RL with Function Approximation

- Why use function approximation for RL?
- Function approximators and basis functions
- Linear approximators and coding
- Linear functions and gradient descent
- The mountain car problem
- Solving the mountain car problem
- Q-learning and function approximation
- Deep Q Network algorithm
- Double DQN algorithm
- Prioritize relay

Why Use Function Approximation?

- Up to now, we have only used **tabular algorithms**
- Tabular algorithms have limits of scalability
  - Value function needs table entry for every state
  - Action-value function need table entry for every action-value pair.
- Problems with large number of discrete states and actions
  - Backgammon: 10^{20} states
  - Go: 10^{170} states
- Problems with continuous variables
  - Examples; temperature, pressure, acceleration, velocity, etc.
  - Have an infinite number of states

Why Use Function Approximation?

- How to scale beyond **tabular algorithms?**
- Need a better **representation**
- Use **function approximation**
  - Agent **learns value function**

    v(s) ~ f(s,w) = function of features w
  - Or, agent **learns action-value function**

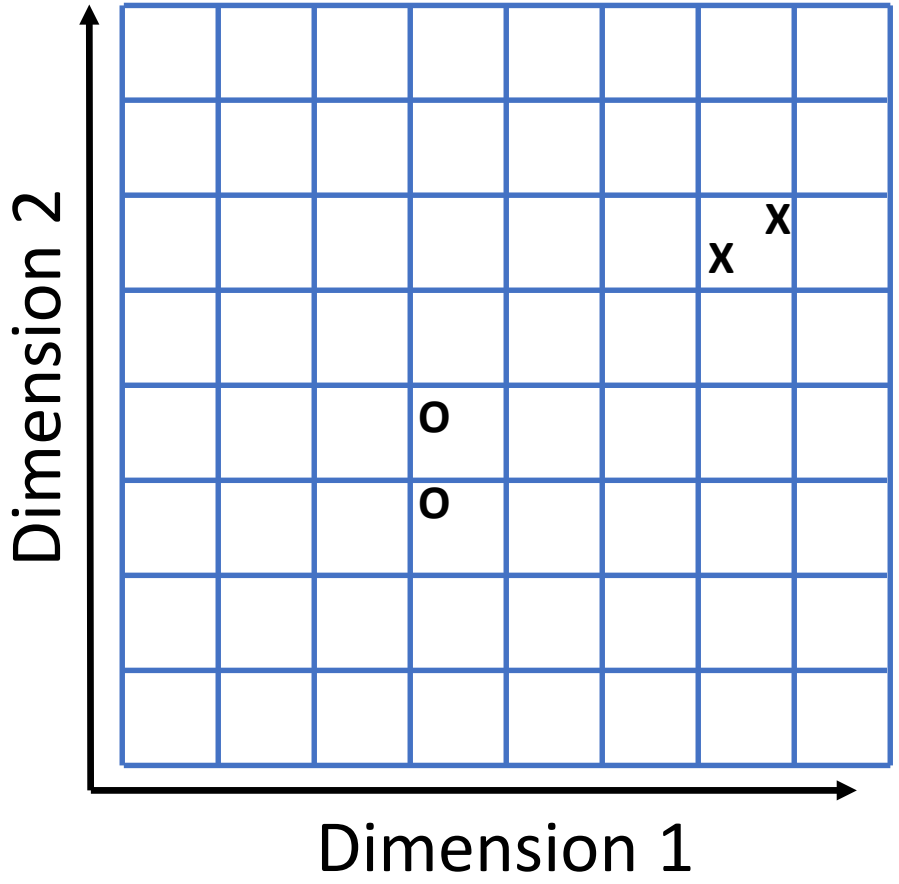
    q(s,a) ~ f(s,a,w)
  - f(s,w), f(s,a,w) have **sparse number of parameters** compared to original space

Function Approximators

- Which **function approximators** to use?
- Linear function approximators
  - Grid coding
  - Coarse coding
  - Fourier and wavelet basis function
  - Radial basis function – e.g. Gaussian
- Nonlinear function approximators
  - Nearest neighbors
  - Decision trees
  - Neural networks
  - More on these later

Coarse Coding

Example, **tile coding**



Two continuous variables

Divided values on rectangular grid

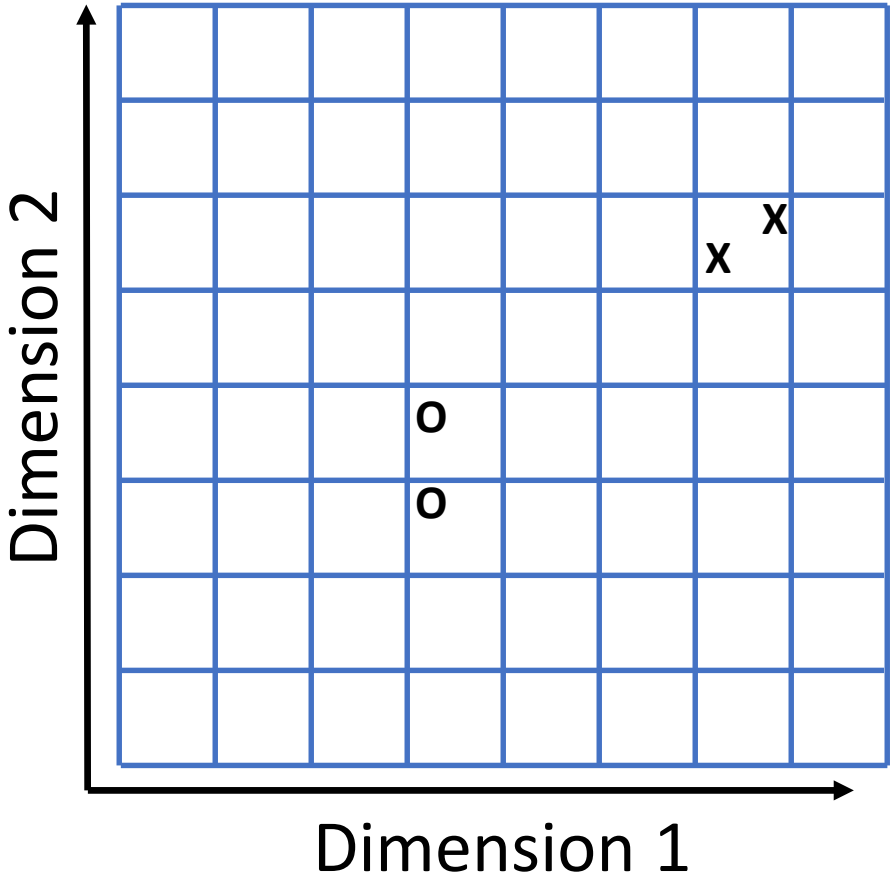
Values coded on the grid

O coded in two tiles

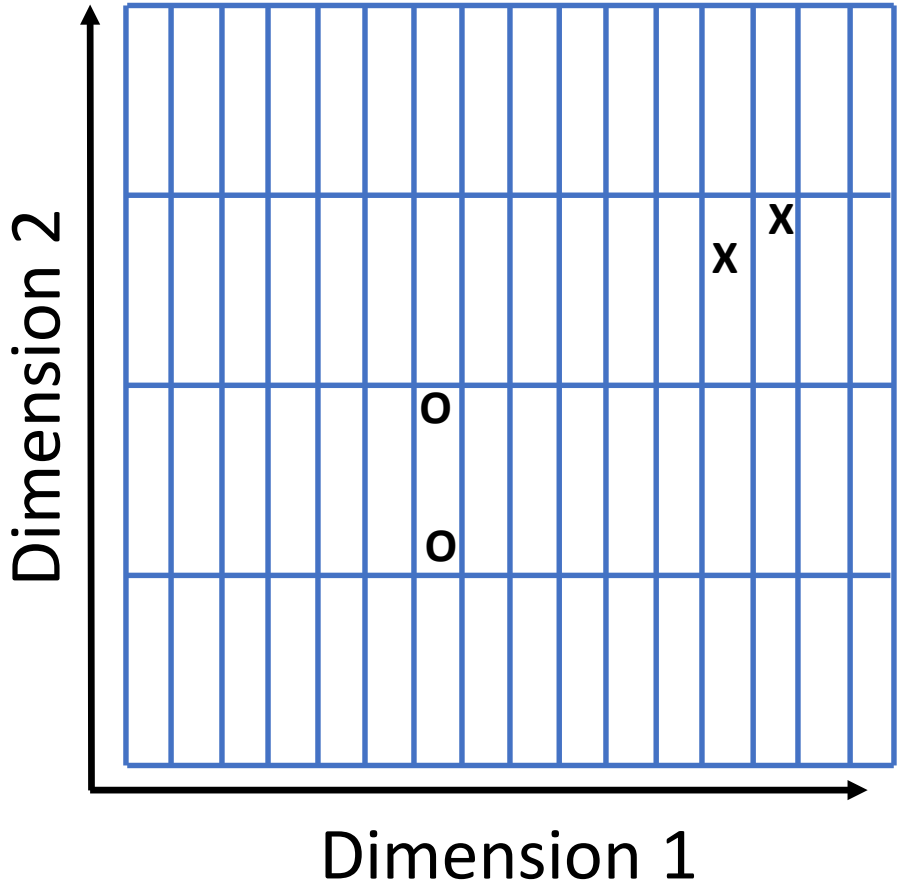
Xs coded in one tile

Coarse Coding

Example, **tile coding**



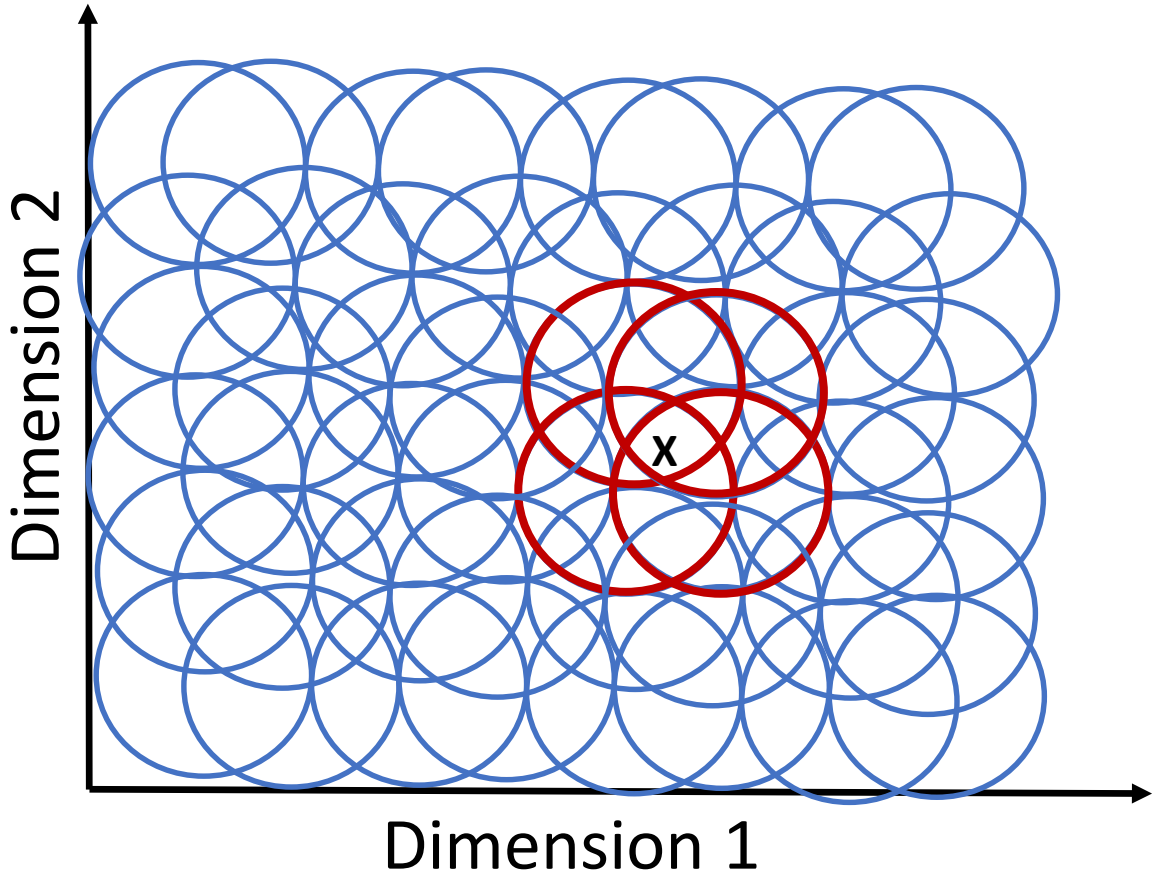
Os coded in two tiles
Xs coded in one tile



Os coded in one tile
Xs coded in one tile

Coarse Coding

Example**, overlapping coarse coding**



Point coded in 4 circles

Linear Functions and Gradient Descent

Grid coding is a linear function approximator

- Each **tile is a feature**
- Each linear **feature has one parameter** or **model weight**
- Function approximator is **linear in the parameters**

Linear Functions and Gradient Descent

The value function as a **linear function of model weights**

- The approximate value function: $\hat{v}(s, \mathbf{w})$
- The weight vector, **w**
- Coding is a function of state: $\mathbf{x}(s)$
- The coded approximate value function is then:

$$\hat{v}(s, \mathbf{w}) = \mathbf{w}^T \mathbf{x}(s) = \sum_{i=1}^{d} w_i x_i(s)$$

Linear Functions and Gradient Descent

How good is the approximation?

- Like any approximator there is a difference between the estimated and actual values
- Can use **value function approximation**, $\hat{v}(S_t, \mathbf{w}_t)$, of the true state value given a policy π, $v_\pi(s)$
- OR, an **action-value function approximation**, $\hat{q}(s, a)$, of the true action-value value given a policy π, $q_\pi(s, a)$

Linear Functions and Gradient Descent

How good is the approximation?

- Compute the **mean square value error** between $v_\pi(s)$ and $\hat{v}(S_t, \mathbf{w}_t)$ as the metric or **loss function**:

$$\overline{VE}(w) = \sum_{s \in S} \mu(s) \left[v_\pi(s) - \hat{v}(s, \mathbf{w}) \right]^2$$

- Where, $\mu(s)$ is the **probability of being in state, s**
- For on-policy algorithms $\mu(s)$ is known as the **on-policy distribution**

Linear Functions and Gradient Descent

Solve linear system of equations with **gradient descent**

- Must compute a **gradient with respect to the d-dimensional weight vector, w**

$$\nabla_w \hat{v}(S_t, \mathbf{w}_t) = \begin{bmatrix} \frac{\partial \hat{v}(S_t, \mathbf{w}_t)}{\partial w_1} \\ \frac{\partial \hat{v}(S_t, \mathbf{w}_t)}{\partial w_2} \\ \vdots \\ \frac{\partial \hat{v}(S_t, \mathbf{w}_t)}{\partial w_d} \end{bmatrix}$$

Linear Functions and Gradient Descent

Solution with **stochastic gradient descent**

- For a **random sample of the data**, $\hat{p}data$, and a **loss function** $J(\mathbf{w}_t)$ the **stochastic gradient descent update** is:

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha \, E_{\hat{p}data}\left[\nabla_w J(\mathbf{w}_t)\right]$$

- Using the estimated state-value function, $\hat{v}(S_t, \mathbf{w}_t)$, the stochastic gradient descent update becomes

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha\left[v_\pi(s) - \hat{v}(S_t, \mathbf{w}_t)\right]\nabla_w \hat{v}(S_t, \mathbf{w}_t)$$

- Since $\hat{v}(S_t, \mathbf{w}_t)$ is a **bootstrap estimate** of $v_\pi(s)$, $\nabla_w \hat{v}(S_t, \mathbf{w}_t)$ is a **semi-gradient**

Mountain Car Problem

- The mountain car problem is a canonical control problem used to test RL algorithms
  - An under-powered car must travel to the top of a hill
  - The car can accelerate forward, +1, backward, -1, and neutral, 0
  - The agent must learn a policy to get the car to the top
- There are two state variables
  - Position
  - Velocity

Mountain Car Problem

- The state equation for car position is:

$$x' = x + \dot{x}$$

- The state equation for car velocity is:

$$\dot{x}' = \dot{x} + 0.001 * \ddot{x} - 0.0025 * cos(3 * x)$$

- Acceleration, is the action, determined by the agent from the set:

$$\ddot{x} = \{-1.0, 0.0, 1.0\}$$

Mountain Car Problem

- Car reward function:
  - Each time step, -1
  - A goal, 100
- The position and velocity of the car are limited

$$-1.2 \leq x \leq 0.5$$
$$-0.07 \leq \dot{x} \leq 0.07$$

- Starting position of car is random

$$p(x_0) = uniform(-0.6 \leq x_0 \leq -0.4)$$

Mountain Car Problem

Why is the Mountain Car Problem Hard?

- Several characteristics make this problem difficult for an agent to learn a good policy
- Positive reward delayed to end of episode
- State variables, position and velocity, are continuous
- The relationship between the state variables is nonlinear

Solving the Mountain Car Problem

Use 3-dimensional tile coding

- First dimension divides the position interval, $-1.2 \leq x \leq 0.5$
- Section dimension divides the velocity interval, $-0.07 \leq \dot{x} \leq 0.07$
- Third dimension has three steps for the acceleration state, $\{-1.0, 0.0, 1.0\}$

- Coding function $x_i(s, a)$ has values:

$$x_i(s, a) = 1 \; if \; in \; tile \; i$$
$$x_i(s, a) = 0 \; otherwise$$

Solving the Mountain Car Problem

Use an approximate linear function for action-values

$$q(s,a) \approx \hat{q}(s,a,\mathbf{w}) = \sum_{i=1}^{d} w_i x_i(s,a)$$

- Where,

$\hat{q}(s,a,\mathbf{w})$ = the approximate state-value function

$x_i(s,a)$ = state-action tile coding function

w_i = function weights, which must be learned

s = state variable tuple, position and velocity

Solving the Mountain Car Problem

n-step SARSA for control

- The n-step gain with function approximation:

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^{n} \hat{q}(S_{t+n}, A_{t+n}, w_{t+n-1})$$

- The weight update:

$$w_{t+n}(S_t, A_t) = w_{t+n-1} + \alpha \left[G_{t:t+n} - \hat{q}(S_{t+n}, A_{t+n}, w_{t+n-1}) \right] \nabla \hat{q}(S_t, A_t, w_{t+n-1})$$

Where:

$$\delta_t = G_{t:t+n} - \hat{q}(S_{t+n}, A_{t+n}, w_{t+n-1}) = \text{the n-step TD error}$$

$$\nabla \hat{q}(S_t, A_t, w_{t+n-1}) = \text{the gradient of the action value approximation}$$

Solving the Mountain Car Problem

Need to find the semi-gradient of the linear function approximation

$$\nabla \hat{q}(S_t, A_t, w_{t+n-1})$$

- The action-value function, $\hat{q}(S_t, A_t, w_{t+n-1})$, is linear in w_i, so:

$$\nabla \hat{q}(S_t, A_t, w_{t+n-1}) = 1, \text{ if } x_i(s, a) = 1$$

else, $\nabla \hat{q}(S_t, A_t, w_{t+n-1}) = 0$

Q-Learning and Function Approximation

What happens if Q-learning is used in tabular case?

- Recall basic Q-learning update

$$Q(S_t, A_t) = Q(S_t, A_t) + \alpha\left[R_{t+1} + \gamma\, max_a Q(S_{t+1}, a) - Q(S_t, A_t)\right]$$

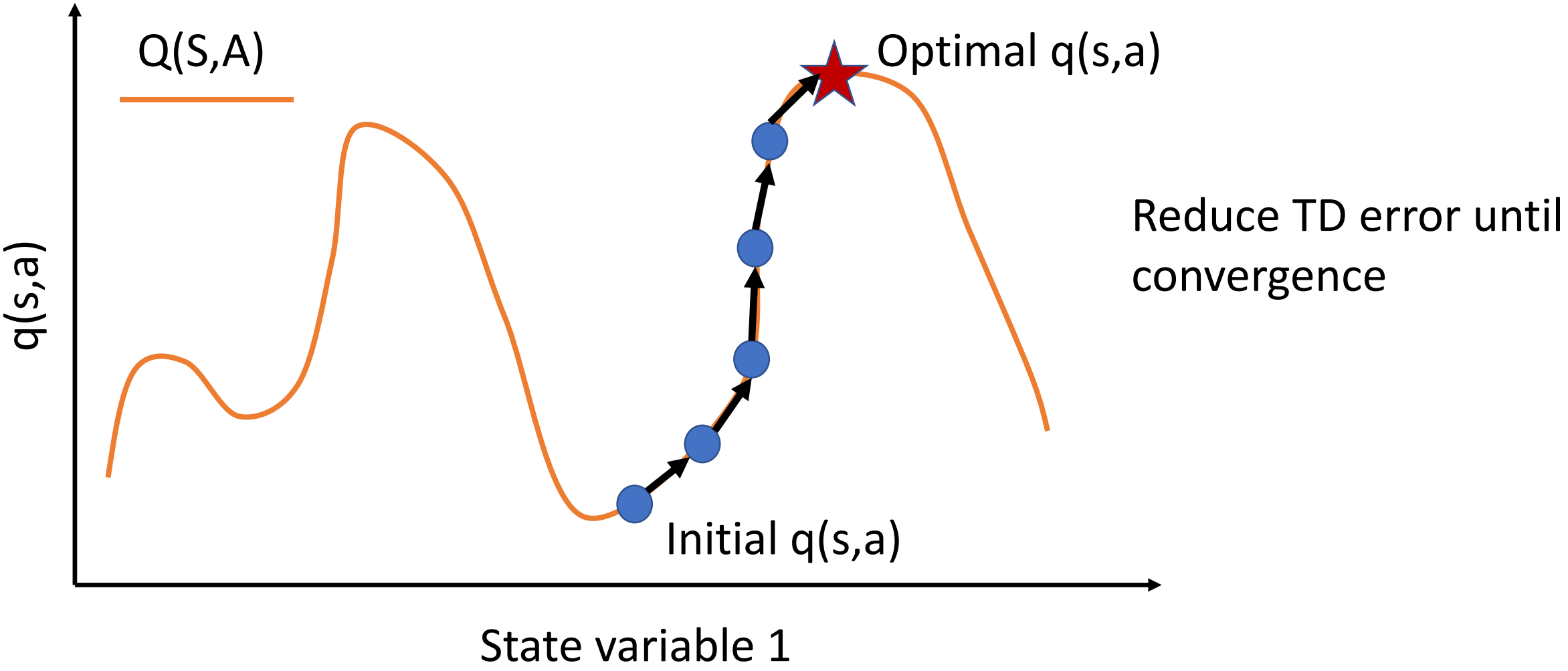
- The TD error is then,

$$\delta_t = R_{t+1} + \gamma max_a Q(S_{t+1}, a) - Q(S_t, A_t)$$

- For tabular case, convergence as δ_t approaches 0

Q-Learning and Function Approximation

Q-learning converges in the tabular case – discrete state-actions



Q-Learning and Function Approximation

What happens if Q-learning is used with function approximation?

- The TD error is uses an approximation of action value function, $\hat{Q}(s_t, a_t, \mathbf{w}_t)$

$$\delta_t = G_t - \hat{Q}(s_t, a_t, \mathbf{w}_t)$$

- The gain depends on the nonlinear $\max_a$ operator

$$G_t = R_{t+1} + \gamma \max_a Q_\pi(s_{t+1}, a, \mathbf{w})$$

- But, the action-value function approximator minimizes least squares error

$$\overline{VE}(\mathbf{w}) = \sum_{s \in S} \mu(s) \left[q_\pi(s, a) - \hat{q}(s, a, \mathbf{w}) \right]^2$$

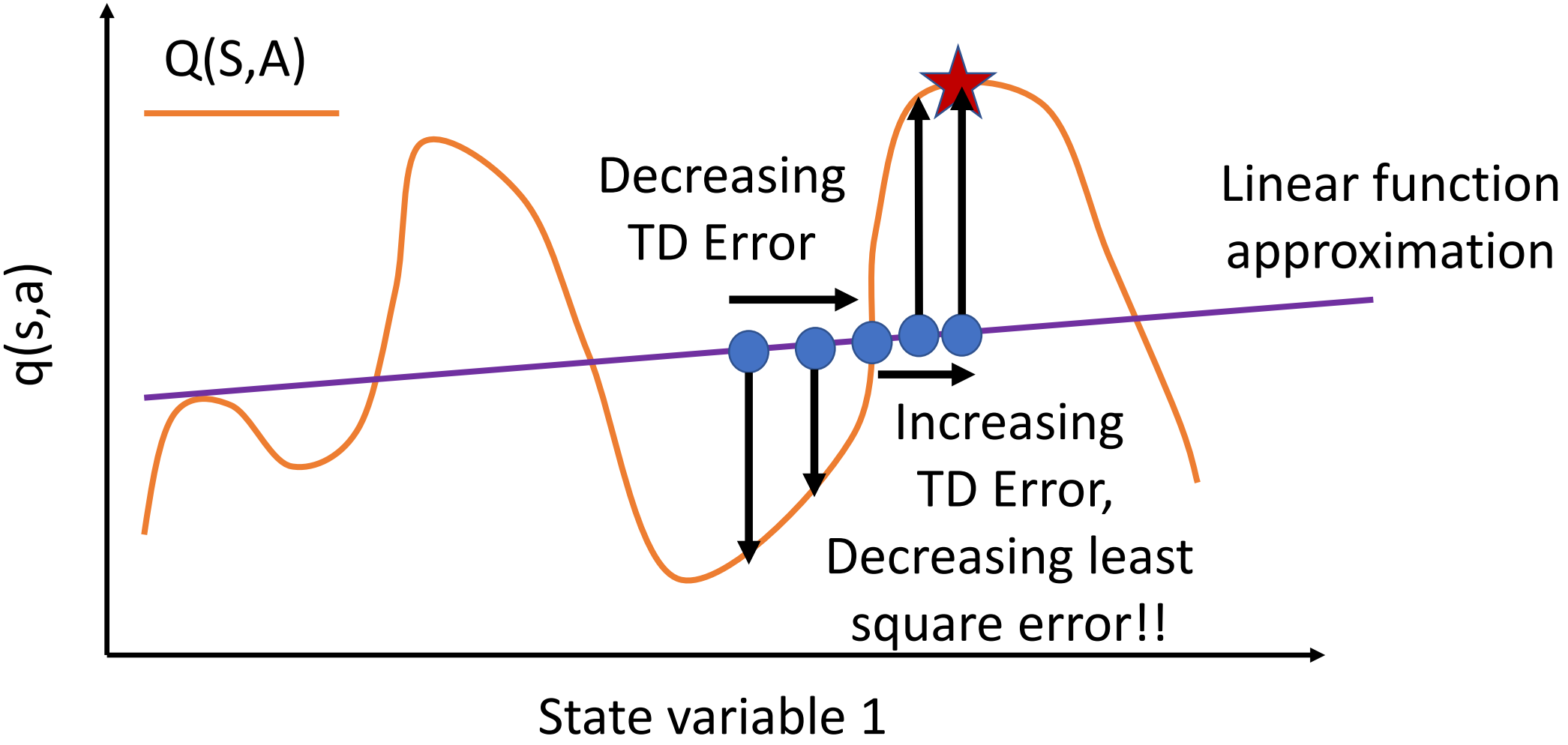
Q-Learning and Function Approximation

What happens if Q-learning is used with function approximation?

- Using TD error based on max operator with function approximation using least squares error leads to **instability and poor convergence**!
- Problem arises went three conditions are met:
  - Function approximation; $\hat{Q}(s_t, a_t, \mathbf{w}_t)$
  - Off-policy algorithm
  - Bootstrapping with an approximate action-value
  - The **deadly triad!**
- Monte Carlo control algorithms do not bootstrap and always converge

Q-Learning and Function Approximation

Q-learning will not converge with function approximation



Q-Learning and Function Approximation

Convergence of control algorithms

Algorithm	Tabular	Linear	Nonlinear
Monte Carlo Control	✓	✓	✓
SARSA	✓	✓	✗
Q-learning	✓	✗	✗

DQN Algorithm

Deep Q Network as a function approximator

- The **DQN algorithm** learns the weights of $\hat{Q}(s_t, a_t, \mathbf{w}_t)$ as a **regression problem**
- The regression estimator learns $\mathbf{w}_t$ minimizing loss function

$$J(\mathbf{w}_t) = \frac{1}{2} \| G_t - \hat{Q}(s_t, a_t, \mathbf{w}_t) \|^2$$

Where

$$G_t = R_{t+1} + \gamma \max_a Q_\pi(s_{t+1}, a, \mathbf{w})$$

DQN Algorithm

Deep neural network as a function approximator for differentiable function

- The gradient descent weight update is:

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha\left[R_{t+1} + \gamma \, max \, Q_\pi(s_{t+1}, a_{t+1}) - \hat{Q}(s_t, a_t, \mathbf{w}_t)\right] \nabla_w \hat{Q}(s_t, a_t, \mathbf{w}_t)$$

- With semi-gradient $\hat{\nabla}_w q(S_t, A_t, \mathbf{w}_t) = \begin{bmatrix} \frac{\partial \hat{v}(S_t, A_t, \mathbf{w}_t)}{\partial w_1} \\ \frac{\partial \hat{v}(S_t, A_t, \mathbf{w}_t)}{\partial w_2} \\ \vdots \\ \frac{\partial \hat{v}(S_t, A_t, \mathbf{w}_t)}{\partial w_d} \end{bmatrix}$

DQN Algorithm

Deep neural network as a function approximator

- The regression estimator learns $\mathbf{w}_t$ minimizing loss function

$$J(\mathbf{w}_t) = \frac{1}{2} \| G_t - \hat{Q}(s_t, a_t, \mathbf{w}_t) \|^2$$

- $\hat{Q}(s_t, a_t, \mathbf{w}_t)$ is the **online model** with weights, $\mathbf{w}_t$, to be learned
- $\mathbf{w}_t$ is updated on each training epoch, as typical with deep neural networks

DQN Algorithm

Deep neural network as a function approximator

- The regression estimator learns $\mathbf{w}_t$ minimizing loss function

$$J(\mathbf{w}_t) = \frac{1}{2} \parallel G_t - \hat{Q}(s_t, a_t, \mathbf{w}_t) \parallel^2$$

- The gain is: $G_t^{DQN} = R_{t+1} + \gamma \, max \, Q_\pi(s_{t+1}, a_{t+1}, \mathbf{w}_t^-)$
- $Q_\pi(s_{t+1}, a_{t+1}, \mathbf{w}_t^-)$ is the **target model** with **fixed weights** $\mathbf{w}_t^-$
- Weights $\mathbf{w}_t^-$ must be frozen so gain, G_t^{DQN}, bootstrap estimate is stable
- Every T epochs the target model weights are updated, $\mathbf{w} \rightarrow \mathbf{w}^-$

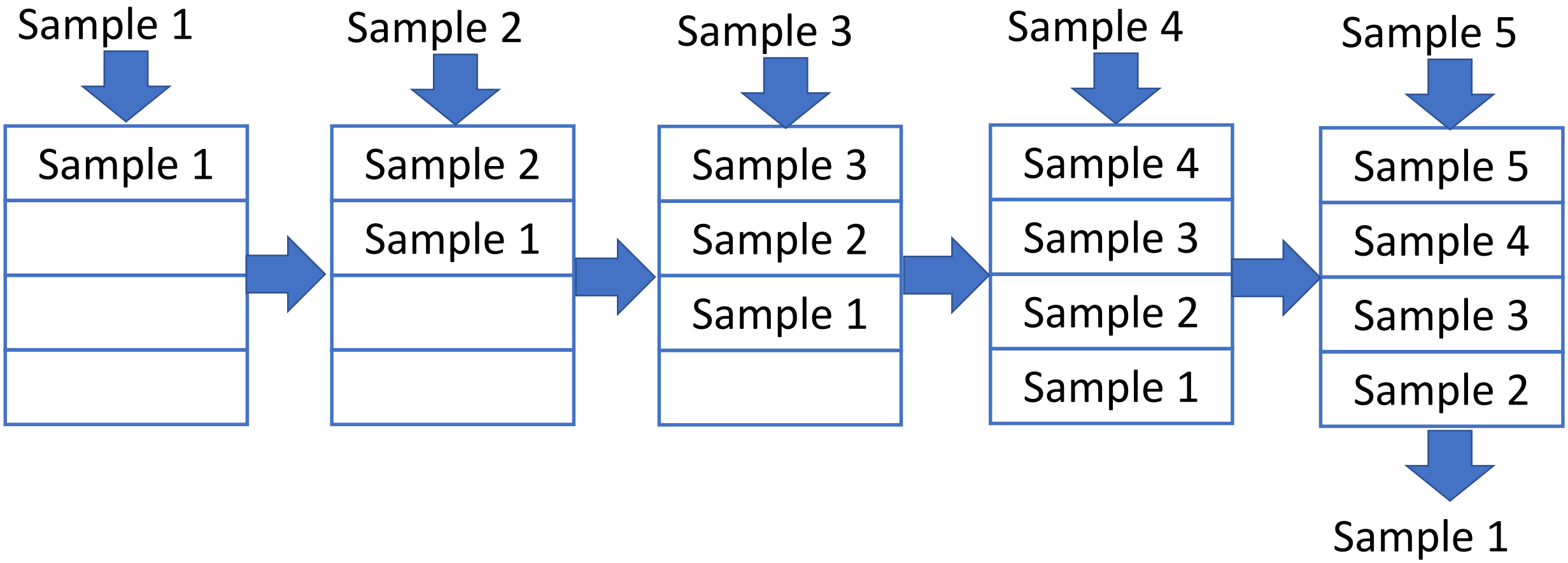
DQN Algorithm

The Replay Buffer

- Deep neural networks **trained with stochastic gradient descent**
- Gradient descent requires **mini-batch samples** from data
- How is this done?
- Use **replay buffer**

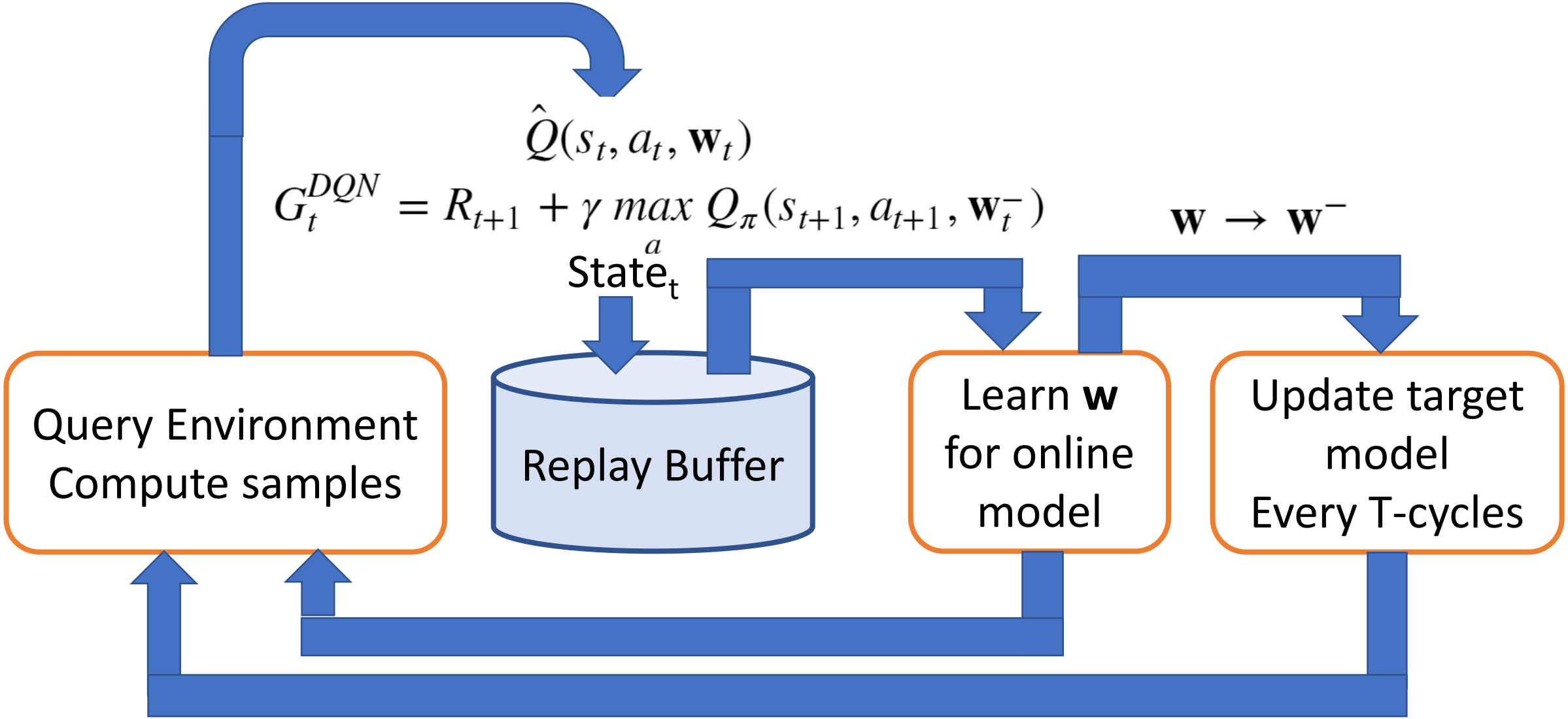
DQN Algorithm

The Replay Buffer



DQN Algorithm

Putting everything together



Double DQN Algorithm

- How to deal with the **bias in the DQN algorithm?**
- The **Double DQN algorithm** eliminates the bias
- The DDQN algorithm uses **two online models**
  1. Initially one model acts as the **online model** and the other as the **target model**
  2. Samples are added to the same **replay buffer**
  3. The **online model is updated**
  4. The **roles of the models are switched** and return to step 1
- Alternating models and sampling eliminates bias

Prioritized Replay

- For DQN and DDQN is **uniform probability of sampling** (learning) from case from replay buffer
- Uniform sampling can lead to slow learning
  - Samples with little information sampled
  - Samples with high information may not be sampled
- Prioritized replay buffer implements importance sampling
  - Samples with high information more likely to be sampled

Prioritized Replay

Importance sampling for prioritized replay

- Unfortunately no direct way to measure importance or information
- Use **absolute value of TD error as proxy:**

$$p_i = |\delta_t| = |R_{t+1} + \gamma \max_a Q_\pi(s_{t+1}, a_{t+1}) - \hat{Q}(s_t, a_t, \mathbf{w}_t)|$$

- This importance measure is both **deterministic** and **subject to noise**

Prioritized Replay

Importance sampling for prioritized replay

- Use **absolute value of TD error as proxy:**

$$p_i = |\delta_t| = |R_{t+1} + \gamma \max_a Q_\pi(s_{t+1}, a_{t+1}) - \hat{Q}(s_t, a_t, \mathbf{w}_t)|$$

- Use probability, P(i):

$$P(i) = \frac{p_i^a}{\sum_k p_k^a}$$

- Exponent, α, determines degree of prioritization
  - $\alpha = 0$, sampling is uniform
  - $\alpha = 1$, sampling sensitive to TD error and P(i) distribution is softmax

Prioritized Replay

Importance sampling for prioritized replay

- Importance sampling introduces a biased estimate of the TD errors
  - Cases with larger error more likely to be sampled
- Use **normalized importance sample weighting**:

$$w_i = \frac{1}{max_i(w_i)} \left(\frac{1}{N \; P(i)} \right)^{\beta}$$

- With normalization, $\frac{1}{max_i(w_i)}$
- The adjusted TD error is then, $w_i \delta_i$
- If exponent $\beta = 1$, bias is fully adjusted