A technical guide to water mass transformation (WMT) analysis

STEPHEN.GRIFFIES@NOAA.GOV

November 25, 2020

NOAA/GFDL + Princeton University

Some general points

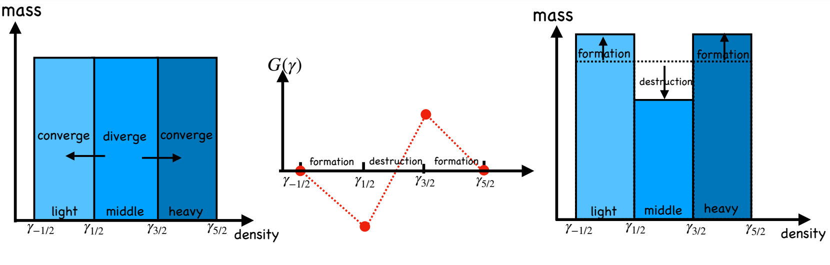
- Water mass transformation (WMT) analysis is the study of budgets for seawater mass and/or tracer mass within layers or *water mass classes*.
  - This approach contrasts to local (Eulerian) budget analysis that focuses on fixed regions.
  - It also contrasts with Lagrangian or quasi-Lagrangian approaches that follow fluid elements.
- Layers or classes are defined by a scalar field λ, which is a function of space and time, $\lambda(\boldsymbol{x}, t)$.
  - tracer concentrations: $\lambda = S, \Theta, CO_2$
  - buoyancy: $\lambda = \gamma$ = potential density or neutral density
- WMT analysis is concerned with the *mass distribution* of seawater binned according to λ-classes.
  - By extension, WMT analysis is concerned with the physical and biogeochemical processes leading to modifications of the mass distribution.
- *Transformation* refers to the transport of seawater mass and tracer mass across a λ-isosurface, thus leading to a modification of the λ mass distribution.
  - Transformation across a λ-isosurface is measured by a nonzero material change to λ.
  - Written mathematically, transformation occurs when

$$\dot{\lambda} = \frac{\mathrm{D}\lambda}{\mathrm{D}t} = \frac{\partial \lambda}{\partial t} + \boldsymbol{v} \cdot \nabla \lambda \neq 0.$$

- *Formation* refers to the net accumulation of water into a layer or class.
  - Formation is computed by taking the difference of the transformation across the layer boundaries.
  - *Formation = convergence of transformation*.

Mass distributions, transformation, and formation

- $\mathrm{d}M = m(\lambda)\,\delta\lambda =$ mass within the infinitesimal λ-layer $[\lambda - \delta\lambda/2, \lambda + \delta\lambda/2]$
- $G(\lambda) =$ *transformation* = mass transport (mass per time) of seawater crossing λ-isosurface.
- $G(\lambda) > 0$ (conventionally) means water moves to larger λ.
- We here consider a three-class example with $\lambda = \gamma =$ buoyancy.



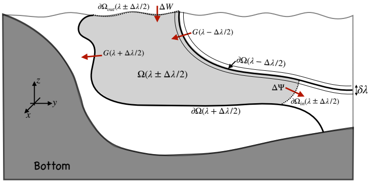
$$G(\gamma') = \left[\begin{array}{lll} 0 & \gamma' = \gamma - \delta\gamma/2 & \text{boundary assumed to be closed} \\ < 0 & \gamma' = \gamma + \delta\gamma/2 & \text{mass moves to light density from middle} \\ > 0 & \gamma' = \gamma + 3\,\delta\gamma/2 & \text{mass moves from middle density to heavy} \\ 0 & \gamma' = \gamma + 5\,\delta\gamma/2 & \text{boundary assumed to be closed} \end{array}\right.$$

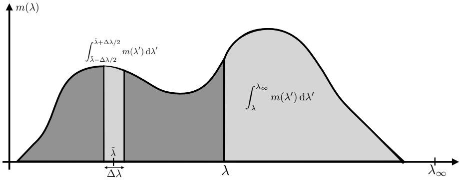
The convergence of G yields the formation into a layer:

$$\text{light-formation} = -[G(\gamma + \delta\gamma/2) - G(\gamma - \delta\gamma/2)] > 0$$
$$\text{middle-formation} = -[G(\gamma + 3\,\delta\gamma/2) - G(\gamma + \delta\gamma/2)] < 0$$
$$\text{heavy-formation} = -[G(\gamma + 5\,\delta\gamma/2) - G(\gamma + 3\,\delta\gamma/2)] > 0.$$

Seawater mass budget for a $\Delta\lambda$-layer

$$\underbrace{\frac{\mathrm{d}\Delta M}{\mathrm{d}t} + \Delta\Psi}_{\text{storage + outflow}} = \underbrace{\Delta W - [G(\lambda+\Delta\lambda/2) - G(\lambda-\Delta\lambda/2)]}_{\text{formation into layer } \Omega(\lambda\pm\Delta\lambda/2)}$$





$$\underbrace{\Delta M(\lambda \pm \Delta\lambda/2)}_{\text{layer mass}} = \int_{\Omega(\lambda\pm\Delta\lambda/2)} \rho\,\mathrm{d}V = \int_{\lambda-\Delta\lambda/2}^{\lambda+\Delta\lambda/2} m(\lambda')\,\mathrm{d}\lambda'$$

$$\underbrace{\Delta\Psi(\lambda \pm \Delta\lambda/2)}_{\text{interior bdy mass transport}} = + \int_{\partial\Omega_{\text{in}}(\lambda\pm\Delta\lambda/2)} \rho\,(\boldsymbol{v} - \boldsymbol{v}^{(b)}) \cdot \hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S} = \int_{\lambda-\Delta\lambda/2}^{\lambda+\Delta\lambda/2} \dot{m}_\Psi(\lambda')\,\mathrm{d}\lambda'$$

$$\underbrace{\Delta W(\lambda \pm \Delta\lambda/2)}_{\text{surface bdy mass transport}} = - \int_{\partial\Omega_{\text{out}}(\lambda\pm\Delta\lambda/2)} \rho\,(\boldsymbol{v} - \boldsymbol{v}^{(\eta)}) \cdot \hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S} = \int_{\partial\Omega_{\text{out}}(\lambda\pm\Delta\lambda/2)} \mathcal{Q}_{\text{m}}\,\mathrm{d}\mathcal{S} = \int_{\lambda-\Delta\lambda/2}^{\lambda+\Delta\lambda/2} \dot{m}_{\text{W}}(\lambda')\,\mathrm{d}\lambda'$$

$m(\lambda)\,\delta\lambda$ = mass contained within the increment $[\lambda - \delta\lambda/2, \lambda + \delta\lambda/2]$

$\dot{m}_\Psi(\lambda)\,\delta\lambda$ = mass per time crossing $\partial\Omega_{\text{in}}$ within the increment $[\lambda - \delta\lambda/2, \lambda + \delta\lambda/2]$

$\dot{m}_{\text{W}}(\lambda)\,\delta\lambda$ = mass per time crossing $\partial\Omega_{\text{out}}$ within the increment $[\lambda - \delta\lambda/2, \lambda + \delta\lambda/2]$.

Calculating the transformation $G(\lambda)$

Transformation across a λ-surface is given by the equivalent mathematical expressions

$$G(\lambda) = \int_{\partial\Omega(\lambda)} \rho\, \hat{\boldsymbol{n}} \cdot (\boldsymbol{v} - \boldsymbol{v}^{(\lambda)})\, \mathrm{d}\mathcal{S} = \int_{\partial\Omega(\lambda)} \frac{\rho\, \dot{\lambda}}{|\nabla\lambda|}\, \mathrm{d}\mathcal{S} = \lim_{\delta\lambda\to 0} \frac{1}{\delta\lambda} \int_{\Omega(\lambda\pm\delta\lambda/2)} \rho\, \dot{\lambda}'\, \mathrm{d}V.$$

The final equality is conducive to estimations via partitioning the contributions to $\rho\, \dot{\lambda}$ into $\delta\lambda$-bins.

There are two conceptually distinct but mathematically equivalent methods to compute $\rho\, \dot{\lambda}$.

- ⋆ The *process method* tells us why there is transformation across a λ-surface. This is the method most commonly used in practice.

$$\rho\, \mathrm{D}\lambda/\mathrm{D}t = \underbrace{-\nabla \cdot \boldsymbol{J}}_{\text{mixing + boundary fluxes}} + \underbrace{\rho\, \dot{\Upsilon}}_{\text{sources}}$$

- ⋆ The *kinematic method* tells us how there is transformation across a λ-surface.

$$\rho\, \mathrm{D}\lambda/\mathrm{D}t = \underbrace{\partial(\rho\, \lambda)/\partial t}_{\text{local tendency}} + \underbrace{\nabla \cdot (\rho\, \boldsymbol{v}^{\dagger}\, \lambda)}_{\text{residual mean advection}}$$

- ⋆ Sometimes simpler to diagnose the kinematic method and then to infer mixing.
- ⋆ Experience with MOM5 suggests that it is not trivial to ensure the two methods yield identical results since the diagnostic methods must be carefully coded.
- ⋆ We focus on the process method in the remainder of this tutorial.
- ⋆ The source term, $\rho\, \dot{\Upsilon}_{\text{sources}}$, is zero for S and Θ, so we ignore it for the remainder.

The process method for computing $G(\lambda)$

In the absence of interiors sources, the process method consists of estimating the following volume integral

$$G(\lambda) = \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \int_{\Omega(\lambda \pm \delta\lambda/2)} \rho\, \dot{\lambda}'\, \mathrm{d}V = \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \int_{\Omega(\lambda \pm \delta\lambda/2)} [-\nabla \cdot \boldsymbol{J}]\, \mathrm{d}V$$

It is convenient to decompose the flux divergence into an interior mixing term plus surface boundary fluxes

$$\nabla \cdot \boldsymbol{J} = \underbrace{\nabla \cdot \boldsymbol{J}_{\text{interior}}}_{\text{interior processes}} + \underbrace{\boldsymbol{J}_{\text{out}} \cdot \hat{\boldsymbol{n}}\, \delta(z - \eta)}_{\text{surface boundary fluxes}} + \underbrace{\boldsymbol{J}_{\text{bot}} \cdot \hat{\boldsymbol{n}}\, \delta(z - \eta_{\text{b}})}_{\text{bottom boundary processes}}$$

$$G(\lambda) = \underbrace{-\lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[\int_{\Omega(\lambda \pm \delta\lambda/2)} \nabla \cdot \boldsymbol{J}_{\text{interior}}\, \mathrm{d}V \right]}_{\text{interior transformation = volume integral of convergence}}$$

$$\underbrace{-\lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[\int_{\partial\Omega_{\text{out}}(\lambda \pm \delta\lambda/2)} \boldsymbol{J}_{\text{out}} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}\mathcal{S} \right]}_{\text{surface transformation = area integral of surface fluxes}} \underbrace{-\lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[\int_{\partial\Omega_{\text{bot}}(\lambda \pm \delta\lambda/2)} \boldsymbol{J}_{\text{bot}} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}\mathcal{S} \right]}_{\text{bottom transformation = area integral of bottom fluxes}}$$

- ⋆ The first term on the right hand side is the binned tendency from interior processes such as mixing and shortwave heating.
- ⋆ *Interior* refers to anything not arising from fluxes crossing $z = \eta$ or $z - \eta_{\text{b}}$. Hence, it includes strong boundary layer mixing processes.
- ⋆ The second term is the binned non-advective surface transport and third is binned non-advective bottom transport.

Non-advective surface boundary fluxes

$$\text{surface transformation} = G(\lambda)_{\text{surface}} \equiv \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[\int_{\partial\Omega_{\text{out}}(\lambda \pm \delta\lambda/2)} [-\boldsymbol{J}_{\text{out}} \cdot \hat{\boldsymbol{n}}] \, \mathrm{d}\mathcal{S} \right]$$

- SALINITY: $\lambda = S$. If salt is dissolved in the surface mass flux, then the advective salt flux crossing the ocean surface is $S_{\text{m}} \, \mathcal{Q}_{\text{m}}$. There may also be a non-advective salt flux, $\mathcal{Q}_{\text{S}}^{\text{non-adv}}$, from turbulent transfer and/or surface salt restoring in an OMIP simulation so that the net salt flux is

$$\mathcal{Q}_{\text{S}} = S_{\text{m}} \, \mathcal{Q}_{\text{m}} + \mathcal{Q}_{\text{S}}^{\text{non-adv}}.$$

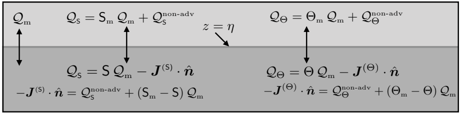
  This salt flux is met on the ocean side by an advective and non-advective ocean flux so that

$$\mathcal{Q}_{\text{S}} = S_{\text{m}} \, \mathcal{Q}_{\text{m}} + \mathcal{Q}_{\text{S}}^{\text{non-adv}} = S \, \mathcal{Q}_{\text{m}} - \hat{\boldsymbol{n}} \cdot \boldsymbol{J}^{(S)} \qquad \text{with } S \text{ the salinity at } z = \eta.$$

  Rearrangement leads to the induced non-advective flux on the ocean side of the surface boundary

$$\boxed{-\hat{\boldsymbol{n}} \cdot \boldsymbol{J}^{(S)} = (S_{\text{m}} - S) \, \mathcal{Q}_{\text{m}} + \mathcal{Q}_{\text{S}}^{\text{non-adv}}.}$$

  Typically $S_{\text{m}} = 0$ except for exchanges of salt with sea ice.



- CONSERVATIVE TEMPERATURE: $\lambda = \Theta$. The same considerations hold for Conservative Temperature so that its net flux is

$$\mathcal{Q}_{\Theta} = \Theta_{\text{m}} \, \mathcal{Q}_{\text{m}} + \mathcal{Q}_{\Theta}^{\text{non-adv}},$$

  where $\mathcal{Q}_{\Theta}^{\text{non-adv}}$ arises from turbulent and radiative heat fluxes. This flux is met on the ocean side by an advective and non-advective ocean flux, $\mathcal{Q}_{\Theta} = \Theta \, \mathcal{Q}_{\text{m}} - \hat{\boldsymbol{n}} \cdot \boldsymbol{J}^{(\Theta)}$, so that the induced non-advective flux is

$$\boxed{-\hat{\boldsymbol{n}} \cdot \boldsymbol{J}^{(\Theta)} = (\Theta_{\text{m}} - \Theta) \, \mathcal{Q}_{\text{m}} + \mathcal{Q}_{\Theta}^{\text{non-adv}} \qquad \text{with } \Theta \text{ the Conservative Temp at } z = \eta.}$$

  In most applications we set $\Theta_{\text{m}} - \Theta = 0$ so that $-\hat{\boldsymbol{n}} \cdot \boldsymbol{J}^{(\Theta)} = \mathcal{Q}_{\Theta}^{\text{non-adv}}$.

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Buoyancy water mass transformation: $\lambda = \gamma$

$\lambda = \gamma$ is potential density or neutral density, used here to measure buoyancy water mass transformation.

$$\rho\,\dot{\gamma} = \frac{\partial\gamma}{\partial S}\,\rho\,\dot{S} + \frac{\partial\gamma}{\partial\Theta}\,\rho\,\dot{\Theta} = \gamma\,(\beta_{\mathrm{S}}\,\rho\,\dot{S} - \alpha_{\Theta}\,\rho\,\dot{\Theta})$$

$$\rho\,\dot{S} = -\nabla\cdot\boldsymbol{J}^{(S)}_{\text{interior}} - \boldsymbol{J}^{(S)}_{\text{out}}\cdot\hat{\boldsymbol{n}}\,\delta(z-\eta) - \boldsymbol{J}^{(S)}_{\text{bot}}\cdot\hat{\boldsymbol{n}}\,\delta(z-\eta_{\mathrm{b}})$$

$$\rho\,\dot{\Theta} = -\nabla\cdot\boldsymbol{J}^{(\Theta)}_{\text{interior}} - \boldsymbol{J}^{(\Theta)}_{\text{out}}\cdot\hat{\boldsymbol{n}}\,\delta(z-\eta) - \boldsymbol{J}^{(\Theta)}_{\text{bot}}\cdot\hat{\boldsymbol{n}}\,\delta(z-\eta_{\mathrm{b}})$$

$$G(\gamma) = \underbrace{-\lim_{\delta\gamma\to 0}\frac{1}{\delta\gamma}\left[\int_{\Omega(\gamma\pm\delta\gamma/2)}\left(\frac{\partial\gamma}{\partial S}\,\nabla\cdot\boldsymbol{J}^{(S)}_{\text{interior}} + \frac{\partial\gamma}{\partial\Theta}\,\nabla\cdot\boldsymbol{J}^{(\Theta)}_{\text{interior}}\right)\mathrm{d}V\right]}_{\text{interior buoyancy transformation = volume integral of convergence}}$$

$$\underbrace{-\lim_{\delta\gamma\to 0}\frac{1}{\delta\gamma}\left[\int_{\partial\Omega_{\text{out}}(\gamma\pm\delta\gamma/2)}\left(\frac{\partial\gamma}{\partial S}\,\boldsymbol{J}^{(S)}_{\text{out}} + \frac{\partial\gamma}{\partial\Theta}\,\boldsymbol{J}^{(\Theta)}_{\text{out}}\right)\cdot\hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S}\right]}_{\text{surface buoyancy transformation = area integral of surface boundary fluxes}}$$

$$\underbrace{-\lim_{\delta\gamma\to 0}\frac{1}{\delta\gamma}\left[\int_{\partial\Omega_{\text{bot}}(\gamma\pm\delta\gamma/2)}\left(\frac{\partial\gamma}{\partial S}\,\boldsymbol{J}^{(S)}_{\text{bot}} + \frac{\partial\gamma}{\partial\Theta}\,\boldsymbol{J}^{(\Theta)}_{\text{bot}}\right)\cdot\hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S}\right]}_{\text{bottom buoyancy transformation = area integral of bottom boundary fluxes}}$$

$$G(\gamma)^{\mathrm{S}}_{\text{surface}} = \lim_{\delta\gamma\to 0}\frac{1}{\delta\gamma}\left[\int_{\partial\Omega_{\text{out}}(\gamma\pm\delta\gamma/2)}\beta_{\mathrm{S}}\,[\mathcal{Q}^{\text{non-adv}}_{S} + (S - S_{\mathrm{m}})\,\mathcal{Q}_{\mathrm{m}}]\gamma\,\mathrm{d}\mathcal{S}\right]$$

$$G(\gamma)^{\Theta}_{\text{surface}} = -\lim_{\delta\gamma\to 0}\frac{1}{\delta\gamma}\left[\int_{\partial\Omega_{\text{out}}(\gamma\pm\delta\gamma/2)}\alpha_{\Theta}\,[\mathcal{Q}^{\text{non-adv}}_{\Theta} + (\Theta_{\mathrm{m}} - \Theta)\,\mathcal{Q}_{\mathrm{m}}]\gamma\,\mathrm{d}\mathcal{S}\right]$$

- ⋆ Note multiplication of terms by $(\partial\gamma/\partial S)$ and $(\partial\gamma/\partial\Theta)$. Offline now (online requires MOM6 code).
- ⋆ $\Theta_{\mathrm{m}} - \Theta = 0$ assumed for surface advective heat fluxes.
- ⋆ $S_{\mathrm{m}} = 0$ assumed for precip, evap, and rivers.

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Example surface buoyancy transformation

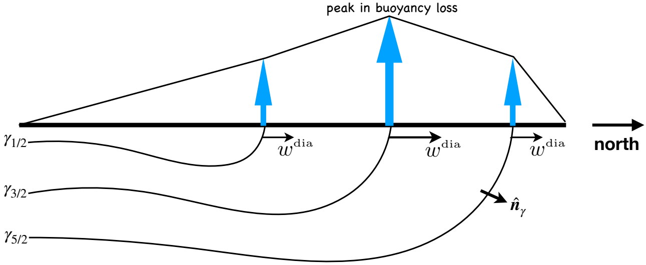

Figure: Transformation due to meridional gradient in the surface buoyancy loss. Buoyancy loss is denoted by the blue vertical arrows indicating the removal of buoyancy from the ocean. The surface buoyancy loss causes γ interfaces to migrate to the south, which in turn causes dianeutral mass flux to move from lighter layers to denser layers (black vectors pointed to the north, w^{dia}). With a peak in the buoyancy loss at a particular latitude, more entrainment is driven into the layer to the north of the peak (water converges to the layer $\gamma_{3/2} \le \gamma \le \gamma_{5/2}$) and less entrainment into the layer to the south (water diverges from the layer $\gamma_{1/2} \le \gamma \le \gamma_{3/2}$).

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Further study

- Much of the formalism presented here is taken from the review paper:
  - The water mass transformation framework for ocean physics and biogeochemistry, 2019: S. Groeskamp, S.M. Griffies, D. Iudicone, R. Marsh, A.J.G. Nurser, and J.D. Zika, *Annual Review of Marine Science*, **11**, 21.1-21.35, doi:10.1146/annurev-marine-010318-095421.
- Further material is provided in Chapter 50 of the book
  - Elements of Geophysical Fluid Mechanics with Ocean Applications, 2020: S.M. Griffies.