Introduction to Markov Processes

- What is a Markov process?
- Markov reward processes
- Markov decision processes

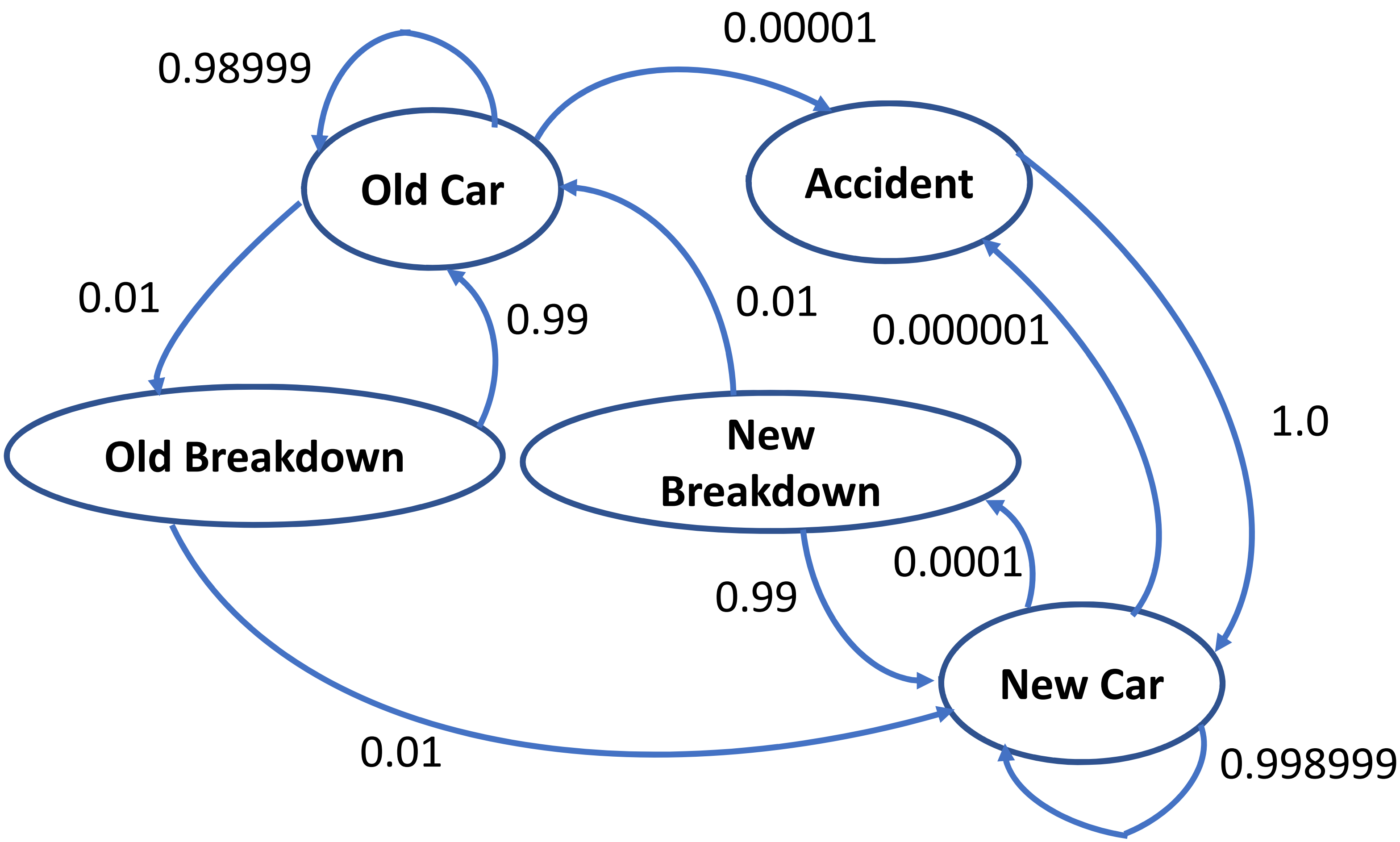
Introduction to Markov Processes

- A **Markov process** has **states**
- The probability of transition from one state to another for a **first order Markov process** is determined only by the **current state**:

$$p[S_{t+1} \mid S_1, \ldots, S_t] = p[S_{t+1} \mid S_t]$$

- Where, the history of states is $S_1, \ldots, S_t$.
- And, the current state is S_t.

Example of Markov Processes



Introduction to Markov Processes

- A Markov process is characterized by a **state probability transition matrix**:

$$\mathcal{P}_{ss'} = \begin{bmatrix} P_{11} & \dots & P_{1n} \\ \vdots & \vdots & \vdots \\ P_{n1} & \dots & P_{nn} \end{bmatrix}$$

- Where, $\mathcal{P}_{ij} =$ probability of transition from state s_i to s_j

Introduction to Markov Processes

- The probability of transition from one state to the next state is computed with the state transition probability matrix:

$$S' = \mathcal{P}_{ss'} S$$

- Or,

$$\begin{bmatrix} s'_1 \\ \vdots \\ s'_n \end{bmatrix} = \begin{bmatrix} P_{11} & \dots & P_{1n} \\ \vdots & \vdots & \vdots \\ P_{n1} & \dots & P_{nn} \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$$

Introduction to Markov Processes

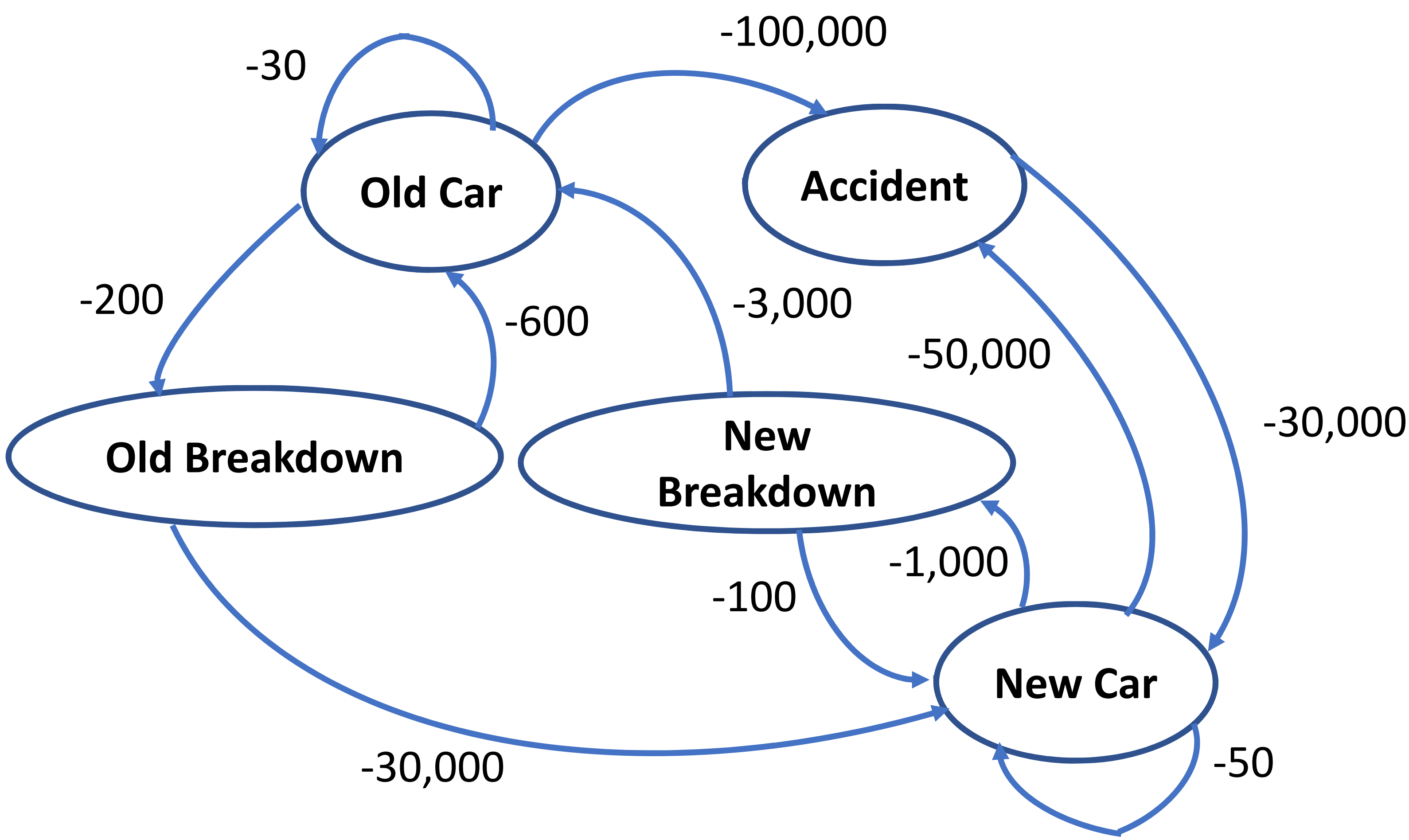
- A **Markov chain** is a sequence of Markov state transition processes
  - E.g. running a Markov process over several time steps creates a Markov chain
- If the state transition probability matrix, $\mathcal{P}_{ss'}$, does not change with time, the Markov chain is **stationary**
- Stationary Markov chains **converge to a steady state**
  - At steady state the state probabilities are unchanged

Introduction to Markov Reward Processes

- A **Markov reward process** generates a reward or change in utility for each state transition
- Reward can be positive or negative
- Reward may not follow economic value
  - The inconvenience of a car breakdown may exceed the cost of repair
  - A piece of art has aesthetic value
- The **reward function** for a transition from state S_t to state S_{t+1} is defined:

$$\mathcal{R}_{ss'} = E\left[R_{t+1} \mid S_t = s\right]$$

Example of Markov Reward Process



Introduction to Markov Reward Processes

- **Utility** is the sum of reward in a Markov chain
- Since the rewards are **additive**:

$$U([s_o, s_1, \ldots, s_T]) = R(s_o) + R(s_1) + \ldots + R(s_T) = \sum_{t=0}^{T} R(s_t)$$

- The above formulation works for an **episodic process**
- An episodic process has a **terminal state**

Introduction to Markov Reward Processes

- What happens to the **Utility** for a **continuous process**
- A continuous process has **no termination state** and the **utility is unbounded**

As $T \rightarrow \infty$ $U(s_t) \rightarrow \infty$

- So, apply a **discount factor** at each time step:

$$U([s_o, s_1, s_2, s_3 \ldots]) = R(s_o) + \gamma R(s_1) + \gamma^2 R(s_2) + \gamma^3 R(s_3) \ldots = \sum_{t=0}^{\infty} \gamma^t R(s_t)$$

- Properties of discounted returns

As $\gamma \rightarrow 0$, the reward process becomes myopic, only counting near term rewards

As $\gamma \rightarrow 1$, the reward process becomes far sighted, valuing distant rewards highly.

Introduction to Markov Reward Processes

- The **Gain** at time t is the sum of future rewards in a Markov chain:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} \dots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- We can define the **state value function**:

$$v(s) = E[G_t \mid S_t = s]$$

Introduction to Markov Decision Processes

A **Markov decision process** is a tuple $<S, A, P, R, \gamma>$:

- S is a finite **set of states**
- A is a finite **set of actions**
- P is the **state transition probability matrix**

  $$P^a_{ss'} = \mathbb{P}[S_{t+1} = s' \mid S_t = s, A_t = a]$$

- R is a **reward function** with expectation of reward given the state and action

  $$R^a_s = \mathbb{E}[S_{t+1} = s' \mid S_t = s, A_t = a]$$

Introduction to Markov Decision Processes

A **policy, π,** is probability distribution over actions given states:

$$\pi(a|s) = \mathcal{P}[A_t = s' \mid S_t = s]$$

- The policy **fully defines agent behavior**
- The MDP **depends only on current state**, not history
- A given policy is **stationary**; does not change in time

Introduction to Markov Decision Processes

Given a **Markov decision process** tuple <S, A, P, R, γ> and policy π:

- *Let* $S_{1,} S_{2,} S_{3,\ldots,} S_t$ be a **state sequence** determined by a **Markov process**
- Then, probability of state transitions and rewards are:

$$P^{\pi}_{ss'} = \sum_A \pi(a|s)\, P^{a}_{ss'}$$

$$R^{\pi}_{s} = \sum_A \pi(a|s)\, R^{a}_{s}$$

Optimal Policy for MDP

- Actions of the agent are determined by a **policy,** π
- The expectation of the policy determines the **action value**

$$q_\pi(a) = \mathbb{E}_\pi[R_t \mid A_t = a]$$

- Goal is to **learn** an **optimal policy**

$$q_{\pi^*}(a) = \mathbb{E}_{\pi^*}[R_t \mid A_t = a]$$

- The optimal policy has an expected action value greater than or equal to all possible policies:

$$q_{\pi^*}(a) \geq q_\pi(a) \; \forall \; \pi$$