Vertical Lagrangian-remapping, generalized vertical coordinates, and spurious diapycnal mixing in ocean models

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Key points in this talk

- ⋆ SPURIOUS DIAPYCNAL MIXING: This problem corrupts many state-of-the-science ocean simulations, particularly those for climate where errors accumulate to degrade stratification and contribute to spurious tracer evolution.
- ⋆ DYNAMICAL CORE FORMULATION: The vertical Lagrangian-remap method and generalized vertical coordinate formulation of ocean equations can be understood through basic notions of fluid mechanics.
- ⋆ CONJECTURE: Vertical Lagrangian-remapping with an appropriate hybrid vertical coordinate provides a suitable (perhaps optimal) framework to simulate the ocean climate system without incurring physically disruptive spurious mixing.

  $\longrightarrow$ We are not there yet, but we will show promising results.
- ⋆ ELEMENTS OF THIS TALK ARE TAKEN FROM:
  - *A primer on ocean generalized vertical coordinate dynamical cores based on the vertical Lagrangian-remap method*, 2020: Griffies, Adcroft, and Hallberg, *in review at JAMES*
  - Adcroft et al, 2019: The GFDL Global Ocean and Sea Ice Model OM4.0: Model Description and Simulation Features, *JAMES*.

Outline

1. The spurious numerical diapycnal mixing problem
2. Vertical Lagrangian-remapping w/ generalized vertical coordinates
3. Closing comments

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Outline

1. The spurious numerical diapycnal mixing problem

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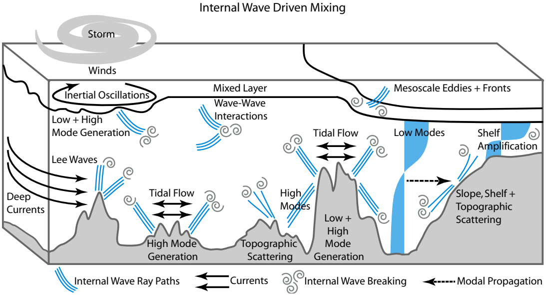
Physically based diapycnal mixing MacKinnon et al (2017)



- ⋆ There are many physical sources for ocean diapycnal mixing.
- ⋆ Diapycnal mixing impacts on vertical stratification, dynamics, tracer ventilation (heat, carbon), sea level, with effects more important as time increases (e.g., climate).
- ⋆ Coordinated efforts such as the US Climate Process Team on internal gravity wave mixing (2010-2015) and ongoing German TRR 181 on energetic transfers have enhanced integrity of physically based mixing parameterizations used by climate and prediction models.
- ⋆ Unfortunately, numerical transport (i.e., advection schemes) can introduce spurious diapycnal mixing that is larger than physics.

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Framing the spurious mixing problem

The numerical representation of advection $= \nabla \cdot (\rho\, C\, \boldsymbol{v})$ generally introduces spurious mixing and unmixing due to truncation errors

$$\nabla \cdot (\rho\, C\, \boldsymbol{v})_{\text{model}} = \nabla \cdot (\rho\, C\, \boldsymbol{v})_{\text{exact}} + \nabla \cdot (\rho\, C\, \boldsymbol{v})_{\text{error}} \tag{1}$$

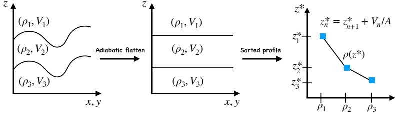
- ⋆ Errors in numerical advection can be interpreted as an extra SGS term

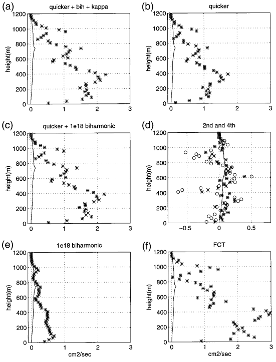
$$\frac{\partial(\rho\, C)}{\partial t} + \nabla \cdot (\rho\, C\, \boldsymbol{v})_{\text{exact}} = -\nabla \cdot \left[\boldsymbol{J} + (\rho\, C\, \boldsymbol{v})_{\text{error}}\right]. \tag{2}$$

- ⋆ Error term is not physical nor is it under our direct control. If large it can corrupt physical integrity of the simulation.
- ⋆ Error term can become larger when refine grid spacing to partially resolve mesoscale eddies, which pump tracer variance to the grid scale.
- ⋆ Spurious mixing from the error term is reduced (but not eliminated) when use higher order accurate advection.
- ⋆ Key concern for climate is spurious diapycnal mixing.
- ⋆ Spurious diapycnal mixing is reduced when use quasi-isopycnal vertical coordinate; errors stopped at layer interface.

Diagnosing spurious diapycnal mixing

Griffies, Pacanowski, Hallberg (2000)





- ⋆ A method based on density sorting to produce a stable background profile, ρ_{back}, following Winters and D'Asaro (1995).
- ⋆ In an adiabatic simulation, evolution of the background state only arises from spurious numerical sources, which we interpret as an effective diffusivity, κ_{eff}

$$\frac{\partial \rho_{\text{back}}}{\partial t} = \frac{\partial}{\partial z^*}\left[\kappa_{eff}\,\frac{\partial \rho_{\text{back}}}{\partial z^*}\right] \tag{3}$$

- ⋆ Diagnosed levels of κ_{eff} from numerical advection can be 10-100x larger than ocean measurements.
- ⋆ Problems can be enhanced in mesoscale eddying simulations where tracer variance is pumped to the gridscale.
- ⋆ Spurious mixing scales with lateral grid Reynolds number (see also Ilicak, Adcroft, Griffies, Hallberg, (2012)):

$$\text{Re}_{\text{grid}} = U\,\Delta t/\Delta < \mathcal{O}(10). \tag{4}$$

Larger Re_{grid} allows for noisy vertical velocity from noisy horizontal convergences: recipe for spurious mixing.

- ⋆ But $\text{Re}_{\text{grid}} > 10$ is very common, thus incurring spurious mixing.

Measures of spurious (diapycnal) mixing

- Sorting along with passive tracer releases
  - Hill, Ferreira, Campin, Marshall, Abernathey, Barrier, 2012: Controlling spurious diapycnal mixing in eddy-resolving height-coordinate ocean models-insights from virtual deliberate tracer release experiments
  - Getzlaff, Nurser, Oschlies, 2012: Diagnostics of diapycnal diffusion in z-level ocean models. Part II: 3-Dimensional OGCM
- BACKGROUND/REFERENCE POTENTIAL ENERGY: Global number (though Ilicak, 2016 suggests local)
  - Ilicak, Adcroft, Griffies, Hallberg, 2012: Spurious dianeutral mixing and the role of momentum closure
  - Petersen, Jacobsen, Ringler, Hecht, Maltrud, 2015: Evaluation of the arbitrary Lagrangian-Eulerian vertical coordinate method in the MPAS-Ocean model
  - Zhao and Liu, 2016: Spurious dianeutral mixing in a global ocean model using spherical centroidal voronoi tessellations
  - Ilicak, 2016: Quantifying spatial distribution of spurious mixing in ocean models
  - Gibson, Hogg, Kiss, Shakespeare, Adcroft, 2017: Attribution of horizontal and vertical contributions to spurious mixing in an Arbitrary Lagrangian-Eulerian ocean model
- VARIANCE METHODS: provides a map for all mixing (no distinction between diapycnal versus isopycnal).
  - Morales-Maqueda and Holloway, 2006: Second-order moment advection scheme applied to Arctic Ocean simulation
  - Burchard, Rennau, 2008: Comparative quantification of physically and numerically induced mixing in ocean models
  - Klingbeil, Mohammadi-Aragh, Grawe, Burchard, (2014): Quantification of spurious dissipation and mixing-discrete variance decay in a Finite-Volume framework
- WATERMASS ANALYSIS
  - Lee, Coward, Nurser 2002: Spurious Diapycnal Mixing of the Deep Waters in an Eddy-Permitting Global Ocean Model
  - Urakawa, Hasumi, 2014: Effect of numerical diffusion on the water mass transformation in eddy-resolving models
  - Megann, 2018: Estimating the numerical diapycnal mixing in an eddy-permitting ocean model
  - Holmes, Zika, England, 2019: Diathermal heat transport in a global ocean model

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Some general points emerging from the studies

- ⋆ Diagnosing the spurious mixing is useful for understanding its character but insufficient to remove the problem.
- ⋆ Higher order numerics helps [e.g., Hill et al (2012)], though realistic simulatons need flux limiters that add mixing.
- ⋆ Maintenance of modest grid Reynolds number [$\mathrm{Re}_{\mathrm{grid}} < \mathcal{O}(10)$] is key to suppress velocity noise that translates into spurious mixing [e.g., Ilicak, Adcroft, Griffies, Hallberg (2012)].
- ⋆ Problem arises from advection in both vertical and horizontal (since isopycnals slope) [e.g., Gibson et al (2017)].
- ⋆ Claims that the problem is solved by certain advection schemes [e.g., Hill et al (2012)] have ignored flux limiters, which add mixing and yet are needed to ensure positive definite tracer concentrations [e.g., Morales-Maqueda and Holloway (2006)].
- ⋆ Ilicak et al (2012) and Megann (2018) and Adcroft et al (2019) suggest that 1/4 degree $Z-$coordinate climate models are poorly situated:
  - Admitting mesoscale eddies w/ 1/4-degree generally requires $\mathrm{Re}_{\mathrm{grid}} > \mathcal{O}(10)$.
  - Suggestions (anecdotal) that 1/10-degree Z-model climate simulations have far smaller spurious mixing; perhaps the grid resolves enough of the variance cascade that its dissipation does not require excessive mixing.

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Outline

2. Vertical Lagrangian-remapping w/ generalized vertical coordinates

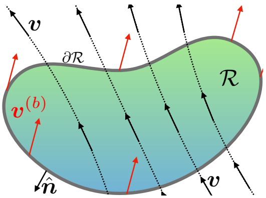
Focusing on the vertical solution method

- ⋆ Isopycnal models have direct control over diapycnal transport.
- ⋆ But isopycnal models are have problems for climate modeling (weakly stratified high latitudes) and coastal (strong vertical mixing).
- ⋆ Hybrid vertical coordinates is a strategy to reduce spurious mixing while allowing for global coverage.
- ⋆ Vertical Arbitrary-Lagrangian-Eulerian (ALE) method, including the special case of vertical Lagangian-remapping, is a strategy to realize hybrid vertical coordinates.
- ⋆ Proposed terminology:
  - Quasi-Eulerian: restricted set of generalized vertical coordinates, where coordinate is directly related to free surface (Boussinesq) or bottom pressure (non-Boussinesq).
  - Vertical Arbitrary-Lagrangian-Eulerian (ALE) without remapping: as in NEMO-$\tilde{z}$ and MPAS-O.
  - Vertical ALE with remapping, aka Vertical Lagrangian Remapping: as in HYCOM and MOM6.
- ⋆ Here we present some of the fundamentals, aiming to develop intuition based on fluid mechanics rather than numerical algorithm details.

Finite volume (weak formulation): Leibniz-Reynolds

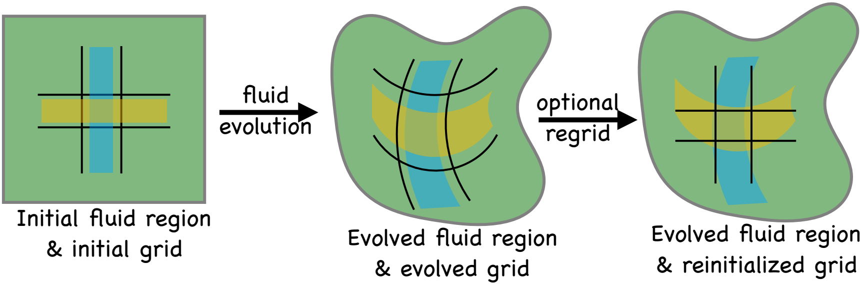
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{R}} \rho\, C \,\mathrm{d}V\right] = -\oint_{\partial\mathcal{R}} \left[\rho\, C\,(\boldsymbol{v} - \boldsymbol{v}^{(b)}) + \boldsymbol{J}\right] \cdot \hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{R}} \rho\,\boldsymbol{v}\,\mathrm{d}V\right] = -\int_{\mathcal{R}} [2\,\boldsymbol{\Omega} \wedge \rho\,\boldsymbol{v} + \rho\,\nabla\Phi]\,\mathrm{d}V + \oint_{\partial\mathcal{R}} [-p\,\mathbb{I} - \rho\,\boldsymbol{v} \otimes (\boldsymbol{v} - \boldsymbol{v}^{(b)}) + \mathbb{T}] \cdot \hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S}$$



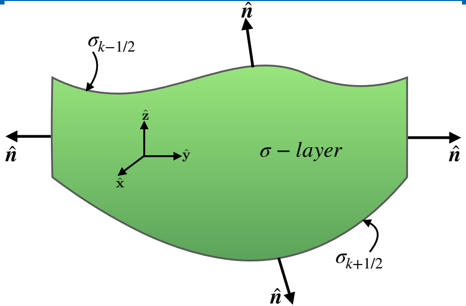
- ⋆ Discrete equations follow by specializing $\mathcal{R}$ to a model grid cell.
- ⋆ Models typically formulate scalar prognostic budgets for extensive quantities: heat, salt, and thus diagnose intensive quantities as in $C = (C\,\rho\,\Delta V)/(\rho\,\Delta V)$.
- ⋆ However, models often formulate a discrete velocity equation, $\partial_t \boldsymbol{v}$, rather than a discrete momentum equation, $\partial_t(\boldsymbol{v}\,\rho\,\Delta V)$, since there are advantages to the vector-invariant velocity equation (e.g., Sadourny energy-enstrophy) over advective-form momentum equation.
- ⋆ Advection refers to the transport of fluid relative to the grid. Fully Lagrangian has zero advection. Fully Eulerian has all motion leading to advection.

The Arbitrary Lagrangian-Eulerian method (ALE)



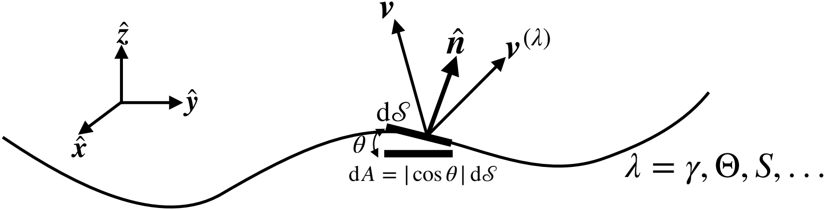
- ALE broadly refers to any method that considers moving cell boundaries.
  - STEP ONE: If grid moves with the flow it is a Lagrangian step. Non-Lagrangian grid motion is also considered by certain ALE approaches.
  - STEP TWO: The regrid/remap step ideally does not alter the ocean state. Rather, it moves the grid ("regrid") and estimates the ocean state on the new grid ("remap").
- Remap step operationally equals to advection (transport relative to grid).
- Ocean models restrict their moving meshes to be just in the vertical and formulate equations using generalized vertical coordinates.

Distinguishing solution methods according to $\boldsymbol{v}^{(b)}$



- ⋆ LATERAL BOUNDARIES: $\boldsymbol{v}^{(b)} \cdot \hat{\boldsymbol{n}}_{\text{sides}} = 0$ (no lateral cell movement).
- ⋆ RIGID-LID Z-MODELS: $\boldsymbol{v}^{(b)} \cdot \hat{\boldsymbol{n}} = 0$ for all boundaries.
- ⋆ FREE SURFACE MODELS: $\boldsymbol{v}^{(b)} \cdot \hat{\boldsymbol{n}}_{k=1/2} \neq 0$.
- ⋆ ANALYTICALLY SPECIFIED COORDINATES: barotropic motion specifies $\boldsymbol{v}^{(b)} \cdot \hat{\boldsymbol{n}}_{k\pm 1/2} \neq 0$ for $\sigma_{\text{terrain}} = (z - \eta)/(H + \eta), z^* = H\,\sigma_{\text{terrain}}$, others.
- ⋆ ISOPYCNAL LAYERS: $\boldsymbol{v}^{(b)} \cdot \hat{\boldsymbol{n}}_{k\pm 1/2} \neq 0$ determined by following layer interfaces.
- ⋆ MORE GENERAL ALE: $\boldsymbol{v}^{(b)} \cdot \hat{\boldsymbol{n}}_{k\pm 1/2} \neq 0$ is arbitrary.

Generalized vertical coordinates & dia-surface transport (Section 6.7 of Griffies 2004)



Dia-surface velocity component, u^{dia}, is defined by seawater transport through moving λ surface

$$\mathcal{T} \equiv u^{\text{dia}}\,\mathrm{d}\mathcal{S} \equiv \hat{\boldsymbol{n}} \cdot (\boldsymbol{v} - \boldsymbol{v}^{(\lambda)})\,\mathrm{d}\mathcal{S} \Longrightarrow \text{can have non-zero transport even if } \hat{\boldsymbol{n}} \cdot \boldsymbol{v} = 0. \tag{5}$$

$$\hat{\boldsymbol{n}} = \nabla\lambda\,|\nabla\lambda|^{-1} = \text{ normal direction pointing to larger } \lambda. \tag{6}$$

$$\boldsymbol{v} = \text{(barycentric) velocity of fluid element and } \left(\partial_t + \boldsymbol{v}^{(\lambda)} \cdot \nabla\right)\lambda = 0. \tag{7}$$

Following from these definitions we have

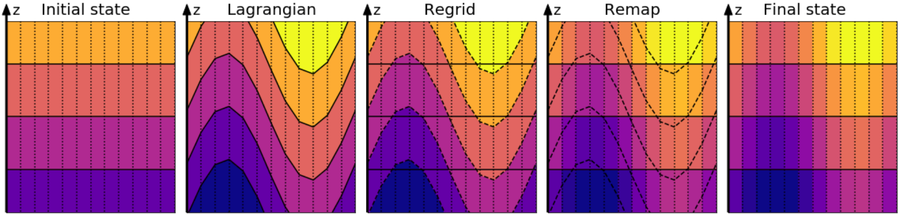
$$\frac{\mathrm{D}\lambda}{\mathrm{D}t} = (\partial_t + \boldsymbol{v} \cdot \nabla)\,\lambda = [\partial_t + \boldsymbol{v}^{(\lambda)} \cdot \nabla + (\boldsymbol{v} - \boldsymbol{v}^{(\lambda)}) \cdot \nabla]\,\lambda = 0 + u^{\text{dia}}\,|\nabla\lambda| \tag{8}$$

$$\Longrightarrow |\nabla\lambda|\,u^{\text{dia}} = \frac{\mathrm{D}\lambda}{\mathrm{D}t} = \dot{\lambda} \Longrightarrow \text{material changes in } \lambda \Longleftrightarrow \text{dia-surface transport.} \tag{9}$$

For stably stratified λ-surfaces as in generalized vertical coordinates with $\lambda = \sigma$, we define

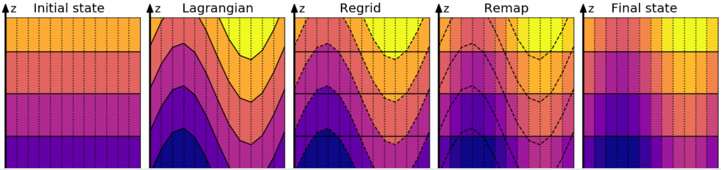
$$\mathcal{T} \equiv u^{\text{dia}}\mathrm{d}\mathcal{S} \equiv w^{(\dot{\sigma})}\mathrm{d}A = \frac{\partial z}{\partial \sigma}\,\dot{\sigma} \Longrightarrow \frac{\mathrm{D}}{\mathrm{D}t} = \left[\frac{\partial}{\partial t}\right]_{\sigma} + \boldsymbol{u} \cdot \nabla_{\sigma} + w^{(\dot{\sigma})}\,\frac{\partial}{\partial z} \tag{10}$$

Vertical Lagrangian-Remapping method



- ⋆ A flavor of ALE where grid cell sides are rigid but top and bottom move.
- ⋆ HYCOM and MOM6 implement the regrid/remap step, constituting the *vertical Lagrangian remap method*.
- ⋆ HYCOM and MOM6 implement the method so that grid layers can vanish and inflate (useful for estuaries and moving ice-shelf grounding lines).
- ⋆ MPAS-O and NEMO algorithms do not implement the vertical regrid/remap step. They are ALE but not Lagrangian.

Comments on vertical Lagrangian-remapping



- WHERE IS DIA-SURFACE ADVECTION? It is part of the evolution of the grid cell thicknesses. Cell interfaces move and carry the state.
  - Z-COORDINATE EXAMPLE: Define h^* according to fixed z-levels. Remapping moves the state onto the fixed $z-$grid, a step that is the operationally same as vertical advection.
- To diagnose the full advection operator, we need to diagnose the contribution from remapping so that

$$\nabla \cdot (\rho\, C\, \boldsymbol{v}) = \underbrace{\nabla_\sigma \cdot (\rho\, C\, \boldsymbol{u})}_{\text{horizontal layer advection}} + \text{ remapping.} \tag{11}$$

- There is no CFL associated with vertical remapping; useful for fine vertical grid spacing. But remember stability does not imply accuracy.
- The vertical remapping algorithm can be used for diagnostic purposes to remap and bin grid cell tendencies according to arbitrary surfaces.

Distinguishing three solution algorithms

- ⋆ We consider three solution algorithms used by ocean models:
  - quasi-Eulerian (e.g., MITgcm, MOM5, NEMO)
  - vertical ALE without remapping (e.g., NEMO-$\tilde{z}$, MPAS-O)
  - vertical Lagrangian-remapping (e.g., HYCOM, MOM6)
- ⋆ To exemplify the rudiments of the algorithms, consider the following bare-bones suite of model equations.

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{A} - w^{(\dot{\sigma})} \, \frac{\partial \boldsymbol{u}}{\partial z} \qquad \text{velocity} \qquad (12a)$$

$$\frac{\partial h}{\partial t} = -\nabla_{\sigma} \cdot (h \, \boldsymbol{u}) - \Delta_{\sigma} w^{(\dot{\sigma})} \qquad \text{thickness} \qquad (12b)$$

$$\frac{\partial (h \, C)}{\partial t} = -\nabla_{\sigma} \cdot [h \, \boldsymbol{u} \, C] - \Delta_{\sigma} [C \, w^{(\dot{\sigma})}] \qquad \text{thickness weighted tracer,} \qquad (12c)$$

where $\boldsymbol{A}$ encompasses accelerations other than dia-surface advection.

Quasi-Eulerian algorithm

- Vertical coordinate analytically determined by barotropic motion that then determines thickness tendency as in

$$z^* = \frac{z - \eta}{H + \eta} \Longrightarrow \mathrm{d}z = (1 + \eta/H)\,\mathrm{d}z^* \Longrightarrow \partial_t(\mathrm{d}z) = \frac{\mathrm{d}z^*}{H}\,\partial_t\eta. \qquad (13)$$

- Example codes: MITgcm, MOM5, NEMO-basic, ROMS
- There is no realization of this algorithm with vanishing layers that maintains machine precision conservation.

ALGORITHM STEPS

1. $[\Delta_\sigma w^{\text{grid}}]^{(n)} = [\partial h/\partial t]^{(n)} \propto [\partial\eta/\partial t]^{(n)}$ — grid motion $\propto$ free surface
2. $h^{(n+1)} = h^{(n)} + \Delta t\,[\Delta_\sigma w^{\text{grid}}]^{(n)}$ — update layer thickness
3. $[\Delta_\sigma w^{(\dot\sigma)}]^{(n)} = -[\Delta_\sigma w^{\text{grid}}]^{(n)} - \nabla_\sigma \cdot [h\,\boldsymbol{u}]^{(n)}$ — diagnose dia-surface velocity
4. $\boldsymbol{u}^{(n+1)} = \boldsymbol{u}^{(n)} + \Delta t\left[\boldsymbol{A} - w^{(\dot\sigma)}\,\partial\boldsymbol{u}/\partial z\right]^{(n)}$ — update horizontal velocity
5. $h^{(n+1)}\,C^\dagger = [h\,C]^{(n)} - \Delta t\,[\nabla_\sigma \cdot (h\,\boldsymbol{u}\,C)]^{(n)}$ — incremental tracer step I
6. $h^{(n+1)}\,C^{n+1} = h^{(n+1)}\,C^\dagger - \Delta t\,\Delta_\sigma\left[C^\dagger\,(w^{(\dot\sigma)})^{(n)}\right]$ — incremental tracer step II

Algorithm for vertical ALE without remapping

- General thickness tendency with general vertical coordinates.
- Vertical coordinate need not be analytically defined.
- Example codes: MPAS-O and NEMO-$\tilde{z}$
- There is no realization of this algorithm with vanishing layers that maintains machine precision conservation.

ALGORITHM STEPS (ONLY STEP 1 DIFFERS FROM QUASI-EULERIAN)

1. $[\Delta_\sigma w^{\text{grid}}]^{(n)} = (h^{\text{target}} - h^{(n)})/\Delta t$ — general layer motion
2. $h^{(n+1)} = h^{(n)} + \Delta t\, [\Delta_\sigma w^{\text{grid}}]^{(n)}$ — update layer thickness
3. $[\Delta_\sigma w^{(\dot\sigma)}]^{(n)} = -[\Delta_\sigma w^{\text{grid}}]^{(n)} - \nabla_\sigma \cdot [h\, \boldsymbol{u}]^{(n)}$ — diagnose dia-surface velocity
4. $\boldsymbol{u}^{(n+1)} = \boldsymbol{u}^{(n)} + \Delta t \left[\boldsymbol{A} - w^{(\dot\sigma)}\, \partial \boldsymbol{u}/\partial z\right]^{(n)}$ — update horizontal velocity
5. $h^{(n+1)}\, C^{\dagger} = [h\, C]^{(n)} - \Delta t\, [\nabla_\sigma \cdot (h\, \boldsymbol{u}\, C)]^{(n)}$ — incremental tracer step I
6. $h^{(n+1)}\, C^{n+1} = h^{(n+1)}\, C^{\dagger} - \Delta t\, \Delta_\sigma \left[C^{\dagger}\, (w^{(\dot\sigma)})^{(n)}\right]$ — incremental tracer step II

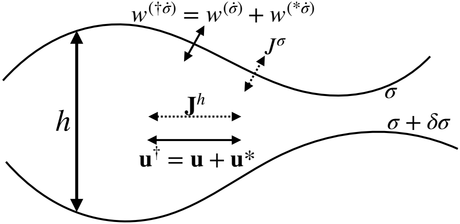
Algorithm for Vertical Lagrangian-remapping

- Pioneered by Bleck (2002).
- General thickness tendency with general vertical coordinates.
- Vertical coordinate need not be analytically defined.
- Lagrangian step includes horizontal dynamics/physics + vertical physics
- Example codes: HYCOM and MOM6
- MOM6 allows for vanishing layers with machine precision conservation.

ALGORITHM STEPS

1. $[\Delta_\sigma w^{\text{grid}}]^{(n)} = -\nabla_\sigma \cdot [h\,\boldsymbol{u}]^{(n)}$ layer motion as per horizontal convergence
2. $h^\dagger = h^{(n)} + \Delta t\,[\Delta_\sigma w^{\text{grid}}]^{(n)}$ Lagrangian thickness
3. $[\Delta_\sigma w^{(\dot\sigma)}]^{(n)} = 0$ zero dia-surface velocity
4. $\boldsymbol{u}^\dagger = \boldsymbol{u}^{(n)} + \Delta t\,\boldsymbol{A}^{(n)}$ Lagrangian velocity
5. $h^\dagger\,C^\dagger = h^{(n)}\,C^{(n)} - \Delta t\,[\nabla_\sigma \cdot (h\,C\,\boldsymbol{u})]^{(n)}$ Lagrangian tracer
6. $h^{(n+1)} = h^{\text{target}}$ regrid to the target
7. $\Delta_\sigma w^{(\dot\sigma)} = -(h^{\text{target}} - h^\dagger)/\Delta t$ dia-surface velocity
8. $\boldsymbol{u}^{(n+1)} = \boldsymbol{u}^\dagger + \Delta t\,w^{(\dot\sigma)}\,(\partial \boldsymbol{u}^\dagger/\partial z)$ remap velocity
9. $h^{(n+1)}\,C^{(n+1)} = h^\dagger\,C^\dagger - \Delta t\,\Delta_\sigma(w^{(\dot\sigma)}\,C^\dagger)$ remap tracer.

Scalar equations for algorithms without remapping



- ⋆ Resolved vertical velocity, $w^{(\dot\sigma)}$, and parameterized vertical velocity (e.g., eddy-induced velocity), $w^{(*\dot\sigma)}$, are diagnosed via continuity equations.
- ⋆ Both $w^{(\dot\sigma)}$ & $w^{(*\dot\sigma)}$ penetrate layer interfaces as per traditional advection.

$$\frac{\partial(h\,\rho)}{\partial t} + \nabla_\sigma \cdot (h\,\rho\,\boldsymbol{u}^{\dagger}) + \delta_\sigma(\rho\,w^{(\dagger\dot\sigma)}) = 0 \;\text{ and }\; \nabla_\sigma \cdot (h\,\rho\,\boldsymbol{u}^{*}) + \delta_\sigma(\rho\,w^{(*\dot\sigma)}) = 0 \qquad (14)$$

$$\frac{\partial(h\,\rho\,C)}{\partial t} + \nabla_\sigma \cdot (h\,\rho\,C\,\boldsymbol{u}^{\dagger}) + \delta_\sigma(\rho\,C\,w^{(\dagger\dot\sigma)}) = -\left[\nabla_\sigma \cdot (h\,\boldsymbol{J}^{h}) + \delta_\sigma J^{\sigma}\right] \qquad (15)$$

$$z_\sigma = \frac{\partial z}{\partial \sigma} \qquad h = z_\sigma\,\mathrm{d}\sigma \qquad w^{(\dagger\dot\sigma)} = \frac{\partial z}{\partial \sigma}\,\frac{\mathrm{D}^{\dagger}\sigma}{\mathrm{D}t} = w^{(\dot\sigma)} + \boldsymbol{v}^{*} \cdot z_\sigma \nabla\sigma = w^{(\dot\sigma)} + w^{(*\dot\sigma)} \qquad (16)$$

$$\delta_\sigma = \mathrm{d}\sigma\,\frac{\partial}{\partial \sigma} \qquad \nabla_\sigma = \nabla_z + \boldsymbol{S}\,\partial_z \qquad \boldsymbol{S} = \nabla_\sigma z = -z_\sigma \nabla_z \sigma \qquad J^{\sigma} = z_\sigma \nabla\sigma \cdot \boldsymbol{J} \qquad (17)$$

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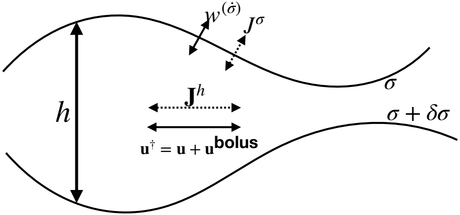
Scalar equations with vertical Lagrangian remapping



- Vertical advection from $w^{(\dot\sigma)}$ is handled during the remap step.
- Vertical parameterized advection from $w^{(*\dot\sigma)}$ is handled during the Lagrangian step via the horizontal convergence $-\nabla_\sigma \cdot [h\,\boldsymbol{u}^{\dagger}]$.
- Use of $\boldsymbol{u}^{\text{bolus}}$ ensures that horizontal advective transport retains constant layer integrated mass just as in an adiabatic isopycnal layer.

$$\frac{\partial(h\,\rho)}{\partial t} + \nabla_\sigma \cdot (h\,\rho\,\boldsymbol{u}^{\dagger}) + \delta_\sigma(\rho\,w^{(\dot\sigma)}) = 0 \qquad (18)$$

$$\frac{\partial(h\,\rho\,C)}{\partial t} + \nabla_\sigma \cdot (h\,\rho\,C\,\boldsymbol{u}^{\dagger}) + \delta_\sigma(\rho\,C\,w^{(\dot\sigma)}) = -\left[\nabla_\sigma \cdot (h\,\boldsymbol{J}^{h}) + \delta_\sigma J^{\sigma}\right] \qquad (19)$$

$$w^{\sigma} = \frac{\partial z}{\partial \sigma}\frac{\mathrm{D}\sigma}{\mathrm{D}t} \qquad \boldsymbol{u}^{\dagger} = \boldsymbol{u} + \boldsymbol{u}^{\text{bolus}} = \text{horizontal residual mean velocity.} \qquad (20)$$

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3 Closing comments

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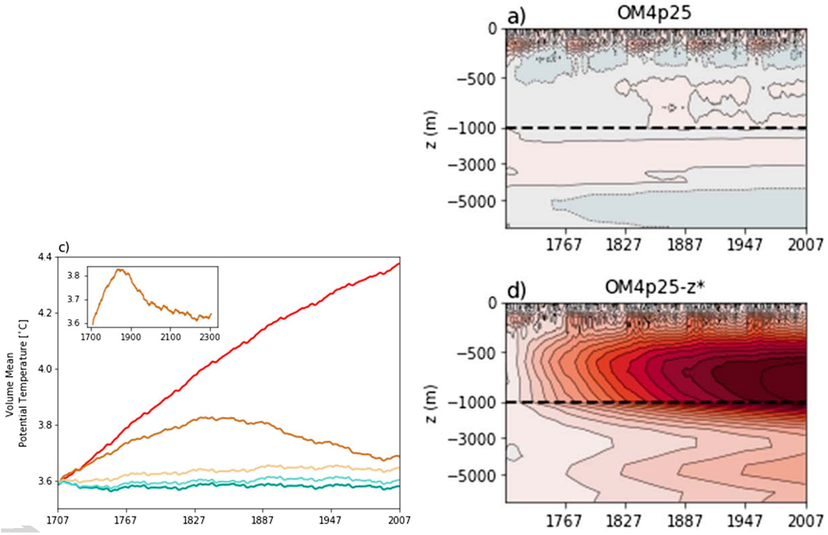
Choice of vertical coordinate matters for heat uptake!



- ⋆ MOM6/SIS2 at $0.25^\circ \times 75$-layers forced by interannual CORE.
- ⋆ z^* dominated by spurious mixing relative to hybrid isopycnal-z^*.

Summary points

- ⋆ We understand a great deal about ocean mixing and how to parameterize it. However, spurious numerical diapycnal mixing remains a nontrivial problem with many simulations that can corrupt their physical fidelity.
- ⋆ There are a handfull of methods for diagnosing spurious mixing. They all point to the need for improved numerical accuracy and maintenance of modest [$\mathrm{Re}_{\mathrm{grid}} < \mathcal{O}(10)$] grid Reynolds number.
- ⋆ Vertical Lagrangian-remapping offers a framework for incorporating hybrid/generalized vertical coordinates.
- ⋆ The design of hybrid coordinates should be targeted at minimizing spurious diapycnal mixing while allowing for an accurate representation of the ocean's multiple regimes of flow.
- ⋆ More work is needed to improve the choice for vertical coordinate, with no optimal coordinate having been found that satisfies all needs (sometimes subjective needs) for climate and coastal applications.

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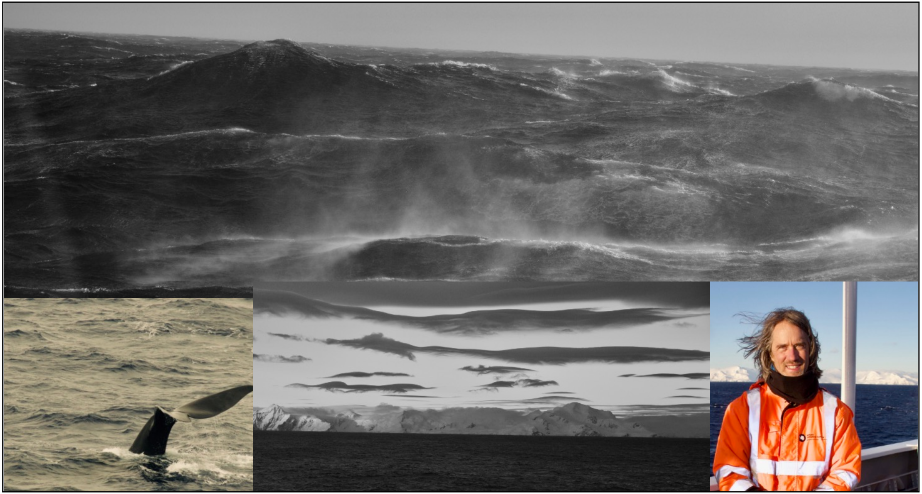
Many thanks for your time and attention



From the Weddell Sea and Scotia Sea, autumn 2017 on the RRS James Clark Ross