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Political Districting: from classical models to recent approaches

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Abstract The Political Districting problem has been studied since the 60's and many different models and techniques have been proposed with the aim of preventing districts' manipulation which may favor some specific political party (*gerrymandering*). A variety of Political Districting models and procedures was provided in the Operations Research literature, based on single- or multiple-objective optimization. Starting from the forerunning papers published in the 60's, this article reviews some selected optimization models and algorithms for Political Districting which gave rise to the main lines of research on this topic in the Operations Research literature of the last five decades.

Keywords Political Districting · Territory design · Survey

1 Introduction

Political Districting (PD) consists in subdividing a given territory into a fixed number of districts in which the election is performed. A given number of seats, generally established on the basis of the population, is allocated to each district. These seats must be assigned to parties within the district according to the adopted electoral system that prescribes how the citizens' votes are transformed into seats.

Federica Ricca and Andrea Scozzari dedicate this work to their dear friend Bruno Simeone, who passed away unexpectedly on October 10, 2010. He was the prime mover of their common research in this field. He started productive research on electoral systems with political districting which remained throughout the years one of his main interests. This paper is meant to be both a tribute to the deep scientific *oeuvre* of Bruno and a wish for further developments in this area.

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To be *fair* the districts must be *neutral*, in the sense that no political party should be able to take advantage from the territorial subdivision in order to gain seats. This bad malpractice is known as *gerrymandering* and it has motivated the big interest for the study of PD since the very beginning of the modern democracies. In the design of neutral electoral districts, attention must be paid to the *size* (in terms of number of resident citizens) and to the *shape* of the districts, since these are the elements that—combined with the electoral formula—may have an indirect effect on the final seats distribution to the parties. These considerations lead to the adoption of a well defined set of districting criteria, such as "population balance", "spatial contiguity" and "compactness", which are the guiding principles for the design of a fair set of districts. In particular, gerrymandering affects the first-past-the-post electoral systems with single-member districts, in which only one seat is at stake and even a one-vote difference between two competing candidates may decide for the allocation of the seat. This explains why PD was studied in the literature from different viewpoints, attracting the interest of scholars coming from very different areas, such as computer, social, and political scientists, but also operations researchers. The big production of optimization models and procedures started in the 60's gave the possibility of describing and solving a PD problem through an optimization model, where the neutrality requirements can be pursued either by imposing constraints, or via the optimization of a suitable objective function. This explains the wide variety of works and papers on this topic that appeared in the Operations Research (OR) literature of the last fifty years. The PD problem can be seen as a particular type of the more general *territorial districting* (or *territory design*) problem which, on the other hand, may refer also to other application fields, such as the design of school and hospital districts, the regionalization of a territory for transportation service purposes, sales districting, and so on. Recent surveys appeared in the literature related to this more general problem, such as the paper by Duque et al. (2007) and the one by Kalcsics et al. (2005). In these surveys PD is frequently discussed among the many applications of the territory design problem, but generally it is not analyzed in detail. A first review completely dedicated to PD is provided in Ricca and Simeone (1997) where the main aspects of the PD problem are analyzed and a variety of different solution approaches are described. Some of them, such as Hess et al. (1965) and Garfinkel and Nemhauser (1970), nowadays can be considered milestones for PD, since, they were the starting point for many other authors who, in the following years, provided some variants and new procedures which were also applied to real-life PD problems in various countries. Starting from Ricca and Simeone (1997), the present paper provides an updated review of the OR literature on PD, focusing the attention on the main contributions of the most recent years. It is not intended to be an exhaustive review, but its aim is to give an account of the classical and more recent lines of research in this area. It must be pointed out that, besides the works reported here, there exists an enormous production of papers on PD motivated by the renewed interest for this problem in the last years, and by the general tendency to suggest semi-automatic procedures to guarantee a recognizably fair and transparent districting process.

The focus of the present paper is on PD models and solution algorithms which seem to be the main issues in the recent literature on this topic. On the contrary, the discussion about PD criteria at the moment seems to be not so lively as in the past, since the community of researchers involved in PD basically agreed on a set of requirements that are traditionally recognized to be useful to pursue neutrality of the districts. Typical PD criteria are *population balance* (or *equality*), *contiguity* and *compactness*. The first criterion is often enforced by bounds on the district population; contiguity promotes districts formed by only one parcel of land, without holes or isolated parts; it is particularly difficult to deal with and, sometimes, it is even discarded from the PD models and considered only *a posteriori*. The compactness criterion is related to the shape and the extent of a district, and requires a district to be packed into a relatively small space. Actually, there are many ways to take into account these criteria, and there exists a variety of indices that can be adopted to measure them. In particular, this is true for the district compactness on which, in the past, the discussion was wide and controversial (see, for example, Grilli di Cortona et al. 1999; Horn et al. 1993; Niemi et al. 1990; Ricca and Simeone 1997). There are other criteria which were considered in the literature, but, since they do not fit every PD problem, they cannot be considered as "basic" PD criteria. Some details on these criteria are given in Sect. 2.

Due to the difficulty of the PD problem and to its multicriteria nature, the contributions in this research field in the last years were mainly concentrated on the production of heuristic and metaheuristic methods. For a detailed discussion about the computational complexity of the PD problem, the reader is referred to Altman (1997) and Ricca and Simeone (2008). The exact solution of the problem in not manageable in practice and, in fact, it does not help much in the real-life applications. Actually, a PD procedure is required to be fast and to produce solutions meeting a given set of PD criteria as much as possible.

The present paper was first published in 2011 in *4OR A Quarterly Journal of Operations Research* (Ricca et al. 2011). The review follows a chronological development and the surveyed papers are classified according to the districting models and approaches they adopt. In Sect. 2, we introduce the PD problem formally, also providing some notation and definitions. In Sect. 3, we recall the most important papers already described in Ricca and Simeone (1997) which represent the starting works in this research field, but we also update this section including papers appeared in the late 90's. In Sect. 4, we present the most recent works on PD, focusing our attention to those that seem to introduce relevant theoretical or methodological innovation. Finally, in Sect. 5, we draw some conclusions.

2 The Political Districting problem

We suppose that political districts must be designed for a territory that is divided into n elementary units (which may correspond to counties, townships, wards, census tracts, etc.). This assumption provides a discretization of the territory where each unit can be identified by its (geographical) center. As we shall see, even if not all the authors adopt a discrete model for the representation of the territory, the above elementary units are always taken into account as *population units*. Let $k < n$ be the total number of districts, we denote by p_i the size of the population in unit i, $i = 1, \dots, n$, and by $P = \sum_{i=1}^{n} p_i$ the total population of the territory; d_{ij} is any distance measure between unit i and unit j. The average district population is given by $\bar{P} = \frac{P}{k}$. The PD problem can be formulated as *finding a partition of the n units into k districts according to a set of suitable criteria*.

The main criteria for PD are the following:

Integrity—Each territorial unit cannot be split between two or more districts.

Contiguity—The units of each district should be geographically contiguous, that is, one can walk from any point in the district to any other point of it without ever leaving the district.

Population equality (or *population balance*)—Under the assumption that the electoral system is a plurality one, all districts should have the same portion of representation (according to the one person-one vote principle); in particular, if they are single-member districts, they should have nearly the same populations.

Compactness—Each district should be compact, that is, "closely and neatly packed together" (Oxford Dictionary). Thus, a round-shaped district is deemed to be acceptable, while an octopus- or an eel-like one is not.

An additional criterion frequently used in PD is the *respect of existing administrative subdivisions of the territory* (also known as *conformity to administrative boundaries* or *administrative conformity*), which avoids to split already existing official or normative regions (such as, counties or states) between two electoral districts.

There are other PD criteria which are seldom used. Since there is no global consensus on their legitimacy, they are used only in specific situations. This is the case of the *respect of natural boundaries* criterion, which is useful in countries like Italy, where the territory is characterized by many rivers, mountains and lakes that may be obstacles for the districts' contiguity. Also *representation of ethnic minorities* and *respect of integrity of communities* could be applied when the population of the territory is composed by many different cultural or ethnic groups of citizens. Broad discussions about political districting criteria can be found in many papers, such as in Bozkaya et al. (2003), Grilli di Cortona et al. (1999), Kalcsics et al. (2005), Ricca and Simeone (1997).

Many authors adopt a graph-theoretic model for representing the territory on which districts must be designed. The territory can be represented as a connected n-node graph $G=(N,E)$, where the nodes correspond to the elementary territorial units and an edge between two nodes exists if and only if the two corresponding units are neighboring (i.e., they share a portion of boundary). The graph G is generally known as the *contiguity graph*. To the best of our knowledge, it was independently introduced by Bodin (1973) and by Simeone (1978), but, in the following years, it was often considered again by many other authors (see, for example, Nygreen 1988). The nodes of the graph are weighted with the population of the associated territorial units, while the weights of the edges represent distances between adjacent units. Note that in the different models these distances may be measured in different ways. Usually, they correspond to road distances, but it is also possible to adopt Euclidean ones or other types of distances.

Considering only the four main criteria listed above, and relying on the contiguity graph G, the PD problem can be stated as *finding a compact partition of G into k connected components such that the weight of each component (i.e., the sum of the weights of its nodes) is as close as possible to $\bar{P}$*. If, on the one hand, this graph-theoretic formulation requires the definition of a suitable measure both for compactness and population balance, on the other hand, the use of G guarantees integrity of the solutions and allows to manage contiguity of the districts via the connectedness of the partition's components. For this reason, the above graph model is used in a number of papers on PD, especially those that follow the improving search approach. Note that the representation of the territory via the contiguity graph G is so natural that the idea of this underlying graph is always present in the papers even when it is not explicitly referred to.

The rest of the paper is devoted to the presentation of the most important PD models and methods proposed in the literature in the last fifty years. We classify the different papers according to the algorithmic approach they adopt. The possible strategies to solve PD, already discussed in Ricca and Simeone (1997), are many. Looking at the papers in the literature, some approaches, such as location techniques and local search, seem to have attracted the attention of the researchers more than others, but, especially in the recent years, also new ideas were suggested for the solution of the problem. An example is Kalcsics et al. (2005) where a successive dichotomies algorithm is proposed in a computational geometry framework; in Ricca et al. (2008) and Miller (2007) the use of Voronoi regions is suggested in order to deal with compactness and contiguity of the districts; some authors try to exploit the graph underlying the territory in different ways (see, for example, Nemoto and Hotta 2003; Yamada 2009). Following Garfinkel and Nemhauser (1970), other papers use an exact solution approach by means of integer (linear and non-linear) programming models for PD which are solved with standard optimization packages. In spite of our classification effort, we notice here that not all papers can be easily classified w.r.t. a unique approach among those listed in Ricca and Simeone (1997), due to the fact that sometimes a combination of more than one strategy is adopted to obtain efficient solution methods.

In the procedures described in the following, it frequently happens that the "center" of a set of units (i.e., the center of a district) must be computed, but, in different papers, it is computed in different ways. According to the traditional definition, given a set N of units and the distances d_{ij} for all pairs i, j in N, the *center* of N corresponds to the unit in N that minimizes $\max_{i\in N} d_{ij}$ (Harary 1994). In this sense, the center of a set is identified with the most central unit. Alternatively, the center of a set of units can be defined as the unit that minimizes $\sum_{i\in N} d_{ij}$. Although the max criterion is usually adopted in graph theory to define a center, the sum criterion is generally preferred, since it leads to linear functions in the corresponding algebraic models. It must be also pointed out that in the papers described in the following the terms "center" and "centroid" are often used without distinction.

3 Classical Political Districting models

In this section we review those papers in the literature of the 60's and 70's that provided the seminal PD models and approaches, laying the foundations for the main lines of research in this field. Starting from these ideas, many other works followed, providing new methodological developments and improvements. Basically, this section traces Sect. 4 of Ricca and Simeone (1997), but it also includes new updates related to the papers published in the second part of the 90's.

3.1 Multi-kernel growth

According to the multi-kernel growth strategy, a district map can be obtained in an incremental fashion. A set of territorial units is generally selected at the beginning as the set of centers (or potential centers) of the districts and the algorithm proceeds by adding neighboring units to the district under construction in order of increasing distance, until a certain population level is reached, and stopping when all units are assigned to some district.

3.1.1 Vickrey (1961)

In Vickrey (1961) a first multi-kernel growth procedure is suggested in which districts are built one at a time. The author considers the territory subdivided into population units corresponding to census tracts and takes into account contiguity, population equality, compactness and conformity to administrative boundaries. At the beginning, the procedure fixes a reference unit, say h. At each iteration, a new district is formed starting from a center $i \neq h$ corresponding to the unassigned unit that is the farthest from h. Once $k-1$ districts are formed, all the unassigned units are included in the district centered in h. This is done in order to avoid forming enclaves or isolated areas that, at some stage of the procedure, cannot be assigned to any neighbor district. Suppose that the district centered in i is under construction. The procedure adds to i the neighboring unassigned population units in order of increasing distances from i, and it halts only when a fixed value is reached for the *population quota*, which is defined as the ratio between the unassigned population and the number of remaining districts. This population quota is updated at each iteration in order to avoid that population excesses and deficiencies cumulate, producing a very unbalanced district centered in h. During the districts' growth also administrative conformity is taken into account by the following test: if the added unit belongs to an already existing normative zone, then the next units to be added should be taken among those belonging to the same zone. Notice that, by following a multi-kernel growth scheme, compactness is pursued indirectly since the districts are formed by adding to the center i its closest neighbors.

3.1.2 *Bodin (1973)*

In Bodin (1973) the author provides a two-step algorithm for PD, introducing for the first time in the PD literature a contiguity graph for the representation of the territory. Actually, the graph considered by Bodin has the same structure as the contiguity graph described in Sect. 2, but it is directed with two arcs (i,ℓ) and (ℓ,i) for each pair of nodes i and ℓ in N. In addition, Bodin suggests to associate the population weights to the arcs of the graph, instead of to its nodes; thus, each arc (i,ℓ) has a weight equal to the population of unit ℓ. The first step of the algorithm follows the strategy of the classical node-labeling procedures in Network Flow Theory and consists of the growth of districts around a set of k centers that are assumed to be already given. Starting from the k centers, the procedure constructs k trees (districts), each rooted at one of the centers, by successively selecting a unassigned node and including it in the most advantageous district w.r.t. population balance. Each node ℓ has a three-dimensional label: the first indicates the center of the district to which ℓ is assigned, say j; the second corresponds to the current population of the district centered in j; the third label denotes the predecessor of ℓ in the tree rooted at j. The algorithm proceeds by adding to the tree centered in j a node ℓ adjacent to some nodes of the tree, in order to increase the population of the corresponding district. It may happen that ℓ was already assigned to another district h; in this case, the label of ℓ is changed only if the global population balance improves, provided that moving ℓ to j does not disconnect h. The algorithm terminates when any further visit of the nodes does not change any label. In this algorithm population equality is the guiding criterion, and, at each iteration, the objective function is given by the difference in population between the largest and the smallest district divided by $\bar{P}$. The second step applies local search in order to further improve population equality. The local search strategy follows the idea of the well known exchange algorithm provided in Lin and Kernighan (1973) for the Traveling Salesman Problem. Given two districts j and h, the procedure tries to make one of the following moves: (i) move a node i from district j to district h; (ii) move a node i from district j to district h and a node ℓ from h to j.

It must be noticed that in the above procedure only contiguity and population equality are considered. On the other hand, there is no guarantee for compactness, since the algorithm does not control the "shape" of the resulting trees.

The PD algorithm was applied to the 1960 census data for the State of Arkansas with $n = 75$ territorial units and testing three different sizes for the district map, with $k = 3, 5, 9$ respectively.

3.1.3 *Arcese et al. (1992)*

In Arcese et al. (1992) the authors propose a PD model based on a contiguity graph G slightly different from the one introduced by Bodin (1973): here the graph is not directed and population weights are associated to each vertex. The problem is to find a partition of G into k connected components so as to maximize the three criteria of population equality, compactness and administrative conformity. The study refers to the Italian electoral districts for the elections of the Chamber of Deputies in 1994. The application is based on data from the 1991 population census and the procedure is applied to each of the 20 Italian Regions separately. The territorial units correspond to the census tracts and their number ranges from $n = 74$ to $n = 1546$ (for Valle D'Aosta and Lombardia Regions, respectively), thus providing a variety of test problems with very different sizes. The optimization is accomplished by three heuristics in cascade. The first stage concerns the location of the k district centers: after locating a preliminary set of possible centers distributed on the territory, the k most populated centers are selected. An iterative procedure is then applied in which the following two steps are repeated until no modification of the centers is observed: (i) for each center u compute its *influence zone* $I(u)$ (i.e., the set of those units for which u is the closest center); (ii) for each influence zone $I(u)$, the center is updated with the unit h in $I(u)$ that minimizes the sum of the distances from all the units in $I(u)$ to h. The second stage consists of a multi-kernel growth procedure to compute the k districts in which the three above PD criteria are taken into account simultaneously. Finally, a third stage is planned in which local search is applied to further improve the optimal solution w.r.t. the same three criteria. For this phase, the authors introduce three indices to measure lack of population equality, compactness and administrative conformity, respectively, and define an objective function given by a weighted combination of them.

3.2 Location

The study of PD through a mathematical formulation can be dated in the early 60's, when Weaver and Hess (1963) published their first work on this topic. In a second paper, Hess et al. (1965) presented their PD model and solution approach more formally, and published it in an OR journal. The approach suggested in these papers basically proposes to adapt techniques for warehouse location-allocation problems to PD. The years that followed were characterized by a rich production of OR papers on PD.

3.2.1 Hess et al. (1965)

The paper by Hess et al. (1965) is generally considered the earliest OR paper in political districting. Let n be the total number of territorial units and k the number of districts, the problem is formalized as a discrete location problem. The idea is to identify k units representing the centers of the k districts, so that each territorial unit must be assigned to exactly one district center. The model has n^2 binary variables, x_{ij}, $i, j = 1, \dots, n$: for $i \neq j$, x_{ij} is equal to 1 when unit i is assigned to center j, and 0 otherwise; the variable x_{jj} takes the value 1 whenever unit j is chosen as one of the centers. The political districting model is the following:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 p_i x_{ij} & \\
& \sum_{j=1}^{n} x_{ij} = 1 & i = 1, \dots, n \\
& \sum_{j=1}^{n} x_{jj} = k & \\
& a\bar{P}x_{jj} \le \sum_{i=1}^{n} p_i x_{ij} \le b\bar{P}x_{jj} & j = 1, \dots, n \\
& x_{ij} \in \{0, 1\}, & i, j = 1, \dots, n
\end{aligned}
\tag{1}
$$

where p_i is the population of unit i, d_{ij} is the distance between unit i and center j, and a and b are the minimum and the maximum allowable district population fractions, calculated as a percentage of the average district population $\bar{P}$. The first n constraints mean that each unit must belong exactly to one district. The next one imposes that the total number of districts is exactly k. The other two groups of n constraints represent the conditions on the maximum and the minimum allowable population for a district with respect to chosen parameters $a < 1$ and $b > 1$ (population balance). Finally, the objective function (total inertia) is a measure of compactness.

The above integer programming model does not consider contiguity of the units belonging to the same district, so that a revision for spatial contiguity is required *a posteriori*.

Due to the computational difficulty in solving the above model, in Hess et al. (1965) an iterative heuristic procedure is proposed as an alternative solution approach. Essentially, the generic iteration of the algorithm consists of four steps: (1) guess the district centers; (2) solve a transportation problem to assign population equally to these centers at minimum cost (defined in terms of distances between units and centers of the districts); (3) adjust the solution of the transportation problem so that each territorial unit is entirely within one district; (4) compute centroids of the current districts and use them to update the district centers. Steps 1–4 are repeated until the procedure converges (i.e., the centers do not change in two successive iterations).

The main step of the above procedure is step 2 in which a transportation problem must be formulated and solved. The formulation of the problem is the following. The origins in the transportation graph represent the current centers, all with supplies equal to $\bar{P}$. The destinations represent the territorial units, with demands equal to their populations. Each edge (i, j) of the graph has a weight equal to d_{ij}^2.

With the above iterative procedure it may happen that, in the solution of the transportation problem, a territorial unit i is split between two or more districts; in this case, in step 3, unit i is entirely assigned to the district to which the largest quota of its population was already assigned. The convergence of the procedure is not guaranteed in theory, but the authors report that, in real-life applications, the heuristic converges very fast to a local minimum. The paper provides an application related to the design of Delaware legislative districts in 1964, where $n = 650$ and $k = 35$.

3.2.2 Hojati (1996)

Starting from Hess et al. (1965), other authors developed political districting methods based on a location approach. Hojati (1996) suggests a three-phases procedure, in which phase 2 corresponds to the algorithm by Hess et al. (1965). Actually, instead of adopting an iterative strategy based on successive adjustments of the centers, they are located only once at the beginning of the procedure and this choice is permanent. To solve this problem, the author introduces a (mixed integer) warehouse location model, similar to the one in Hess et al. (1965), but based on two different sets of variables, namely, x_{ij}, $i, j = 1, \ldots, n$, representing the proportion of population of unit i assigned to district j, and indicator variables y_j, $j = 1, \ldots, n$, such that $y_j = 1$ if unit j is chosen as the center of a district and $y_j = 0$ otherwise. The mixed integer program of phase 1 is the following:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 p_i x_{ij} \\
& \sum_{j=1}^{n} x_{ij} = 1 && i = 1, \dots, n \\
& \sum_{i=1}^{n} p_i x_{ij} = \bar{P} y_j && j = 1, \dots, n \\
& \sum_{j=1}^{n} y_j = k \\
& x_{ij} \le y_j && i, j = 1, \dots, n \\
& 0 \le x_{ij} \le 1, && i, j = 1, \dots, n \\
& y_j \in \{0, 1\} && j = 1, \dots, n
\end{aligned} \tag{2}
$$

where d_{ij} is the Euclidean distance between i and j. A Lagrangian relaxation of the resulting model is derived and it is solved by a subgradient optimization algorithm.

After phase 1, the solution of the above program provides the k districts centers, and step 2 of the procedure of Hess et al. (1965), is applied to assign each unit to exactly one center (phase 2). When in the solution of the transportation problem there are split territorial units (i.e., units fractionally assigned to more than one center) the author introduces the *Split Resolution Problem* (SRP) which is formulated as a graph-theoretic model. Actually, he takes into consideration the subgraph of the transportation graph whose vertices are given, on the one hand, by the split units and, on the other hand, by those centers to which some split units have been (partially) assigned. The author provides a linear programming formulation for SRP and shows that SRP is NP-hard by a reduction from the Partition Problem (Garey and Johnson 1979). He also suggests an heuristic procedure to solve SRP (phase 3 of his algorithm) based on the solution of a sequence of capacitated transportation problems defined over a suitable modified network (see, for details Hojati 1996).

Notice that, applying this heuristic does not guarantee integrity of territorial units but only reduces as much as possible the number of split ones. It must be also pointed out that, as in Hess et al. (1965), Hojati (1996) does not consider the contiguity criterion, so that non-contiguous district maps may arise requiring additional efforts.

This method is applied to the territory of the city of Saskatoon (Canada) for the provincial elections in 1993, and the obtained district map is compared with the institutional one showing good results for compactness.

3.2.3 George et al. (1997)

The procedure proposed in George et al. (1997) follows the iterative location/allocation approach pioneered by Hess et al. (1965), but with the main difference that a new method for assigning territorial units to districts is adopted. For this step, the authors introduce a minimum cost network flow problem defined on the following network. The nodes of the network are given by the territorial units, the district centers and an additional sink node t. Each unit-node i has a supply equal to p_i, while the sink-node has a demand equal to $P = \sum_{i=1}^{n} p_i$. Besides the arcs (i, j), corresponding to all the unit-center pairs, $i = 1, \dots, n$, $j = 1, \dots, k$, there exists an arc (j, t) for each district center j.

The flow on an arc (i, j) corresponds to the population of unit i assigned to district j; the flow on an arc (j, t) is equal to the population of district j. The authors introduce different cost functions to define the costs associated with the arcs of the network. The aim is to take into account the basic PD criteria, making an optimal solution of the above flow problem correspond as much as possible with a district map with good population equality and compactness. In a first version of the algorithm the costs are set so as to reproduce the model in Hess et al. (1965), i.e., only costs associated to unit-center pairs are different from 0. In the other three versions also costs associated to center-sink pairs are introduced. They are functions of the flow passing through the corresponding arc and consist of penalties graduated according to the actual deviation of the district population from the target value (for details about the cost functions, see, George et al. 1997, p. 20, Table 1).

The proposed iterative procedure alternates the location of the centers of the k districts with the allocation of the population of the units to such districts via the solution of the above minimum cost network flow problem. It stops when the difference between the value of two successive optimal solutions of this problem is sufficiently small. The authors observe that with convex cost functions the minimum cost network flow problems can be solved efficiently, but they do not care much about the fact that contiguity may not be satisfied by this solutions. Actually, in one application they found districts that are not contiguous due to the presence of natural barriers and, thus, in one version of their algorithm (Version 4) they try to avoid this drawback by introducing an additional penalty in the costs of some arcs. They also observe that split units may arise and for this problem they suggest to follow the same rule adopted by Hess et al. (1965) (a split unit is completely assigned to the district in which it already has the highest proportion of population).

The authors apply their algorithm to the design of parliamentary districts in New Zeland in 1993 and study the performance of the four versions of their PD procedure; they also compare the results obtained with their automated algorithm with the New Zealand institutional districts obtained by a manual procedure.

The above approaches are widely discussed in Kalcsics et al. (2005). Although this paper deals with the more general territory design problem, the authors explicitly analyze—among others—the application related to PD, and, even if the article is organized as a review, it includes several new contributions by the authors themselves. They mainly discuss the solution approach to the districting problem introduced by Hess et al. (1965) and developed in Hojati (1996) and in George et al. (1997), but they also provide new results for the Split Resolution Problem. In particular, they formulate the SRP with different objective functions, providing solution algorithms and results about the computational complexity for each version of the problem.

3.3 Exact approaches

In Garfinkel and Nemhauser (1970) the authors propose to formulate PD as a linear optimization problem with binary decision variables. In the literature this paper is usually considered as the starting point in the analysis of PD via exact solution approaches based on an algebraic optimization model. In principle, the solution of such model would require the enumeration of all feasible solutions. However, different techniques are suggested in the literature in order to reduce the computational effort to find an optimal solution. Actually, this kind of approach can be applied only to real-life PD problems in which the number of territorial units is very small.

3.3.1 Garfinkel et al. (1970)

Garfinkel and Nemhauser (1970) propose a two-phase PD algorithm based on a set partitioning approach: in phase I, they generate all possible feasible districts w.r.t. three types of constraints related to contiguity, population equality and compactness, respectively, and denote this set by J; in phase II, they formulate a set partitioning model to select a set of feasible districts in J providing a partition of the territorial units that minimizes the overall deviation of districts' populations from $\bar{P}$. An additional cardinality constraint is introduced in the model to take into account the fixed number of districts to be drawn.

The authors represent the set of possible districts of a given territory through a binary matrix A, with as many rows as the total number of territorial units and a column for each possible district. In order to generate all possible feasible districts, in phase I they take into account population equality by imposing a minimum and a maximum level for the district population. Two different measures are introduced for compactness: the first one is based on the notion of "exclusion distance" between two units (see, Garfinkel and Nemhauser 1970 for details), the second is related to the shape of the districts and takes into account both the maximum distance between two units within a district and the district area. Starting from an arbitrary territorial unit, a feasible district is generated following an aggregation strategy in which contiguous units are added to the current district until the population becomes feasible (*unit annexation*). Compactness is checked during the procedure. Feasible districts w.r.t. all criteria are recorded, while a backtrack step is executed when every further annexation makes the district population exceed the upper limit. The whole procedure consists of an implicit enumeration using a binary search tree; some "reduction tests" are performed during the procedure to decrease the number of districts under construction, discarding those that would lead to unfeasible district maps (these reduction tests are based on some theoretical results reported in the Appendix of Garfinkel and Nemhauser 1970).

Once the set J of all the feasible districts is available, in phase II, the authors formulate the following set partitioning model in order to choose a final district map:

$$
\begin{aligned}
\min \quad & \sum_{j\in J} f_j x_j \\
& \sum_{j\in J} a_{ij} x_j = 1 \quad i = 1,\dots,n \\
& \sum_{j\in J} x_j = k \\
& x_j \in \{0,1\} \qquad j \in J
\end{aligned} \tag{3}
$$

where $f_j = \frac{|P_j - \bar{P}|}{\alpha \bar{P}}$ ($\alpha \in [0,1]$ is the tolerance on the percentage of deviation for the population of a district from $\bar{P}$); $a_{ij} = 1$ if unit i is in district j and $a_{ij} = 0$ otherwise; $x_j = 1$ if district $j \in J$ is included in the partition and $x_j = 0$ otherwise.

The solution technique starts from a subset D of feasible districts in J that can be fixed definitively after phase I (see, Theorem 2 in Garfinkel and Nemhauser 1970). The idea is to enlarge D by including additional feasible districts taken from $J \backslash D$, until k districts are selected. As in phase I, the authors propose an implicit enumeration strategy where improving annexation steps alternate to backtracking ones.

Some experimental results are provided by the authors in order to test the actual performance and efficiency of their method. A set of twenty test problems of different sizes is considered: some of them are randomly generated problems (with n ranging from 100 to 1790, and k from 20 to 100); three problems are adapted from the districting problems of Sussex County, Delaware and the State of Washington.

3.3.2 Nygreen (1988)

A two-stage approach similar to the one in Garfinkel and Nemhauser (1970) is proposed by Nygreen (1988). In the first stage, all feasible districts are generated; in the second one, districts are combined to form a district map. The PD problem is formalized through a graph-theoretic model relying on the contiguity graph G in which a district j is represented by a tree that spans the units belonging to j. Contiguity is naturally guaranteed by the connectedness of the tree, while a district is considered compact if the corresponding tree has depth at most two. Also administrative conformity is taken into account. The objective function is related to the population equality criterion and it is defined as the sum of square deviations of district populations from $\bar{P}$. Thus, the PD problem is formulated as finding in G a spanning forest with k trees, each of depth at most two, that minimizes the above objective function. In the following, we provide some details on the generation of the feasible districts. A district j is *valid* if its corresponding tree has depth at most two and the following administrative conformity condition is satisfied: if a territorial unit is in j, then all the other units in the same city are in j, too. The original contribution in the paper is the procedure for generating all valid districts which includes some tests aimed to reduce as much as possible the number of such districts. The procedure generates the districts by considering one unit j at a time, and searching for all the valid districts rooted at j. Two main considerations lie under this procedure. The first is related to the fact that the same valid district may be represented by many different trees rooted at different nodes. If, for a given node j, there exists a $j' \neq j$ such that all valid districts represented by trees rooted at j can be also represented by trees rooted at j', then j can be discarded as a root (this is called the *root rule*). The second consideration is based on the fact that a valid district will never be chosen if it cannot be suitably combined with other $k-1$ valid districts (*the graph that remains test*). Thus, the first stage of the procedure can be summarized in the following steps: (i) Apply the root rule to reduce the possible roots for a district; (ii) for each possible root j, generate all possible valid districts rooted at j and apply the graph that remains test; (iii) delete duplicated districts generated by different roots; (iv) compute the objective function value for each valid district.

At the end of the first stage one has a set of valid districts, each weighted by the corresponding objective function value. The second stage consists of the solution of a set partitioning problem to find the optimal districting map.

Nygreen applies his method to the PD problem for designing the European Assembly constituencies for Wales in 1984. The problem size is $n = 38$ and $k = 4$ and, therefore, the set partitioning problem is solved exactly by implicit enumeration in few seconds.

3.3.3 Mehrotra et al. (1998)

Following Garfinkel and Nemhauser (1970), Mehrotra et al. (1998) suggest again a set partitioning approach for PD, but adopt a different objective function that takes into account the overall non-compactness of the districts. Based on the contiguity graph G, the authors compute the distance $d_{i\ell}$ between two units i and ℓ as the number of edges in a shortest path from i to ℓ in G. For any given connected partition of G into k components, the center of district j corresponds to the node u_j for which the sum of the distances to all the other nodes in the same district is minimized. Then, in the objective function of the set partitioning model (3) the cost f_j of a district $j \in J$ is replaced by the cost c_j given by $c_j = \sum_{i \in j} d_{u_j i}$. To solve the above model the authors apply a column generation methodology in order to avoid to enumerate all the possible feasible districts in J. At each iteration, a subset of feasible districts $\bar{J} \in J$ is available and the model is restricted to $\bar{J}$. The authors denote the master problem by $PLAN(\bar{J})$. At the beginning $\bar{J}$ consists of a set of feasible districts corresponding to either a past districting, or to a district map obtained using a clustering heuristic. The authors follow a *branch-and-price* approach in which at each iteration they solve the continuous relaxation of the current problem $PLAN(\bar{J})$ and then check if there are columns in $J \backslash \bar{J}$ that prices out favorably. If this test succeeds, $\bar{J}$ is updated and the relaxation of the new $PLAN(\bar{J})$ is solved. Otherwise, if the solution of the relaxation of $PLAN(\bar{J})$ is integral, it is a solution of $PLAN(\bar{J})$ and they update the bounds and the active node list in the search tree. If the solution of relaxation of $PLAN(\bar{J})$ is not integral, a suitable branching step is performed generating new active nodes in the tree.

A post-processing phase is introduced in order to meet the population balance requirement, which is not always satisfied by the solution obtained with the above procedure. In this phase a transshipment problem is formulated on a network in which the nodes correspond to the districts of the previous solution and there is a pair of arcs between any two adjacent districts. A district j is a source if $P_j > \bar{P}$, with supply $P_j - \bar{P}$; it is a sink if $P_j < \bar{P}$, with demand equal to $\bar{P} - P_j$; if $P_j = \bar{P}$ the district corresponds to a transshipment node. The aim is to shift population from overpopulated districts to underpopulated ones. In order to discourage flow transshipment between two overpopulated (or two underpopulated) districts, positive costs are associated to the corresponding arcs, while the remaining arcs have cost equal to 0. For obvious reasons, a unit of flow cannot coincide with a single person. Thus, the districts are further divided into subunits (smaller than the original elementary territorial units) and only those shifts involving whole subunits are performed.

The paper provides results for the state of South Carolina based on 1990 census data. The results seem to be quite satisfactory, since compact ad contiguous districts can be found also matching the required restriction on their population. However, it must be pointed out that also in this case the size of the problem is very small ($n = 51$ territorial units and $k = 6$ districts), so that it can be solved exactly in short time.

3.4 Local search

An alternative approach to the solution of PD is the use of local search techniques. They are very general methods which are usually adopted to find solutions for computationally difficult combinatorial problems when an exact algorithm cannot be applied. The first paper suggesting this kind of approach for PD was published in 1981 (Bourjolly et al. 1981), but, in the following twenty years, these techniques have been widely developed, especially with the coming of local search meta-heuristics of the new generation, such as, for example, simulated annealing and tabu search (Aarts and Lenstra 2003; Glover and Laguna 1997). In this section, we report two articles on this topic published in 1981 and 1996 (Bourjolly et al. 1981; Ricca 1996), but in Sect. 4 we will discuss also other—more recent—papers following this approach: they either deal with the application of some already known meta-heuristics, or propose new ones.

3.4.1 Bourjolly et al. (1981)

A first application of local search to PD is found in Bourjolly et al. (1981) where the districting problem is formalized as an integer programming model and the PD criteria taken into account are population equality, compactness, conformity to administrative boundaries and socio-economic homogeneity. In this paper the use of the last two criteria is motivated by the particular nature of the territory under study which corresponds to l'Ile de Montréal (with $n = 418$ and $k = 31$). The variables are x_{ij}, $i = 1, \dots, n$, $j = 1, \dots, k$, with $x_{ij} = 1$ if unit i is in district j and $x_{ij} = 0$ otherwise. The model is the following:

$$\begin{aligned} \min \quad & \sum_{j=1}^{k} \gamma_{pop}\phi_{pop}(j) + \gamma_{dist}\phi_{dist}(j) + \gamma_{rev}\phi_{rev}(j) \\ & \sum_{j=1}^{k} x_{ij} = 1 \quad i = 1, \dots, n \\ & x_{ij} \in \{0, 1\} \quad i = 1, \dots, n, \ j = 1, \dots, k \end{aligned} \tag{4}$$

where ϕ_{pop}, ϕ_{dist} and ϕ_{rev} are functions that measure the lack of population equality, compactness (based on the distances between units) and socio-economic homogeneity (in terms of income), respectively; γ_{pop}, γ_{dist} and γ_{rev} are nonnegative weights corresponding to penalties for each of them. The administrative conformity criterion is not explicitly considered in the model, but the authors state that it can be taken into account by solving the districting problem separately for the different normative regions in which the territory is divided. The above model follows a multicriteria approach and includes as constraints only the conditions that each territorial unit must belong to exactly one district.

According to the typical framework of local search, the procedure starts from a given initial solution (that could be, for example, the already existing district map), and proceeds by successive improvements in the quality of the solution w.r.t. the objective function in (4). In an initialization step, the algorithm computes for each unit i a list of districts in which i may be included. Then, the algorithm goes on by moving units from their current district to a new one contained in the corresponding list when this is advantageous for the objective function, and until no advantageous move exists any more. The peculiarity of the approach is in the lists associated to the units. They are computed once at the beginning and each contains at most all the k districts. Thus, for a given unit i, computing such list and all the possible variations of the objective function when i is moved to another district is not expensive and requires a polynomial number of operations. Since the lists are never updated during the procedure, the number of moves that the algorithm performs is at most $(k - 1)$ for each unit and, thus, the convergence of the algorithm is guaranteed in finite and polynomial number of steps. In spite of the above good results, it must be pointed out that, in model (4) all the requirements on the districts, except one, are encompassed in the objective function as a unique measure, implying that the specific value of each quality measure is not controlled by the procedure. However, the authors report that good results were obtained with the proposed method for compactness and population balance. On the contrary, since contiguity is not explicitly formalized in the model, no guarantee exists that the final district map is contiguous. In fact, the authors themselves report that isolated units are always present in the optimal solutions found, even if the number of such units is generally small. In these cases they suggest to meet the contiguity requirement by modifying the district map manually.

3.4.2 Ricca (1996)

In Ricca (1996) the author provides an experimental comparison of three different local search heuristics for PD, namely, descent, tabu search and simulated annealing. The application refers to the design of the single-member districts for the 1994 election of the Chamber of Deputies in Italy, and it is based on census data of the 1991 population census.[1] The motivation of the work is to show that the institutional districts adopted for that elections could

[1] This work is based on the same electoral data already analyzed in Arcese et al. (1992).

be improved by applying automated procedures. Relying on the graph-theoretic model representing the territory with the contiguity graph G, the author suggests a multicriteria model in which, besides contiguity, three criteria are considered, namely, lack of population equality, compactness and conformity to administrative boundaries. They are measured through the same indices proposed in Arcese et al. (1992), and the objective function (to be minimized) is a convex combination of the three. A specialized implementation of each local search technique is provided in order to test the performance of these class of algorithms for PD. The work is a follow up of the study proposed in Arcese et al. (1992) in which it was already established that the quality of the solutions should be improved via local search (this correspond to the third stage of the algorithm in Arcese et al. 1992). The comparative analysis showed that all local search techniques—even the basic descent algorithm—are able to improve the district maps w.r.t. all the considered criteria, but among the others, tabu search seems to be the most effective.

Other authors in the literature report on the application of local search methods to real-life PD problems, such as, Kaiser (1967) who applied a basic descent search, Bussamra et al. (1996) who adopted tabu search for the districting of the city of Campinas in Brazil, and Browdy (1990) who suggests (in a law journal!) to use simulating annealing to draw the federal legislative districts in the USA States.

4 Recent advances in Political Districting

In this section we give an account of optimization models and approaches for PD that arose in the OR literature in the last decade. We try to focus on works proposing innovative models and techniques, while we voluntarily skip the description of the wide variety of papers related to the solution of a specific application of PD. When possible, we follow the same classification scheme already adopted in Sect. 3, even if not all the approaches were still carried on in the last ten years. For example, we did not find new contributions on multi-kernel growth and location.

4.1 Exact approaches

4.1.1 Nemoto and Hotta (2003)[2]

A new interesting modeling approach for PD is suggested in Nemoto and Hotta (2003) where the authors provide a reformulation of the PD problem in terms of network flows. Starting from the contiguity graph $G = (N, E)$, the authors construct the following network $H = (\bar{N}, \bar{A})$. First, each edge connecting node i and node j in G is replaced by the pair of arcs (i, j) and (j, i); k copies of this graph are considered, and node i in the h-th copy of the graph is denoted by v_i^h, $i = 1, \dots, n$; $h = 1, \dots, k$. In addition, k source-nodes and n sink-nodes are introduced. Each source-node s^h, $h = 1, \dots, k$ is connected to all the nodes of the h-th copy of the graph with an arc (s^h, v_i^h), for all $i = 1, \dots, n$, while for each sink-node t_i, $i = 1, \dots, n$ there exists an arc (v_i^h, t_i), for all $h = 1, \dots, k$ (see, Nemoto and Hotta 2003, Fig. 1). Two sets of binary variables are introduced. The first set corresponds to the usual

[2]We want to thank our Ph.D. colleague and friend Andrea De Vitis for his precious help in the translation of this paper which was published only in Japanese. Besides knowing Japanese, Andrea is an expert in Operations Research and a former student of Bruno Simeone. We are sure that he patiently supported us not only out of friendship, but also in high regard for Bruno.

unit-district assignment variables so that $z_{ih} = 1$ if unit i is assigned to district h, and $z_{ih} = 0$ otherwise. In the proposed model the flow from each source s^h enters the h-th copy of the original graph G through only one node v_i^h (i.e., only one arc (s^h, v_i^h) has flow different from 0); to indicate this node, additional variables y_{ih} are introduced with $y_{ih} = 1$ if in the h-th copy of G the flow enters through node i, and $y_{ih} = 0$ otherwise. In the formulation of the problem, the authors explicitly consider the population equality criterion. An upper and a lower bound on the population of each district are introduced and denoted by u and l, respectively. They are both variables of the model and, in an ideal case, they should coincide; thus, the objective function of the model is to minimize the difference $u - l$. Notice that this formulation for population equality does not generate those infeasibility problems which may arise with the usual constraint that fixes a given tolerance for the deviation of the district population from $\bar{P}$. The model provided in Nemoto and Hotta (2003) is the following:

$$
\begin{aligned}
\min \quad & u - l && \\
& l \le \sum_{i=1}^{n} p_i z_{ih} \le u && h = 1, \dots, k \\
& \sum_{a \in \delta^-(v_i^h)} f(a) = \sum_{a \in \delta^+(v_i^h)} f(a) && i = 1, \dots, n,\ h = 1, \dots, k \\
& f(a) \ge 0 && a \in \bar{A} \\
& f\left(s^h, v_i^h\right) = \beta y_{ih} && i = 1, \dots, n,\ h = 1, \dots, k \\
& \sum_{i=1}^{n} y_{ih} = 1 && h = 1, \dots, k \\
& y_{ih} \in \{0, 1\} && i = 1, \dots, n,\ h = 1, \dots, k \\
& \sum_{a \in \delta^-(v_i^h)} f(a) \le \beta z_{ih} && i = 1, \dots, n,\ h = 1, \dots, k \\
& z_{ih} \le f\left(v_i^h, t_i\right), && i = 1, \dots, n,\ h = 1, \dots, k \\
& \sum_{h=1}^{k} z_{ih} = 1 && i = 1, \dots, n \\
& z_{ih} \in \{0, 1\} && i = 1, \dots, n,\ h = 1, \dots, k
\end{aligned}
\tag{5}
$$

In the above model, $f(a)$ is the flow passing through arc $a \in \bar{A}$, and $\delta^-(v_i^h)$, $\delta^+(v_i^h)$ are the sets of arcs entering and leaving node v_i^h, respectively. The first constraint refers to the condition on the population of the districts. The second set of constraints are the usual flow-balance constraints. Constraints from the 4-th to the 6-th establish the volume of the flow leaving each source s^h and the unique arc (s^h, v_i^h) through which such flow passes. The other constraints establish the relations between the z and the flow variables. In this model the variable β is the volume of the flow from each source s^h, $h = 1, \dots, k$. It must be noticed that the model takes into account integrity, contiguity and population equality, but compactness of the districts is not guaranteed. The PD formulation in Nemoto and Hotta (2003) is a mixed integer linear program. The authors apply their approach for the design of electoral districts in Japan where the whole territory is already subdivided into administrative regions, such as, prefectures and metropolitan areas. Each region must be considered separately and the districting problem is solved in each of them independently. The nodes of the contiguity graph of each administrative region correspond to the wards of that region. In their application, the authors uses a standard optimization software for solving the problem. No results are provided by the authors related to any new (heuristic or exact) solution procedure which might exploit the network flow structure underlying model (5).

4.1.2 Li et al. (2007)

A quadratic model for PD is proposed in Li et al. (2007). The authors first state the problem in graph theoretical terms, but, then, they solve it via the formulation of a quadratic program. Population equality and compactness criteria are considered in the model. Even if the authors use the contiguity graph G to represent the territory, they do not consider contiguity explicitly as a constraint, implying that the optimal solution of their model could turn out to be non-contiguous. For each pair of nodes i and j in G, they compute the minimum cardinality path connecting i and j, and consider such cardinality as the distance d_{ij} between the two corresponding units. Then, they introduce a new complete graph G' with the same set of nodes of G, and such that weights equal to the populations are associated to the vertices in N, and a weight equal to d_{ij} is assigned to each arc of G'. The PD problem is formulated as finding a partition into k components of the nodes of G' such that: (i) the sum of the node-weights of each component is exactly equal to $\bar{P}$; (ii) the sum of the weights of all the edges connecting two nodes in the same component (multiplied by the populations of such nodes) is a minimum. Since it is difficult to find a feasible solution when population balance is formulated through an equation, the model is relaxed by imposing a bound $0 < \delta < 1$ on the maximum absolute deviation of the district population from $\bar{P}$, thus obtaining the following quadratic program:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{k} p_i p_j d_{ij} x_{ih} x_{jh} && \\
& \sum_{i=1}^{n} p_i x_{ih} \le (1+\delta)\bar{P} && h = 1,\dots,k \\
& \sum_{i=1}^{n} p_i x_{ih} \ge (1-\delta)\bar{P} && h = 1,\dots,k \\
& \sum_{h=1}^{k} x_{ih} = 1 && i = 1,\dots,n \\
& 0 \le x_{ih} \le 1, && i = 1,\dots,n,\ h = 1,\dots,k
\end{aligned}
\tag{6}
$$

In the above model the variables are continuous, meaning that x_{ih} is the percentage of the population of unit i that is assigned to district h. Thus, except for the special case of an integer optimal solution, it may happen that the elementary territorial units are split between more than one district. Suppose, for example, that $x_{ih} = \alpha < 1$, then a portion of unit i containing α percent of the population of i is (manually) cut from the side of i closest to district h and added to h. Note that, to perform this operation, the authors assume that the population density within territorial units is constant. Additional problems with the above model are related to contiguity. Actually, it is taken into account only indirectly via the quantities d_{ij} in the objective function, but this is not sufficient to guarantee that the final district map is contiguous.

The authors provide an application for the design of the congressional districts of the State of New York in 2002. Since the problem has $n = 62$ territorial units (counties) and $k = 29$ districts, an optimal solution for model (6) is found by using a commercial optimization software. However, it must be pointed out that, since $d_{ii} = 0$ for all $i = 1, \dots, n$, (6) is a non-convex quadratic programming problem (Carlson and Nemhauser 1966) and it is NP-hard (Sahni 1974).

To conclude this section, we report an integer linear programming formulation of the PD problem provided in Apollonio et al. (2008) in the special case when the contiguity graph G is a tree. Even if this paper focuses on a more general problem called *Minimum Cost Centered Partition Problem* (MCP), the motivation of the study originates from the application to PD. Given a connected graph $G = (N, E)$, a subset $S \subset N$ and costs c_{is} for every pair $i \in N$ and $s \in S$, a *centered partition* of G is defined as a connected partition of the nodes in N such that each component contains exactly one node in S. MCP can be stated as finding a connected centered partition of G that minimizes the sum of the costs c_{is} between each node i and the node $s \in S$ contained in the same component of i. On a general graph G, MCP can be formulated as the problem of finding a spanning forest F of G such that each tree in F contains exactly one node $s \in S$ and the total cost is minimized. When G is a tree, MCP can be formulated as a linear program based on order constraints which guarantee that the partition of the tree is connected. For this problem the authors provide a polynomial time solution algorithm (Apollonio et al. 2008). Consider the contiguity graph G of a given territory; suppose that a set $S \subset N$ of k centers are already selected, and that for every pair $i \in N$ and $s \in S$, c_{is} corresponds to the distance between unit i and the district center s. Then, the PD problem can be formulated as an MCP with side population balance constraints. If such constraints are included into a Lagrangian objective function, the resulting PD problem with updated values for the costs c_{is} corresponds to an MCP. Unfortunately, MCP is NP-hard on general graphs (Apollonio et al. 2008; Cordone 2001), thus it cannot be exploited for solving PD in real cases.

4.2 Local search

In the last decade, the use of local search techniques took a growing place in the study of PD problems. The main contributions in the literature are aimed to evaluate the performance of those meta-heuristics, like tabu search, simulated annealing, genetic algorithms, etc., that have already shown to be highly successful in the solution of other difficult combinatorial optimization problems (Aarts and Lenstra 2003; Glover and Laguna 1997; Goldberg 1989; Osman and Laporte 1996). We point out that local search techniques are general purpose methods whose power relies on the capability of evaluating a huge number of different solutions in short times. Thus, as in the paper by Bourjolly et al. (1981) already discussed in Sect. 3, many of the works on local search follow a multicriteria approach in which all the PD requirements are formalized in a unique objective (or fitness) function which is generally given by a weighted combination of different indices. In the following we report the main contributions related to the design of electoral districts via local search techniques.

4.2.1 Ricca and Simeone (2008)

Starting from Ricca (1996), Ricca and Simeone (2008) propose a follow-up paper in which the comparative analysis between local search techniques is extended to include also an additional heuristic introduced in Hu et al. (1995) and called *old bachelor acceptance*. This heuristic belongs to the class of threshold algorithms, with a non-monotonic update of the threshold. At each step the threshold value specifies the maximum acceptable change in the objective function. When the algorithm moves from the current solution to a neighbor, the objective function may either improve, or worsen within a fixed threshold limit. Each time the threshold is automatically updated in a non-monotonic way. In particular, it decreases after an improvement in the objective function and increases when the objective function worsens. This updating strategy has shown to be an efficient way to avoid premature halts in bad local optima. This algorithm has the characteristic of becoming "pretentious" when improving moves are easily found, while, when it is hard to find improving moves, it accepts also bad solutions in order to escape from local minima and find new promising search directions. The paper adopts the classical graph-theoretic formulation of the political districting problem and takes into account the traditional PD criteria: contiguity, population balance, compactness, and conformity to administrative boundaries. The aim is to evaluate the performance of three different local search methods, tabu search, simulated annealing, and old bachelor acceptance, taking as a benchmark the results that can be obtained for the same problem with a basic descent algorithm. The application is related to a sample of five Italian Regions whose size ranges from $n = 246$, $k = 8$ to $n = 1208$, $k = 28$. The results show that very good solutions can be obtained with all the local search algorithms (except descent), but in particular a good performance is observed for old bachelor acceptance which produces significant improvements w.r.t. the institutional district maps adopted in Italy in 1994, 1996 and 2001 for the elections of the Chamber of Deputies.

In the above paper the focus is on the intrinsic nature of the different search strategies when they are applied to the specific PD problem. This is why the comparison is related to streamlined versions of all the heuristics, removing as far as possible all those enhancements that can be implemented in order to improve their performance. As a matter of fact, if properly designed, all the above procedures may provide efficient and effective tools for political districting. Actually, this is further shown by other papers in the literature which concentrate on the study of only one of such heuristics trying to improve their performance through the incorporation of specific features.

4.2.2 Bozkaya et al. (2003)

In Bozkaya et al. (2003) the authors propose a new heuristic procedure for PD based on a tabu search algorithm which was developed within a more general search framework known as "adaptive memory procedure". Similar to what happens with genetic algorithms, the basic idea is that parts (districts) of high quality solutions (district maps) can be used to obtain other high quality solutions. Beside the basic PD criteria, i.e., population equality, compactness, and contiguity, the authors consider also socio-economic homogeneity, conformity to already existing plans, and integrity of communities. Except for contiguity, all the criteria are measured by proper indices and then combined into a unique weighted objective function. Contiguity is treated as a hard constraint on the basis of adjacency lists computed for the territorial units. The algorithm starts from an initial map of contiguous districts and this feature is preserved in each new generated district by performing only migrations of a unit from its district to an adjacent one. The adaptive memory procedure in Bozkaya et al. (2003) consists of repeatedly applying tabu search starting from different solutions. The first solution is obtained by a multi-kernel growth method similar to the one proposed in Vickrey (1961). At each iteration tabu search provides a new optimal district map. The quality of the districts belonging to an optimal solution is measured in terms of the corresponding optimal value. The algorithm records a set (of fixed cardinality γ) of good districts that, in genetic algorithms' terminology, is called *pool*. Whenever a new district plan is obtained by tabu search, its districts become candidates for entering the pool. The pool is updated at each iteration by selecting the best γ among the generated districts. To initialize each successive tabu search, a starting solution is generated by selecting districts from the pool with a probability that depends on the quality of the district map they belong to. At most k non-overlapping districts are selected, thus obtaining a *partial solution* that is completed in a multi-kernel growth fashion and then optimized. The algorithm stops after a pre-specified number of iterations.

The application in Bozkaya et al. (2003) refers to the City of Edmonton in Canada, with $n = 828$ and $k = 19$. The data are extracted from the 1996 population census. An experimental analysis of the procedure is provided in which different combinations of values for the tabu search parameters are tested. The experimental results show that good levels of population equality and compactness can be obtained simultaneously, also with good values for the index related to the integrity of communities. In addition, the authors show that, by tuning the weights of the terms in the objective function, their method is able to make the optimal district map meeting any preferred PD criterion.

The idea of borrowing techniques from the field of genetic algorithms is followed also by other authors, even if it seems that this line of research is still not fully exploited for the PD problem. In the following, we give details about the results of two recent papers on this subject (Forman and Yue 2003; Bação et al. 2005), but we want to point out here that other authors in the literature suggest districting procedures belonging to the more general family of evolutionary algorithms (see, for example, Tavares-Pereira et al. 2007).

4.2.3 Forman and Yue (2003)

Forman and Yue (2003) propose a genetic algorithm based on encoding and genetic operators generally adopted to solve the Traveling Salesman Problems. The PD criteria considered in the model are contiguity, population balance, and compactness. All these criteria are taken into account into a single fitness function that is defined as a combination of two indices, one measuring lack of compactness and contiguity, and the other related to population equality. In particular, the compactness index takes into account the number of non-contiguous pieces of a district, basing on the idea that, in this way, in a district map lack of contiguity can be controlled via the fitness function. Although the contiguity graph is not explicitly introduced, a representation of the territory through adjacency lists is considered and the encoding of a solution corresponds to a sequence of territorial units which, in principle, may be contiguous or not. Thus a solution is encoded by an n-vector corresponding to a permutation of the territorial units. The partition into districts can be obtained by scanning such vector and summing the populations of the units until a fixed threshold is met. The initial set of solutions is obtained by generating permutations as follows: starting from a territorial unit lying on the border of the territory, unvisited units are selected at random. Even if the authors report that their algorithm first checks if there are unvisited units which are adjacent to some units already included in the district under construction, non-contiguous solutions may arise during this step. The generated solutions are ranked according to their fitness value, from the best to the worst, and the selection process is performed by choosing solutions (with replacement) according to probabilities proportional to the solutions' ranks. A copy of the best solution in the ranking is always chosen. Usual genetic crossover and mutation operators (Goldberg 1989) are adopted to generate a variety of new solutions (for a detailed description, see Forman and Yue 2003). The procedure heavily relies on the fact that a serious lack of contiguity of a solution is well detected by the fitness function, but, since a big number of solutions is generated at each step and non-contiguity may always arise, an additional "clean-up" process is systematically applied to delete districts with too many enclaves.

Applying the above procedure may lead to a situation in which, in the final set, no solution satisfies the fixed threshold on the population error for all the districts. To bring the district populations within the desired range both pre- and a post-processing phases are performed. Generally, in a given territory, there are at least some units with very large populations; in the pre-processing each large unit is partitioned into a set of smaller subunits by applying the same genetic algorithm to the single units. When also this additional step is not sufficient to reach the desired population level in the final district map, a post-processing is applied which consists of solving a transshipment problem as in Mehrotra et al. (1998).

Basing on data from the 2000 population census, the authors apply their procedure to the three States of North Carolina ($n = 147$, $k = 13$), South Carolina ($n = 69$, $k = 6$) and Iowa ($n = 108$, $k = 5$). Due to the pre-processing phase, n may increase and different values for n are actually tested in order to obtain solutions with a satisfying population balance. Even if the size of the territories under study is generally small, for some of them the results are good. However, it must be pointed out that the main problem of the proposed genetic method is the difficulty in reaching the required population balance when using a unique fitness function summarizing all the PD criteria together. In addition, even if the provided experimental results generally yield to contiguous maps of districts, the procedure is not able to control this feature, and a final check for contiguity is always required on each solution.

4.2.4 Bação et al. (2005)

In Bação et al. (2005) the authors provide an implementation of a genetic algorithm for PD and describe an application to the design of electoral districts in Portugal in 1998 ($k = 93$). Each territorial unit i is represented by its centroid, for which the geographical coordinates are known, and the Euclidean distances between each pair of units are also given. To generate a solution, the approach is to select k territorial units as centers, and then construct the districts around them by assigning each territorial unit to its closest center. The evaluation of a given fitness function and a check for contiguity is performed on each solution. Starting from an initial set of solutions, the algorithm proceeds by generating new ones via the application of genetic operators of selection, crossover and mutation, and it stops when a prespecified maximum number of generation steps without improvements has been observed. The selection, crossover and mutation operators adopted to generate new solutions are well known in the literature (for details, see Bação et al. 2005 and Goldberg 1989), while some of the functions adopted to measure the fitness of a solution are new. A first fitness function is given by $\sum_{j=1}^{k} |P_j - \bar{P}|$, but also hybrid functions are suggested which combine population equality and compactness into a single objective. The proposed functions (to be minimized) are the following: (i) $\sum_{j=1}^{k}(|P_j - \bar{P}| + \sum_{i \in j} d_{u_j i})$; (ii) $\sum_{j=1}^{k}(|P_j - \bar{P}| \cdot \sum_{i \in j} d_{u_j i})$, where u_j is the unit corresponding to the center of district j and $d_{u_j i}$ is the Euclidean distance between the centroids of unit i and unit u_j.

The novel issue in this approach consists of the particular encoding of the solutions. Actually, a solution is encoded through an array with k elements corresponding to "reference points" for the k districts: each of them represents the location of one district. The authors suggest and test two different encodings. A first encoding (*encoding 1*) identifies each district with the centroid of a territorial unit, thus, recording in the array the indices of the k units corresponding to the districts' centers. According to the second encoding (*encoding 2*), the k district reference points may be located everywhere in the territory under study, so that they do not necessarily coincide with the centroid of a territorial unit. In this case, each element of the array is given by the pair of geographical coordinates of the reference point of the corresponding district.

The authors report that the territory of Portugal is subdivided into 18 regions, and that their algorithm was applied to each of them. However, only the results related to the region of Lisbon ($n = 53$, $k = 7$) are discussed in the paper. They test different combinations of the two encodings and of the suggested fitness functions. The experimental results show that the use of the hybrid fitness functions generally does not lead to satisfying trade-offs between population balance and compactness of the districts. The application confirms that the performance of the genetic algorithm strongly depends on the adopted encoding: in particular, better solutions can be found with encoding 2, due to the fact that it allows a much larger number of solution to be evaluated. In spite of this, looking at the district maps reported in Bação et al. (2005), it must be pointed out that some of them show a good level of compactness, but some others—namely, those obtained via encoding 2—contain elongated or octopus shaped districts. Thus, when encoding 2 is adopted, the fitness function must be chosen carefully because it is the only available tool to control compactness.

4.2.5 Yamada (2009)

An interesting heuristic approach to PD is provided in Yamada (2009) where the author formulates PD as a minimax spanning forest problem which is then solved via a local search based heuristic. The region under consideration is modeled by the usual contiguity graph G. The author defines the weight of a tree T as the sum of the weights of all the vertices and of all the edges of T, and adopts an objective function that, for a given forest F, is defined as the weight of the heaviest tree in F, $w(F)$. However, in Yamada (2009) only the weights associated to the vertices of G are taken into account, while those assigned to the edges of G are set to 0. This means that, in the PD problem, the distances between territorial units are never considered. Then, the weight of a tree corresponds only to the sum of the weights of its vertices. Under a graph-theoretic viewpoint, a connected component of G (district) can be always spanned by a tree and, thus, in the model a set U of k vertices of G (corresponding to the k roots of such trees) is assumed to be given. The PD problem is then formulated as finding a spanning forest of G w.r.t. the given set of roots (*U-rooted spanning forest*), that minimizes $w(F)$. To solve the problem, the author refers to an algorithm already provided in Yamada et al. (1996) for the particular case where weights are assigned only to the edges of G. To reduce the PD problem to this case, a new graph G' is introduced by adding to N a copy v' of each vertex v, and to E the edge (v, v'). The weight originally assigned to v is now associated to (v, v'). Thus, in G' all the vertices and the original edges of G have weight equal to zero, while nonnegative weights are assigned only to the new edges (v, v'). The algorithm consists of two phases. In the first one, a greedy method is applied which, starting from the set U, constructs a U-rooted spanning forest by including at each step the edge that produces the least increase in the objective function. The second phase consists of a sequence of adjustments performed by an hill-climbing method. At each iteration, a move is defined as follows. The procedure first selects a tree in the current forest F, say T_i; then, it identifies all the edges (u, v) not included in F, with $u \in T_i$ and $v \notin T_i$; a move consists of cutting from T_i the subtree rooted at one of the above defined u, and connecting it via the edge (u, v) to the tree T_j which vertex v belongs to. Among the possible moves related to a given T_i, the algorithm evaluates the best one and performs it only if it improves the objective function. The author suggests two versions of the algorithm: in the *myopic* version, at each iteration the heaviest tree T_i is selected; in the *hyperopic* version all possible pairs of trees T_i and T_j are evaluated at each step. According to the general hill-climbing scheme, in both versions the algorithm halts when no further improvement is observed.

The author provides experimental results related to the Kanagawa Prefecture in Japan in 1994 with $n = 49$ and $k = 17$. Although the provided districts are contiguous and show a good level of population balance, there is no guarantee of compactness and no control over the districts' shapes. This is due to the fact that the model does not take into account this criterion at all.

4.3 Computational geometry

Due to the spatial dimension of the PD problem, a new class of methods arose borrowing notions and techniques from the computational geometry area. It must be pointed out that this idea was mentioned for the first time by Forrest (1964), but only recently some authors started again to study and develop this approach. Specifically, some papers refer to Voronoi regions or diagrams that are already known in the literature (Aurenhammer and Edelsbrunner 1984): Ricca et al. (2008) performs a discretization of the territory and use the (weighted) discrete version of the Voronoi regions; on the other hand, in Miller (2007) standard Voronoi diagrams are applied to the territory considered as a continuous area. A novel approach is also the one suggested in Kalcsics et al. (2005) that follows a successive dichotomies strategy. All these techniques are heuristics and generally take into account the basic contiguity, compactness and population balance criteria.

4.3.1 Kalcsics et al. (2005)

Even if Kalcsics et al. (2005) concerns the more general territory design problem, the procedure suggested in this paper can be applied to PD. The criteria taken into account are contiguity, population equality and compactness. Starting from the complete set of elementary units, along with their relative position in the plane, the general idea of the algorithm is to repeatedly partition the current set of units into two half-spaces by drawing a straight line, thus generating two new subsets of units. This strategy relies on a binary tree search where the root corresponds to the original problem, while the nodes correspond to the subproblems generated at each successive partitioning. Since the total number of districts is fixed at k, in each subproblem the number of districts to be drawn is reduced and, as soon as this number becomes equal to 1, the corresponding node in the search tree is not explored any more (and it is declared a leaf). When all the active nodes of the tree are leaves the procedure outputs the solution in which the subset of units associated to each leaf corresponds to a district. The performance of the heuristic depends on two main issues, namely, how to perform the partitioning of a problem into two subproblems and how to explore the tree. We do not provide the details of these procedures (for a complete discussion see Kalcsics et al. 2005). However, some attention must be paid to some geometrical aspects related to the generation of a straight line in the plane. In order to reduce the computational effort of each partitioning step, the authors observe that only a small number of possible directions of the lines can be considered. We point out that the quality of the final solution depends on these directions. When choosing a line, it must be taken into account that its direction, together with the spatial distribution of the units in the plane, may have a different impact on the compactness of the resulting subsets. The problem here is to match the discrete nature of the set of units located in the plane with the continuous nature of the half-spaces generated by a straight line. Basing on the convex hull of the units, the authors suggest different methods to obtain compact subsets by means of separating straight lines, also considering the cases in which the units are not uniformly distributed in the plane.

4.3.2 Ricca et al. (2007, 2008)

An approach based on weighted discrete Voronoi regions (or diagrams) is proposed in Ricca et al. (2007, 2008). Although Voronoi diagrams are well known in the field of computational geometry, in these papers it is used for the first time in the context of political districting. The underlying idea is that Voronoi regions are "inherently compact", so that one may overcome the problem of choosing a suitable measure of compactness among the wide variety provided in the literature, which is one of the problems that typically affects PD (Grilli di Cortona et al. 1999; Horn et al. 1993; Niemi et al. 1990; Young 1988). Basing on the representation of the territory through the contiguity graph, Ricca et al. (2008) propose an iterative procedure that alternates the computation of the discrete Voronoi regions (which can be seen as the graph-theoretic counterpart of the ordinary Voronoi diagram in continuous space) with the updating of the distances between the territorial units and each district center. This updating is aimed at taking into account the population equality criterion, since, taking as districts the Voronoi regions, generally guarantees compactness, but a poor population balance might ensue. The algorithm consists of two phases: in the first one, k units are chosen as centers for the districts by applying location techniques, and a starting district map is computed by drawing on G the discrete Voronoi regions w.r.t. the distances d_{ij}, for all units i and centers j; these regions form a connected partition of G. In the second phase, in order to re-balance population, the algorithm performs an iterative procedure where, at the beginning of iteration t, the distance between unit i and the center of district j is updated through a weight w_j^{t-1} proportional to the population of district j in the Voronoi regions computed at iteration $t-1$. The distance updating promotes the unit migration out of "heavier" districts (population-wise) and into lighter ones. Two different updating strategies are implemented; the updating is called *static* if, at iteration t, $d_{ij}^t = w_j^{t-1} d_{ij}$, while it is *dynamic* if $d_{ij}^t = w_j^{t-1} d_{ij}^{t-1}$. The authors propose three different approaches for the migration of the units, all seen as population transfers, namely, the *full* transfer, the *single* transfer, and the *path* transfer. According to the full transfer, at each iteration all sites are allowed to move from their current district to a new one, and, thus, the discrete Voronoi regions are repeatedly recomputed w.r.t. the updated distances. On the other hand, with the single and the path transfers, the Voronoi regions are calculated only at the beginning, and, at each iteration only one unit or a selected set moves to a new district. Altogether six possible variants of the weighted Voronoi algorithm (static/dynamic updating, full/partial/single transfer) are proposed and theoretical results are presented showing properties and pathologies of all variants. In particular, finite termination is proved for the single and the path transfer variants, both in their static and dynamic version, together with other theoretical properties.

The single and the path transfer versions of the algorithm are applied to different test problems: some are taken from real-life data (Italian Regions), while others correspond to randomly generated grid graphs, which, as the contiguity graphs of real territories, are planar and have low vertex degree. The results show a good performance of the Discrete Voronoi approach in finding good compromises between population equality and compactness.

To conclude this section, we point out that other authors in the literature referred to the use of Voronoi diagrams for PD. Even if in her Master thesis, in Miller (2007) the author suggests to use Centroidal Voronoi diagrams to draw congressional districts in the State of Washington. The procedure applied in this work computes a Voronoi Tessellation in the continuous space with the main objective of guaranteeing contiguity and compactness of the districts. The territory is seen as a continuous area Ω, but census tracts are taken into account as population units. A population based probability density function $\rho(y)$ is introduced w.r.t. all the points $y \in \Omega$. Details are not given on how this density is defined, but the general idea of the PD procedure is to repeatedly locate k centers in Ω and compute the corresponding Voronoi diagram. At the beginning, the k district centers are the geographical centroids of the k most populated units. Using the above density function, the procedure repeatedly selects at random a population unit i, finds the district center closest to i, say u_j, and updates u_j by a weighted average of its geographical coordinates and those of the center of i. The procedure is then iterated until some stopping conditions are met. The proposed conditions may be related to a fixed number of total iterations, or to the stability of the current district centers, that is, in two successive iterations, the coordinates of all centers do not change more than a (small) given value.

Even if the above approach seems to be of some interest, it does not consider population equality at all, and the author herself explicitly states that there is no guarantee on the quality of the solutions in terms of population balance.

5 Conclusions

In the last fifty years, the design of electoral districts attracted the attention of many scholars from different research areas. The study of this problem was originally motivated by the fact that a political party may take advantage from the shape and the size of unfair districts in order to gain more seats. This makes PD a very controversial issue and the discussion on this problem is always lively in the literature of Political and Social Science on one side, and in Operations Research, and Computer Science on the other side. The wide interest for PD in the latter fields is due to the fact that this problem can be described rigorously through formal models, and automated techniques can be adopted to solve it. The idea is that an algorithmic approach may protect against politically motivated manipulations of the district map. Therefore, this kind of approach, based on the satisfaction of criteria such as population equality and compactness, is strongly advised in order to provide neutral districts. Actually, pursuing such criteria may help in avoiding party manipulation, but it must be pointed out that they cannot guarantee that no party is advantaged by the designed districts. However, the use of automated districting procedures is at least able to ensure that systematic distortions of the electoral outcome are avoided, and, denying political parties the opportunity to manipulate districts, they can be very useful to provide fair district maps.

In this survey we provided a review of papers dealing with PD in the OR literature of the last five decades, and classified them according to their solution approach. The first methodologies suggested in the 60's, such as, multi-kernel growth strategies or location models and techniques, were widely exploited till the 90's, but, after this, they were abandoned and it seems that no further improvement can be obtained in these directions. In the following years, there was a strong use of local search techniques for PD, with a particular interest for the performance of meta-heuristics, such as tabu search and simulated annealing. Also the exact solution approach is treated in the literature and different integer linear and non-linear models are suggested for the correct formulation of the problem. In applications of very small size, that frequently characterize the papers review above, the exact solution of these models can be obtained even by the use of standard optimization software. However, as other real-world applications show, the territory under study may have a much larger size so that the exact approach often fails. Some recent papers deal with the use of notions and methods from computational geometry, leading to the definition of a new generation of heuristic approaches. The PD works reported in this review are summarized in Table 1 where they are

Table 1 PD algorithmic approaches

Approach	60's	70's	80's	90's	2000's
Multi kernel growth	Vickrey (1961)	Bodin (1973)		Arcese et al. (1992)	
Location	Hess et al. (1965)		Hojati (1996) George et al. (1997)		
Exact appr.		Garfinkel and Nemhauser (1970)	Nygreen (1988)	Mehrotra et al. (1998)	Nemoto and Hotta (2003); Li et al. (2007)
Local search				Bourjolly et al. (1981); Ricca (1996)	Ricca and Simeone (2008); Bozkaya et al. (2003); Forman and Yue (2003); Bação et al. (2005); Yamada (2009)
Comput. geometry					Kalcsics et al. (2005); Ricca et al. (2007, 2008); Miller (2007)

classified according to the year of publication and the adopted solution approach. We point out here that, besides the contributions discussed in detail in this survey, there is a variety of other papers in which real-life applications of PD are solved by traditional techniques. Finally, also some very particular results can be found for PD in the literature, such as, for example, the paper by Chou and Li (2006) which proposes a study of PD based on q-state Potts models that are well known models in physics.

Besides the different solution methods, it seems that two main difficult issues characterize PD, namely, guarantee spatial contiguity of the units in the same district, and accomplish a good population balance. The papers reviewed in this article often share strategies for dealing with them. Whatever the algorithmic approach, it is usually computationally expensive to check contiguity of the districts in each generated solution; on the other hand, formulating contiguity as an explicit constraint in an algebraic model generally makes the problem much harder to solve. This is the reason why many authors do not take contiguity into account at all in their models, but they plan a post-processing phase in which contiguity is forced, often manually. The achievement of a good population balance between the districts is also a difficult task. When integrity of the population units is not considered as a hard constraint, split units may arise and a post-processing phase is required. Many authors make use of transportation or transshipment models in order to fix this problem. Others try to avoid this trouble by preliminary subdividing the elementary territorial units into smaller entities. Table 2 reports the criteria adopted in each reviewed article. The basic PD criteria are present in the majority of the papers, even if there are examples in which important criteria are not taken into account. In particular, contiguity is not considered by Hojati (1996), George et al. (1997) and Li et al. (2007); Bodin (1973), Nemoto and Hotta (2003) and Yamada (2009) do not take compactness into account; Miller (2007) does not consider population balance. In this table, a double asterisk for contiguity means that no explicit constraint is included in the model, but the criterion is taken into account only *a posteriori*. A double asterisk for population equality means that split units may arise.

Table 2 Districting algorithms and criteria

PD model	Population equality	Compactness	Contiguity	Other criteria
Vickrey (1961)	*	*	*	*
Bodin (1973)	*	–	*	–
Arcese et al. (1992)	*	*	*	*
Hess et al. (1965)	**	*	**	–
Hojati (1996)	**	*	–	–
George et al. (1997)	**	*	–	–
Garfinkel and Nemhauser (1970)	*	*	*	–
Nygreen (1988)	*	*	*	*
Mehrotra et al. (1998)	*	*	*	–
Nemoto and Hotta (2003)	*	–	*	–
Li et al. (2007)	*	*	–	–
Bourjolly et al. (1981)	*	*	**	*
Ricca (1996)	*	*	*	*
Ricca and Simeone (2008)	*	*	*	*
Bozkaya et al. (2003)	*	*	*	*
Forman and Yue (2003)	**	*	*	–
Bação et al. (2005)	*	*	**	–
Yamada (2009)	*	–	*	–
Kalcsics et al. (2005)	*	*	*	–
Ricca et al. (2007, 2008)	*	*	*	–
Miller (2007)	–	*	*	–

To conclude, we believe that the review proposed in the present paper gives an idea of how much interest and effort was dedicated to the design of electoral districts in the OR literature in the last fifty years. The study of this problem seems to be a living matter for the OR researchers, always driven by the aim of promoting intelligent problem solving procedures to support the decision makers. Among the classical problems related to electoral systems, PD remains perhaps the most studied and the most challenging one, as it is shown by the abundance and the variety of papers discussed above. It must be pointed out that, besides the useful tools that applied mathematics is able to provide for the practical solution of real-life problems, many of these papers contain also new theoretical results in OR, showing how the study of real-life applications like PD is able to give rise to "good" mathematics.

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