THE COMPOSITION AND DERIVED LENGTHS OF A SOLUBLE GROUP

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Abstract. Given a finite soluble group with derived length d and composition length n, the present paper investigates upper bounds for d in terms of n. An elementary argument is used to show that $d \leq \lceil 2n/3 \rceil$ where $\lceil 2n/3 \rceil$ denotes the least integer greater than or equal to $2n/3$. The sharper bound $d \leq \lceil (n+3)/2 - 3/(n+2) \rceil$ is obtained by using properties of soluble subgroups of two-dimensional general linear groups. Finally, arguments like those used by Hall and Higman are used in conjunction with bounds for the derived length of soluble linear groups to show that $d \leq f(n) < 3\log_2 n + 9$.

Introduction

Let G be a finite soluble group of derived length $d(G)$ and composition length $n(G)$. If $|G| = p_1^{k_1} \dots p_r^{k_r}$ is the decomposition of the order of G into distinct primes, then $n(G) = k_1 + \cdots + k_r$ because the composition factors of G have prime order. Hence, $n(G)$ is easily computed and depends only on $|G|$. By comparison, $d(G)$ depends on the structure of G and is frequently difficult to compute. It is therefore useful to have upper bounds for $d(G)$ in terms of $n(G)$. The groups discussed in the sequel are all finite and soluble. When no confusion arises, the symbols d and n will be used in preference to $d(G)$ and $n(G)$.

Given a group G of order p^n, Burnside [1] showed that $n \geq 3(d-1)$, and subsequently improved the bound to $n \geq d(d+1)/2$, see [2]. Hall [4] noted that the containment $G^{(i)} \leq \gamma_{2^i}(G)$ may be used to show that $n \geq 2^{d-1}$ and in the same paper improved the bound to $n \geq 2^{d-1}+d-1$. The first inequality may be rewritten as $d \leq \lfloor \log_2 n \rfloor + 1$ where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. This bound is not best possible, however, it differs from the best possible bound by at most a factor of two. This is shown by considering a Sylow p-subgroup of the general linear group $GL(m,p)$ over the field of p elements. For this group has $d(G) = \lfloor \log_2 (m-1) \rfloor + 1$ and $n(G) = m(m-1)/2$.

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In addition, Hall [4] showed that $d \leq \lfloor \log_2 c \rfloor + 1$ if G is a non-trivial p-group with nilpotency class c. Since a Sylow p-subgroup of $GL(m,p)$ has class $c = m-1$, this bound is best possible.

It is well-known that a soluble group with composition length n may have much larger derived length than a nilpotent group with the same composition length. It is proved in Theorems 2 and 7 that $d \leq \lceil 2n/3 \rceil$ and $d \leq \lceil (n+3)/2 - 3/(n+2) \rceil$. The sharper bound $d \leq f(n)$ is proved in Theorem 8 where f is a certain recursively defined function which satisfies $f(n) < \alpha \log_2 n + \beta$ and where $\alpha \doteq 2.578$ and $\beta \doteq 8.785$. The bound $d \leq f(n)$, however, does not make the linear bounds obsolete. The function f is difficult to compute and the estimate $\alpha \log_2 n + \beta$ is only useful for large values of n. By comparison, the linear bounds provide excellent bounds which are comparable to the bound $d \leq f(n)$ for small values of n.

Linear Upper Bounds for $d(G)$

Given a soluble group G of derived length d, denote by $a(G)$ the sequence $a(G) = (a_1, a_2, \ldots, a_d)$ where $a_i = n(G^{(i-1)}/G^{(i)})$. Clearly n equals $a_1 + \cdots + a_d$.

The following lemma is used to prove Theorems 2 and 6.

Lemma 1. *Let N be a normal subgroup of the group G.*
(a) If $N \leq Z(G)$ and G/N is cyclic, then G is abelian.
(b) If $d(G) = 3$ and G'' is cyclic, then G'/G'' is not cyclic.
(c) If $d(G) = 3$ and G'' is cyclic, then G' is nilpotent.
(d) If $a(G) = (a, b, 1)$, then G' is nilpotent and $b > 1$.
(e) If $G^{(i)} \not\leq N$, then $d(G/N) > i$.

Proof. If $G = HZ(G)$ where H is abelian, then it is well-known that G is abelian [7] and hence part (a) is true.

Suppose that $d(G) = 3$ and G'' is cyclic. Since $\mathrm{Aut}(G'')$ is abelian, $G/C_G(G'')$ is abelian and so G' centralizes G''. If G'/G'' is cyclic, then (a) implies that G' is abelian. Hence, G'/G'' is not cyclic and part (b) follows. Part (c) follows from the fact that G' centralizes G''.

If $a(G) = (a, b, 1)$, then G'' is cyclic and so part (d) follows directly from parts (b) and (c). Finally, if $G^{(i)} \not\leq N$, then $(G/N)^{(i)} = G^{(i)}N/N$ is non-trivial and hence $d(G/N) > i$. □

Theorem 2. *If G is a soluble group of derived length d and composition length n, then $d \leq \lceil 2n/3 \rceil$. (Here $\lceil x \rceil$ denotes the least integer greater than or equal to x.)*

Proof. Let $a(G) = (a_1, a_2, \ldots, a_d)$. It follows from Lemma 1(d) that $a_i + a_{i+1} \geq 3$ for $2 \leq i \leq d-1$. Consider separately the cases when d is odd and even.
Case (a). If $d = 2k+1$, then $n = a_1 + (a_2 + a_3) + \cdots + (a_{d-1} + a_d) \geq 3k+1$. Hence $\lceil 2n/3 \rceil \geq \lceil 2(3k+1)/3 \rceil = 2k+1 = d$.
Case (b). If $d = 2k+2$, then $n = a_1 + (a_2 + a_3) + \cdots + (a_{d-2} + a_{d-1}) + a_d \geq 3k+2$. Hence $\lceil 2n/3 \rceil \geq \lceil 2(3k+2)/3 \rceil = 2k+2 = d$. □

This simple linear upper bound is very sharp for groups of small derived length. For example, let $G = \Gamma U(3,2)$ be the group of semilinear transformations that preserve a non-degenerate unitary form on the three-dimensional vector space over the field of four elements. Then $G/G^{(4)} \cong GL(2,3)$ and $G^{(4)}$ is extraspecial of order 3^3 and exponent 3. Hence $a(G) = (1,1,2,1,2,1)$ and $d(G) = \lceil 2n(G)/3 \rceil$ holds. Indeed, if $0 \leq i \leq 6$, then $i = d(G/G^{(i)}) = \lceil 2n(G/G^{(i)})/3 \rceil$ holds.

A sharper linear upper bound for d may be obtained by considering the derived lengths of p-groups and of two-dimensional soluble linear groups.

Theorem 3. [4, Theorem 2.57] *If G is a p-group of order p^n and derived length d, then $n \geq 2^{d-1} + d - 1$. Consequently, any group of order p^5 is metabelian.*

Theorem 4. *If F is an arbitrary (commutative) field and G is a soluble subgroup of $GL(2,F)$, then $d \leq 4$.*

Proof. Let $\rho(m)$ be the maximum value of $d(G)$ where G ranges over the soluble subgroups of m-dimensional general linear groups. It is well-known that $\rho(m)$ is finite for finite m. Indeed, Newman [6] has calculated a formula for $\rho(m)$. This theorem is a corollary of the fact that $\rho(2) = 4$. A short proof of Theorem 4 is included here for the reader's convenience.

The heart of the proof relies on work due to Suprunenko [8, Theorems 4 and 11]: *Let K be an algebraically closed field, and let G be a primitive soluble subgroup of $GL(m,K)$. Let $p_1^{k_1} \dots p_r^{k_r}$ be the decomposition of m into distinct positive primes. If $Fit(G)$ is the Fitting subgroup of G, then $Fit(G)/Z(G)$ is abelian. Also $G/Fit(G)$ is isomorphic to a subgroup of the direct product of symplectic groups $Sp(2k_i, p_i)$, $1 \leq i \leq r$.*

Let K be the algebraic closure of F. Since $GL(2,F) \leq GL(2,K)$, it is sufficient to prove the result for $GL(2,K)$. Let G be a soluble subgroup of $GL(2,K)$ which acts on a two-dimensional space V over K.

Case (a). If G acts reducibly on V, then G stabilizes a maximal flag $V > W > \langle 0 \rangle$. Since the stabilizer in $GL(2,K)$ of this flag is metabelian, it follows that $d(G) \leq 2$.

Case (b). If G acts imprimitively (as a linear group) on V, then there exists a decomposition $V = V_1 \oplus V_2$ of V into one-dimensional subspaces where G acts on the set $\{V_1, V_2\}$. Hence G is isomorphic to a subgroup of the metabelian group $GL(1,K)$ wr S_2 where S_2 is the symmetric group on 2 letters.

Case (c). If G acts primitively on V, then $G/\mathrm{Fit}(G)$ is a subgroup of $\mathrm{Sp}(2,2)$ which is metabelian of order 6. Therefore $d(G) \leq d(G/\mathrm{Fit}(G)) + d(\mathrm{Fit}(G)/Z(G)) + d(Z(G)) \leq 2 + 1 + 1 = 4$. □

Lemma 5. *If $G^{(5)}$ is an elementary abelian q-group of order q^2, then $G^{(4)}$ centralizes $G^{(5)}$ and $G^{(4)}/G^{(5)}$ is not cyclic.*

Proof. If $G^{(4)}$ is not contained in $M = C_G(G^{(5)})$, then $d(G/M) > 4$ by Lemma 1(e). This, however, contradicts Theorem 4 as G/M is isomorphic to a subgroup of $GL(2,q)$. Therefore $G^{(4)} \leq M$ and $G^{(4)}$ is nilpotent. If $G^{(4)}/G^{(5)}$ is cyclic, then by Lemma 1(a), $G^{(4)}$ is abelian. Hence $G^{(4)}/G^{(5)}$ is not cyclic. □

Theorem 6. *If $d(G) = 7$, then $n(G^{(4)}) \geq 6$.*

Proof. Let $a(G) = (a_1, \dots, a_7)$, and let $N = G^{(4)}$. Suppose that $n(N) = a_5 + a_6 + a_7$ is less than 6. Then $a(N)$ equals one of (1,1,1), (1,1,2), (1,2,1), (2,1,1), (1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2), (2,2,1). By Lemma 1(d), there can not be two consecutive 1's in $a(N)$. Therefore $a(N)$ equals one of (1,2,1), (1,3,1), (1,2,2), (2,1,2), (2,2,1).

Suppose that $a(N) = (1, 2, 1)$ or $(1, 2, 2)$. By Lemma 1(b), N'/N'' is not cyclic and so must be elementary abelian. Hence the group G'/N'' contradicts Lemma 5. Similarly, if $a(N) = (2, 1, 2)$, then N'' is elementary abelian and G'' contradicts Lemma 5. Therefore these three cases never occur.

Suppose that $a(N) = (1, 3, 1)$. If there exists a characteristic subgroup M of N' which satisfies $N'' < M < N'$, then $a(G/M) = (a_1, a_2, a_3, a_4, 1, a)$ where a equals 1 or 2. By Lemma 1(b), N'/M is not cyclic and is therefore elementary abelian. This, however, contradicts Lemma 5. Hence no such characteristic subgroup exists and the abelian group N'/N'' must be elementary abelian of order q^3 for some prime q. By Lemma 1(d), N' is nilpotent and hence $|N'| = q^4$. Since $Z(N')$ is characteristic and $N'' \leq Z(N') < N'$, it follows that $Z(N') = N''$. In summary, the Frattini subgroup $\mathrm{Frat}(N')$ and the centre $Z(N')$ both equal N''. Therefore N' is an extraspecial group of order q^4. However, extraspecial groups have odd composition lengths, therefore the case $a(N) = (1, 3, 1)$ never arises.

Finally, assume that $a(N) = (2, 2, 1)$. By Lemma 1(d), N' is nilpotent of order q^3 and hence N'/N'' is elementary abelian. By Lemma 5, N/N' is not cyclic and N centralizes N'/N''. Therefore N/N' is elementary abelian of order p^2. If $p \neq q$, then N/N'' must be abelian. This contradiction shows that $p = q$. Therefore N is a group of order q^5 with derived length 3 contrary to Theorem 3. Hence the case $a(N) = (1, 3, 1)$ never arises. In summary, none of the ten possibilities for $a(N)$ occur and so $n(N) \geq 6$ as claimed. □

Theorem 7. *If G is a finite soluble group of derived length d and composition length n, then $d \leq \lceil (n + 3)/2 - 3/(n + 2) \rceil$.*

Proof. If d equals 1, 2 or 3, then it follows from Lemma 1(d) that n is at least 1, 2 or 4, respectively. Therefore the inequality holds for $d \leq 3$. If $d \geq 4$, then $n \geq 5$ and so $3/(n + 2) < 1/2$. Since $(n + 3)/2$ is either an integer or half an odd integer, $\lceil (n + 3)/2 \rceil = \lceil (n + 3)/2 - 3/(n + 2) \rceil$ holds in this case.

It is therefore sufficient to prove that $d \leq \lceil (n + 3)/2 \rceil$ when $d = 3k + 4$, $3k + 5$ or $3k + 6$ and $k = 0, 1, 2, \dots$. Let $a(G) = (a_1, \dots, a_d)$ and let $d = 3k + 4$. Then n equals $(a_1 + a_2 + a_3 + a_4) + (a_5 + a_6 + a_7) + \cdots + (a_{d-2} + a_{d-1} + a_d)$. Therefore by Theorem 6, $n \geq 5+6+\cdots+6 = 6k+5$ and so $\lceil (n+3)/2 \rceil \geq \lceil (6k+8)/2 \rceil = 3k+4 = d$. If $d = 3k+5$, then $n \geq 6k + 6$ and $\lceil (n + 3)/2 \rceil \geq \lceil (6k + 9)/2 \rceil = 3k + 5 = d$. If $d = 3k + 6$, then $a_1 + \cdots + a_{d-2} \geq 6k + 5$ by the first case. Also $a_{d-1} + a_d \geq 3$ by Lemma 1(d), therefore $n \geq 6k + 8$ and $\lceil (n + 3)/2 \rceil \geq \lceil (6k + 11)/2 \rceil = 3k + 6 = d$. Hence the inequality holds for all values of d. □

The techniques used in the proofs of Theorems 4 and 6 may be used to prove structure theorems about soluble groups with large derived lengths. For example, one may show that if $d(G) = 6$ and $n(G) = 8$, then $G/G^{(4)}$ is isomorphic to one of the two covering groups of the symmetric group S_4 and that $G^{(4)}$ is isomorphic to an extraspecial group of order p^3 and exponent p.

A Logarithmic Upper Bound for $d(G)$

The bounds given in the previous section are useful for small values of n, typically $n \leq 20$. In this section, a function f is given which satisfies $d(G) \leq f(n(G))$ for all finite soluble groups G. The function f satisfies $f(n) \leq \min\{\lceil (n+3)/2 - 3/(n+2)\rceil, \lceil 2n/3\rceil\}$ for all n, however, the values of $f(n)$ are very difficult to compute by hand. A computer program was used to obtain the values of $f(n)$ listed in Table 1 below.

Theorem 8. *Let N be the set of non-negative integers and let p be a prime number. Let ρ and δ be non-decreasing functions which satisfy $d(G) \leq \rho(m)$ if G is a soluble subgroup of $GL(m,p)$, and $d(G) \leq \delta(m)$ if G is a p-group of order p^m. Recursively define a function $f : N \to N$ by $f(0) = 0$ and*

$$f(n) = \max\{\, g(a,b,c) \mid a,b,c \in N,\ n = a+b+c,\ b \geq 1\,\}$$

where $g : N^3 \to N$ maps (a,b,c) to $\min\{\, f(a), \rho(b)\,\} + \min\{\, \delta(b+c),\ \delta(c)+1\,\}$. Then $d(G) \leq f(n(G))$ for all finite soluble groups G. If ρ is the function defined by Newman [6] and δ is defined by the rule $\delta(m) = \max\{\, d \mid m \geq 2^{(d-1)} + (d-1)\,\}$, then $f(n) < \alpha \log_2 n + \beta$ for $n > 0$, where $\alpha \doteq 2.578$ and $\beta \doteq 8.785$.

Proof. Since ρ and δ are non-decreasing, $g(\alpha,\beta,\gamma) \leq g(\alpha,\beta+1,\gamma)$ and hence

$$\begin{aligned} f(n) &\leq \max\{\, g(\alpha,\beta+1,\gamma) \mid \alpha,\beta,\gamma \in N,\ \alpha+\beta+\gamma = n, \beta \geq 1\,\} \\ &\leq \max\{\, g(a,b,c) \mid a,b,c \in N, a+b+c = n+1, b \geq 1\,\} \\ &= f(n+1). \end{aligned}$$

The inequality $d(G) \leq f(n(G))$ may be proved by using induction on the composition length of G. It is certainly true when $n = 0$. Henceforth assume that $n > 0$. Suppose that G has two distinct non-trivial minimal normal subroups M and N. Then there is a monomorphism from G into $G/M \times G/N$ defined by $x \mapsto (xM, xN)$. However, $d(G) \leq \max\{d(G/M), d(G/N)\}$ which, by induction, is at most $\max\{f(n(G/M)), f(n(G/N))\}$. But f is non-decreasing and so $d(G) \leq f(n(G))$ as required.

Assume now that G has a unique minimal normal subgroup M. Since G is soluble, M is a p-group for some prime p. Therefore $P = O_p(G)$ is non-trivial and $O_{p'}(G)$ is trivial. Let $a = n(G/P)$, $b = n(P/\mathrm{Frat}(P))$ and $c = n(\mathrm{Frat}(P))$. Since P is non-trivial, b is positive and it follows from the inductive hypothesis that $d(G/P) \leq f(a)$. Now G/P acts faithfully on the b-dimensional vector space $P/\mathrm{Frat}(P)$ (see [5]) and so $d(G/P) \leq \rho(b)$. In addition, $d(P) \leq \delta(b+c)$ and $P' \leq \mathrm{Frat}(P)$ so that $d(P) \leq \delta(c)+1$. Combining these facts gives $d(G) \leq d(G/P) + d(P) \leq g(a,b,c)$ and hence that $d(G) \leq f(n(G))$.

It follows from [6, Theorem A] that $\rho(b) \leq 5\log_9(b/8) + 14$. Therefore, if $n > 1$

$$\begin{aligned} f(n) &\leq \max\{\, \rho(b) + \delta(n-b) + 1 \mid 1 \leq b \leq n-1\,\} \\ &\leq \max\{\, 5\log_9(b/8) + \log_2(n-b) + 16 \mid 1 \leq b \leq n-1\,\}. \end{aligned}$$

One may use calculus to maximize the function $5\log_9(b/8)+\log_2(n-b)+16 = \lambda\log_e b + \mu\log_e(n-b)+\nu$ subject to the constraint $0<b<n$ where $\lambda = 5/\log_e 9, \mu = 1/\log_e 2$, $\nu = 16-5\log_9 8$ and e is the base for the natural logarithm. The maximum occurs when $b=\lambda n/(\lambda+\mu)$ and hence it follows that $f(n)\leq \alpha\log_2 n+\beta$, where $\alpha = (\lambda+\mu)\log_e 2$ and $\beta = \lambda\log_e(\lambda/(\lambda+\mu)) + \mu\log_e(\mu/(\lambda+\mu))+\nu$. □

Table 1 shows some values for the function $f(n)$ where ρ and δ are the functions mentioned in Theorem 8. Since $f(n)$ is a non-decreasing function, only the smallest values of n for which $f(n)=m$ are listed. The relative merits of the three bounds $d\leq f(n)$, $d\leq \lceil (n+3)/2-3/(n+2)\rceil = g(n)$ and $d\leq\lceil 2n/3\rceil = h(n)$ may be determined by comparing the last three rows of Table 1.

Table 1

n	1	2	4	5	7	8	11	13	15	19	22	27	30	34	40	51	60	73	84
$f(n)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$g(n)$	1	2	3	4	5	6	7	8	9	11	13	15	17	19	22	27	32	38	44
$h(n)$	1	2	3	4	5	6	8	9	10	13	15	18	20	23	27	34	40	49	56

Concluding Remarks

The author has constructed a soluble group G of order $2^{11}3^{13}$ which satisfies $a(G) = (1,1,2,1,2,1,6,1,8,1)$. If $n_i = n(G/G^{(i)})$ and $d_i = d(G/G^{(i)})$, $0\leq i\leq 10$, then the following table may be used to compare d_i to $f(n_i)$

Table 2

n_i	0	1	2	4	5	7	8	14	15	23	24
d_i	0	1	2	3	4	5	6	7	8	9	10
$f(n_i)$	0	1	2	3	4	5	6	8	9	11	11.

The group G is a subgroup of the holomorph of the extraspecial group E of order 3^9 and of exponent 3. Let E be defined by the presentation below where the trivial commutators of the form $[e_j, e_i]$, $1\leq i<j\leq 9$, have been omitted

$$E = \langle\, e_1,\ldots,e_9 \mid e_i^3 = 1,\ 1\leq i\leq 9,\ [e_8,e_1]=[e_5,e_4]=e_9,\ [e_7,e_2]=[e_6,e_3]=e_9^{-1}\rangle.$$

Then every element of E may be uniquely expressed as a product $e_1^{a_1}\ldots e_9^{a_9}$ where $0\leq a_j<3$ and $1\leq j\leq 9$. An automorphism α of E may be identified with the 9×9 matrix (a_{ij}) where $(e_i)\alpha = e_1^{a_{i1}}\ldots e_9^{a_{i9}}$, $1\leq i\leq 9$. Let H be the subgroup of $\mathrm{Aut}(E)$

which is generated by the automorphisms

$$\begin{pmatrix} 2&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&2&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&2&0&0&0\\ 0&0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1&0&0&0&2&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 1&0&1&0&2&0&2&0&0\\ 0&0&0&0&1&0&1&0&0\\ 0&1&0&0&0&2&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&1&0&1&0&2&0&2&0\\ 0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

Then G is defined to be the split extension of E by H. The structure of soluble groups, such as G, which are built from extraspecial groups is described by Glasby and Howlett [3].

The bound $d \le f(n)$ is not best possible since $f(11) = 7$ and one may show that there is no group with composition length 11 and derived length 7. However, certain wreath products may be used to show that the logarithmic upper bound is close to being best possible. For example, if G is the $(r-1)$-fold permutational wreath product of the symmetric group S_4 on four letters, then G is a permutation group of degree 4^r for which $d(G) = 3r$ and $n(G) = 4(4^r - 1)/3$. Therefore if $d(G) \le \alpha' \log_2 n(G) + \beta'$ holds for all soluble groups G, then it follows that $\alpha' \ge 3/2$.

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