and $q^R n = 1664$. Omitting from the comparison the terms logarithmic and sublogarithmic in n ($n = 8$, hence they would only help the hypothesis of Theorem 7 to be satisfied), we see that the 1664-th ranked disadvantaged student performs at par with the 6848-th advantaged student, hence well within the hypothesis of the theorem.

C.4. From Section 4.3

Proof of Proposition 9. To prove the proposition, instead of carrying out the deferred acceptance algorithm as we introduced in Section 2 based on Roth (1984) for choice function models, we consider an equivalent execution of the algorithm when choice functions $\mathcal{C}$ are responsive. This

algorithm was introduced by McVitie and Wilson (1971) and it similarly runs in rounds. The algorithm starts with all students unmatched. In every round, one student s who is not (temporarily) matched applies to his or her most preferred school c that has not yet rejected him or her. Let S_c denote the set of students c has temporarily accepted at the end of the previous round. School c temporarily accepts $\mathcal{C}_c(S_c \cup \{s\})$ and rejects the rest. Note that during the algorithm, at every round, the student s can be arbitrarily selected. Hence, we now consider a particular execution of the algorithm on the auxiliary instance (i.e., the order in which students are selected). The execution has three stages, and they match exactly to the three stages of the discovery program mechanism. In the first stage, the algorithm can only select students who would apply to schools of type c'. Since after this stage, students will only apply to schools of type c'', the students who are temporarily matched in the first stage would not be rejected in later stages. That is, the temporary assignment at the end of the first stage becomes permanent, and it is matching $\mu_1^{\texttt{DISC}}$. For the second stage, the algorithm can only select disadvantaged students. Since schools of type c'' prefers disadvantaged students to advantaged students, the temporary assignment at the end of the second stage is also permanent and it corresponds to $\mu_2^{\texttt{DISC}}$. In the last stage, the algorithm continues without restriction until it terminates. Since there are only advantaged students applying to schools of type c'' at this final stage, the matching finalized at this stage is $\mu_3^{\texttt{DISC}}$. $\square$

Proof of Proposition 10. We first show that matchings in the original instance $I_1 := (G, >, \mathbf{q})$ and matchings in the auxiliary instance $I_2 := (G^{\texttt{aux}}, >_S^{\texttt{JSA-a}}, >_C^{\texttt{aux}}, \mathbf{q})$ can be transformed from each other. One direction is straightforward. Given a matching μ_2 in instance I_2, its corresponding matching μ_1 in instance I_1 has $\mu_1(s) = \omega(\mu_2(s))$ for all students $s \in S$. For the other direction, let μ_1 be a matching in instance I_1, we can construct its corresponding matching μ_2 in instance I_2 as follows. For every school c, $\mu_2(c') = \max(\mu_1(c), >_c, q_c^G)$ and $\mu_2(c'') = \mu_1(c) \setminus \mu_2(c')$. Let ψ denote the above mapping from matchings in I_2 to matchings in I_1, and let ψ^{-1} denote the above mapping for the reverse direction. By construction, a matching μ of I_1 is stable in I_1 if and only if $\psi^{-1}(\mu)$ is stable in I_2. Therefore, the student-optimal stable matching in I_1 can be obtained from the student-optimal stable matching in I_2 via mapping ψ^{-1}, and the claim follows. $\square$

Appendix D: Additional Figures for all Academic Years
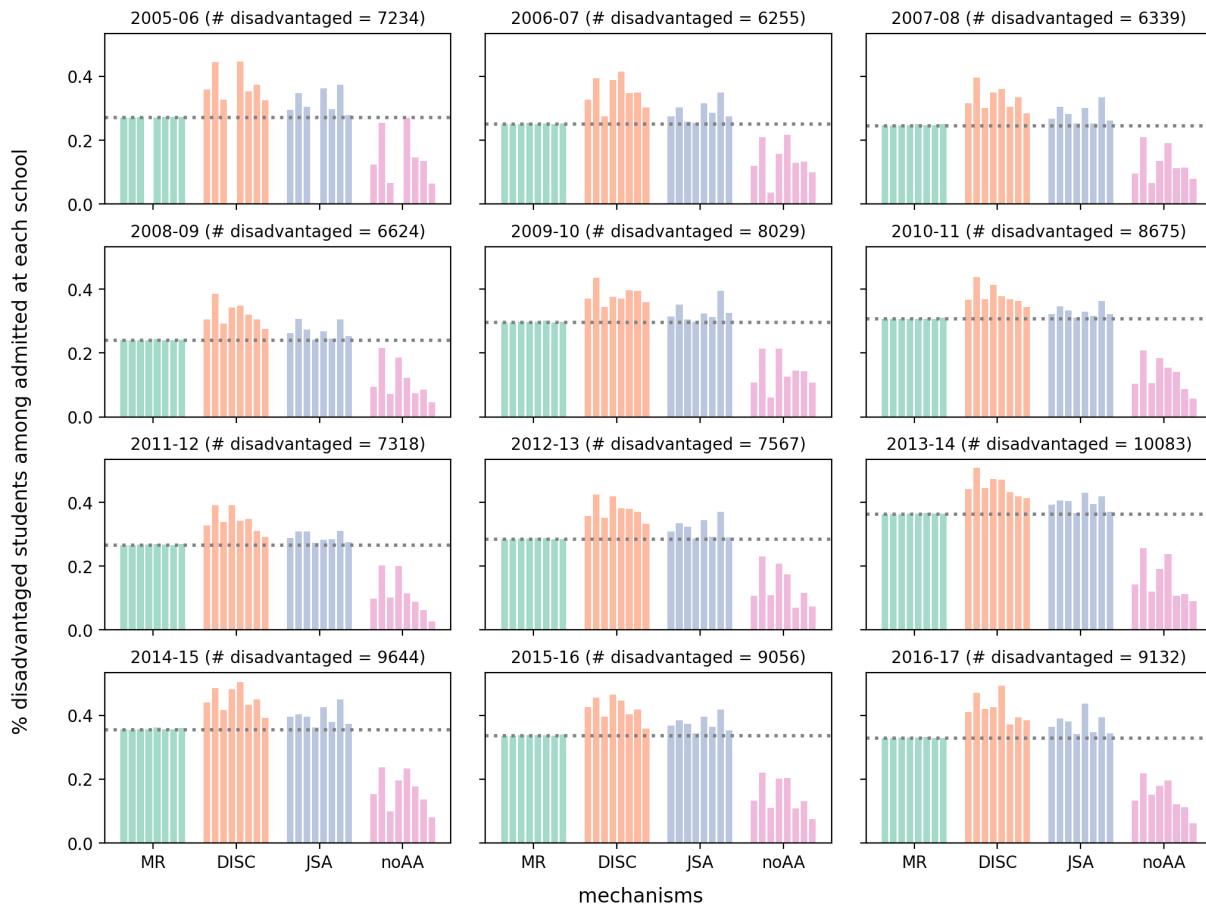
Figure 5   All academic years of Figure 3a.
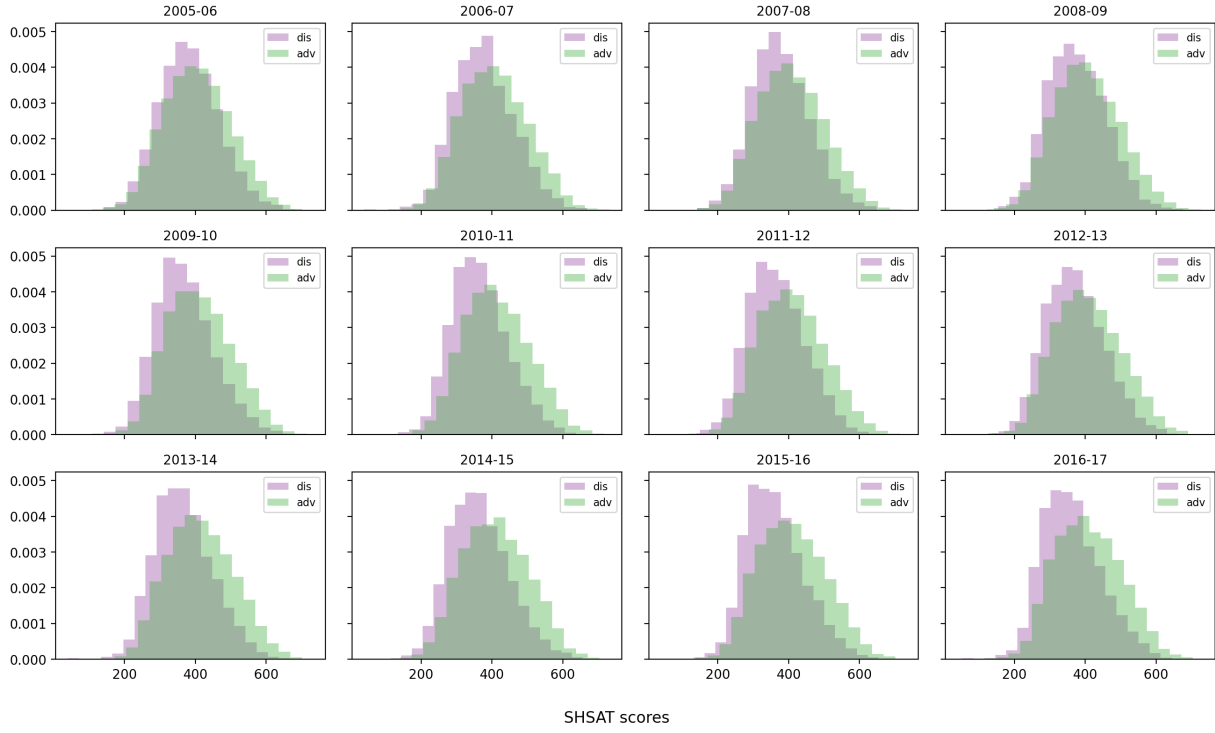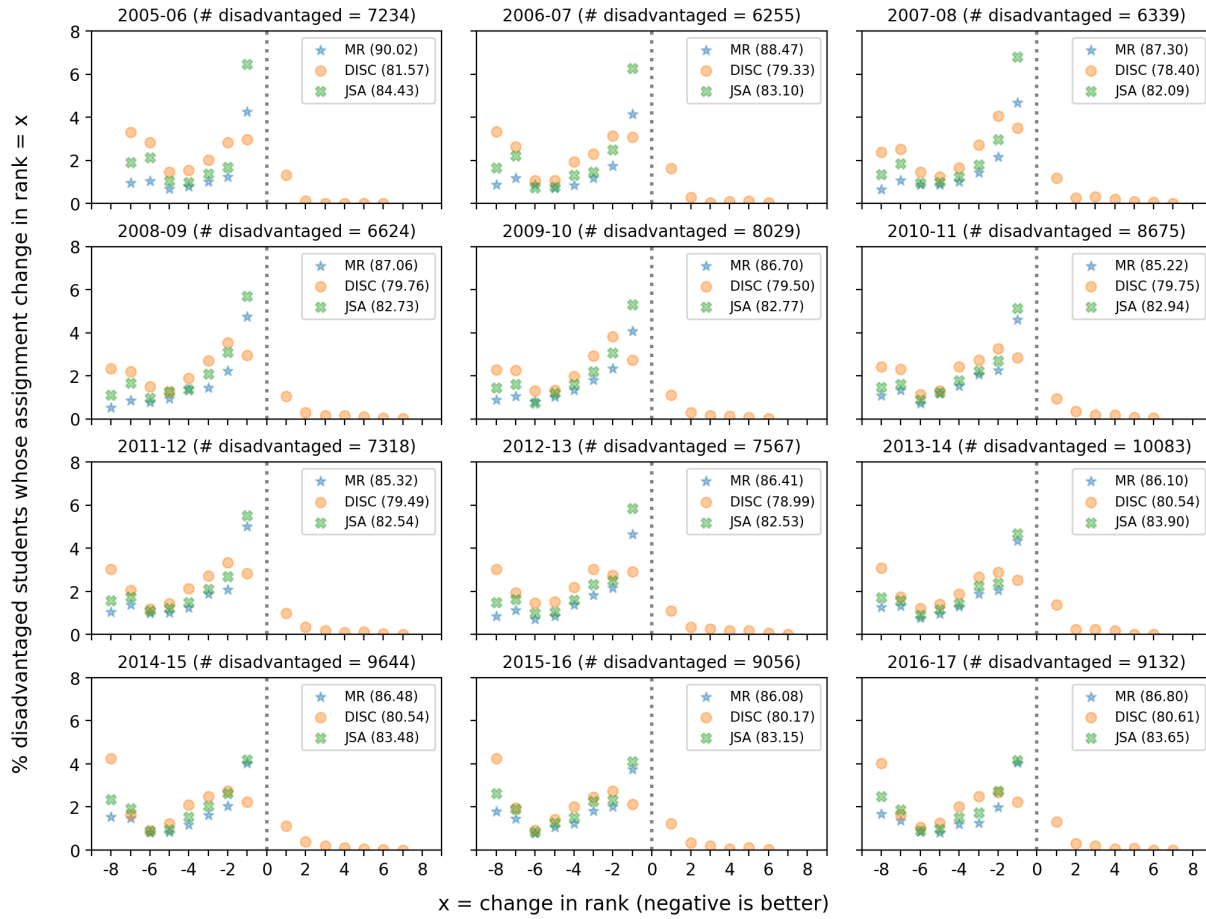
Figure 6 All academic years of Figure 3b.
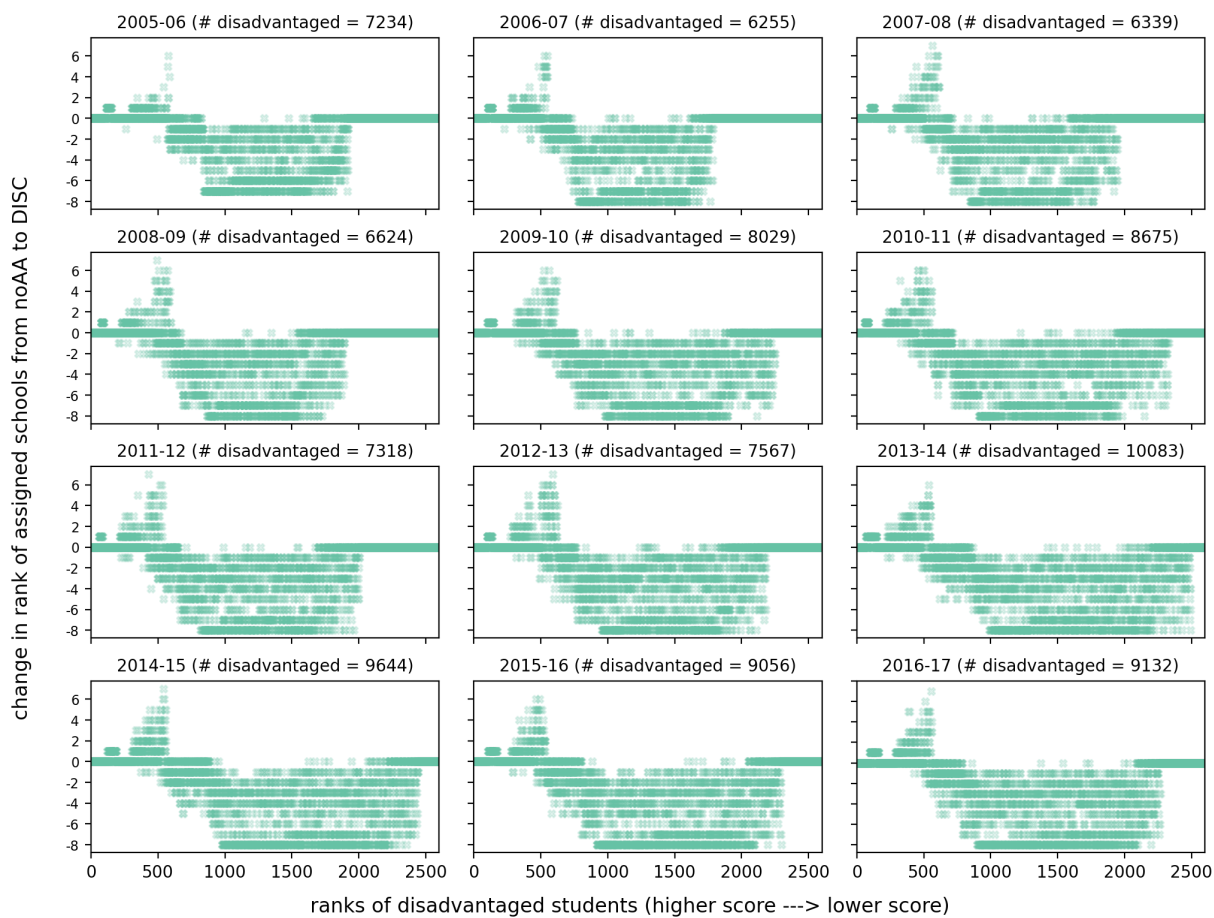
Figure 7 All academic years of Figure 4a.
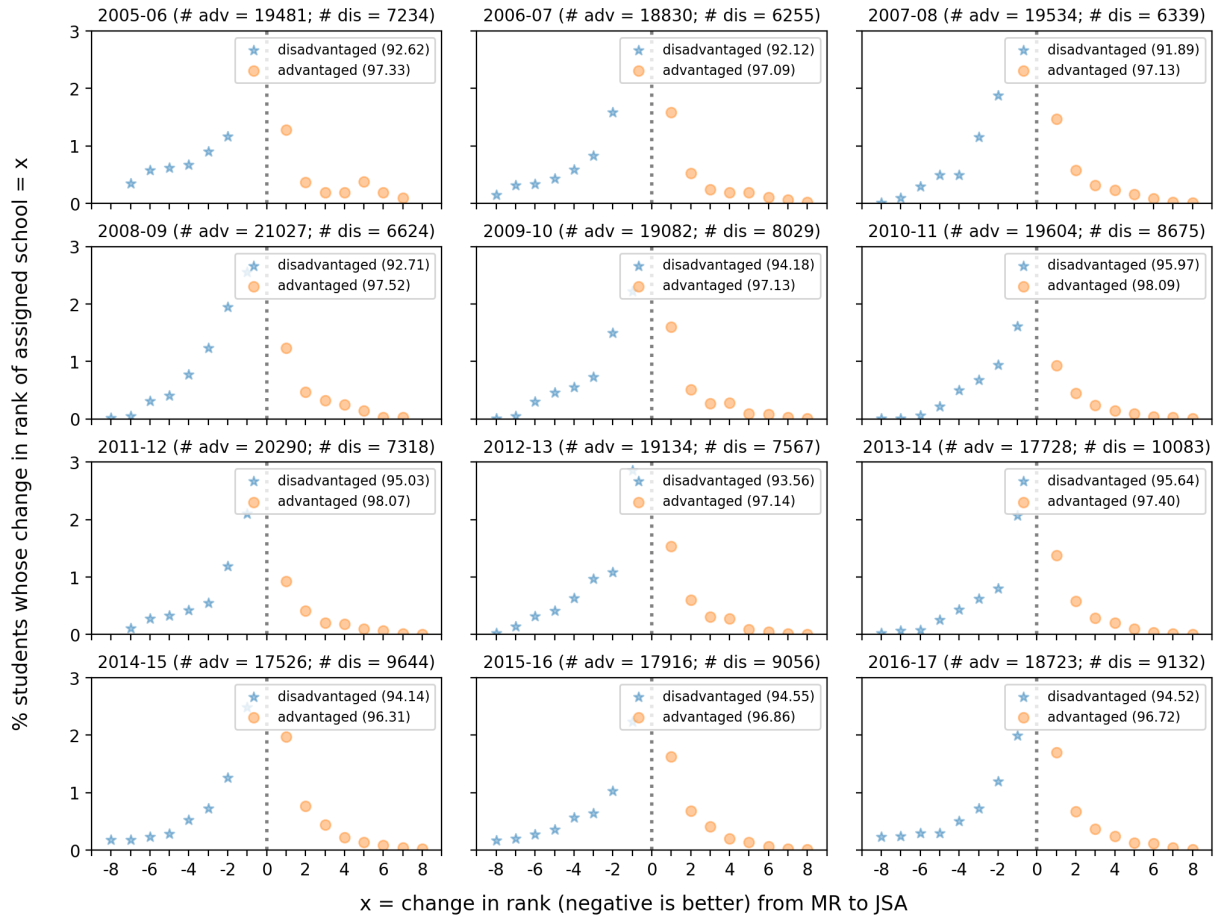
Figure 8 All academic years of Figure 2.

Figure 9 All academic years of Figure 4b.

all students   students in district 31

	S	T	B	Q	R	M	L	A
S	0.0	-0.03	0.24	0.45	0.56	0.32	0.46	0.41
T	0.03	0.0	0.18	0.52	0.65	0.39	0.66	0.47
B	-0.24	-0.18	0.0	0.25	0.39	0.13	0.23	0.28
Q	-0.45	-0.52	-0.25	0.0	0.28	-0.22	-0.01	-0.02
R	-0.56	-0.65	-0.39	-0.28	0.0	-0.41	-0.25	-0.27
M	-0.32	-0.39	-0.13	0.22	0.41	0.0	0.19	0.23
L	-0.46	-0.66	-0.23	0.01	0.25	-0.19	0.0	-0.02
A	-0.41	-0.47	-0.28	0.02	0.27	-0.23	0.02	0.0

	S	T	B	Q	R	M	L	A
S	0.0	0.08	0.48	0.49	-0.87	0.43	0.46	0.48
T	-0.08	0.0	0.48	0.49	-0.94	0.42	0.49	0.47
B	-0.48	-0.48	0.0	0.02	-0.99	-0.13	-0.02	0.02
Q	-0.49	-0.49	-0.02	0.0	-0.99	-0.15	-0.06	-0.01
R	0.87	0.94	0.99	0.99	0.0	0.98	0.98	0.99
M	-0.43	-0.42	0.13	0.15	-0.98	0.0	0.08	0.15
L	-0.46	-0.49	0.02	0.06	-0.98	-0.08	0.0	0.04
A	-0.48	-0.47	-0.02	0.01	-0.99	-0.15	-0.04	0.0

Figure 10   Each cell in this table represents the extent to which students prefer the row school to the column school. Specifically, the number is calculated as the percentage of students in each district who prefer the row school to the column school minus the percentage of students who prefer the column school to the row school. The cells are color-formatted with numbers in $[-1, 1]$ mapped to a spectrum from red to green.

Appendix E: Discussion on the school-over-seat hypothesis

In this section, we delve into some empirical observations of students' preference lists and we do so for two reasons. The first one is to investigate the school-over-seat hypothesis. Since students are not asked to report their preferences over different types of seats, we can only make some inferences based on the pattern of the preferences submitted by students. For the second reason, recall that in Section 4.3, we show how different mechanisms expand differently students' original preferences over schools to their preferences over reserved and general seats. Hence, our observations aim to shed some light on the validity of these expansions. For the following discussion, we forgo the assumption that participation in the summer enrichment program does not affect students' preference for schools.

The second table in Figure 10 indicates that geographic proximity could lead to a strong preference for some schools. We observe that students in district 31 strongly prefer Staten Island Tech (S) to any other schools. This is because district 31 is the only school district on Staten Island, and Staten Island Tech is the only specialized high school on Staten Island. Hence, for students residing in Staten Island, since transportation to other boroughs are extremely limited and lengthy, it is reasonable to assume the school-over-seat hypothesis when comparing Staten Island Tech to any other specialized high school. From the same type of tables for other school districts which we include in Appendix F, we observe similar patterns: students in district 10 strongly prefers Bronx Science (B) and students in district 29 strongly prefers Queens High School for the Sciences at York (Q). The difference in preferences towards Stuyvesant and Brooklyn Tech seems to be more nuanced. The complete map of school districts in New York City can and the map of specialized high schools can be found in Appendix G and H.

Lastly, we would like to point out some concerns that are not directly observable from our data. Aygun and Turhan (2020) noted that for admissions to Indian Institutes of Technology (IIT), there is often social stigma associated with reserved seats and thus, many students prefer to not be admitted via reserved seats. We also note that NYC DOE defines disadvantaged students based on their social economic status instead of a caste system as in the case of IIT admission. Hence, the severity of the social stigma associated with reserved seats might differ between these two markets.

In sum, we believe more study is needed to understand students' preference structure over reserved and general seats for the NYC SHS market. Moreover, as a future direction, it would be interesting to design and study mechanisms which incorporate students' preferences over general and reserved seats at all schools, possibly in orders that are not consistent with those interpreted by the mechanisms.
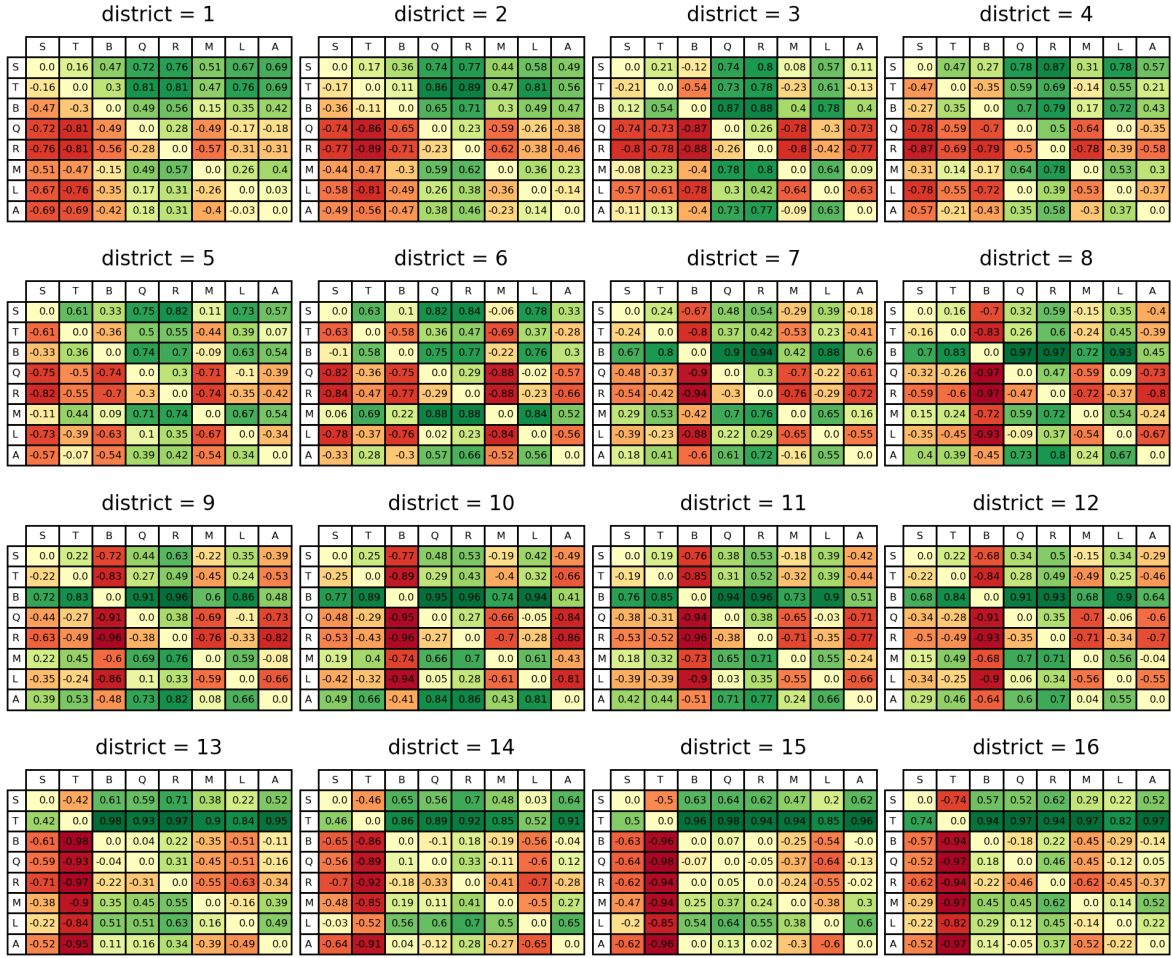
Appendix F: Additional Figures for Section E

district = 1, district = 2, district = 3, district = 4, district = 5, district = 6, district = 7, district = 8, district = 9, district = 10, district = 11, district = 12, district = 13, district = 14, district = 15, district = 16

Figure 11   These tables are the same as those in Figure 10, but for districts 1 – 16.
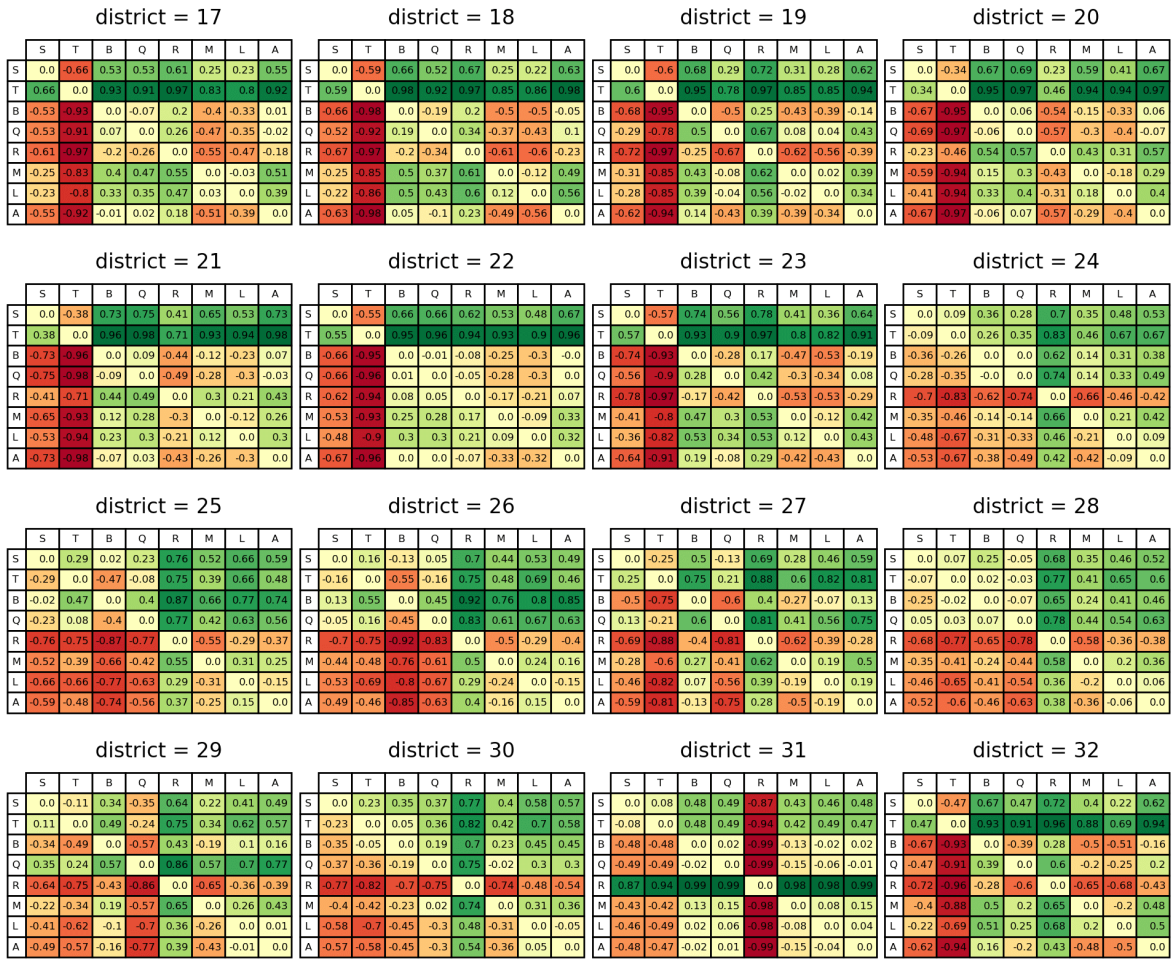
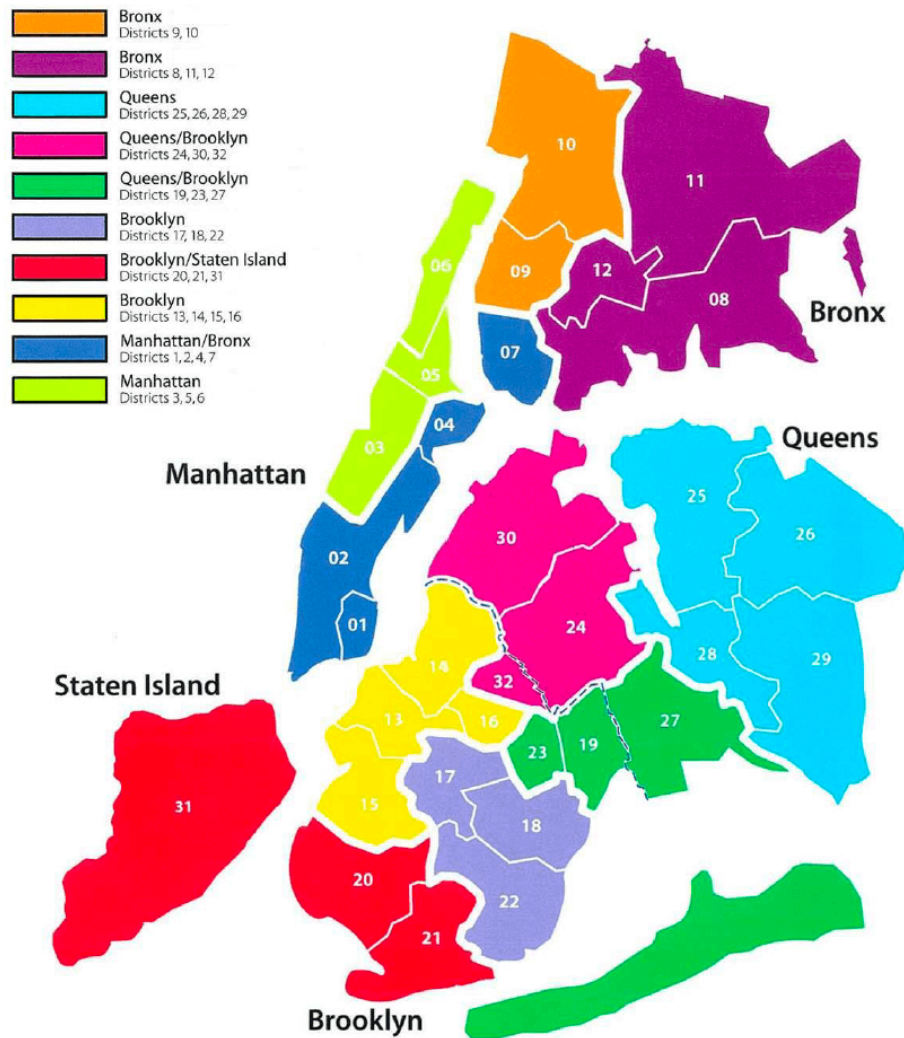**Figure 12** These tables are the same as those in Figure 10, but for districts 17 – 32.

Appendix G: Map of NYC School Districts

Figure 13   Map of school districts in New York City, compiled by NYC DOE and available online at
https://video.eschoolsolutions.com/udocs/DistrictMap.pdf

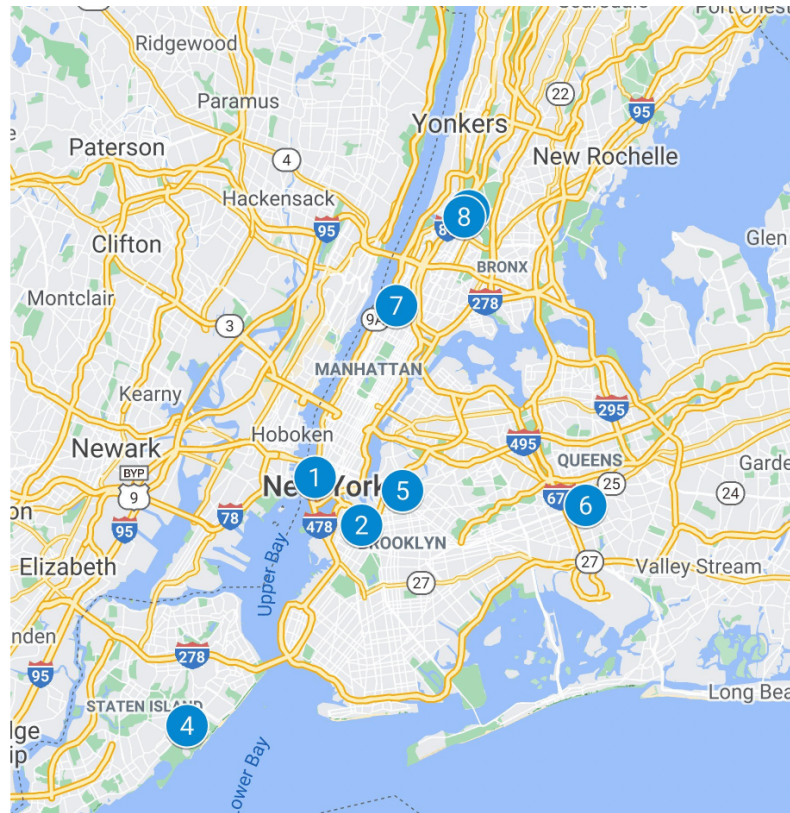## Appendix H: Map of NYC Specialized High Schools

Figure 14   Map of specialized high schools in New York City. In Bronx, the two schools numbered by 3 and 8 are overlapping on the map. The map is generated by *Google My Maps*.